Lecture 9. Convexity, Concavity and Curve Sketching. Procedure for Graphing.

Convexity (Concavity) and Curve Sketching: We have seen how the first derivative tells us where a function is increasing, where it is decreasing, and whether a local maximum or local minimum occurs at a critical point. Here we see that the second derivative gives us information about how the graph of a differentiable function bends or turns. With this knowledge about the first and second derivatives, coupled with our previous understanding of symmetry and asymptotic behavior, we can now draw an accurate graph of a function. By organizing all of these ideas into a coherent procedure, we give a method for sketching graphs and revealing visually the key features of functions.

The graph of a differentiable function $y = f(x)$ is

(a) **concave (concave up)** on an open interval I if f' is increasing on I;

(b) **convex (concave down)** on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity: Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave.
2. If $f'' < 0$ on I, the graph of f over I is convex.

If $y = f(x)$ is twice-differentiable, we will use the notations f'' and y'' interchangeably when denoting the second derivative.

Example 1. (a) The curve $y = x^3$ is convex on $(-\infty, 0)$, where $y'' = 6x < 0$ and concave on $(0, \infty)$ where $y'' = 6x > 0$.

(b) The curve $y = x^2$ is concave on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive.

Points of Inflection: A point $(c, f(c))$ where the graph of a function has a tangent line and where the convexity (concavity) changes is a **point of inflection**.

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

Example 2. The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)x^{2/3} = 0$ when $x = 0$. However, the second derivative $f''(x) = \dfrac{10}{9}x^{-1/3}$ fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin.

Example 3. The curve $y = x^4$ has no inflection point at $x = 0$. Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign.

Theorem 1 (Second Derivative Test for Local Extrema). Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Proof. Part (1). If $f''(c) < 0$, then $f''(x) < 0$ on some open interval I containing the point c, since f'' is continuous. Therefore, f' is decreasing on I. Since $f'(c) = 0$, the sign of f' changes from positive to negative at c so f has a local maximum at c by the First Derivative Test.

The proof of Part (2) is similar.

For Part (3), consider the three functions $y = x^4, y = -x^4$, and $y = x^3$. For each function, the first and second derivatives are zero at $x = 0$. Yet the function $y = x^4$ has a local minimum there,

$y=-x^4$ has a local maximum, and $y=x^3$ is increasing in any open interval containing $x = 0$ (having neither a maximum nor a minimum there). Thus, the test fails. □

Example 4. Sketch a graph of the function $f(x)=x^4-4x^3+10$ using the following steps.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave and where it is convex.

(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

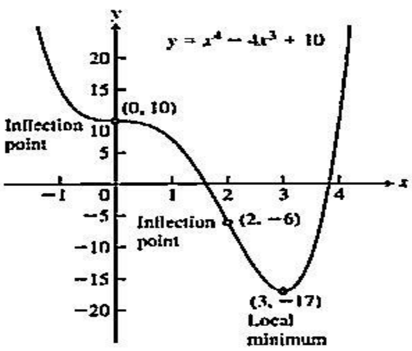
Solution. The function f is continuous since $f'(x)=4x^3-12x^2$ exists. The domain of f is $(-\infty,\infty)$, and the domain of f' is also $(-\infty,\infty)$. Thus, the critical points of f occur only at the zeros of f'. Since $f'(x)=4x^3-12x^2=4x^2(x-3)$, the first derivative is zero at $x = 0$ and $x = 3$. We use these critical points to define intervals where f is increasing or decreasing.

(a) Using the First Derivative Test for local extrema, we see that there is no extremum at $x=0$ and there is a local minimum at $x=3$.

(b) Also we see that f is decreasing on $(-\infty, 0]$ and $[0, 3]$, and f is increasing on $[3,\infty)$.

(c) $f''(x)=12x^2-24x=12x(x-2)$ is zero at $x=0$ and $x=2$. We use these points to define intervals where f is concave or convex. We see that f is concave on the intervals $(-\infty, 0)$ and $(2, \infty)$, and convex on $(0, 2)$.

(e) Plot the curve's intercepts (if possible) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape as a guide to sketch the curve. Plot additional points as needed.



Procedure for Graphing $y = f(x)$:

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

Example 5. Sketch the graph of $f(x)=\dfrac{(x+1)^2}{1+x^2}$.

Solution: 1. The domain of f is $(-\infty,\infty)$ and there are no symmetries about either axis or the origin.

2. Find f' and f'': $f'(x)=\dfrac{2(1-x^2)}{(1+x^2)^2}$; $f''(x)=\dfrac{4x(x^2-3)}{(1+x^2)^3}$.

3. The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f. At $x = -1$, $f''(-1) = 1 > 0$, yielding a local minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$, yielding a local maximum by the Second Derivative test.

4. *Increasing and decreasing*. We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.

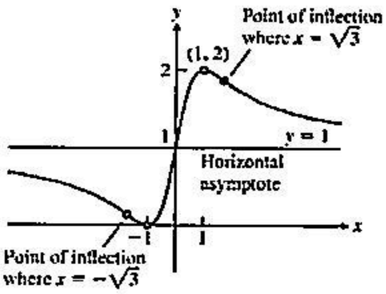
5. *Inflection points*. Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}$, 0 and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $(-\infty, -\sqrt{3})$, positive on $(-\sqrt{3}, 0)$, negative on $(0, \sqrt{3})$, and positive again on $(\sqrt{3}, \infty)$. Thus, each point is a point of inflection. The curve is convex on the interval $(-\infty, -\sqrt{3})$, concave on $(-\sqrt{3}, 0)$, convex on $(0, \sqrt{3})$, and concave again on $(\sqrt{3}, \infty)$.

6. *Asymptotes*. Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives $f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} = \frac{1+(2/x)+(1/x^2)}{(1/x^2)+1}$.

We see that $f(x) \to 1^+$ as $x \to \infty$ and that $f(x) \to 1^-$ as $x \to -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since f decreases on $(-\infty, -1)$ and then increases on (-1, 1), we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$, it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$, approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $0 \le y \le 2$).

7. Notice how the graph is convex as it approaches the horizontal asymptote $y = 1$ as $x \to -\infty$, and concave in its approach to $y = 1$ as $x \to \infty$.



Glossary

non-overlapping – непересекающийся; **reveal** – раскрыть; **plot** – построить
point of inflexion – точка перегиба; **intercept** – отрезок прямой; **to bend** – сгибать
coherent – последовательный

Exercises for Seminar 9

1. (Graphing Functions). Identify the coordinates of any local and absolute extreme points and inflection points of the following functions. Graph the function.

a) $y = x^2 - 4x + 3$; b) $y = x^3 - 3x + 3$;

c) $y = x + \sin x,\ 0 \le x \le 2\pi$; d) $y = \dfrac{x}{\sqrt{x^2 + 1}}$.

2. (Sketching the General Shape, Knowing y'). Find y'' and then use Steps 2-5 of the graphing procedure to sketch the general shape of the graph of f:

a) $y' = 2 + x - x^2$; b) $y' = x(x-3)^2$; c) $y' = \sec^2 x,\ -\pi/2 < x < \pi/2$;
d) $y' = (x+1)^{-2/3}$.

3. (Graphing Rational Functions) Graph the rational functions using all the steps in the graphing procedure:

a) $y = \dfrac{2x^2 + x - 1}{x^2 - 1}$; b) $y = \dfrac{x^4 + 1}{x^2}$; c) $y = \dfrac{1}{x^2 - 1}$; d) $y = \dfrac{x^2 - 2}{x^2 - 1}$.

Exercises for Homework 9

1. (Graphing Functions). Identify the coordinates of any local and absolute extreme points and inflection points of the following functions. Graph the function.

a) $y = 6 - 2x - x^2$; b) $y = x(6-2x)^2$;
c) $y = x - \sin x,\ 0 \le x \le 2\pi$; d) $y = \sqrt{1 - x^2}$.

2. (Sketching the General Shape, Knowing y'). Find y'' and then use Steps 2-5 of the graphing procedure to sketch the general shape of the graph of f:

a) $y' = x^2 - x - 6$; b) $y' = x^2(2-x)$;
c) $y' = \tan x,\ -\pi/2 < x < \pi/2$; d) $y' = (x-2)^{-1/3}$.

3. (Graphing Rational Functions) Graph the rational functions using all the steps in the graphing procedure:

a) $y = \dfrac{x^2 - 49}{x^2 + 5x - 14}$; b) $y = \dfrac{x^2 - 4}{2x}$; c) $y = \dfrac{x^2}{x^2 - 1}$; d) $y = \dfrac{x^2 - 4}{x^2 - 2}$.