Solutions to Exercise Set 2.

3.2. (a) The moment generating function of $X \in \mathcal{N}(0,1)$ is easy to compute: $M_X(z) = \mathrm{E}e^{zX} = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} e^{zx}e^{-x^2/2}\,dx = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} e^{(x-z)^2}\,dx e^{z^2/2} = e^{z^2/2}$. Its value on the pure imaginary line, $z = it$, is the characteristic function, $\varphi_X(t) = e^{(it)^2/2} = e^{-t^2/2}$.

(b) If $X \in \mathcal{P}(\lambda)$, then $\varphi_X(t) = \mathrm{E}e^{itX} = \sum_{x=0}^{\infty} e^{itx}e^{-\lambda}\lambda^x/x! = e^{-\lambda}\sum_{x=0}^{\infty}(\lambda e^{it})^x/x! = e^{-\lambda}e^{\lambda e^{it}} = e^{-\lambda(1-e^{it})}$.

(c) $\varphi_{a+bX}(t) = \mathrm{E}e^{it(a+bX)} = e^{iat}\mathrm{E}e^{ibtX} = e^{iat}\varphi_X(bt)$.

(d) $\varphi_Y(t) = \varphi_{(X-\lambda)\sqrt{\lambda}}(t) = e^{-i\sqrt{\lambda}t}\varphi_X(t/\sqrt{\lambda}) = e^{-i\sqrt{\lambda}t}\exp\{-\lambda(1-e^{-it/\sqrt{\lambda}})\} = \exp\{-\lambda(1+(it/\sqrt{\lambda}) - e^{-it/\sqrt{\lambda}})\}$. Now expand $e^{-it/\sqrt{\lambda}}$ in a power series, and note that the first two terms cancel exactly, leaving

$$\begin{aligned}\varphi_Y(t) &= \exp\{-\lambda(-\frac{1}{2}(\frac{-it}{\sqrt{\lambda}})^2 - O(1/\lambda^{3/2})\} \\ &= \exp\{-\frac{1}{2}t^2 + O(1/\sqrt{\lambda})\} \to \exp\{-\frac{1}{2}t^2\}\end{aligned}.$$

(e) We may conclude that the Poisson distribution with large parameter λ is approximately normal with mean λ and variance λ. More precisely, $(X-\lambda)/\sqrt{\lambda} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$ as $\lambda \to \infty$.

3.4. (a) Since $\sum_0^\infty \binom{r+x-1}{x}(1-p)^r p^x = 1$, we have $\sum_0^\infty \binom{r+x-1}{x}p^x = (1-p)^{-r}$, for all p. Therefore,

$$\begin{aligned}\varphi(t) = \mathrm{E}e^{itX} &= \sum_{x=0}^{\infty} e^{itx}\binom{r+x-1}{x}(1-p)^r p^x \\ &= (1-p)^r\sum_{x=0}^{\infty}\binom{r+x-1}{x}(pe^{it})^x = \frac{(1-p)^r}{(1-pe^{it})^r}.\end{aligned}$$

(b) If $r \to \infty$ and $p \to 0$ in such a way that $rp \to \lambda$, then $(1-p)^r \to e^{-\lambda}$ and $(1-pe^{it})^r \to e^{-\lambda e^{it}}$. Therefore, $\varphi(t) \to e^{\lambda(e^{it}-1)}$. Since this is the characteristic finction of the Poisson distribution, $\mathcal{P}(\lambda)$, we have $X \xrightarrow{\mathcal{L}} \mathcal{P}(\lambda)$.

4.1. (a) The least squares estimate of λ is that value of λ that minimizes $\sum_1^n(X_i - \lambda z_i)^2$. Taking a derivative of this sum with respect to λ, setting to zero and solving gives $\hat\lambda_{LS} = \sum_1^n X_i z_i/\sum_1^n z_i^2$. The weighted least squares estimate of λ is that value of λ that minimizes $\sum_1^n(X_i-\lambda z_i)^2/z_i$. Similarly, we find $\hat\lambda_W = \sum_1^n X_i/\sum_1^n z_i$.

(b) Since $\mathrm{E}\hat\lambda_{LS} = \lambda$, it is sufficient to find for what values of the z_i we have $\mathrm{Var}(\hat\lambda_{LS}) \to 0$ as $n \to \infty$ (see Exercise 1.5). But

$$\mathrm{Var}\hat\lambda_{LS} = \mathrm{Var}\frac{\sum_1^n X_i z_i}{\sum_1^n z_i^2} = \frac{\mathrm{Var}\sum_1^n X_i z_i}{(\sum_1^n z_i^2)^2} = \frac{\sum_1^n z_i^2 \mathrm{Var}X_i}{(\sum_1^n z_i^2)^2} = \frac{\lambda\sum_1^n z_i^3}{(\sum_1^n z_i^2)^2}$$

Therefore, λ_{LS} is consistent in quadratic mean if and only if $\sum_1^n z_i^3/(\sum_1^n z_i^2)^2 \to 0$ as $n \to \infty$.

(c) $\hat{\lambda}_W$ is also unbiased, and $\mathrm{Var}(\hat{\lambda}_W) = \lambda \sum_1^n z_i/(\sum_1^n z_i)^2 = \lambda/\sum_1^n z_i$. So $\hat{\lambda}_W$ is consistent in quadratic mean if and only if $\sum_1^n z_i \to \infty$ as $n \to \infty$.

(d) If $\hat{\lambda}_W$ is not consistent in quadratic mean, i.e. if $\sum_1^\infty z_i < \infty$, then both $\sum_1^\infty z_i^2 < \infty$ and $\sum_1^\infty z_i^3 < \infty$, so $\hat{\lambda}_{LS}$ is not consistent either. However if $z_i = 1/i$, then $\sum_1^\infty z_i = \infty$ so that $\hat{\lambda}_W$ is consistent in quadratic mean, but $\sum_1^\infty z_i^2 < \infty$ and $\sum_1^\infty z_i^3 < \infty$ so that $\hat{\lambda}_{LS}$ is not consistent.