Lecture 7. Linearization and Differentials.

Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications and easier to work with. The approximating functions discussed here are called *linearizations*, and they are based on tangent lines. We introduce new variables dx and dy, called differentials, and define them in a way that makes Leibniz's notation for the derivative dy/dx a true ratio. We use dy to estimate error in measurement, which then provides for a precise proof of the Chain Rule.

Linearization. If f is differentiable at $x=a$, then the approximating function $L(x)=f(a)+f'(a)(x-a)$ is the **linearization** of f at a. The approximation $f(x)\approx L(x)$ of f by L is the **standard linear approximation** of f at a. The point $x=a$ is the **center** of the approximation.

Example 1. Find the linearization of $f(x)=\sqrt{1+x}$ at $x=0$.

Solution: Since $f'(x)=\frac{1}{2}(1+x)^{-1/2}$, we have $f(0)=1$ and $f'(0)=1/2$, giving the linearization

$$L(x)=f(a)+f'(a)(x-a)=1+\frac{1}{2}(x-0)=1+\frac{x}{2}.$$

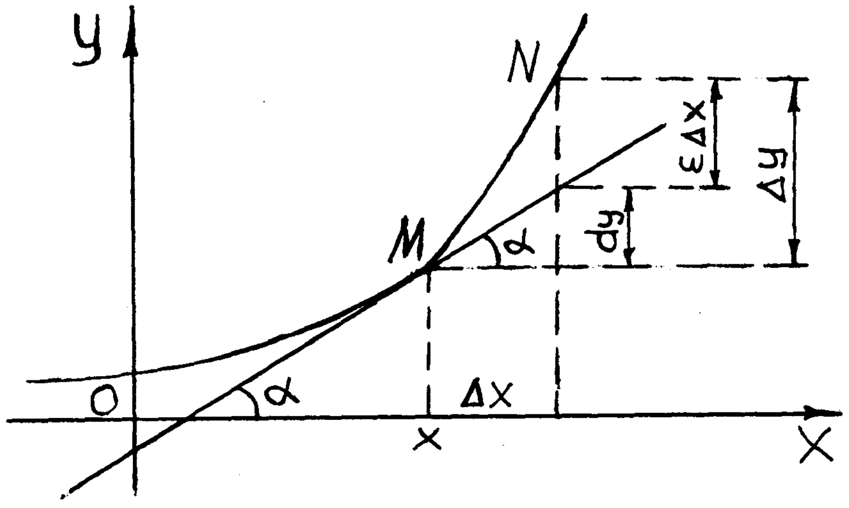
Differentials. Let $y=f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is $dy=f'(x)dx.$

Unlike the independent variable dx, the variable dy is always a dependent variable. It depends on both x and dx. If dx is given a specific value and x is a particular number in the domain of the function f, then these values determine the numerical value of dy. Often the variable dx is chosen to be Δx, the change in x.

Example 2. (a) Find dy if $y=x^5+37x$. (b) Find the value of dy when $x=1$ and $dx=0{,}2$.

Solution: (a) $dy=(5x^4+37)dx$. (b) Substituting $x=1$ and $dx=0{,}2$ in the expression for dy, we have $dy=(5\cdot 1^4+37)\cdot 0{,}2=8{,}4$.

The differential geometrically represents the increment of the ordinate of the tangent to the graph of a function in a point *M(x; y)*.



Here $dx=\Delta x$, $\Delta y=f(x+\Delta x)-f(x)$.

We sometimes write $df=f'(x)dx$ in place of $dy=f'(x)dx$, calling df the **differential of** f. For instance, if $f(x)=3x^2-6$, then $df=d(3x^2-6)=6xdx$.

Basic properties of a differential

1. $dC = 0$ where C is a constant. 2. $d(Cu) = Cdu$.
3. $d(u \pm v) = du \pm dv$. 4. $d(uv) = udv + vdu$.
5. $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$ $(v \neq 0)$. 6. $df(u) = f'(u)du$.

Estimating with Differentials. Suppose we know the value of a differentiable function $f(x)$ at a point a and want to estimate how much this value will change if we move to a nearby point $a + dx$. If $dx = \Delta x$ is small, then we can see from Figure that Δy is approximately equal to the differential dy. Since $f(a + dx) = f(a) + \Delta y,\ \Delta x = dx$, the differential approximation gives $f(a + dx) = f(a) + dy$ when $dx = \Delta x$. Thus the approximation $\Delta y \approx dy$ can be used to estimate $f(a + dx)$ when $f(a)$ is known, dx is small, and $dy = f'(a)dx$.

Example 3. The radius r of a circle increases from $a = 10m$ to $10{,}1m$. Use dA to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

Solution: Since $A = \pi r^2$, the estimated increase is $dA = A'(a)dr = 2\pi\, adr = 2\pi \cdot 10 \cdot 0{,}1 = 2\pi\, m^2$.

Thus, since $A(r + \Delta r) = A(r) + dA$, we have $A(10 + 0{,}1) = A(10) + 2\pi = \pi \cdot 10^2 + 2\pi = 102\pi$.

The area of a circle of radius $10{,}1m$ is approximately $102\pi\, m^2$.

The true area is $A(10{,}1) = \pi \cdot (10{,}1)^2 = 102{,}01\pi\, m^2$.

The error in our estimate is $0{,}01\pi\, m^2$, which is the difference $\Delta A - dA$.

Example 4. Use differentials to estimate (a) $7{,}97^{1/3}$ and (b) $\sin(\pi/6 + 0{,}01)$.

Solution: (a) The differential associated with the cube root function $y = x^{1/3}$ is $dy = \frac{dx}{3x^{2/3}}$. We set $a = 8$, the closest number near $7{,}97$ where we can easily compute $f(a)$ and $f'(a)$. To arrange that $a + dx = 7{,}97$, we choose $dx = -0{,}03$. Approximating with the differential gives

$$f(7{,}97) = f(a + dx) \approx f(a) + dy = 8^{1/3} + \frac{-0{,}03}{3 \cdot 8^{2/3}} = 2 - \frac{0{,}03}{12} = 1{,}9975.$$

This gives an approximation to the true value of $7{,}97^{1/3}$, which is $1{,}997497$ to 6 decimals.

(b) The differential associated with $y = \sin x$ is $dy = \cos x dx$. To estimate $\sin(\pi/6 + 0{,}01)$, we set $a = \pi/6$ and $dx = 0{,}01$. Then

$$f(\pi/6 + 0{,}01) = f(a + dx) \approx f(a) + dy = \sin(\pi/6) + \cos(\pi/6) \cdot 0{,}01 = 1/2 + (\sqrt{3}/2) \cdot 0{,}01 \approx 0{,}5087$$

For comparison, the true value of $\sin(\pi/6 + 0{,}01)$ to 6 decimals is 0,508635.

Remark: We will use differentials mainly as a bookkeeping device to aid in the process of finding indefinite integrals.

Error in Differential Approximation. Let $f(x)$ be differentiable at $x = a$ and suppose that $dx = \Delta x$ is an increment of x. We have two ways to describe the change if f as x changes from a to $a + \Delta x$:

The true change: $\Delta f = f(a + \Delta x) - f(a)$

The differential estimate: $df = f'(a)\Delta x$.

How well does df approximate Δf?

We measure the approximation error by subtracting df from Δf:

Approximation error $= \Delta f - df = \Delta f - f'(a)\Delta x = f(a + \Delta x) - f(a) - f'(a)\Delta x =$

$$= \left(\frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a)\right) \cdot \Delta x = \varepsilon \cdot \Delta x.$$

As $\Delta x \to 0,$ the difference quotient $\dfrac{f(a+\Delta x)-f(a)}{\Delta x}$ approaches $f'(a)$ (remember the definition of $f'(a)$), so the quantity in parentheses becomes a very small number (which is why we called it ε). In fact, $\varepsilon \to 0$ as $\Delta x \to 0.$

Change in $y = f(x)$ **near** $x = a$**:** If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x,$ the change Δy in f is given by $\Delta y = f'(a)\Delta x + \varepsilon \Delta x$ (1)
in which $\varepsilon \to 0$ as $\Delta x \to 0.$
In Example 3 we found that $\Delta A = \pi \cdot (10{,}1)^2 - \pi \cdot 10^2 = (102{,}01 - 100)\pi = (2\pi + 0{,}01\pi)\,m^2$, so the approximation error is $\Delta A - dA = \varepsilon\, \Delta r = 0{,}01\pi$ and $\varepsilon = 0{,}01\pi / \Delta r = 0{,}01\pi / 0{,}1 = 0{,}1\pi\, m.$

Proof of the Chain Rule. Equation (1) enables us to prove the Chain Rule correctly. Our goal is to show that if $f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of $x,$ then the composite $y = f(g(x))$ is a differentiable function of $x.$ Since a function is differentiable iff it has a derivative at each point in its domain, we must show that whenever g is differentiable at x_0 and f is differentiable at $g(x_0),$ then the composite is differentiable at x_0 and the derivative of the composite satisfies the equation $\left.\dfrac{dy}{dx}\right|_{x=x_0} = f'(g(x_0)) \cdot g'(x_0).$
Let Δx be an increment in x and let Δu and Δy be the corresponding increments in u and $y.$ Applying Equation (1) we have $\Delta u = g'(x_0)\Delta x + \varepsilon_1 \Delta x = (g'(x_0) + \varepsilon_1)\Delta x,$ where $\varepsilon_1 \to 0$ as $\Delta x \to 0.$
Similarly, $\Delta y = f'(u_0)\Delta u + \varepsilon_2 \Delta u = (f'(u_0) + \varepsilon_2)\Delta u,$ where $\varepsilon_2 \to 0$ as $\Delta u \to 0.$
Notice also that $\Delta u \to 0$ as $\Delta x \to 0.$ Combining the equations for Δu and Δy gives

$$\Delta y = (f'(u_0) + \varepsilon_2)(g'(x_0) + \varepsilon_1)\Delta x,$$

so $\dfrac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) + \varepsilon_2 g'(x_0) + f'(u_0)\varepsilon_1 + \varepsilon_2\varepsilon_1.$
Since ε_1 and ε_2 go to zero as Δx goes to zero, the last three terms on the right vanish in the limit, leaving $\left.\dfrac{dy}{dx}\right|_{x=x_0} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) = f'(g(x_0)) \cdot g'(x_0).$

Inverse Trigonometric Functions.
$y = \tan^{-1} x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x.$
$y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x.$
$y = \sec^{-1} x$ is the number in $[0, \pi/2) \cup (\pi/2, \pi]$ for which $\sec y = x.$
$y = \csc^{-1} x$ is the number in $[-\pi/2, 0) \cup (0, \pi/2]$ for which $\csc y = x.$

The Derivative of $y = \sin^{-1} x$
We find the derivative of $y = \sin^{-1} x$ by applying Theorem 2 (Lecture 6) with $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$: $\left(f^{-1}\right)'(x) = \dfrac{1}{f'\left(f^{-1}(x)\right)} = \dfrac{1}{\cos(\sin^{-1} x)} = \dfrac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} = \dfrac{1}{\sqrt{1 - x^2}}.$
Here we use that $f'(u) = \cos u,$ $\cos u = \sqrt{1 - \sin^2 u},$ $\sin(\sin^{-1} x) = x.$
If u is a differentiable function of x with $|u| < 1,$ we apply the Chain Rule to get the general formula

$$\frac{d}{dx}\left(\sin^{-1} u\right)=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx},\ |u|<1.$$

Example 5. Using the Chain Rule, we calculate the derivative

$$\frac{d}{dx}\left(\sin^{-1} x^2\right)=\frac{1}{\sqrt{1-(x^2)^2}}\cdot\frac{d}{dx}\left(x^2\right)=\frac{2x}{\sqrt{1-x^4}}.$$

The Derivative of $y=\tan^{-1} x$

$$\left(f^{-1}\right)'(x)=\frac{1}{f'\left(f^{-1}(x)\right)}=\frac{1}{\sec^2(\tan^{-1} x)}=\frac{1}{1+\tan^2(\tan^{-1} x)}=\frac{1}{1+x^2}.$$

Here we use that $f'(u)=sec^2 u, sec^2 u=1+tan^2 u, tan(tan^{-1} x)=x.$

The Derivative of $y=\sec^{-1} x$

Instead of applying the formula in Theorem 2 directly, we find the derivative of $y=\sec^{-1} x$, using implicit differentiation and the Chain Rule as follows:

$y=\sec^{-1} x \Leftrightarrow \sec y=x$. Then $\frac{d}{dx}(\sec y)=\frac{d}{dx}x \Rightarrow \sec y\tan y\frac{dy}{dx}=1\Rightarrow \frac{dy}{dx}=\frac{1}{\sec y\tan y}.$

To express the result in terms of x, we use the relationships

$\sec y=x$ and $\tan y=\pm\sqrt{\sec^2 y-1}=\pm\sqrt{x^2-1}$ to get $\frac{dy}{dx}=\pm\frac{1}{x\sqrt{x^2-1}}.$

With the absolute value symbol, we can write a single expression that eliminates the "±" ambiguity: $\frac{d}{dx}(\sec^{-1} x)=\frac{1}{|x|\sqrt{x^2-1}}.$

Derivatives of the Other Three Inverse Trigonometric Functions

We use the following identities: **Inverse Function-Inverse Cofunction Identities**

$$\cos^{-1} x=\pi/2-\sin^{-1} x,\ \cot^{-1} x=\pi/2-\tan^{-1} x,\ \csc^{-1} x=\pi/2-\sec^{-1} x$$

For example, the derivative of $\cos^{-1} x$ is calculated as follows:

$$\frac{d}{dx}(\cos^{-1} x)=\frac{d}{dx}\left(\frac{\pi}{2}-\sin^{-1} x\right)=-\frac{d}{dx}(\sin^{-1} x)=-\frac{1}{\sqrt{1-x^2}}.$$

Similarly, we find the derivatives of $\cot^{-1} x$ and $\csc^{-1} x$:

$$\frac{d}{dx}(\cot^{-1} x)=-\frac{1}{1+x^2} \text{ and } \frac{d}{dx}(\csc^{-1} x)=-\frac{1}{|x|\sqrt{x^2-1}}.$$

Glossary

bookkeeping device – устройство (приспособление) для подсчета

lateral – боковой

Exercises for Seminar 7

1. Find the linearization $L(x)$ of $f(x)$ at $x=a$:

a) $f(x)=x^3-2x+3,\ a=2$; b) $f(x)=x+1/x,\ a=1.$

2. Use the linear approximation $(1+x)^k\approx 1+kx$ to find an approximation for the function $f(x)$ for values of x near zero:

a) $f(x)=(1-x)^6$; b) $f(x)=1/\sqrt{1+x}$.

3. Find dy:

a) $y = x^3 - 3\sqrt{x}$; b) $y = 2x/(1+x^2)$.

4. Each function $f(x)$ changes value when x changes from x_0 to $x_0 + dx$. Find

a. the change $\Delta f = f(x_0 + dx) - f(x_0)$; **b.** the value of the estimate $df = f'(x_0)dx$; and

c. the approximation error $|\Delta f - df|$.

a) $f(x) = x^2 + 2x, x_0 = 1, dx = 0{,}1$. b) $f(x) = x^3 - x, x_0 = 1, dx = 0{,}1$.

5. Write a differential formula that estimates the given change in volume or surface area.

a) The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from r_0 to $r_0 + dr$

b) The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from x_0 to $x_0 + dx$.

6. Find the derivative of y with respect to the appropriate variable:

a) $y = \cos^{-1}(x^2)$; b) $y = \sin^{-1}\sqrt{2t}$; c) $y = \sec^{-1}(2s+1)$;

d) $y = \csc^{-1}(x^2+1)$; e) $y = \cot^{-1}\sqrt{t}$; fl) $y = \ln(\tan^{-1} x)$.

Exercises for Homework 7

1. Find the linearization $L(x)$ of $f(x)$ at $x = a$:

a) $f(x) = \sqrt{x^2+9},\ a = -4$; b) $f(x) = \sqrt[3]{x}, a = -8$

2. Use the linear approximation $(1+x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of x near zero:

a) $f(x) = 2/(1-x)$; b) $f(x) = \sqrt[3]{1+x}$.

3. Find dy:

a) $y = x\sqrt{1-x^2}$; b) $y = 2\sqrt{x}/(3(1+\sqrt{x}))$.

4. Each function $f(x)$ changes value when x changes from x_0 to $x_0 + dx$. Find

a. the change $\Delta f = f(x_0 + dx) - f(x_0)$; **b.** the value of the estimate $df = f'(x_0)dx$; and

c. the approximation error $|\Delta f - df|$.

a) $f(x) = 2x^2 + 4x - 3, x_0 = -1, dx = 0{,}1$. b) $f(x) = x^4, x_0 = 1, dx = 0{,}1$.

5. Write a differential formula that estimates the given change in volume or surface area.

a) The change in the volume $V = x^3$ of a cube when the edge lengths change from x_0 to $x_0 + dx$

b) The change in the lateral surface area $S = \pi r\sqrt{r^2+h^2}$ of a right circular cone when the radius changes from r_0 to $r_0 + dr$ and the length does not change.

6. Find the derivative of y with respect to the appropriate variable:

a) $y = \cos^{-1}(1/x)$; b) $y = \sin^{-1}(1-t)$; c) $y = \sec^{-1} 5s$;

d) $y = \csc^{-1}\dfrac{x}{2}$; e) $y = \cot^{-1}\sqrt{t-1}$; f) $y = \tan^{-1}(\ln x)$.