1

Demostración:

$$-\int_{\gamma^{\sim}} f(z)\,dz = \int_a^b - f(\gamma^{\sim}(t))\,\gamma'^{\sim}(t)\,dt$$

$$= \int_a^b \cancel{-} f(\gamma(a+b-t)) \cancel{-} \gamma'(a+b-t)\,dt$$

$$= \int_b^a - f(\gamma(u))\,\gamma'(u)\,du \quad \leftarrow \quad \begin{array}{l} u = a+b-t \\ du = -dt \end{array}$$

$$= \int_a^b f(\gamma(u))\gamma'(u)\,du = \int_\gamma f(z)\,dz$$

2 **Demostración**

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \quad \text{definición}$$

$$\leq \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

$$\leq \int_a^b M\,|\gamma'(t)|\,dt$$

$$= M\int_a^b |\gamma'(t)|\,dt = ML$$

3 **Demostración:** Como $f: D \to \mathbb{C}$ tiene una derivada continua f', es claro que f es una primitiva de f', entonces

$$\int_\gamma f'(z)\,dz = \int_a^b f'(\gamma(t))\gamma'(t)\,dt$$

$$= \int_a^b \left(f(\gamma(t))\right)' \, dt$$

$$= f(\gamma(t))\Big|_a^b \quad \square$$

④ **Demostración:** Sea $I = \int_\gamma f(z)\,dz \in \mathbb{C}$, existe $w = e^{i\theta}$ tq $wI \in \mathbb{R}^{\geq 0}$, luego

$$|I| = e^{i\theta}\int_\gamma f(z)\,dz = \int_0^{2\pi} f(e^{it})\, i e^{i(t+\theta)}\,dt$$

$$= \operatorname{Re}\left(\int_0^{2\pi} f(e^{it})\, i e^{i(t+\theta)}\,dt\right)$$

$$= \int_0^{2\pi} - f(e^{it}) \sin(t+\theta)\,dt$$

$$\leq \int_0^{2\pi} |\sin(t+\theta)|\,dt = \int_0^{2\pi} |\sin t|\,dt \quad \square$$

⑤ **Demostración** $\gamma(t) = z_0 + re^{it}$ con $t \in [a,b]$,

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0} = \frac{1}{2\pi \cancel{i}}\int_a^b \frac{dt}{\cancel{re^{it}}}\cancel{ire^{it}} = \frac{b-a}{2\pi}$$

⑥

- $\int_\gamma e^z\,dz = e^z\Big|_{z_1}^{z_2} \quad \square$
- $\int_\gamma \cos z\,dz = \sin\Big|_{z_1}^{z_2} \quad \square$
- $\int_\gamma z^n\,dz = \frac{z^{n+1}}{n+1}\Big|_{z_1}^{z_2} \quad \square$

7 Demostración:

$$fg\big|_{\gamma} = f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a))$$

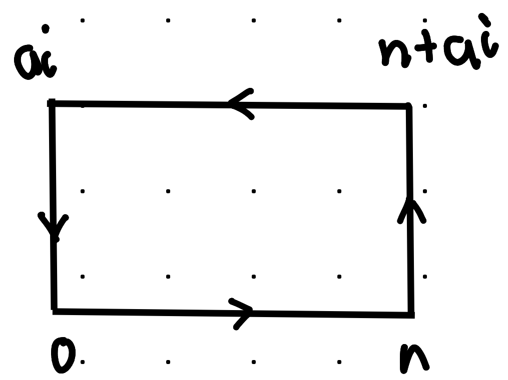
$$= \int_a^b (fg)'(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_a^b f'(\gamma(t))g(\gamma(t))\gamma'(t)\,dt + \int_a^b f(\gamma(t))g'(\gamma(t))\gamma'(t)\,dt$$

$$= \int_{\gamma} f'(z)g(z)\,dz + \int_{\gamma} f(z)g'(z)\,dz \quad \boxed{\checkmark}$$

8 Demostración

$\gamma_n =$



a) $$\int_{\gamma_n} e^{-z^2}\,dz = 0 = \int_0^n e^{-t^2}\,dt + \int_0^a e^{-(n+ti)^2}\, i\,dt$$

$$+\int_n^0 e^{-(t+ai)^2}\,dt + \int_a^0 e^{-(ti)^2}\, i\,dt$$

$$= J_1 + J_2 + J_3 + J_4 = 0$$

$$\lim_{n\to\infty} \int_{\gamma_n} e^{-z^2}\,dz = \cancelto{\frac{\sqrt{\pi}}{2}}{\int_0^\infty e^{-t^2}\,dt} + \int_0^a \cancelto{0}{\lim_{n\to\infty} e^{-(n+ti)^2}}\, i\,dt$$

A

$$-\left(\int_0^\infty e^{-t^2-2tai+a^2}\,dt + \int_0^a e^{t^2}\, i\,dt\right)$$

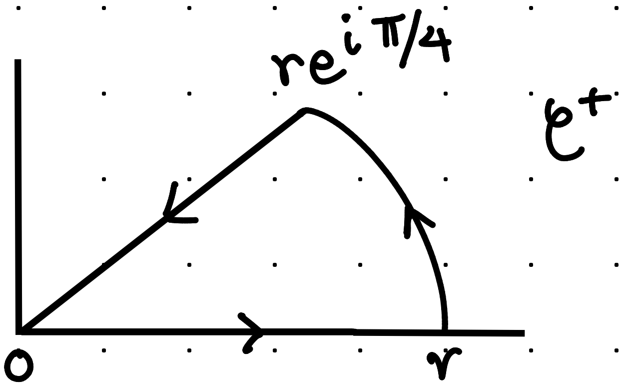
$$A = \sqrt{\pi}/2 - \left(\int_0^\infty e^{-t^2+a^2} e^{-2tai}\,dt + \int_0^a e^{t^2} i\,dt\right)$$

$$= \sqrt{\pi}/2 - \left(\int_0^\infty e^{-t^2+a^2}\left(\cos(2ta) - i\sin(2ta)\right)dt + \int_0^a e^{t^2} i\,dt\right)$$

$$= \sqrt{\pi}/2 - \left(\int_0^\infty e^{-t^2+a^2}\cos(2ta)\,dt - i\int_0^\infty e^{-t^2+a^2}\sin(2ta)\,dt + \int_0^a e^{t^2} i\,dt\right)$$

$$= \frac{\sqrt{\pi}}{2} - \underbrace{e^{a^2}\int_0^\infty e^{-t^2}\cos(2at)\,dt}_{\to 0} + i\underbrace{\left(e^{a^2}\int_0^\infty e^{-t^2}\sin(2ta)\,dt - \int_0^a e^{t^2}\,dt\right)}_{\to 0}$$

9 Demostración:



$$\underbrace{\int_{C^+} e^{iz^2}\,dz}_{B} = 0 = \int_0^r e^{it^2}\,dt + \int_0^{\pi/4} e^{ir^2(\cos(2t)+i\sin(2t))}\, ire^{it}\,dt$$

$$+ \int_1^0 e^{i(t^2 e^{i\pi/2} r^2)}\, re^{i\pi/4}\,dt$$

$$= \int_0^r e^{it^2}\,dt + \int_0^{\pi/4} e^{ir^2(\cos(2t)+i\sin(2t))}\, ire^{it}\,dt$$

$$- \int_0^r e^{-t^2} e^{i\pi/4}\,dt$$

Note q'

C

$$\left|\int_0^{\pi/4} e^{ir^2\cos(2t)} e^{-r^2\sin(2t)}\, ire^{it}\,dt\right| \le \int_0^{\pi/4} \left|e^{-r^2\sin(2t)}\right| |r|\,dt$$

$r \to \infty$

$$|C| \leq \int_0^{\pi/4} \lim_{r\to\infty} \left|e^{-r^2\sin(2t)}\right| |r|\, dt = \int_0^{\pi/4} 0\, dt \to 0$$

Ahora

$$B = 0 = \int_0^\infty \cos(t^2)\,dt + i\int_0^\infty \sin(t^2)\,dt - \frac{\sqrt{\pi}}{2} e^{i\pi/4}$$

$$= \int_0^\infty \cos(t^2)\,dt + i\int_0^\infty \sin(t^2)\,dt - \frac{\sqrt{\pi}}{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$\to \int_0^\infty \cos(t^2)\,dt = \frac{\sqrt{2\pi}}{4} = \int_0^\infty \sin(t^2)\,dt$$

(10) Demonstração:

$$z^{-1}\left(z+\frac{1}{z}\right)^{2n} = z^{-1}\sum_{k=0}^{2n}\binom{2n}{k} z^{2n-k} z^{-k} = \sum_{k=0}^{2n} \frac{2n!\; z^{2n-2k-1}}{k!\,(2n-k)!}$$

luego

$$\oint z^{-1}\left(z+\frac{1}{z}\right)^{2n} dz = \sum_{k=0}^{2n}\frac{2n!}{k!\,(2n-k)!}\oint z^{2n-2k-1}\,dz$$

$$= 2\pi i\left(\frac{(2n)!}{(n!)^2}\right)$$

y ahora

$$\int_0^{2\pi}(2\cos(t))^{2n}\,dt = \int_0^{2\pi}\left(e^{it}+e^{-it}\right)^{2n} dt$$

(Salmo 23) $$=\int_0^{2\pi} \left(e^{it}+e^{-it}\right)^{2n}\left(-ie^{-it}\right) ie^{it}$$

$$=\oint -iz^{-1}\left(z+\frac{1}{z}\right)^{2n} dz = \frac{2\pi\,(2n)!}{(n!)^2} \quad \boxtimes$$

⑪ **Demostración** Note q

$$\int_0^{\pi/2} \cos^{2n}(t)\,dt = \frac{1}{4}\int_0^{2\pi} \cos^{2n}(t)\,dt = \frac{1}{4\,2^{2n}}\int_0^{2\pi} (2\cos(t))^{2n}\,dt$$

$$=\frac{1}{4}\,\frac{2\pi\,(2n)!}{(n!)^2\,2^{2n}} = \pi/2\,\frac{(2n)!}{(n!)^2 2^{2n}} \quad \boxtimes$$

$u=\pi/2-t$

$$\int_0^{\pi/2} \sin^{2n}(t)\,dt = \int_0^{\pi/2} \cos^{2n}(\pi/2-t)\,dt = -\int_{\pi/2}^{0} \cos^{2n}(u)\,du$$

$$=\int_0^{\pi/2} \cos^{2n}(u)\,du \quad \boxtimes$$

⑫ **Demostración:**

$$\underbrace{\oint \frac{e^z}{z}\,dz}_{B} = \int_0^{2\pi} e^{\cos(t)+i\sin t}\, \cancel{e^{-it}}\; i\cancel{e^{it}}\,dt$$

$$=i\int_0^{2\pi} e^{\cos t}\left(\cos(\sin t)+i\sin(\sin t)\right)dt$$

$$B=\oint \sum_{k=0}^{\infty} \frac{z^{k-1}}{k!}\,dz = \sum_{k=0}^{\infty}\oint \frac{z^{k-1}}{k!} = 2\pi i$$

$$\to 2\pi i = i\int_0^{2\pi} e^{\cos t} \cos(\sin t)\, dt \quad \square$$