PHYC90045 Introduction to Quantum Computing

Week 1

Lecture 1
1.1 Non-technical overview of the quantum world (& QC)
1.2 Qubits: mathematical preliminaries I

Lecture 2
2.1 Qubits: mathematical preliminaries II
2.2 Single qubit logic gates

Lab 1
QUI, Single qubit gates, BB84 protocol

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2.1 Qubits: mathematical preliminaries II

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Lecture 2

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Lecture 2 overview

In this lecture:

2.1 Qubits: mathematical preliminaries II

- Operations
- Expectations values
- The Bloch sphere
- Pauli matrices and the axes of the Bloch sphere

2.2 Single qubit operations

- Qubit operations as rotations on the Bloch sphere
- X, Y, Z, H, S, T gate operations, QUI representation
- Arbitrary rotation gate
- Euler angle rotations
- Quantum circuit diagrams
- Using the QUI

- Rieffel, Chapter 3
- Kaye, 2.6
- Nielsen & Chuang, 1.3.2-1.3.4

Operations

In quantum mechanics, *unitary* operators acting on quantum states produce new quantum states. These operators can be described by *unitary* matrices.

The new state is given by: $|\psi'\rangle = U\,|\psi\rangle$

Unitary operations are ones for which: $U^{\dagger}U = I$

Where the dagger represents taking the transpose (t) and complex conjugate (*).

$$U^{\dagger} = U^{t*}$$

In quantum mechanics, all unitary operations are **reversible**.

It's possible to efficiently express every classical computation using equivalent reversible logic gates, but there can be a cost in terms of additional bits and operations.

Operations Don't Commute!

For operators (e.g. matrices), remember

$$AB \neq BA$$

Order matters!

Expectation Values

The *expectation value* of an operator is given by:

$$\langle A\rangle = \langle\psi|A|\psi\rangle$$

Some operators correspond to physical observables – the expectation value then gives the average value of that quantity when measured in a given state.

For example, consider measuring the the total energy of the system represented by the energy operator, i.e. the "Hamiltonian" $\mathcal{H}$ which in matrix representation is

$$\mathcal{H} = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}$$

Expectation values:

$$\langle 0|\,\mathcal{H}\,|0\rangle = E_0 \qquad \langle 1|\,\mathcal{H}\,|1\rangle = E_1$$

Expectation values in a superposition state

For example, consider measuring the the total energy of the system

$$\mathcal{H} = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}$$

For the equal superposition state:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

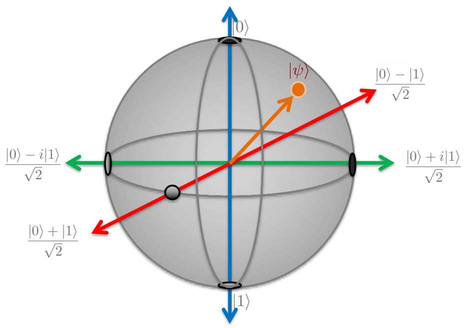
The expectation value is:

$$\langle +| H |+\rangle = \frac{E_0 + E_1}{2}$$

i.e. in this superposition (X basis) the energy is an average of the two values, and it generalizes for other superpositions (ex).

The Bloch Sphere

A convenient geometric representation of single qubit states is the Bloch sphere:



Polar co-ordinates and global phase

Recall, an arbitrary qubit state: $|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle = |a_0| e^{i\theta_0} |0\rangle + |a_1| e^{i\theta_1} |1\rangle$

$$|a_0|^2 + |a_1|^2 = 1$$

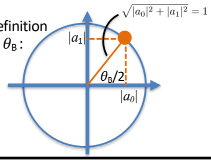
Which we can rearrange to be:

$$|\psi\rangle = e^{i\theta_0} \left(|a_0| |0\rangle + |a_1| e^{i(\theta_1 - \theta_0)} |1\rangle \right) = e^{i\theta_{\text{global}}} \left(\cos\frac{\theta_{\text{B}}}{2} |0\rangle + \sin\frac{\theta_{\text{B}}}{2} e^{i\phi_{\text{B}}} |1\rangle \right)$$

The global phase is unimportant, only the relative phase matters. The real variables θ_B and ϕ_B dictate the position of this state on the Bloch sphere.

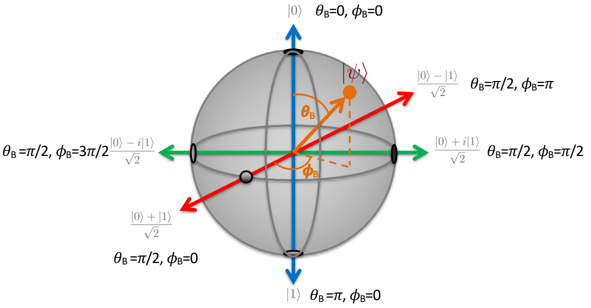
$$|\psi\rangle = a_0 |0\rangle + a_1 |a_1\rangle$$
$$\to \cos\frac{\theta_{\text{B}}}{2} |0\rangle + \sin\frac{\theta_{\text{B}}}{2} e^{i\phi_{\text{B}}} |1\rangle$$

Definition of θ_B:



States on the Bloch sphere

$$|\psi\rangle = a_0\,|0\rangle + a_1\,|a_1\rangle \rightarrow \cos\frac{\theta_{\rm B}}{2}\,|0\rangle + \sin\frac{\theta_{\rm B}}{2}e^{i\phi_{\rm B}}\,|1\rangle$$



The Pauli Matrices

Operations transform states and are equivalent to moving around on the Bloch sphere.

Qubits: the most important operators are the generators of rotations under the SU(2) group. In matrix representation these are the Pauli matrices:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Useful identity for (possible) future reference (**n** is a spatial vector):

$$\exp(i\,\theta\,\hat{\mathbf{n}}\cdot\mathbf{X}) = \cos\theta + i\,\hat{\mathbf{n}}\cdot\mathbf{X}\sin\theta \qquad \mathbf{X} = (X, Y, Z)$$

Expectation of Pauli Matrices

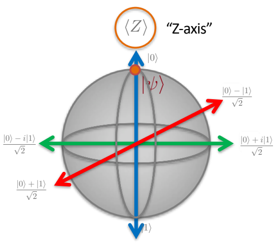
Recall: expectation value for an operator: $\langle A\rangle = \langle\psi|A|\psi\rangle$

For the state: $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\langle X\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$\langle Y\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$\langle Z\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$$



Expectation of Pauli Matrices

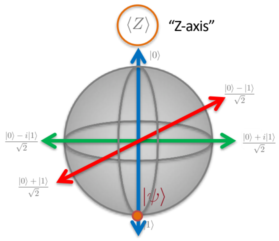
Recall: expectation value for an operator: $\langle A \rangle = \langle \psi | A | \psi \rangle$

For the state: $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\langle X \rangle = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

$$\langle Y \rangle = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

$$\langle Z \rangle = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1$$



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Expectation values of Pauli matrices

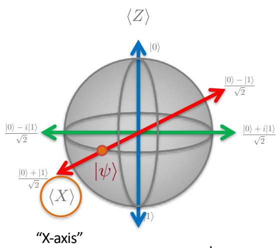
Recall: expectation value for an operator: $\langle A \rangle = \langle \psi | A | \psi \rangle$

For the state: $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$$\langle X \rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$$

$$\langle Y \rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$\langle Z \rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$



…and so on.

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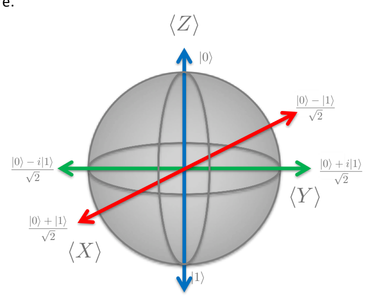
Pauli matrices & Bloch sphere axes

The expectation values of the Pauli matrices (operators) define the axes on the Bloch sphere.



2.2 Single qubit operations

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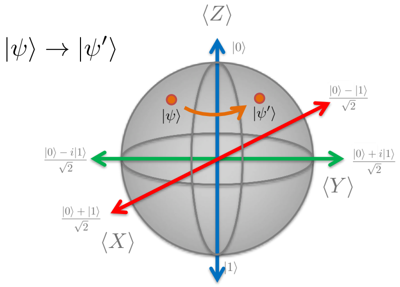
Lecture 2

Operations as rotations on Bloch sphere

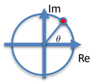
Now we have the full Bloch sphere laid out, we will look at how single qubit operations have a convenient geometric interpretation as rotations about a specific axis, or combination of axes.



Recap: amplitudes in the QUI

Quantum mechanics represents the *wave function.* Complex numbers represent the amplitude *and phase* of this wave.

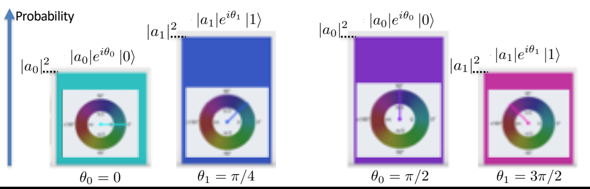
$$|\psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle \;\rightarrow |\psi\rangle = |a_0|e^{i\theta_0}\,|0\rangle + |a_1|e^{i\theta_1}\,|1\rangle$$



Recall: $a = \mathrm{Re}[a] + i\,\mathrm{Im}[a] = |a|e^{i\theta} \rightarrow \begin{array}{l} |a| = \sqrt{\mathrm{Re}[a]^2 + \mathrm{Im}[a]^2} \\ \theta = \tan^{-1}(\mathrm{Im}[a]/\mathrm{Re}[a]) \end{array}$

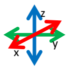
$e^{i\theta} = \cos\theta + i\sin\theta$

In the QUI, phase is represented using the phase wheel colour map, and probability by histogram, e.g. two different states:



X Gate (the X operator): π around X-axis

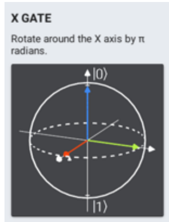
Circuit symbol: X



Matrix representation: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$X\left(a_0|0\rangle + a_1|1\rangle\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$$

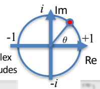
$$\text{i.e.} \quad X\left(a_0|0\rangle + a_1|1\rangle\right) = a_1|0\rangle + a_0|1\rangle$$



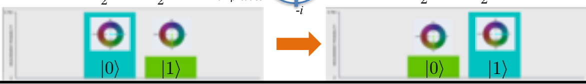
Action on ket states: $a_0|0\rangle + a_1|1\rangle \rightarrow a_1|0\rangle + a_0|1\rangle$

QUI example:

$\frac{\sqrt{3}}{2}|0\rangle + \frac{-i}{2}|1\rangle$

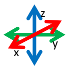


$\frac{-i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$



Y Gate (the Y operator): π around Y-axis

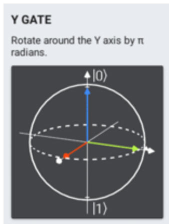
Circuit symbol: Y



Matrix representation: $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$$Y\left(a_0|0\rangle + a_1|1\rangle\right) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} -ia_1 \\ ia_0 \end{bmatrix}$$

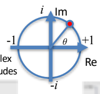
$$\text{i.e.} \quad Y\left(a_0|0\rangle + a_1|1\rangle\right) = -ia_1|0\rangle + ia_0|1\rangle$$



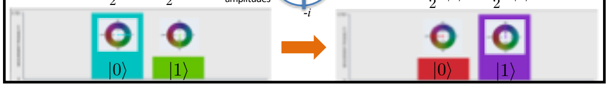
Action on ket states: $a_0|0\rangle + a_1|1\rangle \rightarrow -ia_1|0\rangle + ia_0|1\rangle$

QUI example:

$\frac{\sqrt{3}}{2}|0\rangle + \frac{-i}{2}|1\rangle$

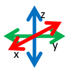


$\frac{-1}{2}|0\rangle + \frac{i\sqrt{3}}{2}|1\rangle$



Z Gate (the Z operator): π around Z-axis

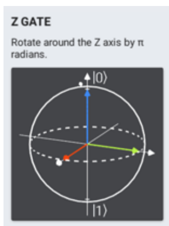
Circuit symbol: Z



Matrix representation: $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$Z\left(a_0|0\rangle + a_1|1\rangle\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} a_0 \\ -a_1 \end{bmatrix}$$

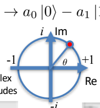
$$\text{i.e.} \quad Z\left(a_0|0\rangle + a_1|1\rangle\right) = a_0|0\rangle - a_1|1\rangle$$



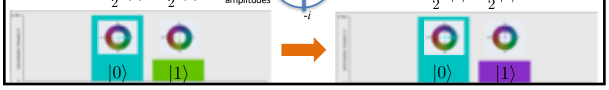
Action on ket states: $a_0|0\rangle + a_1|1\rangle \rightarrow a_0|0\rangle - a_1|1\rangle$

QUI example:

$\frac{\sqrt{3}}{2}|0\rangle + \frac{-i}{2}|1\rangle$

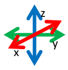


$\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$

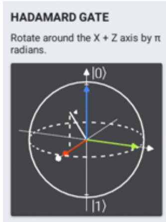


H Gate (the H operator): π around X+Z-axis

Circuit symbol: 



Matrix representation: $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

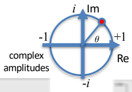


Action on ket states: $|0\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ $|1\rangle \rightarrow \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

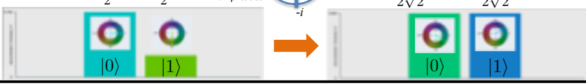
$$a_0 |0\rangle + a_1 |1\rangle \rightarrow \frac{a_0 + a_1}{\sqrt{2}} |0\rangle + \frac{a_0 - a_1}{\sqrt{2}} |1\rangle$$

QUI example:

$\frac{\sqrt{3}}{2}|0\rangle + \frac{-i}{2}|1\rangle$



$\frac{\sqrt{3}-i}{2\sqrt{2}}|0\rangle + \frac{\sqrt{3}+i}{2\sqrt{2}}|1\rangle$



Hadamard gate as change of basis

Normally, we talk about making measurements in the computational Z-basis: “0”, ”1”

NB. “Z-basis” because these are eigenstate of Z: $Z|0\rangle = +|0\rangle\,, \quad Z|1\rangle = -|1\rangle$

But we could equally think of making measurements in the X-basis: “+” state or “–” state. The Hadamard gate takes us from the Z-basis to the X-basis (and vice-versa):

$$H|+\rangle = |0\rangle \qquad H|-\rangle = |1\rangle \qquad H(a_+|+\rangle + a_-|-\rangle) = a_+|0\rangle + a_-|1\rangle$$

A Hadamard gate directly before a measurement “changes the basis” of the measurement from 0/1 to +/-.

NB. The +/- states are eigenstates of the X operator with eigenvalues +/- 1:

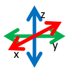
$$X|\pm\rangle = \frac{X(|0\rangle \pm |1\rangle)}{\sqrt{2}} = \frac{(|1\rangle \pm |0\rangle)}{\sqrt{2}} = \frac{\pm(|0\rangle \pm |1\rangle)}{\sqrt{2}} \rightarrow X|\pm\rangle = \pm|\pm\rangle$$

Similarly, the Hadamard gate can be considered to change basis back again:

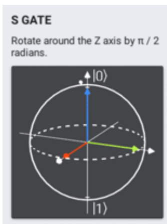
$$H|0\rangle = |+\rangle \qquad H|1\rangle = |-\rangle$$

S Gate (the S operator): $\pi/2$ rotation

Circuit symbol: 



Matrix representation: $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$



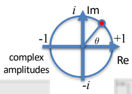
$$S(a_0|0\rangle + a_1|1\rangle) = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} a_0 \\ ia_1 \end{bmatrix}$$

$$\text{i.e.} \quad S(a_0|0\rangle + a_1|1\rangle) = a_0|0\rangle + ia_1|1\rangle$$

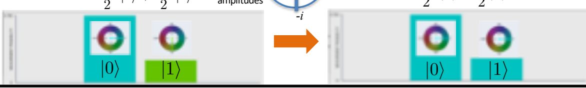
Action on ket states: $a_0|0\rangle + a_1|1\rangle \rightarrow a_0|0\rangle + ia_1|1\rangle$

QUI example:

$\frac{\sqrt{3}}{2}|0\rangle + \frac{-i}{2}|1\rangle$

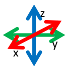


$\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$



T Gate (the T operator): $\pi/4$ rotation

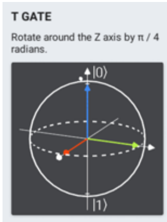
Circuit symbol: 



Matrix representation: $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

$$T\left(a_0 |0\rangle + a_1 |1\rangle\right) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} a_0 \\ e^{i\pi/4} a_1 \end{bmatrix}$$

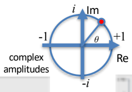
i.e. $T\left(a_0 |0\rangle + a_1 |1\rangle\right) = a_0 |0\rangle + e^{i\pi/4} a_1 |1\rangle$



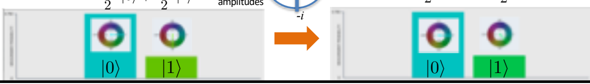
Action on ket states: $a_0 |0\rangle + a_1 |1\rangle \rightarrow a_0 |0\rangle + e^{i\pi/4} a_1 |1\rangle$

QUI example:

$\frac{\sqrt{3}}{2}|0\rangle + \frac{-i}{2}|1\rangle$



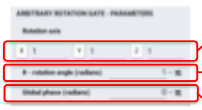
$\frac{\sqrt{3}}{2}|0\rangle + \frac{-ie^{i\pi/4}}{2}|1\rangle$



Arbitrary axis rotation

The QUI provides the ability to code an arbitrary rotation gate: 

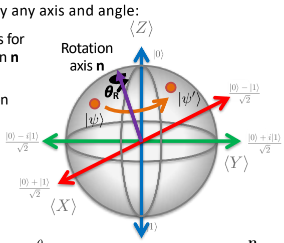
The "parameters" menu allows you to specify any axis and angle:



Cartesian cords for axis of rotation **n**

Angle of rotation θ_R about **n**

Global phase factor: generally can forget about this, but details in next slide



Mathematically, the operation in matrix rep is:

$$|\psi'\rangle = R_{\hat{\mathbf{n}}}(\theta_R)|\psi\rangle \quad R_{\hat{\mathbf{n}}}(\theta_R) = \mathbf{1}\cos\frac{\theta_R}{2} - i\,\hat{\mathbf{n}}\cdot\mathbf{X}\sin\frac{\theta_R}{2} \quad \mathbf{X} = (X, Y, Z) \quad \hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|}$$

$$\langle X\rangle\text{–axis}: \hat{\mathbf{n}} = (1,0,0), \quad \langle Y\rangle\text{–axis}: \hat{\mathbf{n}} = (0,1,0), \quad \langle Z\rangle\text{–axis}: \hat{\mathbf{n}} = (0,0,1)$$

$$\langle X+Z\rangle\text{–axis}: \hat{\mathbf{n}} = (1,0,1)/\sqrt{2}$$

Arbitrary axis rotation – example

Let's see how it works by comparing with a Z gate.

Rotation about Z-axis by π radians – but a global phase choice is made.

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad a_0|0\rangle + a_1|1\rangle \rightarrow a_0|0\rangle - a_1|1\rangle$$

Rotation about Z-axis by π radians, using the R-gate.

$$\hat{\mathbf{n}} = (0,0,1) \quad \theta = \pi \quad \hat{\mathbf{n}}\cdot\mathbf{X} = (0,0,1)\cdot(X,Y,Z) = Z$$

$$R_{\hat{\mathbf{n}}}(\theta) = \mathbf{1}\cos\frac{\theta}{2} - i\,\hat{\mathbf{n}}\cdot\mathbf{X}\sin\frac{\theta}{2}$$

$$\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\frac{\pi}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\pi}{2}$$

$$= -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -iZ$$

$$R_Z(\pi) = -iZ$$

We see the appearance of the global phase – the R-gate is equivalent to the Z- gate with phase $-i$, i.e. for equivalence we need to multiply by a global phase

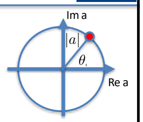
$$e^{i\pi/2} = i$$

So to get same as Z (if we wanted to) we set global phase to $\pi/2$ radians.

Note angles in context – abundant use of θ

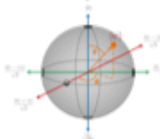
Phase angle of complex amplitudes in polar coordinates:

$$|\psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle \;\; \rightarrow |\psi\rangle = |a_0|e^{i\theta_0}\,|0\rangle + |a_1|e^{i\theta_1}\,|1\rangle$$



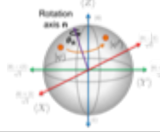
Angle specifying position on the Bloch sphere:

$$|\psi\rangle = a_0\,|0\rangle + a_1\,|a_1\rangle \rightarrow \cos\frac{\theta_B}{2}\,|0\rangle + \sin\frac{\theta_B}{2}e^{i\phi_B}\,|1\rangle$$



Angle of rotation of a qubit state on the Bloch sphere about a specified axis, **n**:

$$|\psi'\rangle = R_{\hat{\mathbf{n}}}(\theta_R)\,|\psi\rangle \qquad R_{\hat{\mathbf{n}}}(\theta_R) = \mathbf{1}\cos\frac{\theta_R}{2} - i\,\hat{\mathbf{n}}\cdot\mathbf{X}\sin\frac{\theta_R}{2}$$



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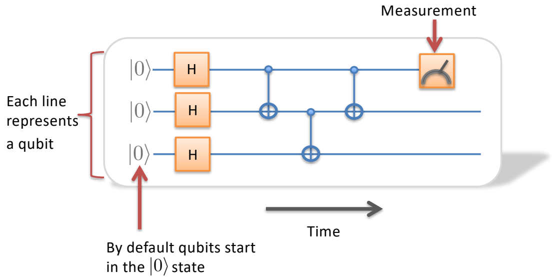
Quantum Circuit Diagrams



Several qubits can be considered as a binary representation (in computational basis).

$$|0000101\rangle = |5\rangle$$

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Using the QUI

In labs you will learn to use the "Quantum User Interface" (QUI) to construct and run circuits. Your first lab will be all about single qubit rotations.

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Week 1 so far

Lecture 1
1.1 Non-technical overview of the quantum world (& QC)
1.2 Qubits: mathematical preliminaries I

Lecture 2
2.1 Qubits: mathematical preliminaries II
2.2 Single qubit logic gates

Lab 1
QUI, Single qubit gates, BB84 protocol