Lecture 1. Numbers. Sets. Functions.

Numbers. Much of calculus is based on properties of the real number system. **Real numbers** are numbers that can be expressed as decimals, such as

$$-\frac{3}{4} = -0{,}75000....; \ \frac{1}{3} = 0{,}33333....; \ \sqrt{2} = 1{,}4142....$$

The dots … in each case indicate that the sequence of decimal digits goes on forever. Every conceivable decimal expansion represents a real number, although some numbers have two representations. For instance, the infinite decimals 0,999… and 1,000… represent the same real number 1.

The real numbers can be represented geometrically as points on a number line called the **real line**. The symbol R denotes either the real number system or, equivalently, the real line. The properties of the real number system fall into three categories: algebraic properties, order properties, and completeness. The **algebraic properties** say that the real numbers can be added, subtracted, multiplied, and divided (except by 0) to produce more real numbers under the usual rules of arithmetic. *You can never divide by* 0. The **completeness property** of the real number system is deeper and harder to define precisely. However, the property is essential to the idea of limit. Roughly speaking, it says that there are enough real numbers to "complete" the real number line, in the sense that there are no "holes" or "gaps" in it. Many theorems of calculus would fail if the real number system were not complete.

We distinguish three special subsets of real numbers: the **natural numbers**, namely 1, 2, 3, 4, …; the **integers**, namely $0, \pm 1, \pm 2, \pm 3, ...;$ the **rational numbers**, namely the numbers that can be expressed in the form of a fraction $m/n,$ where m and n are integers and $n \neq 0.$ The rational numbers are precisely the real numbers with decimal expansions that are either terminating (ending in an infinite string of zeros), for example, ¾ = 0,75000…= 0,75 or eventually repeating (ending with a block of digits that repeats over and over), for example, $\frac{23}{11} = 2{,}090909... = 2{,}\overline{09}.$ A terminating decimal expansion is a special type of repeating decimal, since the ending zeros repeats. The set of rational numbers has all the algebraic and order properties of the real numbers but lacks the completeness property. For example, there is no rational number whose square is 2; there is a "hole" in the rational line where $\sqrt{2}$ should be. Real numbers that are not rational are called **irrational numbers**.

Sets and Elementary Operations on them. If *x* is an object, *P* is a property, and *P(x)* denotes the assertion that *x* has property *P*, then the class of objects having the property *P* is denoted $\{x \mid P(x)\}$.

The objects that constitute a class or set are called the *elements* of the class or set.

The set consisting of elements $x_1, ..., x_n$ is usually denoted by $\{x_1, ..., x_n\}.$ Wherever no confusion can arise we allow ourselves to denote the one-element set $\{a\}$ simply as *a*.

As has already been pointed out, the objects that comprise a set are usually called the *elements* of the set. We tend to denote sets by uppercase letters and their elements by the corresponding lowercase letters.

The statement, "*x* is an element of the set X " is written briefly as $x \in X$, and its negation as $x \notin X$.

When statements about sets are written, frequent use is made of the logical operators $\exists$ ("there exists" or "there are") and $\forall$ ("every" or "for any") which are called the *existence* and *generalization* quantifiers respectively.

For example, the string $\forall x((x \in A) \Leftrightarrow (x \in B))$ means that for any object *x* the relations $x \in A$ and $x \in B$ are equivalent. Since a set is completely determined by its elements, this statement is usually written briefly as *A* = *B*, read "*A* equals *B*", and means that the sets *A* and *B* are the same.

Thus, two sets are *equal* if they consist of the same elements.

If every element of *A* is an element of *B*, we write $A \subseteq B$ or $B \supseteq A$ and say that *A* is a *subset* of *B* or that *B* contains *A* or that *B* includes *A*. In this connection, the relation $A \subseteq B$ between sets *A* and *B* is called the *inclusion relation*.
If $A \subseteq B$ and $A \neq B$, we shall say that *A* is a *proper* subset of *B* and denote it by $A \subset B$.
Using these definitions, we can now conclude that $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$.
If *M* is a set, any property *P* distinguishes in *M* the subset $\{x \in M \mid P(x)\}$ consisting of the elements of *M* that have the property.
On the other hand, if *P* is taken as a property that no element of the set *M* has, for example, $P(x) := (x \neq x)$, we obtain the set $\varnothing = \{x \in M \mid x \neq x\}$, called the *empty* subset of M.
Let *A* and *B* be subsets of a set M .
The *union* of *A* and *B* is the set $A \cup B := \{x \in M \mid x \in A \vee x \in B\}$, consisting of precisely the elements of *M* that belong to at least one of the sets *A* and *B*.
The *intersection* of *A* and *B* is the set $A \cap B := \{x \in M \mid x \in A \wedge x \in B\}$, formed by the elements of *M* that belong to both sets *A* and *B*.
The *difference* between *A* and *B* is the set $A \setminus B := \{x \in M \mid x \in A \wedge x \notin B\}$, consisting of the elements of *A* that do not belong to *B*.
The difference between the set *M* and one of its subsets *A* is usually called the *complement* of *A* in *M* and denoted by $C_M A$, or *CA, or* $\overline{A}$, when the set in which the complement of *A* is being taken is clear from the context.
Now let *X* and *Y* be arbitrary sets. The set $X \times Y := \{(x, y) \mid x \in X \wedge y \in Y\}$, formed by the ordered pairs *(x, y)* whose first element belongs to *X* and whose second element belongs to *Y*, is called the *direct* or *Cartesian product* of the sets *X* and *Y* (in that order!).
It follows obviously from the definition of the direct product and the remarks made above about the ordered pair that in general $X \times Y \neq Y \times X$. Equality holds only if *X* = *Y*. In this last case, we abbreviate $X \times X$ as X^2. The familiar system of Cartesian coordinates in the plane makes this plane precisely into the direct product of two real axes. This familiar object shows vividly why the Cartesian product depends on the order of the factors. For example, different points of the plane correspond to the pairs (0, 1) and (1, 0).
In the ordered pair $z = (x_1, x_2)$, which is an element of the direct product $Z = X_1 \times X_2$ of the sets X_1 and X_2, the element x_1 is called the *first projection* of the pair *z*, while the element x_2 is the *second projection* of *z*.
By analogy with the terminology of analytic geometry, the projections of an ordered pair are often called the (first and second) *coordinates* of the pair.

Intervals. A subset of the real line is said to be an **interval** if it contains at least two numbers and contains all the real numbers lying between any two its elements. For example, the set of all real numbers x such that $x > 6$ is an interval, as is the set of all x such that $-2 \leq x \leq 5$. The set of all nonzero real numbers is not an interval; since 0 is absent, the set fails to contain every real number between -1 and 1 (for example). Geometrically, intervals correspond to rays and line segments on the real line, along with the real line itself. Intervals of numbers corresponding to line segments are **finite intervals**; intervals corresponding to rays and the real line are **infinite intervals**. A finite interval is said to be **closed** if it contains both of its endpoints, **half-open** if it contains one endpoint but not the other, and **open** if it contains neither endpoint. The endpoints are also called **boundary points**; they make up the interval's **boundary**. The remaining points of the interval are **interior points** and together compose the interval's **interior**. Infinite intervals are closed if they contain a finite endpoint, and open otherwise. The entire real line R is an infinite interval that is both open and closed.

Functions. Let X and Y be certain sets. We say that there is a *function* defined on X with values in Y if, by virtue of some rule f, to each element $x \in X$ there corresponds a unique element $y \in Y$. In this case, the set X is called the *domain* of the function. The symbol x used to denote a general element of the domain is called the *argument* of the function, or the *independent variable.* The element $y_0 \in Y$ corresponding to a particular value $x_0 \in X$ of the argument x is called the *value* of the function at x_0, or the value of the function at the value $x = x_0$ of its argument, and is denoted by $f(x_0)$. As the argument $x \in X$ varies, the value $y = f(x) \in Y$, in general, varies depending on the values of x. For that reason, the quantity $y = f(x)$ is often called the *dependent variable.*

The set $f(X) := \{y \in Y \mid \exists x((x \in X) \wedge (y = f(x)))\}$ of values assumed by a function on elements of the set X is called the *set of values* or the *range* of the function.

The term "function" has a variety of useful synonyms in different areas of mathematics, depending on the nature of the sets X and Y: *mapping, transformation, morphism, operator, functional.* The commonest is *mapping,* and we shall use it frequently.

For a function (mapping) the following notations are standard: $f : X \to Y$, $X \xrightarrow{f} Y$.

If $A \subseteq X$ and $f : X \to Y$ is a function, we denote by $f \mid A$ or $f|_A$ the function $\varphi : A \to Y$ that agrees with f on A. More precisely, $f|_A (x) := \varphi(x)$ if $x \in A$. The function $f|_A$ is called the *restriction* of f to A, and the function $f : X \to Y$ is called an *extension* or a *continuation of* φ *to X.*

When a function $f : X \to Y$ is called a mapping, the value $f(x) \in Y$ that it assumes at the element $x \in X$ *is* usually called the *image* of x.

The *image* of a set $A \subseteq X$ under the mapping $f : X \to Y$ is defined as the set

$$f(A) := \{y \in Y \mid \exists x(x \in A \wedge y = f(x))\}$$

consisting of the elements of Y that are images of elements of A.

The set $f^{-1}(B) := \{x \in X \mid f(x) \in B\}$ consisting of the elements of X whose images belong to B is called the *preimage* (or *complete pre-image)* of the set $B \subseteq Y$.

A mapping $f : X \to Y$ is said to be *surjective* (a mapping of X *onto Y)* if $f(X) = Y$; *injective* (or an *imbedding* or *injection)* if for any elements x_1, x_2 of X $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, that is, distinct elements have distinct images; *bijective* (or a *one-to-one correspondence)* if it is both surjective and injective.

If a mapping $f : X \to Y$ is bijective, that is, it is a one-to-one correspondence between the elements of the sets X and Y, there naturally arises a mapping defined as follows: if $f(x) = y$, then $f^{-1}(y) = x$, that is, to each element $y \in Y$ one assigns the element $x \in X$ whose image under the mapping f is y.

By the surjectivity of f there exists such an element, and by the injectivity of f, it is unique. Hence the mapping f^{-1} is well-defined. This mapping is called the *inverse* of the original mapping f.

The operation of composition of functions is on the one hand a rich source of new functions and on the other hand a way of resolving complex functions into simpler ones.

If the mappings $f : X \to Y$ and $g : Y \to Z$ are such that one of them (in our case g) is defined on the range of the other (f), one can construct a new mapping $g \circ f : X \to Z$, whose values on elements of the set X are defined by the formula $(g \circ f)(x) := g(f(x))$.

The compound mapping $g \circ f$ so constructed is called the *composition (*or **composite***)* of the mapping f and the mapping g (in that order!).

The operation of composition sometimes has to be carried out several times in succession, and in this connection it is useful to note that it is associative, that is, $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof. Indeed, $h\circ(g\circ f)(x)=h(g\circ f)(x)=h(g(f(x)))=(h\circ g)(f(x))=((h\circ g)\circ f)(x)$. □

If all the terms of a composition $f_n\circ\ldots\circ f_1$ are equal to the same function f, we abbreviate it to f^n. We further note that even when both compositions $g\circ f$ and $f\circ g$ are defined, in general $g\circ f\neq f\circ g$.

The mapping $f:X\to X$ that assigns to each element of X the element itself, that is $x\xrightarrow{f}x$, will be denoted id_X and called the *identity mapping* on X.

If f is a function with domain D, its *graph* is $\{(x,f(x))\mid x\in D\}$. Not every curve in the coordinate plane can be the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so no vertical line can intersect the graph of a function more than once (**the vertical line test for a function**). If a is in the domain of the function f, then the vertical line $x=a$ will intersect the graph of f at the single point $(a,f(a))$.

Example. The circle $x^2+y^2=1$ is not the graph of a function; it fails the vertical line test. The upper semicircle is a the graph of a function $f(x)=\sqrt{1-x^2}$. The lower semicircle is the graph of a function $g(x)=-\sqrt{1-x^2}$.

Sometimes a function is described in pieces by using different formulas on different parts of its domain. One example is the **absolute value function:** $|x|=\begin{cases} x,\ x\geq 0\\ -x,\ x<0.\end{cases}$

Piecewise-defined functions often arise when real-world data are modeled. Here are some other examples.

The function whose value at any number x is the greatest integer less than or equal to x is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x\rfloor$.

The function whose value at any number x is the smallest integer greater than or equal to x is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x\rceil$.

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1)<f(x_2)$ whenever $x_1<x_2$, then f is said to be (**strictly**) **increasing** on I.
2. If $f(x_1)>f(x_2)$ whenever $x_1<x_2$, then f is said to be (**strictly**) **decreasing** on I.

A function $y=f(x)$ is an **even function** of x if $f(-x)=f(x)$, and it is an **odd function** of x if $f(-x)=-f(x)$, for every x in the function's domain.

For example, the function $y=x^2$ is even, and the function $y=x^3$ is odd.

The graph of an even function is **symmetric about the y-axis**. The graph of an odd function is **symmetric about the origin**.

A variety of important types of functions is frequently encountered in calculus. We identify and briefly describe then here.

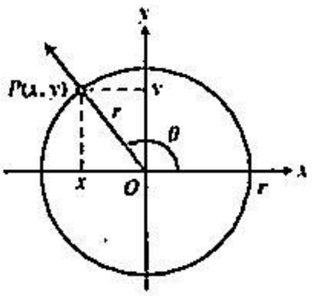
A function of the form $f(x)=mx+b$, for constants m and b, is called a linear function. The function $f(x)=x$ is called the **identity function**. A function $f(x)=x^a$, where a is a constant, is called a **power function**. A function p is a **polynomial** if $p(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ where n is a nonnegative integer and the numbers $a_0,a_1,a_2,\ldots,a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty,\infty)$. If the leading coefficient $a_n\neq 0$ and $n>0$, then n is called the **degree** of the polynomial.

A **rational function** is a quotient or ratio $f(x)=p(x)/q(x)$, where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x)\neq 0$. Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) lies within the class of **algebraic functions**. All rational functions are algebraic, but the class of algebraic functions also included are more complicated functions (for example, $y = x^{1/3}(x-4)$, $y = (3/4)(x^2-1)^{2/3}$, $y = x(1-x)^{2/5}$).

Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called **exponential functions**. All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$, so an exponential function never assumes the value 0. The most important exponential function used for modeling natural, physical, and economic phenomena is the **natural exponential function**, whose base is the special number e. The number e is irrational, and its value is 2,718281828 to nine decimal places.

Logarithmic functions are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions.

Trigonometric functions.



sine: $\sin\theta = \dfrac{y}{r}$; **cosine**: $\cos\theta = \dfrac{x}{r}$; **tangent**: $\tan\theta = \dfrac{y}{x}$; **cotangent**: $\cot\theta = \dfrac{x}{y}$;

cosecant: $\csc\theta = \dfrac{r}{y}$; **secant**: $\sec\theta = \dfrac{r}{x}$.

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x+p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

The tangent and cotangent functions have period π, and the other four functions have period 2π.

We define the arcsine and arccosine as functions whose values are angles (measured in radians) that belong to restricted domains of the sine and cosine functions:

$y = \sin^{-1} x$ (or $y = \arcsin x$) is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$y = \cos^{-1} x$ (or $y = \arccos x$) is the number in $[0, \pi]$ for which $\cos y = x$.

Glossary

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Exercises for Seminar 1

1. Find the domain and range of each function:

a) $f(x) = 1 + x^2$ b) $F(x) = \sqrt{5x+10}$ c) $f(t) = \dfrac{4}{3-t}$

2. Determine whether the function is even, odd, or neither. Give reasons for your answer:

a) $f(x) = x^2 + 1$ b) $g(x) = x^3 + x$ c) $g(x) = \dfrac{1}{x^2-1}$ d) $h(t) = \dfrac{1}{t-1}$ e) $h(t) = 2t + 1$

3. Write the formula for $f \circ g \circ h$:

a) $f(x) = x+1, g(x) = 3x, h(x) = 4-x$ b) $f(x) = \sqrt{x+1}, g(x) = \dfrac{1}{x+4}, h(x) = \dfrac{1}{x}$

4. Write formulas for $f \circ g$ and $g \circ f$ and find the domain and range of each.

$f(x) = \sqrt{x+1}$, $g(x) = 1/x$

5. Evaluate $\sin\dfrac{7\pi}{12}$ as $\sin\left(\dfrac{\pi}{4}+\dfrac{\pi}{3}\right)$. 6. Evaluate $\cos\dfrac{\pi}{12}$.

7. Find the domain and range for the following functions:

a) $f(x)=\dfrac{1}{2+e^x}$ b) $g(t)=\sqrt{1+3^{-t}}$

8. Find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f(f^{-1}(x))=f^{-1}(f(x))=x$.

a) $f(x)=x^5$ b) $f(x)=x^3+1$ c) $f(x)=1/x^2,\ x>0$ d) $f(x)=\dfrac{x+3}{x-2}$

9. Find the exact value of each expression:

a) $\sin^{-1}\left(\dfrac{-1}{2}\right)$ b) $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ c) $\sin^{-1}\left(\dfrac{-\sqrt{3}}{2}\right)$ d) $\arccos(-1)$ e) $\arcsin\left(-\dfrac{1}{\sqrt{2}}\right)$

10. Solve the inequalities, expressing the solution sets as intervals or unions of intervals. Also, show each solution set on the real line:

a) $|t-1|\le 3$ b) $\left|\dfrac{z}{5}-1\right|\le 1$ c) $|2s|\ge 4$ d) $\left|\dfrac{r+1}{2}\right|\ge 1$

Exercises for Homework 1

1. Find the domain and range of each function:

a) $f(x)=1-\sqrt{x}$ b) $g(x)=\sqrt{x^2-3x}$ c) $G(t)=\dfrac{2}{t^2-16}$

2. Determine whether the function is even, odd, or neither. Give reasons for your answer:

a) $f(x)=x^{-5}$ b) $g(x)=x^2+x$ c) $g(x)=x^4+3x^2-1$ d) $h(t)=\dfrac{t}{t^2-1}$ e) $h(t)=|t^3|$

3. Write the formula for $f\circ g\circ h$:

a) $f(x)=3x+4, g(x)=2x-1, h(x)=x^2$ b) $f(x)=\dfrac{x+2}{3-x}, g(x)=\dfrac{x^2}{x^2+1}, h(x)=\sqrt{2-x}$

4. Write formulas for $f\circ g$ and $g\circ f$ and find the domain and range of each.

$f(x)=x^2,\ g(x)=1-\sqrt{x}$

5. Evaluate $\cos\dfrac{11\pi}{12}$ as $\cos\left(\dfrac{\pi}{4}+\dfrac{2\pi}{3}\right)$. 6. Evaluate $\sin\dfrac{5\pi}{12}$.

7. Find the domain and range for the following functions:

a) $f(x)=\dfrac{3}{1-e^{2x}}$ b) $g(t)=\cos(e^{-t})$

8. Find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f(f^{-1}(x))=f^{-1}(f(x))=x$.

a) $f(x)=x^4,\ x\ge 0$ b) $f(x)=(1/2)x-7/2$ c) $f(x)=1/x^3,\ x\ne 0$ d) $f(x)=\dfrac{\sqrt{x}}{\sqrt{x}-3}$

9. Find the exact value of each expression:

a) $\cos^{-1}\left(\dfrac{1}{2}\right)$ b) $\cos^{-1}\left(\dfrac{-1}{\sqrt{2}}\right)$ c) $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ d) $\arccos 0$ e) $\arcsin(-1)$

10. Solve the inequalities, expressing the solution sets as intervals or unions of intervals. Also, show each solution set on the real line:

a) $|3y-7|<4$ b) $|3-1/x|<1/2$ c) $|1-x|>1$