Solutions to Exercise Set 7.

14.1. (a) We have $F(x) = 1/(1+e^{-x})$. Since $(1-F(t+xR))/(1-F(t)) = (1+e^{-t})/(e^{xR}+e^{-t}) \to 1/e^x$ as $t \to \infty$ if $R(t) \equiv 1$, we have $(M_n - a_n)/b_n \xrightarrow{\mathcal{L}} G_3$. Since $1-F(a_n) = 1/n$ implies $a_n = \log(n-1)$, and $b_n = R(a_n) = 1$, we have $M_n - \log(n-1) \xrightarrow{\mathcal{L}} G_3$.

(b) We have $1-F(x) = c(x)x^{-1}$ with $c(x) = \log(x+1)$ slowly varying. Therefore, the hypotheses of case (a) of Theorem 14 are satisfied with $\gamma = 1$. We find $1-F(b_n) = \log(b_n+1)/b_n = 1/n$, or $b_n = n\log(b_n+1)$. We now show that $b_n \sim n\log(n)$. Let $b_n = nc_n$, so that $c_n = \log(nc_n+1)$. Since c_n is clearly greater than 1, this shows $c_n > \log(n)$. But since $g_n(x) = \log(nx+1)/x$ is decreasing in x with $g_n(c_n) = 1$ and $g_n(2\log(n)) = \log(2n\log(n)+1)/(2\log(n)) \to 1/2$, we have $c_n < 2\log(n)$ for n sufficiently large. Then, since $c_n = \log(n) + \log(c_n + 1/n)$, we have $c_n/\log(n) \to 1$. Thus, $c_n \sim \log(n)$ and $b_n \sim n\log(n)$. Therefore, $M_n/(n\log(n)) \xrightarrow{\mathcal{L}} G_{1,1}$.

(c) We have $x_0 = \pi/2$ and $F(x) = (\sin(x)+1)/2$ for $|x| < x_0$. Since $1-F(x) = (1-\sin(x))/2 \sim (x-(\pi/2))^2/4$, we have $\gamma = 2$, and $(b_n)^2/4 = 1/n$, or $b_n = 2/\sqrt{n}$. Hence, $\sqrt{n}(M_n - (\pi/2))/2 \xrightarrow{\mathcal{L}} G_{2,2}$.

15.3. (a) From Theorem 15, $n(U_{(1)}, U_{(2)}) \xrightarrow{\mathcal{L}} (S_1, S_2) = (X_1, X_1 + X_2)$, where X_1 and X_2 are i.i.d. exponential random variables. From Slutsky's Theorem,

$$\frac{U_{(1)}}{U_{(2)}} = \frac{nU_{(1)}}{nU_{(2)}} \xrightarrow{\mathcal{L}} \frac{X_1}{X_1+X_2} \in \mathcal{U}(0,1).$$

(b) The joint density of $U_{(1)}$ and $U_{(2)}$ is $f(u_1,u_2) = n(n-1)(1-u_2)^{n-2}\mathrm{I}(0 < u_1 < u_2 < 1)$. Make the change of variable $R = U_{(1)}/U_{(2)}$ for $U_{(1)}$. Then $u_1 = ru_2$ and $du_2 = u_2\,dr$, and the joint density of R and $U_{(2)}$ is

$$\begin{aligned} f(r,u_2) &= n(n-1)(1-u_2)^{n-2}u_2\mathrm{I}(0 < ru_2 < u_2 < 1) \\ &= n(n-1)(1-u_2)^{n-2}u_2\mathrm{I}(0 < r < 1)\mathrm{I}(0 < u_2 < 1). \end{aligned}$$

We see that R and $U_{(2)}$ are independent, $R \in \mathcal{U}(0,1)$ for all $n \geq 2$, and $X_{(2)}$ has a beta distribution, $\mathcal{B}e(2, n-1)$.