Solutions to Exercise Set 3.

5.1. (a) As in the proof of Chebyshev's inequality, we have $\mathrm{E}|X_{nj}|^{2+\alpha} \geq \mathrm{E}[|X_{nj}|^{2+\alpha}\mathrm{I}(|X_{nj}| \geq \epsilon B_n)] = \mathrm{E}(X_{nj}^2|X_{nj}|^{\alpha}\mathrm{I}(|X_{nj}| \geq \epsilon B_n)) \geq \epsilon^{\alpha} B_n^{\alpha}\mathrm{E}[X_{nj}^2\mathrm{I}(|X_{nj}| \geq \epsilon B_n)]$.

(b) The Lindeberg condition holds using the inequality in (a) since

$$\frac{1}{B_n^2}\sum_{j=1}^{n}\mathrm{E}[X_{nj}^2\mathrm{I}(|X_{nj}| \geq \epsilon B_n) \leq \frac{1}{\epsilon^{\alpha}B_n^{2+\alpha}}\sum_{j=1}^{n}\mathrm{E}|X_{nj}|^{2+\alpha} \to 0$$

as $n \to \infty$ from the assumption.

5.2. For an exponential random variable with mean β, we have $\mathrm{E}X^n = n!\beta^n$. So, $\mathrm{E}(X-\beta)^2 = \beta^2$ and $\mathrm{E}(X-\beta)^4 = \mathrm{E}X^4 - 4\beta\mathrm{E}X^3 + 6\beta^2\mathrm{E}X^2 - 4\beta^3\mathrm{E}X + \beta^4 = 9\beta^4$. Hence, $\mathrm{Var}Z_n = B_n^2 = \sum_{j=1}^n \beta_j^2$, and

$$\frac{1}{B_n^4}\sum_{j=1}^{n}\mathrm{E}(X_j - \mathrm{E}X_j)^4 = \frac{9\sum_1^n \beta_j^4}{(\sum_1^n \beta_j^2)^2} \leq \frac{9(\max_{1\leq j\leq n}\beta_j^2)\sum_1^n \beta_j^2}{(\sum_1^n \beta_j^2)^2} = \frac{9(\max_{1\leq j\leq n}\beta_j^2)}{\sum_1^n \beta_j^2} \to 0.$$

as $n \to \infty$. It follows from Liapounov's Theorem that $(Z_n - \mathrm{E}Z_n)/B_n \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$.

5.5. (a) From the electronic textbook, we find $\mathrm{P}(X \leq 10) = .58304$.

(b) $\mathrm{P}(X \leq 10) = \mathrm{P}(X \leq 10.5)$ (the correction for continuity). When $\lambda = 10$, $\mathrm{E}X = 10$ and $\mathrm{Var}X = 10$. So if $Z = (X - 10)/\sqrt{10}$, then $\mathrm{P}(X \leq 10.5) = \mathrm{P}(Z \leq z)$ where $z = (10.5 - 10)/\sqrt{10} = .15811$. From the normal distribution in the textbook, we find $\mathrm{P}(Z \leq z) \approx .56282$.

(c) For $X \in \mathcal{P}(\lambda)$, we find $\mathrm{E}X = \lambda$, $\mathrm{Var}X = \lambda$, $\mathrm{E}(X-\lambda)^3 = \lambda$, and $\mathrm{E}(X-\lambda)^4 = 3\lambda^2 + \lambda$. Hence $\beta_1 = 1/\sqrt{\lambda}$ and $\beta_2 = 1/\lambda$. The first Edgeworth correction term is $-\beta_1(z^2-1)\varphi(z)/6 = .02025$, so the corresponding extimate of the probability is $.56282 + .02025 = .58307$, amazingly close!

(d) The second Edgeworth correction term is $-\varphi(z)[3\beta_2(z^3 - 3z) + \beta_1^2(z^5 - 10z^3 + 15z)]/72 = -.00052$, so the corresponding estimate of the probability is $.58307 - .00052 = .58255$.

6.4.(a) Let $\hat{\theta}_n = ((n+c)/n)M_n$. If $n(\theta - M_n) \xrightarrow{\mathcal{L}} Z$, where $Z \in \mathcal{G}(1,\theta)$, then $n(\theta - \hat{\theta}_n) = (n+c)(\theta - M_n) - c\theta \xrightarrow{\mathcal{L}} Z - c\theta$. This has an exponential distribution starting at $-c\theta$.

(b) We want to choose c to minimize the expectation of $(Z - c\theta)^2$. This is done by choosing $c\theta$ to be the mean of Z, namely θ, so $c = 1$.

(c) We want to choose c to minimize the expectation of $|Z - c\theta|$. This is done by choosing $c\theta$ to be the median, m, of Z. This is found by solving $F(x) = 1 - \exp\{-x/\theta\} = 1/2$. This gives $c = \log 2 = .693\cdots$.