Solutions to Exercise Set 9.

20.3. (a) $\mu(\theta) = 2^{-1/\theta}$.

(b) $f(\mu(\theta)|\theta) = \theta 2^{-(\theta-1)/\theta}$, so $\sqrt{n}(\hat{\mu}_n - \mu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 2^{-2/\theta}/\theta^2)$.

(c) Fisher information is $1/\theta^2$, and $g(\theta) = 2^{-1/\theta}$, $\dot{g}(\theta) = (2^{-1/\theta}\log 2)/\theta^2$, so an unbiased estimate of μ has variance at least $\dot{g}(\theta)^2/\mathcal{I}(\theta) = 2^{-2/\theta}(\log 2)^2/\theta^2$.

(d) The efficiency is $[2^{-2/\theta}(\log 2)^2/\theta^2]/[2^{-2/\theta}/\theta^2] = (\log 2)^2 = .48\cdots$.

20.6. (a) $\mathrm{P}(Y_1 = n) = \int_n^{n+1}(1/\theta)e^{-x/\theta}\,dx = e^{-n/\theta} - e^{-(n+1)/\theta} = (1-p)p^n$, where $p = e^{-1/\theta}$.

(b) The likelihood function is

$$L_n(\theta) = \frac{1}{\theta^n}\exp\{-\frac{1}{\theta}\sum_1^n X_i\}(1-e^{-1/\theta})^n \exp\{-\frac{1}{\theta}\sum_1^n Y_i\}.$$

We see that $\overline{X}_n + \overline{Y}_n$ is a sufficient statistic. Taking the derivative of the log of L_n gives

$$\dot{\ell}_n(\theta) = \frac{n}{\theta^2}\left[\overline{X}_n + \overline{Y}_n - \theta - \frac{1}{e^{1/\theta}-1}\right]$$

which gives the likelihood equation

$$\theta + \frac{1}{e^{1/\theta}-1} = \overline{X}_n + \overline{Y}_n.$$

One can use $\hat{\theta}_n^{(0)} = \overline{X}_n$ as a preliminary estimate for the method of scoring. A fully efficient estimate of θ is therefore

$$\hat{\theta}_n^{(1)} = \overline{X}_n + \mathcal{I}(\overline{X}_n)^{-1}\frac{1}{n}\dot{\ell}_n(\overline{X}_n)$$

where $\mathcal{I}(\theta)$is found in part (c).

(c) Fisher information for X is $1/\theta^2$. All we need to do is to find Fisher information for Y and add (since X and Y are independent). We have $\log P_\theta(y) = \log(1-p) + y\log(p)$, where $p = \exp\{-1/\theta\}$, so that $(\partial/\partial\theta)\log P_\theta(y) = -(\partial p/\partial\theta)/(1-p) + y(\partial p/\partial\theta)/p$. Fisher information is the variance of this. We have $\mathrm{Var}_\theta(Y) = p/(1-p)^2$, and $\partial p/\partial\theta = p/\theta^2$. Therefore, Fisher information of Y is

$$\mathcal{I}(\theta) = \mathrm{Var}(Y(\partial p/\partial\theta)/p) = \frac{p}{(1-p)^2}\cdot\frac{(\partial p/\partial\theta)^2}{p^2} = \frac{p}{(1-p)^2\theta^4},$$

and Fisher information for X and Y is $1/\theta^2 + p/(1-p)^2\theta^4$. So the asymptotic variance of $\sqrt{n}(\hat{\theta}_n^{(1)} - \theta)$ is $\dfrac{\theta^4(1-p)^2}{(1-p)^2\theta^2 + p}$.

22.3. (a) The likelihood function is $L = \prod_i \prod_j (1/\beta_i) e^{-x_{ij}/\beta_i} = \prod_i (1/\beta_i^n) e^{-x_{i.}/\beta_i}$. Under the general hypothesis, H, the MLE's are $\hat{\beta}_i = \overline{X}_{i.}$. Under hypothesis H_0, the MLE is $\tilde{\beta} + \overline{X}_{..}$. The likelihood ratio test rejects H_0 if the likelihood ratio,

$$\Lambda = \frac{\sup_{H_0} L}{\sup_H L} = \frac{(1/\tilde{\beta}^{nk}) e^{-nk}}{(\prod_i (1/\hat{\beta}_i^n) e^{-nk}} = \left(\frac{\prod_i \overline{X}_{i.}}{\overline{X}_{..}^k} \right)^n$$

is too small.

(b) Under H_0, the statistic $-2 \log \Lambda$ has asymptotically a chi-square distribution with $k-1$ degrees of freedom.