Stat 200C, Spring 2008 Ferguson

Solutions to Exercise Set 1.

1.3. (a) Since $P(X_n = j) = \binom{n}{j} p^j (1-p)^{n-j}$ for $j = 0, 1, \ldots, n$, we have

$$P(X_n \le k-1 | X_n \le k) = \frac{P(X_n \le k-1)}{P(X_n \le k)}$$
$$= \frac{(1-p)^n + np(1-p)^{n-1} + \cdots + \binom{n}{k-1} p^{k-1}(1-p)^{n-k+1}}{(1-p)^n + np(1-p)^{n-1} + \cdots + \binom{n}{k} p^k (1-p)^{n-k}}.$$

If numerator and denominator are divided by $(1-p)^n$, then this ratio become a polynomial of degree $k-1$ in n divided by a polynomial of degree k in n. As n tends to ∞, this tends to zero.

(b) This means that $Y_n \xrightarrow{\mathcal{L}} Y$, where Y is degenerate at k.

2.2. Use the following random sequential method. At stage n, choose a number X_n independently at random from the set $\{-5n, \ldots, 0, \ldots, 5n\}$ with probability $1/(10n+1)$ each, and ask "Are you at X_n?". Suppose he starts at integer N. Since he is able to move at most two up or down at each stage, he will certainly be in the interval $\{-5n, \ldots, 0, \ldots, 5n\}$ at stage n if $5n \ge |N| + 2n$. So at all stages $n \ge |N|/3$, he will answer "Yes" with probability $1/(10n+1)$. Since $\sum_{n \ge |N|/3} 1/(10n+1) = \infty$, the probability is 1 that he will eventually say "Yes" (infinitely often, in fact, from the Borel-Cantelli Lemma).

(Suppose his maximum speed is some integer k, so that he can move up or down at most k per stage, but that you don't know what k is. Can you still find him with probability one?)

2.3. (a) $E(X_n^r | X_{n-1}) = c^r X_{n-1}^r E(X^r) = c^r X_{n-1}^r/(r+1)$ where X has a uniform distribution on $[0,1]$. So $E(X_n^r) = E(E(X_n^r|X_{n-1})) = (c^r/(r+1))E(X_{n-1}^r)$. Then by induction, $E(X_n^r) = (c^r/(r+1))^{n-1}/(r+1)$, since $\mathrm{E}(X_1^r) = 1/(r+1)$.

(b) For $r = 1$, we have $E(X_n) = (c/2)^{n-1}/2 \to 0$ as $n \to \infty$ since $c < 2$; so X_n converges to 0 in mean. For $r = 2$, we have $E(X_n^2) = (c^2/3)^{n-1}/3 \to \infty$, since $c^2/3 > 1$; so X_n does not converge to 0 in quadratic mean.

(c) Let $\epsilon > 0$. By Chebyshev's Inequality with $r = 1$, we have $P(X_n > \epsilon) < EX_n/\epsilon = (c/2)^{n-1}/(2\epsilon)$. Then since $c < 2$, we have $\sum P(X_n > \epsilon) < \infty$. Therefore by the Borel-Cantelli Lemma, we have $P(X_n > \epsilon \quad \text{i.o.}) = 0$. This implies that X_n converges to 0 almost surely.