Cryptography and Network Security

FINITE FIELDS AND NUMBER THEORY

Groups, Rings, and Fields

Session Meta Data

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Revision History

Revision Date	Details	Version no.
		1.0

Agenda

- Introduction
- Divisors
  - Properties
  - Division algorithm
  - GCD
- Modular arithmetic
  - Operations
  - Example
  - Properties
- Euclidean algorith
  - Finding inverses
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Introduction

- will now introduce finite fields
- of increasing importance in cryptography
  - AES, Elliptic Curve, IDEA, Public Key
- concern operations on “numbers”
  - where what constitutes a “number” and the type of operations varies considerably
- start with basic number theory concepts

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Divisors

- say a non-zero number b **divides** a if for some m have a=mb (a,b,m all integers)
- that is b divides into a with no remainder
- denote this b|a
- and say that b is a **divisor** of a
- eg. all of 1,2,3,4,6,8,12,24 divide 24
- eg. 13 | 182; –5 | 30; 17 | 289; –3 | 33; 17 | 0

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Properties of Divisibility

- If $a|1$, then $a = \pm 1$.
- If $a|b$ and $b|a$, then $a = \pm b$.
- Any b /= 0 divides 0.
- If $a \mid b$ and $b \mid c$, then $a \mid c$
  - *e.g.* 11 | 66 and 66 | 198 so 11 | 198
- If $b|g$ and $b|h$, then $b|(mg + nh)$

  for arbitrary integers m and n

  e.g. $b = 7; g = 14; h = 63; m = 3; n = 2$

  $7|14$ and $7|63$ hence $7 \mid 42+126 = 168$

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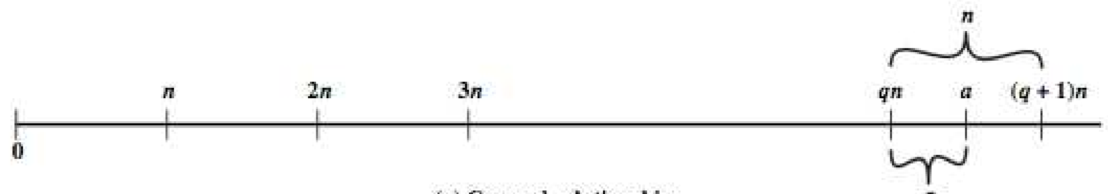
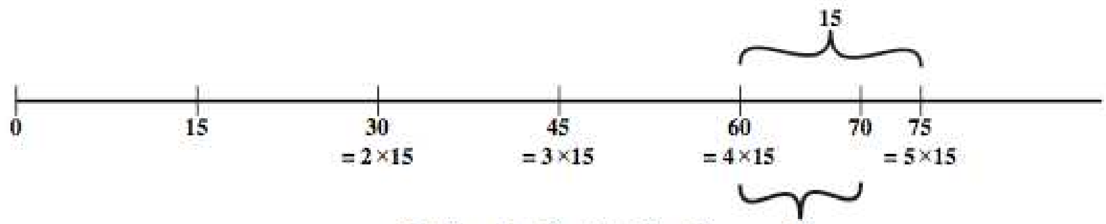
Division Algorithm

- if divide a by n get integer quotient *q* and integer remainder *r* such that:
  - $a = qn + r$ where $0 <= r < n; q = floor(a/n)$
- remainder *r* often referred to as a residue



(a) General relationship



(b) Example: 70 = (4×15) + 10

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Greatest Common Divisor (GCD)

- a common problem in number theory
- GCD (a,b) of a and b is the largest integer that divides evenly into both a and b
  - eg GCD(60,24) = 12
- define gcd(0, 0) = 0
- often want **no common factors** (except 1) define such numbers as **relatively prime**
  - eg GCD(8,15) = 1
  - hence 8 & 15 are relatively prime

Example GCD(1970,1066)

1970 = 1 x 1066 + 904	gcd(1066, 904)
1066 = 1 x 904 + 162	gcd(904, 162)
904 = 5 x 162 + 94	gcd(162, 94)
162 = 1 x 94 + 68	gcd(94, 68)
94 = 1 x 68 + 26	gcd(68, 26)
68 = 2 x 26 + 16	gcd(26, 16)
26 = 1 x 16 + 10	gcd(16, 10)
16 = 1 x 10 + 6	gcd(10, 6)
10 = 1 x 6 + 4	gcd(6, 4)
6 = 1 x 4 + 2	gcd(4, 2)
4 = 2 x 2 + 0	gcd(2, 0)

GCD(1160718174, 316258250)

Dividend	Divisor	Quotient	Remainder
a = 1160718174	b = 316258250	q1 = 3	r1 = 211943424
b = 316258250	r1 = 211943424	q2 = 1	r2 = 104314826
r1 = 211943424	r2 = 104314826	q3 = 2	r3 = 3313772
r2 = 104314826	r3 = 3313772	q4 = 31	r4 = 1587894
r3 = 3313772	r4 = 1587894	q5 = 2	r5 = 137984
r4 = 1587894	r5 = 137984	q6 = 11	r6 = 70070
r5 = 137984	r6 = 70070	q7 = 1	r7 = 67914
r6 = 70070	r7 = 67914	q8 = 1	r8 = 2156
r7 = 67914	r8 = 2156	q9 = 31	r9 = 1078
r8 = 2156	r9 = 1078	q10 = 2	r10 = 0

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Modular Arithmetic

- define **modulo operator** "a mod n" to be remainder when a is divided by n
  - where integer *n* is called the **modulus**
- *b* is called a **residue** of *a* mod *n*
  - since with integers can always write: a = qn + b
  - usually chose smallest positive remainder as residue
    - ie. 0 <= b <= n-1
  - process is known as **modulo reduction**
    - eg. -12 mod 7 = -5 mod 7 = 2 mod 7 = 9 mod 7
- *a* & *b* are **congruent** if: a mod n = b mod n
  - when divided by *n*, a & b have same remainder
  - eg. 100 mod 11 = 34 mod 11

    so 100 is congruent to 34 mod 11

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Modular Arithmetic Operations

- can perform arithmetic with residues
- uses a finite number of values, and loops back from either end

$Z_n = \{0, 1, . . . , (n - 1)\}$

- modular arithmetic is when do addition & multiplication and modulo reduce answer
- can do reduction at any point, ie
  - a+b mod n = [a mod n + b mod n] mod n

Modular Arithmetic Operations

1. [(a mod n) + (b mod n)] mod n = (a + b) mod n
2. [(a mod n) – (b mod n)] mod n = (a – b) mod n
3. [(a mod n) x (b mod n)] mod n = (a x b) mod n

e.g.

[(11 mod 8) + (15 mod 8)] mod 8 = 10 mod 8 = 2 (11 + 15) mod 8 = 26 mod 8 = 2

[(11 mod 8) – (15 mod 8)] mod 8 = –4 mod 8 = 4 (11 – 15) mod 8 = –4 mod 8 = 4

[(11 mod 8) x (15 mod 8)] mod 8 = 21 mod 8 = 5 (11 x 15) mod 8 = 165 mod 8 = 5

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Modulo 8 Addition Example

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

Modulo 8 Multiplication

+	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

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Modular Arithmetic Properties

Property	Expression
Commutative laws	$(w + x) \bmod n = (x + w) \bmod n$ $(w \times x) \bmod n = (x \times w) \bmod n$
Associative laws	$[(w + x) + y] \bmod n = [w + (x + y)] \bmod n$ $[(w \times x) \times y] \bmod n = [w \times (x \times y)] \bmod n$
Distributive law	$[w \times (x + y)] \bmod n = [(w \times x) + (w \times y)] \bmod n$
Identities	$(0 + w) \bmod n = w \bmod n$ $(1 \times w) \bmod n = w \bmod n$
Additive inverse $(-w)$	For each $w \in Z_n$, there exists a z such that $w + z \equiv 0 \bmod n$

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Euclidean Algorithm

- an efficient way to find the GCD(a,b)
- uses theorem that:
  - GCD(a,b) = GCD(b, a mod b)
- Euclidean Algorithm to compute GCD(a,b) is:

```
Euclid(a,b)
    if (b=0) then return a;
    else return Euclid(b, a mod b);
```

Extended Euclidean Algorithm

- calculates not only GCD but x & y:

  $ax + by = d = \gcd(a, b)$

- useful for later crypto computations
- follow sequence of divisions for GCD but assume at each step i, can find x &y:

$r = ax + by$

- at end find GCD value and also x & y
- if GCD(a,b)=1 these values are inverses

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Finding Inverses

EXTENDED EUCLID(m, b)

1. (A1, A2, A3)=(1, 0, m);

(B1, B2, B3)=(0, 1, b)

2. if B3 = 0

return A3 = gcd(m, b); no inverse

3. if B3 = 1

return B3 = gcd(m, b); B2 = b^{-1} mod m

4. Q = A3 div B3

5. (T1, T2, T3)=(A1 – Q B1, A2 – Q B2, A3 – Q B3)

6. (A1, A2, A3)=(B1, B2, B3)

7. (B1, B2, B3)=(T1, T2, T3)

8. goto 2

Inverse of 550 in GF(1759)

Q	A1	A2	A3	B1	B2	B3
—	1	0	1759	0	1	550
3	0	1	550	1	–3	109
5	1	–3	109	–5	16	5
21	–5	16	5	106	–339	4
1	106	–339	4	–111	355	1

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Group

- a set S of elements or “numbers”
  - may be finite or infinite
- with some operation ‘.’ so G=(S,.)
- Obeys CAIN:
  - Closure: a,b in S, then a.b in S
  - Associative law: (a.b).c = a.(b.c)
  - has Identity e: e.a = a.e = a
  - has iNverses a^{-1}: $a.a^{-1} = e$
- if commutative a.b = b.a
  - then forms an **abelian group**

Cyclic Group

- define **exponentiation** as repeated application of operator
  - example: $a^3 = a.a.a$
- and let identity be: $e=a^0$
- a group is cyclic if every element is a power of some fixed element a
  - i.e., $b = a^k$ for some a and every b in group
- a is said to be a generator of the group

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Ring

- a set of "numbers"
- with two operations (addition and multiplication) which form:
- an abelian group with addition operation
- and multiplication:
  - has closure
  - is associative
  - distributive over addition: $a(b+c) = ab + ac$
- if multiplication operation is commutative, it forms a **commutative ring**
- if multiplication operation has an identity and no zero divisors, it forms an **integral domain**

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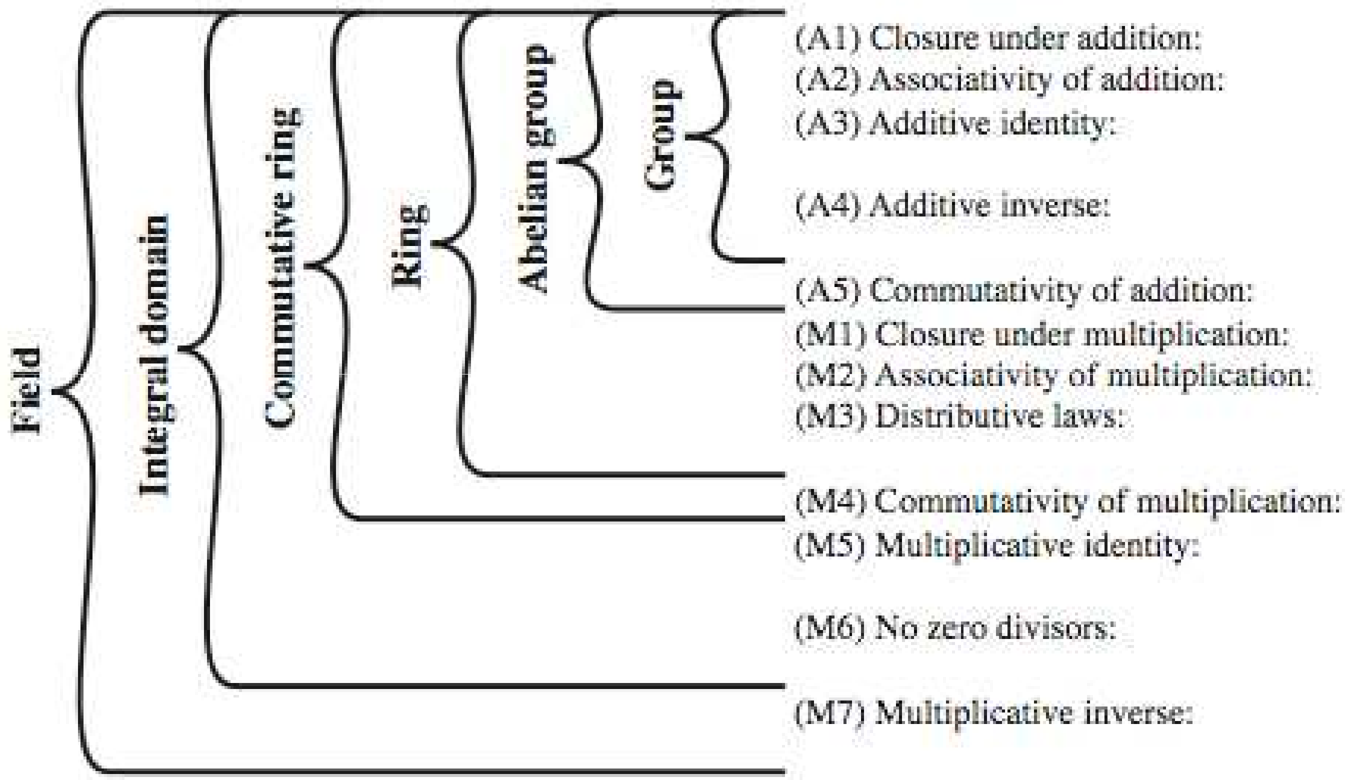
Field

- a set of numbers
- with two operations which form:
  - abelian group for addition
  - abelian group for multiplication (ignoring 0)
  - ring
- have hierarchy with more axioms/laws
  - group -> ring -> field

Group, Ring, Field



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Finite (Galois) Fields

- finite fields play a key role in cryptography
- can show number of elements in a finite field **must** be a power of a prime p^n
- known as Galois fields
- denoted $GF(p^n)$
- in particular often use the fields:
  - $GF(p)$
  - $GF(2^n)$

Galois Fields GF(p)

- GF(p) is the set of integers {0,1, … , p-1} with arithmetic operations modulo prime p
- these form a finite field
  - since have multiplicative inverses
  - find inverse with Extended Euclidean algorithm
- hence arithmetic is "well-behaved" and can do addition, subtraction, multiplication, and division without leaving the field GF(p)

GF(7) Multiplication Example

$\times$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

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Polynomial Arithmetic

- can compute using polynomials

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = \sum a_i x^i$

  - n.b. not interested in any specific value of x
  - which is known as the indeterminate

- several alternatives available
  - ordinary polynomial arithmetic
  - poly arithmetic with coefs mod p
  - poly arithmetic with coefs mod p and polynomials mod m(x)

Ordinary Polynomial Arithmetic

- add or subtract corresponding coefficients
- multiply all terms by each other
- eg

let $f(x) = x^3 + x^2 + 2$ and $g(x) = x^2 - x + 1$

$f(x) + g(x) = x^3 + 2x^2 - x + 3$

$f(x) - g(x) = x^3 + x + 1$

$f(x) \text{ x } g(x) = x^5 + 3x^2 - 2x + 2$

Polynomial Arithmetic with Modulo Coefficients

- when computing value of each coefficient do calculation modulo some value
  - forms a polynomial ring
- could be modulo any prime
- but we are most interested in mod 2
  - ie all coefficients are 0 or 1
  - eg. let $f(x) = x^3 + x^2$ and $g(x) = x^2 + x + 1$

$$f(x) + g(x) = x^3 + x + 1$$

$$f(x) \text{ x } g(x) = x^5 + x^2$$

Polynomial Division

- can write any polynomial in the form:
  - $f(x) = q(x)\, g(x) + r(x)$
  - can interpret $r(x)$ as being a remainder
  - $r(x) = f(x) \bmod g(x)$
- if have no remainder say $g(x)$ divides $f(x)$
- if $g(x)$ has no divisors other than itself & 1 say it is **irreducible** (or prime) polynomial
- arithmetic modulo an irreducible polynomial forms a field

Polynomial GCD

- can find greatest common divisor for polys
  - $c(x) = \text{GCD}(a(x), b(x))$ if $c(x)$ is the poly of greatest degree which divides both $a(x), b(x)$
- can adapt Euclid's Algorithm to find it:

Euclid($a(x)$, $b(x)$)

if ($b(x)$=0) then return $a(x)$;

else return

Euclid($b(x)$, $a(x)$ mod $b(x)$);

- all foundation for polynomial fields as see next

Modular Polynomial Arithmetic

- can compute in field $GF(2^n)$
  - polynomials with coefficients modulo 2
  - whose degree is less than n
  - hence must reduce modulo an irreducible poly of degree n (for multiplication only)
- form a finite field
- can always find an inverse
  - can extend Euclid's Inverse algorithm to find

Example GF(23)

Table 4.7 Polynomial Arithmetic Modulo ($x^3 + x + 1$)

(a) Addition

	+	000 0	001 1	010 x	011 $x+1$	100 x^2	101 x^2+1	110 x^2+x	111 x^2+x+1
000	0	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
001	1	1	0	$x+1$	x	x^2+1	x^2	x^2+x+1	x^2+x
010	x	x	$x+1$	0	1	x^2+x	x^2+x+1	x^2	x^2+1
011	$x+1$	$x+1$	x	1	0	x^2+x+1	x^2+x	x^2+1	x^2
100	x^2	x^2	x^2+1	x^2+x	x^2+x+1	0	1	x	$x+1$
101	x^2+1	x^2+1	x^2	x^2+x+1	x^2+x	1	0	$x+1$	x
110	x^2+x	x^2+x	x^2+x+1	x^2	x^2+1	x	$x+1$	0	1
111	x^2+x+1	x^2+x+1	x^2+x	x^2+1	x^2	$x+1$	x	1	0

(b) Multiplication

	×	000 0	001 1	010 x	011 $x+1$	100 x^2	101 x^2+1	110 x^2+x	111 x^2+x+1
000	0	0	0	0	0	0	0	0	0
001	1	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
010	x	0	x	x^2	x^2+x	$x+1$	1	x^2+x+1	x^2+1
011	$x+1$	0	$x+1$	x^2+x	x^2+1	x^2+x+1	x^2	1	x
100	x^2	0	x^2	$x+1$	x^2+x+1	x^2+x	x	x^2+1	1
101	x^2+1	0	x^2+1	1	x^2	x	x^2+x+1	$x+1$	x^2+x
110	x^2+x	0	x^2+x	x^2+x+1	1	x^2+1	$x+1$	x	x^2
111	x^2+x+1	0	x^2+x+1	x^2+1	x	1	x^2+x	x^2	$x+1$

Computational Considerations

- since coefficients are 0 or 1, can represent any such polynomial as a bit string
- addition becomes XOR of these bit strings
- multiplication is shift & XOR
  - cf long-hand multiplication
- modulo reduction done by repeatedly substituting highest power with remainder of irreducible poly (also shift & XOR)

Computational Example

- in GF(2^3) have (x^2+1) is 101_2 & (x^2+x+1) is 111_2
- so addition is
  - $(x^2+1) + (x^2+x+1) = x$
  - 101 XOR 111 = 010_2
- and multiplication is
  - $(x+1).(x^2+1) = x.(x^2+1) + 1.(x^2+1)$
    $= x^3+x+x^2+1 = x^3+x^2+x+1$
  - 011.101 = (101)<<1 XOR (101)<<0 =
    1010 XOR 101 = 1111_2

Computational Example (con't)

- in $GF(2^3)$ have (x^2+1) is 101_2 & (x^2+x+1) is 111_2
- polynomial modulo reduction (get q(x) & r(x)) is
  - $(x^3+x^2+x+1) \bmod (x^3+x+1) = 1.(x^3+x+1) + (x^2) = x^2$
  - $1111 \bmod 1011 = 1111 \text{ XOR } 1011 = 0100_2$

Using a Generator

- equivalent definition of a finite field
- a **generator** g is an element whose powers generate all non-zero elements
  - in F have 0, g^0, g^1, …, g^{q-2}
- can create generator from **root** of the irreducible polynomial
- then implement multiplication by adding exponents of generator

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Summary

1. Euclid's tabular method allows finding gcd and inverses
2. Group is a set of element and an operation that satisfies closure, associativity, identity, and inverses
3. Abelian group: Operation is commutative
4. Rings have two operations: addition and multiplication
5. Fields: Commutative rings that have multiplicative identity and inverses
6. Finite Fields or Galois Fields have p^n elements where p is prime
7. Polynomials with coefficients in GF(2n) also form a field.

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Test your understanding

- Briefly define a group.
- Briefly define a ring.
- Briefly define a field.
- What does it mean to say that b is a divisor of a?
- What is the difference between modular arithmetic and ordinary arithmetic?
- List three classes of polynomial arithmetic.

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References

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