Solutions to Exercise Set 6.

12.2.(a) By symmetry, $z_j = z_{N+1-j}$, so $\bar{z}_N = 0$. Then using the Riemann approximation to an integral, $\sigma_z^2 = (1/N)\sum z_j^2 = \sum \Phi^{-1}(j/(N+1))^2(1/N) \to \int_0^1 \Phi^{-1}(x)^2\,dx = \int_{-\infty}^{\infty} u^2\,d\Phi(u) = 1$.

(b) As in Example 1, $\max(a_j - \bar{a})^2$ is bounded and $\sum(a_j - \bar{a})^2 = m(N-m)/N$. Therefore δ_N goes to 0 if $\max z_j^2/N \to 0$. But $\max z_j^2 = z_N^2 = \Phi^{-1}(N/(N+1))^2$. Using the approximation $1 - \Phi(x) \approx \phi(x)/x$ for large values of x, we find $z_N^2 < 2\log(N+1)$ for large N. Thus $\delta_N \to 0$ and Theorem 12 gives $S_N/\sqrt{\mathrm{Var}(S_N)} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$, where $\mathrm{Var}(S_N) = N^2\sigma_z^2\sigma_a^2/(N-1) \approx Nr(1-r)$. Unlike Exercise 12.1(b), no further conditions are needed to conclude that $S_N/\sqrt{N} \xrightarrow{\mathcal{L}} \mathcal{N}(0, r(1-r))$.

12.6. (a) $a(i) = b_1$ for $i = 1,\ldots,n$, $a(i) = b_2$ for $i = n+1,\ldots,2n$, and $a(i) = 0$ for $i = 2n+1,\ldots,3n = N$.

(b) $\bar{a}_N = (b_1 + b_2)/3$, and $(1/N)\sum a(i)^2 = (b_1^2 + b_2^2)/3$, so $\sigma_a^2 = (b_1^2 + b_2^2)/3 - (b_1 + b_2)^2/9 = 2(b_1^2 - b_1b_2 + b_2^2)/9$. So $\mathrm{E}S_N = N\bar{z}_n\bar{a}_N = (N(N+1)/2)((b_1+b_2)/3)$, and $\mathrm{Var}(S_N) = (N^2/(N-1))\sigma_z^2\sigma_a^2 = (N^2(N+1)/12)\sigma_a^2$. Since $\max_j(a(j) - \bar{a}_N)^2/\sigma_a^2$ is constant and $\max_j(j - ((n+1)/2))^2/\sigma_z^2 = [(N-1)^2/2]/[(N^2-1)/12] \to 6$, we have

$$\frac{1}{N}\cdot\frac{\max_j(z_j - \bar{z}_N)^2}{\sigma_z^2}\cdot\frac{\max_j(a(j) - \bar{a}_N)^2}{\sigma_a^2} \to 0.$$

Hence, $(S_N - \mathrm{E}S_N)/\sqrt{\mathrm{Var}(S_N)} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$. This reduces to $\sqrt{N}((S_N/N^2) - (b_1 + b_2)/6)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_a^2/12) = \mathcal{N}(0, (b_1^2 - b_1b_2 + b_2^2)/54)$.

(c) Thus for all (b_1, b_2),

$$\sqrt{N}\,(b_1\;\; b_2)\left(\begin{pmatrix} T_N/N^2 \\ V_N/N^2 \end{pmatrix} - \begin{pmatrix} 1/6 \\ 1/6 \end{pmatrix}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \frac{1}{54}(b_1\;\; b_2)\begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix})$$

This implies by the Cramér-Wold device,

$$\sqrt{N}\left(\begin{pmatrix} T_N/N^2 \\ V_N/N^2 \end{pmatrix} - \begin{pmatrix} 1/6 \\ 1/6 \end{pmatrix}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{1}{54}\begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}).$$

13.5.(a) Solving $F(x_p|\theta) = p$, we find $x_p = p\theta/(1-p)$. Therefore,

$$\sqrt{n}(X_{\lceil np\rceil} - \frac{p\theta}{(1-p)}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \frac{p(1-p)}{f(x_p|\theta)^2}) = \mathcal{N}(0, \frac{p\theta^2}{(1-p)^3}).$$

(b) For $g(x) = (1-p)x/p$ and $\hat{\theta}_n = (1-p)X_{\lceil np\rceil}/p$, we have $g'(x) = (1-p)/p$ and

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \frac{p\theta^2(1-p)^2/p^2}{(1-p)^3}) = \mathcal{N}(0, \frac{\theta^2}{p(1-p)}).$$

The asymptotic variance is minimized by choosing p to maximize $p(1-p)$. This gives $p = 1/2$, so the median is the best quantile to use to estimate θ (if you are only going to use one quantile).