Finite difference Burgers Equation and Slpit step Burgers Equation

$\frac{du}{dt} = v\frac{d^2u}{dx^2} - u\frac{du}{dx}$

$u\frac{du}{dx}$ —- Wave stepping term

$v\frac{d^2u}{dx^2}$ —- Diffusion term

$\frac{du}{dt} \approx \frac{u_i^{n+1} - u_i^n}{t^{n+1} - t^n} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$

The above modification is done in forward finite difference scheme

$\frac{d^2u}{dx^2} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$

$\frac{du}{dx} \approx \frac{u_{i+1}^n - u_i^n}{(\Delta x)}$

The above modification is done in central finite difference scheme

The precise numerical method used here is FTCS (forward in time centered in space) method

Now rearranging the terms we get

$u_i^{n+1} = u_i^n + \Delta t * (v\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} - u\frac{u_{i+1}^n - u_i^n}{(\Delta x)})$

Now taking Gaussian function as the initial solution then We get

Part 2 : Split Step Fourier Method

Finite difference method approximate derivatives of a function by local arguments (such as $\acute{u} \approx \frac{u(x+h) - u(x-h)}{2h}$)

In contrast, spectral methods are global. The traditional way to introduce them starts by approximating the function as a sum of very smooth basis functions.

$u(x,t) \approx \sum_{N}^{k=0} u_k(t)\phi_k(x)$

where the $\phi_k(x)$ are polynomials or trigonometric functions. In practice, there are many feasible choices of the basis functions, such as:

$\phi_k(x) = e^{ikx}$ (The fourier spectral method)

$\phi_k(x) = T_K(x)$ ($T_K(x)$ are the Chebyshev polynomials; the Chebyshev spectral method);

$\phi_k(x) = L_K(x)$ ($L_K(x)$ are the Legendre polynomials; the Legendre spectral method);

Now using Fourier spectral method (i.e. the basis functions are chosen as e^{ikx})

The Burgers equation can be written as $\frac{du}{dt} = v\frac{d^2u}{dx^2} - u\frac{du}{dx}$

$\frac{du}{dt} = \sum_{N}^{k=0} u_k'(t)e^{ik(x)}$

$vu_{xx} = v\sum_{N}^{k=0} u_k(t)(ik)^2 e^{ik(x)}$

$uu_x = u\sum_{N}^{k=0} u_k(t)(ik)e^{ik(x)}$

Since the frequencies are uncoupled, we have $u_k'(t) = -vu_k(t)k^2 - u(ik)u_k(t)$

After that we have $u_k(t) = e^{-vk^2t}u_k(0) + e^{-u(ik)t}u_k(0)$