Final Examination

Statistics 200C

T. Ferguson June 11, 2007

1. Let $X_1, X_2, \ldots, X_n$ be independent, with X_i having a Poisson distribution with mean λa_i, where $\lambda > 0$ is unknown and the a_i are known positive numbers. (The density of the Poisson with mean λ is $e^{-\lambda}\lambda^x/x!$ for $x = 0, 1, \ldots$.)

(a) Find $\hat{\lambda}_n$, the maximum likelihood estimate of λ.

(b) Under what conditions on the a_i is it true that $\hat{\lambda}_n \xrightarrow{q.m.} \lambda$ as $n \to \infty$?

2. Let $X_0, X_1, X_2, \ldots$ be independent identically distributed with mean μ and variance σ^2, and let $Y_i = X_{i-1}X_iX_{i+1}$ for $i = 1, 2, \ldots$. Find the asymptotic distribution of $\overline{Y}_n = \sum_1^n Y_i/n$. State the theorem you are using and say why the hypotheses are satisfied in this case.

3. Let a sample of size n be taken from each of three distributions, and let T_N, respectively V_N, denote the sum of the ranks of the observations from the first, respectively second, distribution when all $N = 3n$ observations are ranked in order from 1 to N. Let $S_N = b_1T_N + b_2V_N$, for arbitrary real numbers b_1 and b_2. Let H_0 be the hypothesis that the three distributions are identical.

(a) Show that S_N is a linear rank statistic under H_0 of the form $S_N = \sum_{j=1}^N z_j a(R_j)$ where $z_j = j$; that is, find $a(i)$ for $i = 1, \ldots, N$.

(b) We have $\bar{z}_N = (N+1)/2$ and $\sigma_z^2 = (N^2-1)/12$. Find $\mathrm{E}S_N$ and $\mathrm{Var}(S_N)$.

(c) Show that $(S_N - \mathrm{E}S_N)/\sqrt{\mathrm{Var}(S_N)} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$.

4. Let $X_1, \ldots, X_n$ be a sample from the Pareto distribution with density $f(x|\theta) = \theta/(x+\theta)^2$ for $x > 0$, and distribution function $F(x|\theta) = x/(x+\theta)$ for $x > 0$. Let $x_p(\theta)$ denote the pth quantile of the distribution and let $X_{\lceil np \rceil}$ denote the sample pth quantile.

(a) What is the asymptotic distribution of $X_{\lceil np \rceil}$ as $n \to \infty$?

(b) Find a constant $c(p)$ such that $\hat{\theta}_n = c(p)X_{\lceil np \rceil}$ is a consistent asymptotically unbiased estimate of θ. For what value of p is the asymptotic variance of $\hat{\theta}_n$ a minimum?

5. Let $X_1, \ldots, X_n$ be a sample from a mixture of exponential distributions:

$$f(x|\theta) = (1-\theta)\,e^{-x} + \theta\,(1/2)e^{-x/2} \qquad \text{for} \quad x > 0$$

where $0 < \theta < 1$.

(a) What is the estimate of θ given by the method of moments?

(b) Show how to improve this estimate by one iteration of Newton's method applied to the likelihood equation (the method of scoring).

6.(a) Find Fisher Information for the distribution with density

$$f(x|\theta) = \frac{1}{\sqrt{2\pi}(1+\theta^2)} x^2 \, e^{-(x-\theta)^2/2}$$

where θ is an unknown parameter.

(b) Find the Cramér-Rao lower bound for an unbiased estimate of $\log(1+\theta^2)$, based on a sample of size n from $f(x|\theta)$.

7. For all $i = 1, \ldots, d$ and $j = 1, \ldots, n$, let X_{ij} and Y_{ij} be independent Poisson random variables with means $\mathrm{E}X_{ij} = \lambda_i$ and $\mathrm{E}Y_{ij} = \mu_i$.

(a) What are the maximum likelihood estimates of λ_i and μ_i under the above general hypothesis?

(b) Find the maximum likelihood estimates of the parameters under the hypothesis $H_0 : \lambda_i = \theta\mu_i$ for all $i = 1, \ldots, d$ and some unknown $\theta > 0$.

(c) Describe the likelihood ratio test of H_0 and how to approximate its distribution when n is large.

8. A sample of size n is taken in a multinomial experiment with 4 cells. Let p_i denote the probability of cell i where $\sum_1^4 p_i = 1$, and let n_i denote the number of obsevrations that fall in cell i where $\sum_1^4 n_i = n$.

(a) Let H_1 denote the hypothesis that $p_1 = \pi_1^2$, $p_2 = 2\pi_1(1-\pi_1)$, $p_3 = (1-\pi_1)^2\pi_2$ and $p_4 = (1-\pi_1)^2(1-\pi_2)$, for some probabilities $0 < \pi_1 < 1$ and $0 < \pi_2 < 1$. Find the chi-square test of H_1 against all alternatives. How many degrees of freedom?

(b) Let H_0 denote the hypothesis that $p_1 = \pi_1^2$, $p_2 = 2\pi_1(1-\pi_1)$, $p_3 = (1-\pi_1)^2\pi_1$ and $p_4 = (1-\pi_1)^3$, for some probability $0 < \pi_1 < 1$. Find the chi-square test of H_0 against all alternatives. How many degrees of freedom?

(c) What, then, is the chi-square test of H_0 against H_1, and how many degrees of freedom does it have?

Solutions to the Final Examination, Stat 200C, Spring 2007.

1. (a) The log likelihood is $\ell_n = -\lambda \sum a_i + \sum x_i(\log\lambda + \log a_i) - \sum \log x_i!$. The likelihood equation is $\partial\ell_n/\partial\lambda = -\sum a_i + (1/\lambda)\sum x_i = 0$, from which we find the MLE to be $\hat{\lambda}_n = \overline{X}_n/\bar{a}_n$.

(b) $\mathrm{E}\hat{\lambda}_n = (1/\bar{a}_n)\frac{1}{n}\sum \lambda a_i = \lambda$, so $\hat{\lambda}_n \xrightarrow{q.m.} \lambda$ if and only if $\mathrm{Var}\hat{\lambda}_n \to 0$. But $\mathrm{Var}\hat{\lambda}_n = (1/\bar{a}_n)^2(1/n^2)\sum \lambda a_i = \lambda/\sum a_i$. So $\hat{\lambda}_n \xrightarrow{q.m.} \lambda$ if and only if $\sum_1^n a_i \to \infty$ as $n \to \infty$.

2. The sequence Y_i is stationary and m-dependent for $m = 2$. The mean is $\mathrm{E}Y_1 = \mathrm{E}X_0X_1X_2 = \mu^3$. Therefore, $\sqrt{n}(\overline{Y}_n - \mu^3) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_{00} + 2\sigma_{01} + 2\sigma_{02})$. We have

$$\begin{aligned}
\sigma_{00} &= \mathrm{Var}(X_0X_1X_2) = \mathrm{E}X_0^2X_1^2X_2^2 - (\mu^3)^2 = (\sigma^2+\mu^2)^3 - \mu^6 = \sigma^6 + 3\mu^2\sigma^4 + 3\mu^4\sigma^2 \\
\sigma_{01} &= \mathrm{Cov}(Y_1, Y_2) = \mathrm{E}X_0X_1^2X_2^2X_3 - (\mu^3)^2 = \mu^2\sigma^4 + 2\mu^4\sigma^2 \\
\sigma_{02} &= \mathrm{Cov}(Y_1, Y_3) = \mathrm{E}X_1X_2X_3^2X_4X_5 - \mu^6 = \mu^4\sigma^2.
\end{aligned}$$

Hence, $\sqrt{n}(\overline{Y}_n - \mu^3) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^6 + 3\mu^2\sigma^4 + 3\mu^4\sigma^2 + 2[\mu^2\sigma^4 + 2\mu^4\sigma^2] + 2[\mu^4\sigma^2]) = \mathcal{N}(0, \sigma^6 + 5\mu^2\sigma^4 + 9\mu^4\sigma^2)$.

3. (a) $a(i) = b_1$ for $i = 1, \ldots, n$, $a(i) = b_2$ for $i = n+1, \ldots, 2n$, and $a(i) = 0$ for $i = 2n+1, \ldots, 3n = N$.

(b) $\bar{a}_N = (b_1 + b_2)/3$, and $(1/N)\sum a(i)^2 = (b_1^2 + b_2^2)/3$, so $\sigma_a^2 = (b_1^2+b_2^2)/3 - (b_1 + b_2)^2/9 = 2(b_1^2 - b_1b_2 + b_2^2)/9$. So $\mathrm{E}S_N = N\bar{z}_n\bar{a}_N = (N(N+1)/2)((b_1+b_2)/3)$, and $\mathrm{Var}(S_n) = (N^2/(N-1))\sigma_z^2\sigma_a^2 = (N^2(N+1)/12)\sigma_a^2$.

(c) Since $\max_j(a(j) - \bar{a}_N)^2/\sigma_a^2$ is constant and $\max_j(j - ((n+1)/2))^2/\sigma_z^2 = [(N-1)^2/2]/[(N^2-1)/12] \to 6$, we have

$$\frac{1}{N} \cdot \frac{\max_j(z_j - \bar{z}_N)^2}{\sigma_z^2} \cdot \frac{\max_j(a(j) - \bar{a}_N)^2}{\sigma_a^2} \to 0.$$

Hence, $(S_N - \mathrm{E}S_n)/\sqrt{\mathrm{Var}(S_N)} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$. This reduces to $\sqrt{N}((S_N/N^2) - (b_1 + b_2)/6)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, (b_1^2 - b_1b_2 + b_2^2)/54)$.

4. (a) Solving $F(x_p|\theta) = p$, we find $x_p = p\theta/(1-p)$. Therefore,

$$\sqrt{n}(X_{\lceil np\rceil} - \frac{p\theta}{(1-p)}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \frac{p(1-p)}{f(x_p|\theta)^2}) = \mathcal{N}(0, \frac{p\theta^2}{(1-p)^3}).$$

(b) For $g(x) = (1-p)x/p$ and $\hat{\theta}_n = (1-p)X_{\lceil np\rceil}/p$, we have $g'(x) = (1-p)/p$ and

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \frac{p\theta^2(1-p)^2/p^2}{(1-p)^3}) = \mathcal{N}(0, \frac{\theta^2}{p(1-p)}).$$

The asymptotic variance is minimized by choosing p to maximize $p(1-p)$. This gives $p = 1/2$, so the median is the best quantile to use to estimate θ.

5. (a) $\mathrm{E}_\theta(X) = (1-\theta) + 2\theta = 1+\theta$, so the method of moments estimator of θ is $\hat\theta_n = \overline{X}_n - 1$.

(b) The log likelihood is $\ell_n(\theta) = \sum \log[(1-\theta)e^{-x_i} + \theta(1/2)e^{-x_i/2}]$, so the likelihood equation is

$$\ell_n'(\theta) = \sum \frac{-e^{-x_i/2} + (1/2)}{(1-\theta)e^{-x_i/2} + \theta(1/2)} = 0.$$

One may improve $\hat\theta_n$ by the Newton method formula $\tilde\theta_n = \hat\theta_n - \ell_n'(\hat\theta_n)/\ell_n''(\hat\theta_n)$, where

$$\ell_n''(\theta) = -\sum \frac{(e^{-x_i/2} - (1/2))^2}{[(1-\theta)e^{-x_i/2} + \theta(1/2)]^2}.$$

6. (a) Since

$$\frac{\partial}{\partial\theta}\log(f(x|\theta) = -\frac{2\theta}{1+\theta^2} + (x-\theta)$$

we have

$$\frac{\partial^2}{\partial\theta^2}\log f(x|\theta) = -\frac{2(1+\theta^2) - 4\theta^2}{(1+\theta^2)^2} - 1 = -\frac{\theta^4+3}{(1+\theta^2)^2}.$$

So Fisher Information is $\mathcal{I}(\theta) = (\theta^4+3)/(1+\theta^2)^2$.

(b) If an estimate has expectation $g(\theta) = \log(1+\theta^2)$, then its variance must be at least $g'(\theta)^2\mathcal{I}(\theta)^{-1}/n$. Since $g'(\theta) = 2\theta/(1+\theta^2)$, we have that the variance is at least $4\theta^2/(n(\theta^4+3))$.

7. (a) The MLEs under the general hypothesis are $\hat\lambda_i = \bar X_{i.}$ and $\hat\mu_i = \bar Y_{i.}$.

(b) Under H_0, the likelihood function is

$$L(\boldsymbol\lambda, \boldsymbol\mu) \propto \prod_{i=1}^d e^{-n\theta\mu_i}(\theta\mu_i)^{X_{i.}} e^{n\mu_i}\mu_i^{Y_{i.}} = \prod_{i=1}^d e^{-n(\theta+1)\mu_i}\theta^{X_{i.}}\mu_i^{X_{i.}+Y_{i.}}$$

The likelihood equations are

$$\begin{aligned}\frac{\partial}{\partial\theta}\log L &= -n\sum\mu_i + \frac{1}{\theta}\sum X_{i.} = 0\\ \frac{\partial}{\partial\mu_i}\log L &= -n(\theta+1) + \frac{1}{\mu_i}(X_{i.}+Y_{i.}) = 0 \qquad \text{for } i = 1,\ldots,d.\end{aligned}$$

Solving these equations, we find as the MLEs under H_0,

$$\tilde\theta = X_{..}/Y_{..}, \quad \tilde\mu_i = (\overline{X}_{i.} + \overline{Y}_{i.})\frac{Y_{..}}{X_{..}+Y_{..}}, \quad \tilde\lambda_i = (\overline{X}_{i.} + \overline{Y}_{i.})\frac{X_{..}}{X_{..}+Y_{..}}$$

(c) The likelihood ratio test rejects H_0 if $\Lambda = L(\tilde{\boldsymbol\lambda}, \tilde{\boldsymbol\mu})/L(\hat{\boldsymbol\lambda}, \hat{\boldsymbol\mu})$ is too small, or equivalently if $-2\log\Lambda$ is too large. When n is large this latter has appoximately a chi-square distribution under H_0 with $d-1$ degrees of freedom, since the H_0 puts $d-1$ restrictions on the general hypothesis.

8. The general chi-square (modified for simplicity) is

$$\chi^2(\pi_1, \pi_2) = \frac{(n_1 - n\pi_1^2)^2}{n_1} + \frac{(n_2 - 2n\pi_1(1-\pi_1))^2}{n_2} + \frac{(n_3 - n(1-\pi_1)^2\pi_2)^2}{n_3} + \frac{(n_4 - n(1-\pi_1)^2(1-\pi_2))^2}{n_4}$$

(a) The likelihood is proportional to $(\pi_1^2)^{n_1}(\pi_1(1-\pi_1))^{n_2}((1-\pi_1)^2(\pi_2)^{n_3}((1-\pi_1)^2(1-\pi_2))^{n_3} = \pi_1^{2n_1+n_2}(1-\pi_1)^{n_2+2n_3+2n_4}\pi_2^{n_3}(1-\pi_2)^{n_4}$. Therefore the MLEs are

$$\hat{\pi}_1 = \frac{2n_1 + n_2}{2n_1 + 2n_2 + 2n_3 + 2n_4} \qquad \text{and} \qquad \hat{\pi}_2 = \frac{n_3}{n_3 + n_4}$$

We reject H_1 if $\chi^2(\hat{\pi}_1, \hat{\pi}_2)$ is too large. For large n, this has approximately a chi-square distribution with 1 degree of freedom.

(b) $H_0 : \pi_1 = \pi_2$, so the likelihood is proportional to $\pi_1^{2n_1+n_2+n_3}(1-\pi_1)^{n_2+2n_3+3n_4}$. The MLEs are

$$\tilde{\pi}_1 = \tilde{\pi}_2 = \frac{2n_1 + n_2 + n_3}{2n_1 + 2n_2 + 3n_3 + 3n_4}$$

We reject H_1 if $\chi^2(\tilde{\pi}_2, \tilde{\pi}_2)$ is too large. For large n, this has approximately a chi-square distribution with 2 degrees of freedom.

(c) We reject H_0 vs. H_1 if $\chi^2(\tilde{\pi}_2, \tilde{\pi}_2) - \chi^2(\hat{\pi}_1, \hat{\pi}_2)$ is too large. For large n, this difference has approximately a chi-square distribution with 1 degree of freedom.