Introduction to Algorithms
Massachusetts Institute of Technology — 6.046J/18.410J
Singapore-MIT Alliance — SMA5503
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Diagnostic Test Solutions

Problem 1

Consider the following pseudocode:

```
ROUTINE(n)
if n = 1
   then return 1
   else return n + ROUTINE(n − 1)
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(a) Give a one-sentence description of what ROUTINE(n) does. (Remember, don't guess.)

Solution: The routine gives the sum from 1 to n.

(b) Give a precondition for the routine to work correctly.

Solution: The value n must be greater than 0; otherwise, the routine loops forever.

(c) Give a one-sentence description of a faster implementation of the same routine.

Solution: Return the value $n(n + 1)/2$.

Problem 2

Give a short (1–2-sentence) description of each of the following data structures:

(a) FIFO queue

Solution: A dynamic set where the element removed is always the one that has been in the set for the longest time.

(b) Priority queue

Solution: A dynamic set where each element has an associated priority value. The element removed is the element with the highest (or lowest) priority.

(c) Hash table

Solution: A dynamic set where the location of an element is computed using a function of the element's key.

Problem 3

Using Θ-notation, describe the worst-case running time of the best algorithm that you know for each of the following:

(a) Finding an element in a sorted array.

Solution: $\Theta(\log n)$

(b) Finding an element in a sorted linked-list.

Solution: $\Theta(n)$

(c) Inserting an element in a sorted array, once the position is found.

Solution: $\Theta(n)$

(d) Inserting an element in a sorted linked-list, once the position is found.

Solution: $\Theta(1)$

Problem 4

Describe an algorithm that locates the first occurrence of the largest element in a finite list of integers, where the integers are not necessarily distinct. What is the worst-case running time of your algorithm?

Solution: Idea is as follows: go through list, keeping track of the largest element found so far and its index. Update whenever necessary. Running time is $\Theta(n)$.

Problem 5

How does the height h of a balanced binary search tree relate to the number of nodes n in the tree?

Solution: $h = O(\lg n)$

Problem 6

Does an undirected graph with 5 vertices, each of degree 3, exist? If so, draw such a graph. If not, explain why no such graph exists.

Solution: No such graph exists by the Handshaking Lemma. Every edge adds 2 to the sum of the degrees. Consequently, the sum of the degrees must be even.

Problem 7

It is known that if a solution to Problem A exists, then a solution to Problem B exists also.

(a) Professor Goldbach has just produced a 1,000-page proof that Problem A is unsolvable. If his proof turns out to be valid, can we conclude that Problem B is also unsolvable? Answer yes or no (or don't know).

Solution: No

(b) Professor Wiles has just produced a 10,000-page proof that Problem B is unsolvable. If the proof turns out to be valid, can we conclude that problem A is unsolvable as well? Answer yes or no (or don't know).

Solution: Yes

Problem 8

Consider the following statement:

> If 5 points are placed anywhere on or inside a unit square, then there must exist two that are no more than $\sqrt{2}/2$ units apart.

Here are two attempts to prove this statement.

Proof (a): Place 4 of the points on the vertices of the square; that way they are maximally separated from one another. The 5th point must then lie within $\sqrt{2}/2$ units of one of the other points, since the furthest from the corners it can be is the center, which is exactly $\sqrt{2}/2$ units from each of the four corners.

Proof (b): Partition the square into 4 squares, each with a side of $1/2$ unit. If any two points are on or inside one of these smaller squares, the distance between these two points will be at most $\sqrt{2}/2$ units. Since there are 5 points and only 4 squares, at least two points must fall on or inside one of the smaller squares, giving a set of points that are no more than $\sqrt{2}/2$ apart.

Which of the proofs are correct: (a), (b), both, or neither (or don't know)?

Solution: (b) only

Problem 9

Give an inductive proof of the following statement:

> For every natural number $n > 3$, we have $n! > 2^n$.

Solution: Base case: True for $n = 4$.

Inductive step: Assume $n! > 2^n$. Then, multiplying both sides by $(n + 1)$, we get $(n + 1)n! > (n + 1)2^n > 2 * 2^n = 2^{n+1}$.

Problem 10

We want to line up 6 out of 10 children. Which of the following expresses the number of possible line-ups? (Circle the right answer.)

(a) $10!/6!$

(b) $10!/4!$

(c) $\binom{10}{6}$

(d) $\binom{10}{4} \cdot 6!$

(e) None of the above

(f) Don't know

Solution: (b), (d) are both correct

Problem 11

A deck of 52 cards is shuffled thoroughly. What is the probability that the 4 aces are all next to each other? (Circle the right answer.)

(a) $4!49!/52!$

(b) $1/52!$

(c) $4!/52!$

(d) $4!48!/52!$

(e) None of the above

(f) Don't know

Solution: (a)

Problem 12

The weather forecaster says that the probability of rain on Saturday is 25% and that the probability of rain on Sunday is 25%. Consider the following statement:

> The probability of rain during the weekend is 50%.

Which of the following best describes the validity of this statement?

(a) If the two events (rain on Sat/rain on Sun) are independent, then we can add up the two probabilities, and the statement is true. Without independence, we can't tell.

(b) True, whether the two events are independent or not.

(c) If the events are independent, the statement is false, because the the probability of no rain during the weekend is $9/16$. If they are not independent, we can't tell.

(d) False, no matter what.

(e) None of the above.

(f) Don't know.

Solution: (c)

Problem 13

A player throws darts at a target. On each trial, independently of the other trials, he hits the bull's-eye with probability $1/4$. How many times should he throw so that his probability is 75% of hitting the bull's-eye at least once?

(a) 3

(b) 4

(c) 5

(d) 75% can't be achieved.

(e) Don't know.

Solution: (c), assuming that we want the probability to be ≥ 0.75, not necessarily exactly 0.75.

Problem 14

Let X be an indicator random variable. Which of the following statements are true? (Circle all that apply.)

(a) $\Pr\{X = 0\} = \Pr\{X = 1\} = 1/2$

(b) $\Pr\{X = 1\} = \mathrm{E}[X]$

(c) $\mathrm{E}[X] = \mathrm{E}[X^2]$

(d) $\mathrm{E}[X] = (\mathrm{E}[X])^2$

Solution: (b) and (c) only