Solutions to Exercise Set 4.

7.3. (a) $(X-\lambda)/\sqrt{\lambda} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$ is the same as $\sqrt{\lambda}((X/\lambda)-1) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$. The proof of Cramér's Theorem with $g(x)=\log(x)$, $g'(x)=1/x$, implies that $\sqrt{\lambda}(\log(X/\lambda)-\log(1)) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$, or equivalently, $\sqrt{\lambda}(\log(X)-\log(\lambda)) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$. This is the same as $\log(X) \sim \mathcal{N}(\log(\lambda), 1/\lambda)$.

(b) Similarly, using $g(x)=x^2$, $g'(x)=2x$, we have $\sqrt{\lambda}((X/\lambda)^2-1) \xrightarrow{\mathcal{L}} \mathcal{N}(0,4)$, or $(X^2-\lambda^2)/\lambda^{3/2} \xrightarrow{\mathcal{L}} \mathcal{N}(0,4)$. This is the same as $X^2 \sim \mathcal{N}(\lambda^2, 4\lambda^3)$.

(c) The above method doesn't work for $g(x)=e^x$. In fact, there is no function, $\sigma^2(\lambda)$, for which we can say $e^X \sim \mathcal{N}(e^\lambda, \sigma^2(\lambda))$. If there were, we would have $P(e^X - e^\lambda < x\sigma(\lambda)) \to \Phi(x)$ for all x as $\lambda\to\infty$. But

$$\begin{aligned} P(e^X - e^\lambda < x\sigma(\lambda)) &= P(X < \log(e^\lambda + x\sigma(\lambda))) \\ &= P((X-\lambda)/\sqrt{\lambda} < \log(1+xe^{-\lambda}\sigma(\lambda))/\sqrt{\lambda}). \end{aligned}$$

If this converges, the limit must be independent of x, because $e^{-\lambda}\sigma(\lambda)$ would then have to tend to infinity and so the "1" in the log may be dropped, and $\log(x)/\sqrt{\lambda}\to 0$. Thus we cannot get convergence to $\Phi(x)$ for all x.

7.8. To find the asymptotic distribution of $\hat\sigma^2 = m_2 - (m_1 m_3/m_2)$, we need the asymptotic joint distribution of (m_1, m_2, m_3). From the central limit theorem with $\mathrm{E}X = \mu_1 = 0$, we have

$$\sqrt{n}\left(\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} - \begin{pmatrix} 0 \\ \mu_2 \\ \mu_3 \end{pmatrix}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma\right)$$

where

$$\Sigma = \begin{pmatrix} \mathrm{Var}(X) & \mathrm{Cov}(X,X^2) & \mathrm{Cov}(X,X^3) \\ \mathrm{Cov}(X,X^2) & \mathrm{Var}(X^2) & \mathrm{Cov}(X^2,X^3) \\ \mathrm{Cov}(X,X^3) & \mathrm{Cov}(X^2,X^3) & \mathrm{Var}(X^3) \end{pmatrix}$$

Now apply Cramér's Theorem with $g(m_1,m_2,m_3) = m_2 - (m_1 m_3/m_2) = \hat\sigma^2$. We find $\dot g(m_1,m_2,m_3) = (-m_2/m_3, 1+(m_1 m_3/m_2^2), -m_1/m_2)$ and $\dot g(0,\mu_2,\mu_3) = (-\mu_3/\mu_2, 1, 0)$. Using $\mathrm{Var}(X)=\mu_2$, $\mathrm{Cov}(X,X^2)=\mu_3$ and $\mathrm{Var}(X^2) = \mu_4-\mu_2^2$, we find $\dot g \Sigma \dot g^{-1} = \mu_4 - \mu_2^2 - (\mu_3^2/\mu_2)$. Therefore, $\sqrt{n}(\hat\sigma^2-\mu_2) \xrightarrow{\mathcal{L}} \mathcal{N}(0,\tau^2)$, where $\tau^2 = \mu_4-\mu_2^2-(\mu_3^2/\mu_2)$. This is less than or equal to $\mu_4-\mu_2^2$ with equality if and only if $\mu_3=0$.

All two-point distributions with means zero are equivalent, up to change of scale, to one of the distributions, $\mathrm{P}(X=-1) = a/(a+1)$, $P(X=a) = 1/(a+1)$, for some $a>0$. We find

$$\begin{aligned} \mathrm{E}X^2 &= \frac{a^2}{a+1} + \frac{a}{a+1} = a \\ \mathrm{E}X^3 &= \frac{a^3}{a+1} - \frac{a}{a+1} = a(a-1) \\ \mathrm{E}X^4 &= \frac{a^4}{a+1} + \frac{a}{a+1} = a(a^2-a+1) \end{aligned}$$

So $\tau^2 = \mu_4 - \mu_2^2 - (\mu_3^2/\mu_2) = a(a^2 - a + 1) - a^2 - a(a-1)^2 = 0$.

9.1. (a) The Pearson $\chi^2 = 9.2$. (b) The Neyman $\chi_N^2 = 10.668$. The Hellinger $\chi_H^2 = 9.789$. The 5% cut-off point from the χ^2 distribution with 5 degrees of freedom is 11.09.