Stat 200C, Spring 2008 Ferguson

Solutions to Exercise Set 10.

24.1 (a) The chi-square is $\chi^2(q_1, q_2, q_3) = \sum_{i=1}^r \sum_{j=1}^c (n_{ij} - p_{ij})^2/p_{ij}$, where

$$p_{ij} = \begin{cases} q_3 & \text{if } 2 < i < r \text{ and } 2 < j < c. \\ q_1 & \text{if } i = 1 \text{ and } j = 1, \text{ or } i = 1 \text{ and } j = c, \text{ or } i = r \text{ and } j = 1, \text{ or } i = r \text{ and } j = c. \\ q_2 & \text{otherwise.} \end{cases}$$

There are 4 corner cells, $2r + 2c - 8$ edge cells and $(r - 2)(c - 2)$ central cells, so we have $4q_1 + 2(r+c-4)q_2 + (r-2)(c-2)q_3 = 1$. The likelihood function is $L \propto q_1^{N_1} q_2^{N_2} q_3^{N_3}$, where N_1 is the number of observations falling in the corners, N_2 is the number on the edges, and N_3 is the number in the central cells. Therefore the Maximum Likelihood Estimates are

$$\hat{q}_1 = \frac{1}{4}\frac{N_1}{n}$$
$$\hat{q}_2 = \frac{1}{2(r + c - 4)}\frac{N_2}{n}$$
$$\hat{q}_3 = \frac{1}{(r - 2)(c - 2)}\frac{N_3}{n}$$

The test rejects H_1 if $\chi^2(\hat{q}_1, \hat{q}_2, \hat{q}_3)$ is too large, in reference to the χ^2 distribution with $rc - 3$ degrees of freedom.

(b) Under H_0, we have $q_1 = q_2 = q_3 = 1/(rc)$. So $\chi^2(1/(rc), 1/(rc), 1/(rc))$ is the χ^2 used to test H_0 against all alternatives. It has $rc - 1$ degrees of freedom. For testing H_0 against H_1, we reject H_0 if the difference, $\chi^2(1/(rc), 1/(rc), 1/(rc)) - \chi^2(\hat{q}_1, \hat{q}_2, \hat{q}_3)$ is too large. This has 2 degrees of freedom, the number of restrictions going from H_1 to H_0.

24.5. We find the maximum likelihood estimates of the p_{ij} under the two hypotheses. The likelihood function, $L(\mathbf{p})$, is proportional to $\prod_{i=1}^I \prod_{j=1}^J p_{ij}^{n_{ij}}$.

(a) Under H, we seek to maximize $L(\mathbf{p})$ under the constraints $\sum_{j=1}^J p_{ij} = 1/I$ for all i. This occurs at $\hat{p}_{ij} = n_{ij}/(In_{i\cdot})$ for all i and j, where $n_{i\cdot} = \sum_{j=1}^J n_{ij}$. The χ^2 statistic is

$$\chi_a^2 = \sum_{i=1}^I \sum_{j=1}^J \frac{(n_{ij} - N\hat{p}_{ij})^2}{N\hat{p}_{ij}} = \sum_{i=1}^I \frac{(n_{i\cdot} - (N/I))^2}{N/I}.$$

For each i, $J - 1$ parameters were estimated so the χ^2 has $(IJ - 1) - I(J - 1) = I - 1$ degrees of freedom.

(b) Under H_0, we seek to maximize L when p_{ij} is replaced by p_j and we have the constraint $\sum_{j=1}^J p_j = 1/I$. The maximum likelihood estimates are $p_j^* = n_{\cdot j}/(IN)$ and the chi-square is

$$\chi_b^2 = \sum_{i=1}^I \sum_{j=1}^J \frac{(n_{ij} - Np_j^*)^2}{Np_j^*}.$$

It has $(IJ-1)-(J-1) = IJ-J$ degrees of freedom.

(c) To test H_0 against $H-H_0$, we use $\chi_b^2-\chi_a^2$. It has $(IJ-J)-(I-1) = IJ-I-J+1 = (I-1)(J-1)$ degrees of freedom. Under H_0, $\chi_b^2-\chi_a^2$ is asymptotically equivalent to the χ^2 of Example 24.1 for testing the homogeneity of a contingency table.

24.7. (a) We find the maximum likelihood estimates of the p_{ij} under H_0. The likelihood function, $L(\boldsymbol{p})$, is proportional to $\prod_i \prod_j p_{ij}^{n_{ij}}$. Under H_0, we seek to maximize L when p_{ij} is replaced by p_j and we have the constraint $\sum_{j=1}^J p_j = 1/I$. The maximum likelihood estimates are $p_j^* = \sum_{i=1}^I n_{ij}/(IN)$ and the chi-square is

$$\chi^2 = \sum_{i=1}^I \sum_{j=1}^J \frac{(n_{ij}-Np_j^*)^2}{Np_j^*}.$$

The original chi-square had $(IJ-1)$ degrees of freedom, and we have estimated $(J-1)$ parameters, so the above chi-square has $(IJ-1)-(J-1) = IJ-J$ degrees of freedom.

(b) This χ^2 has approximately a $\chi_J^2(\lambda)$ distribution, where the non-centrality parameter, λ is found by replacing n_{ij} wherever it occurs in χ^2 by Np_{ij}, where p_{ij} are the true values. Here we replace n_{ij} by $N(1+\epsilon_i)/(IJ)$. This also leads to replacing p_j^* by

$$\sum_{i=1}^I \frac{N(1+\epsilon_i)}{IJ}\,\frac{1}{IN} = \frac{1}{IJ}.$$

This gives

$$\begin{aligned}
\lambda &= \sum_{i=1}^I \sum_{j=1}^J \frac{\left(\frac{N(1+\epsilon_i)}{IJ}-\frac{N}{IJ}\right)^2}{N/IJ} = N\sum_{i=1}^I \sum_{j=1}^J \frac{\epsilon_i^2}{IJ} \\
&= N\,\frac{1}{I}\sum_{i=1}^I \epsilon_i^2.
\end{aligned}$$