Lecture 2. Limit of a Function and Limit Laws.

Here we begin with an informal definition of *limit* and show how we can calculate the values of limits. A precise definition will be given later.

Suppose $f(x)$ is defined on an open interval about c, *except possibly at c itself*. If $f(x)$ is arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to c, we say that f approaches the **limit** L as x approaches c, and write $\lim\limits_{x\to c} f(x) = L$, which is read "the limit of $f(x)$ as x approaches c is L." Our definition here is "informal" because phrases like *arbitrarily close* and *sufficiently close* are imprecise; their meaning depends on the context. (To a machinist manufacturing a piston, *close* may mean *within a few thousandths of an inch*. To an astronomer studying distant galaxies, *close* may mean *within a few thousand light-years*.) Nevertheless, the definition is clear enough to enable us to recognize and evaluate limits of many specific functions.

Consider the following three functions: $f(x) = \frac{x^2-1}{x-1}$; $g(x) = \begin{cases} \frac{x^2-1}{x-1}, x \neq 1 \\ 2, x = 1 \end{cases}$; $h(x) = x + 1$. The limits of $f(x), g(x)$ and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$.

A function may not have a limit at a particular point. Some ways that limits can fail to exist are described in the next example.

Example 1. Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$.

(a) $U(x) = \begin{cases} 0, x < 0 \\ 1, x \geq 0 \end{cases}$; (b) $g(x) = \begin{cases} 1/x, x \neq 0 \\ 0, x = 0 \end{cases}$; (c) $f(x) = \begin{cases} 0, x \leq 0 \\ sin\frac{1}{x}, x > 0 \end{cases}$.

Solution. (a) It *jumps*: The **unit step function** $U(x)$ has no limit as $x \to 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \to 0$.

(b) It *grows too "large" to have a limit*: $g(x)$ has no limit as $x \to 0$ because the values of g grow arbitrarily large in absolute value as $x \to 0$ and do not stay close to *any* fixed real number. We say the function is *not bounded*.

(c) It *oscillates too much to have a limit*: $f(x)$ has no limit as $x \to 0$ because the function's values oscillate between +1 and -1 in every open interval containing 0. The values do not stay close to any one number as $x \to 0$.

To calculate limits of functions that are arithmetic combinations of functions having known limits, we can use several fundamental rules.

Theorem 1 (Limit Laws). If L, M, c, and k are real numbers and $\lim\limits_{x\to c} f(x) = L$ and $\lim\limits_{x\to c} g(x) = M$, then 1. *Sum Rule:* $\lim\limits_{x\to c}\big(f(x) + g(x)\big) = L + M$.

2. *Difference Rule:* $\lim\limits_{x\to c}(f(x) - g(x)) = L - M$.

3. *Constant Multiple Rule:* $\lim\limits_{x\to c}(k * f(x)) = kL$.

4. *Product Rule:* $\lim\limits_{x\to c}(f(x) * g(x)) = LM$.

5. *Quotient Rule:* $\lim\limits_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$.

6. *Power Rule*: $\lim\limits_{x\to c}[f(x)]^n = L^n, n$ is a positive integer.

7. *Root Rule*: $\lim\limits_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, n$ is a positive integer.

(If n is even, we assume that $\lim\limits_{x\to c} f(x) = L > 0$).

Example 2. Use the observations $\lim\limits_{x\to c} k = k$ and $\lim\limits_{x\to c} x = c$ and the fundamental rules of limits to find the following limits: (a) $\lim\limits_{x\to c}(x^3 + 4x^2 - 3)$; (b) $\lim\limits_{x\to c}\frac{x^4+x^2-1}{x^2+5}$; (c) $\lim\limits_{x\to -2}\sqrt{4x^2 - 3}$.

Solution. (a) $\lim\limits_{x\to c}(x^3 + 4x^2 - 3) = \lim\limits_{x\to c} x^3 + \lim\limits_{x\to c} 4x^2 - \lim\limits_{x\to c} 3 = c^3 + 4c^2 - 3$.

(b) $\lim\limits_{x\to c}\frac{x^4+x^2-1}{x^2+5}=\frac{\lim\limits_{x\to c}(x^4+x^2-1)}{\lim\limits_{x\to c}(x^2+5)}=\frac{\lim\limits_{x\to c}x^4+\lim\limits_{x\to c}x^2-\lim\limits_{x\to c}1}{\lim\limits_{x\to c}x^2+\lim\limits_{x\to c}5}=\frac{c^4+c^2-1}{c^2+5}$.

(c) $\lim\limits_{x\to -2}\sqrt{4x^2-3}=\sqrt{\lim\limits_{x\to -2}(4x^2-3)}=\sqrt{\lim\limits_{x\to -2}4x^2-\lim\limits_{x\to -2}3}=\sqrt{4(-2)^2-3}=\sqrt{16-3}=\sqrt{13}$.

Theorem 1 simplifies the task of calculating limits of polynomials and rational functions. To evaluate the limit of a polynomial function as x approaches c, merely substitute c for x in the formula for the function. To evaluate the limit of a rational function as x approaches a point c *at which the denominator is not zero*, substitute c for x in the formula for the function. We state these results formally as theorems.

Theorem 2 (Limits of Polynomials) If $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0$, then

$$\lim_{x\to c}P(x)=P(c)=a_nc^n+a_{n-1}c^{n-1}+\ldots+a_0.$$

Theorem 3 (Limits of Rational Functions) If $P(x)$ and $Q(x)$ are polynomials and $Q(c)\neq 0$, then $\lim\limits_{x\to c}\frac{P(x)}{Q(x)}=\frac{P(c)}{Q(c)}$.

Eliminating Common Factors from Zero Denominators

Theorem 3 applies only if the denominator of the rational function is not zero at the limit point c. If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at c. If this happens, we can find the limit by substitution in the simplified fraction.

Example 3. Evaluate $\lim\limits_{x\to 1}\frac{x^2+x-2}{x^2-x}$.

Solution. We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x-1)$ in common with the denominator. Canceling this common factor gives a simpler fraction with the same values as the original for $x\neq 1$: $\frac{x^2+x-2}{x^2-x}=\frac{(x-1)(x+2)}{x(x-1)}=\frac{x+2}{x}$, if $x\neq 1$.

Using the simpler fraction, we find the limit of these values as $x\to 1$ by Theorem 3:

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}=\lim_{x\to 1}\frac{x+2}{x}=\frac{1+2}{1}=3.$$

Example 4. Evaluate $\lim\limits_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}$.

Solution. We can create a common factor by multiplying both the numerator and denominator by the conjugate radical expression $\sqrt{x^2+100}+10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

$$\frac{\sqrt{x^2+100}-10}{x^2}=\frac{\sqrt{x^2+100}-10}{x^2}\cdot\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}=\frac{x^2+100-100}{x^2(\sqrt{x^2+100}+10)}$$
$$=\frac{x^2}{x^2(\sqrt{x^2+100}+10)}=$$
$$=\frac{1}{\sqrt{x^2+100}+10}.$$

Therefore, $\lim\limits_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}=\lim\limits_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}=\frac{1}{\sqrt{0^2+100}+10}=\frac{1}{20}=0{,}05$.

We cannot always algebraically resolve the problem of finding the limit of a quotient where the denominator becomes zero. The next theorems give helpful tools by using function comparisons.

Theorem 4 (the Sandwich Theorem). Suppose that $\boldsymbol{g(x)}\leq \boldsymbol{f(x)}\leq h\boldsymbol{(x)}$ for all x in some open interval containing c, except possibly at $x=c$ itself. Suppose also that $\lim\limits_{x\to c}g(x)=\lim\limits_{x\to c}h(x)=L$. Then $\lim\limits_{x\to c}f(x)=L$.

Example 5. Given that $1-\frac{x^2}{4}\leq u(x)\leq 1+\frac{x^2}{2}$ for all $x\neq 0$, find $\lim\limits_{x\to 0}u(x)$, no matter how complicated u is.

Solution. Since $\lim_{x\to 0}(1-(x^2/4)) = 1$ and $\lim_{x\to 0}(1+(x^2/2)) = 1$, the Sandwich Theorem implies that $\lim_{x\to 0} u(x) = 1$.

Definition of Limit. Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L**, and write $\lim_{x\to c} f(x) = L$, if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x-c| < \delta \Rightarrow |f(x)-L| < \varepsilon.$$

The formal definition of limit does not tell how to find the limit of a function, but it enables us to verify that a conjectured limit value is correct. The following examples show how the definition can be used to verify limit statements for specific functions. However, the real purpose of the definition is not to do calculations like this, but rather to prove general theorems.

Example 6. Show that $\lim_{x\to 1}(5x-3) = 2$.

Solution. Set $c = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\varepsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $c = 1$, that is, whenever

$$0 < |x-1| < \delta,$$

it is true that $f(x)$ is within distance ε of $L = 2$, so $|f(x)-2| < \varepsilon$.

We find δ by working backward from the ε -inequality:

$$|(5x-3)-2| = |5x-5| < \varepsilon \Rightarrow 5|x-1| < \varepsilon \Rightarrow |x-1| < \varepsilon/5.$$

Thus, we can take $\delta = \varepsilon/5$. If $0 < |x-1| < \delta = \varepsilon/5$, then

$|(5x-3)-2| = |5x-5| = 5|x-1| < 5(\varepsilon/5) = \varepsilon$, which proves that $\lim_{x\to 1}(5x-3) = 2$.

The value of $\delta = \varepsilon/5$ is not the only value that will make $0 < |x-1| < \delta$ imply $|5x-5| < \varepsilon$. Any smaller positive δ will do as well. The definition does not ask for a "best" positive δ, just one that will work.

Proof of Theorem 1 (Sum Rule): Let $\varepsilon > 0$ be given. We want to find a positive number δ such that for all x with $0 < |x-c| < \delta \Rightarrow |f(x)+g(x)-(L+M)| < \varepsilon$.

Regrouping terms, we get

$|f(x)+g(x)-(L+M)| = |(f(x)-L)+(g(x)-M)| \le |f(x)-L| + |g(x)-M|$

by triangle inequality $|a+b| \le |a|+|b|$.

Since $\lim_{x\to c} f(x) = L$, there exists a number $\delta_1 > 0$ such that for all x with $0 < |x-c| < \delta_1$ we have $|f(x)-L| < \varepsilon/2$.

Similarly, since $\lim_{x\to c} g(x) = M$, there exists a number $\delta_2 > 0$ such that for all x with $0 < |x-c| < \delta_2$ we have $|g(x)-M| < \varepsilon/2$.

Let $\delta = min\{\delta_1, \delta_2\}$, the smaller of δ_1 and δ_2. If $0 < |x-c| < \delta$ then $|x-c| < \delta_1$, so $|f(x)-L| < \varepsilon/2$, and $|x-c| < \delta_2$, so $|g(x)-M| < \varepsilon/2$. Therefore

$|f(x)+g(x)-(L+M)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. This shows that $\lim_{x\to c}(f(x)+g(x)) = L+M$. □

One-Sided Limits. Here we extend the limit concept to *one-sided limits*, which are limits as x approaches the number c from the left-hand side (where $x < c$) or the right-hand side ($x > c$) only.

Approaching a Limit from One Side. To have a limit L as x approaches c, a function f must be defined on *both sides* of c and its values $f(x)$ must approach L as x approaches c from either side. That is, f must be defined in some open interval about c, but not necessarily at c. Because of this, ordinary limits are called **two-sided**.

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit**. From the left, it is a **left-hand limit**.

The function $f(x) = x/|x|$ has limit 1 as x approaches 0 from the right, and limit –1 as x approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as x approaches 0. Therefore, $f(x)$ does not have a (two-sided) limit at 0.

Intuitively, if $f(x)$ is defined on an interval (c, b), where $c < b$, and approaches arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. We write

$$\lim_{x\to c^+} f(x) = L \quad \text{or} \quad \lim_{x\to c+0} f(x) = L.$$

The symbol "$x \to c^+$" (or $x \to c + 0$) means that we consider only values of x greater than c. Similarly, if $f(x)$ is defined on an interval (a, c), where $a < c$ and approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c. We write

$$\lim_{x\to c^-} f(x) = M \quad \text{or} \quad \lim_{x\to c-0} f(x) = M.$$

The symbol "$x \to c^-$" (or $x \to c - 0$) means that we consider only x-values less than c.

Example 7. The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2,2]$. We have

$$\lim_{x\to -2^+} \sqrt{4 - x^2} = 0 \text{ and } \lim_{x\to 2^-} \sqrt{4 - x^2} = 0.$$

The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have a two-sided limit at either -2 or 2 because each point does not belong to an open interval over which f is defined.

Theorem 5. A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^-} f(x) = L \text{ and } \lim_{x\to c^+} f(x) = L.$$

Precise definitions of one-sided limits:

We say that $f(x)$ has **right-hand limit L at c**, and write $\lim\limits_{x\to c^+} f(x) = L$ if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x with $c < x < c + \delta \Rightarrow |f(x) - L| < \varepsilon$.

We say that f has **left-hand limit L at c**, and write $\lim\limits_{x\to c^-} f(x) = L$ if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x with $c - \delta < x < c \Rightarrow |f(x) - L| < \varepsilon$.

Example 8. Prove that $\lim\limits_{x\to 0^+} \sqrt{x} = 0$.

Solution. Let $\varepsilon > 0$ be given. Here $c = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that for all x with $0 < x < \delta \Rightarrow |\sqrt{x} - 0| < \varepsilon$, or $0 < x < \delta \Rightarrow \sqrt{x} < \varepsilon$. Squaring both sides of this last inequality gives $x < \varepsilon^2$ if $0 < x < \delta$. If we choose $\delta = \varepsilon^2$ we have $0 < x < \delta = \varepsilon^2 \Rightarrow \sqrt{x} < \varepsilon$, or $0 < x < \varepsilon^2 \Rightarrow |\sqrt{x} - 0| < \varepsilon$. According to the definition, this shows that $\lim\limits_{x\to 0^+} \sqrt{x} = 0$.

Example 9. Show that $y = sin(1/x)$ has no limit as x approaches zero from either side.

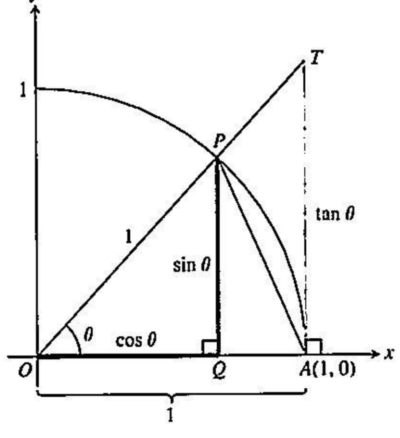
Solution. As x approaches zero, its reciprocal, $1/x$, grows without bound and the values of $sin(1/x)$ cycle repeatedly from -1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$.

Limits Involving $(\boldsymbol{sin\,\theta})/\boldsymbol{\theta}$.

A central fact about $(sin\,\theta)/\theta$ is that in radian measure its limit as $\theta \to 0$ is 1.

Theorem 6 (Limit of the Ratio $sin\,\theta/\theta$ as $\theta \to 0$) $\lim\limits_{\theta\to 0} \frac{sin\,\theta}{\theta} = 1$ (θ in radians).

Proof of Theorem 6. The plan is to show that the right-hand and left-hand limits are both 1.



Then we will know that the two-sided limit is 1 as well. To show that the right-hand limit is 1, we begin with positive values of θ less than $\pi/2$. Notice that

$$\text{Area } \Delta OAP < \text{Area sector } OAP < \text{Area } \Delta OAT.$$

By definition, $TA/OA = tan\,\theta$, but $OA = 1$, so $TA = tan\,\theta$. We can express these areas in terms of θ as follows: Area $\Delta OAP = \frac{1}{2}$ base × height = $\frac{1}{2} \cdot 1 \cdot sin\,\theta = \frac{1}{2} sin\,\theta$.

Area sector $OAP = \frac{1}{2} r^2 \theta = \frac{\theta}{2}$, Area $\Delta OAT = \frac{1}{2}$ base × height = $\frac{1}{2} \cdot 1 \cdot tan\,\theta = \frac{1}{2} tan\,\theta$.

Thus, $\frac{1}{2} sin\,\theta < \frac{1}{2}\theta < \frac{1}{2} tan\,\theta$. The last inequality goes the same way if we divide all three terms by the number $(1/2)\,sin\,\theta$, which is positive, since $0 < \theta < \pi/2$: $1 < \theta / sin\,\theta < 1/cos\,\theta$.

Taking reciprocals reverses the inequalities: $1 > sin\,\theta / \theta > cos\,\theta$.

Since $\lim\limits_{\theta \to 0^+} cos\,\theta = 1$, the Sandwich theorem gives $\lim\limits_{\theta \to 0^+} \frac{sin\,\theta}{\theta} = 1$.

To consider the left-hand limit, we recall that $sin\,\theta$ and θ are both odd functions. Therefore, $f(\theta) = (sin\,\theta)/\theta$ is an even function, with a graph symmetric about the y -axis. This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$\lim\limits_{\theta \to 0^-} \frac{sin\,\theta}{\theta} = 1 = \lim\limits_{\theta \to 0^+} \frac{sin\,\theta}{\theta}$, so $\lim\limits_{\theta \to 0} \frac{sin\,\theta}{\theta} = 1$ by Theorem 5. □

Example 10. Show that (a) $\lim\limits_{h \to 0} \frac{cos\,h - 1}{h} = 0$ and (b) $\lim\limits_{x \to 0} \frac{sin\,2x}{5x} = \frac{2}{5}$.

Solution. (a) Using the half-angle formula $cos\,h = 1 - 2\,sin^2(h/2)$, we calculate

$$\lim_{h \to 0} \frac{cos\,h - 1}{h} = \lim_{h \to 0} \left(- \frac{2\,sin^2(h/2)}{h} \right) = -\lim_{h \to 0} \frac{sin(h/2)\,sin(h/2)}{h/2} = (-1) * 0 = 0.$$

(b) We need a 2*x* in the denominator, not a 5*x*. We produce it by multiplying numerator and denominator by 2/5.

$$\lim_{x \to 0} \frac{sin\,2\,x}{5x} = \lim_{x \to 0} \frac{(2/5)\,sin\,2\,x}{(2/5) * 5x} = \frac{2}{5} \lim_{x \to 0} \frac{sin\,2\,x}{2x} = \frac{2}{5} * 1 = \frac{2}{5}.$$

Glossary

piston – поршень; **inch** – дюйм

oscillate – колебаться, вибрировать; **quotient** – частное

Exercises for Seminar 2

1 (Existence of limits). Explain why the limits do not exist: $\lim\limits_{x \to 0} \frac{x}{|x|}$.

2 (Calculating limits). Find the limits:

a) $\lim\limits_{x \to -3} (x^2 - 13)$; b) $\lim\limits_{x \to 5} \frac{x-5}{x^2-25}$; c) $\lim\limits_{x \to -5} \frac{x^2+3x-10}{x+5}$; d) $\lim\limits_{x \to 1} \frac{x-1}{\sqrt{x+3}-2}$;

e) $\lim\limits_{x \to 0} (2\,sin\,x - 1)$; f) $\lim\limits_{x \to 0} \frac{1+x+sin\,x}{3\,cos\,x}$.

3 (Finding Deltas Algebraically) In each case, find an open interval about c on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds:

a) $f(x) = x + 1, L = 5, c = 4, \varepsilon = 0{,}01$; b) $f(x) = \sqrt{x+1}, L = 1, c = 0, \varepsilon = 0{,}1$.

4 (Using the formal definition of limit). Let a function $f(x)$, a point c, and a positive number ε be given. Find $L = \lim\limits_{x \to c} f(x)$. Then find a number $\delta > 0$ such that for all x with $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$: a) $f(x) = 3 - 2x, c = 3, \varepsilon = 0{,}02$. c) $f(x) = \frac{x^2-4}{x-2}, c = 2, \varepsilon = 0{,}05$.

5. Let $f(x) = \begin{cases} 3 - x, x < 2 \\ x/2 + 1, x > 2 \end{cases}$. Find: a) $\lim\limits_{x \to 2^+} f(x)$ and $\lim\limits_{x \to 2^-} f(x)$. b) Does $\lim\limits_{x \to 2} f(x)$ exist? If so, what is it? If not, why not? c) $\lim\limits_{x \to 4^+} f(x)$ and $\lim\limits_{x \to 4^-} f(x)$. d) Does $\lim\limits_{x \to 4} f(x)$ exist? If so, what is it? If not, why not?

6. Find the following limits:

a) $\lim\limits_{x\to -0,5^-}\sqrt{\frac{x+2}{x+1}}$; c) $\lim\limits_{x\to -2^+}\left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right)$.

7. Using $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ find the limits: a) $\lim\limits_{\theta\to 0}\frac{\sin\sqrt{2}\theta}{\sqrt{2}\theta}$; b) $\lim\limits_{y\to 0}\frac{\sin 3y}{4y}$; c) $\lim\limits_{t\to 0}\frac{2t}{\tan t}$;
d) $\lim\limits_{x\to 0} 6x^2(\cot x)(\csc 2x)$

Exercises for Homework 2

1 (Existence of limits). Explain why the limits do not exist: $\lim\limits_{x\to 1}\frac{1}{x-1}$.

2 (Calculating limits). Find the limits:

a) $\lim\limits_{x\to 2}(-x^2+5x-2)$; b) $\lim\limits_{x\to -3}\frac{x+3}{x^2+4x+3}$; c) $\lim\limits_{x\to 2}\frac{x^2-7x+10}{x-2}$; d) $\lim\limits_{x\to -1}\frac{\sqrt{x^2+8}-3}{x+1}$;
e) $\lim\limits_{x\to \pi/4}\sin^2 x$; f) $\lim\limits_{x\to 0}(x^2-1)(2-\cos x)$.

3 (Finding Deltas Algebraically) In each case, find an open interval about c on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-c|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds:
a) $f(x)=2x-2, L=-6, c=-2, \varepsilon=0{,}02$; b) $f(x)=\sqrt{x}, L=1/2, c=1/4, \varepsilon=0{,}1$.

4 (Using the formal definition of limit). Let a function $f(x)$, a point c, and a positive number εbe given. Find $L=\lim\limits_{x\to c} f(x)$. Then find a number $\delta>0$ such that for all x with $0<|x-c|<\delta\Rightarrow |f(x)-L|<\varepsilon$: a) $f(x)=-3x-2, c=-1, \varepsilon=0{,}03$. b) $f(x)=\frac{x^2+6x+5}{x+5}, c=-5, \varepsilon=0{,}05$.

5. Let $f(x)=\begin{cases}3-x, x<2\\ 2, x=2\\ x/2, x>2\end{cases}$. Find: a) $\lim\limits_{x\to 2^+} f(x)$, $\lim\limits_{x\to 2^-} f(x)$ and $f(2)$. b) Does $\lim\limits_{x\to 2} f(x)$ exist? If so, what is it? If not, why not? c) $\lim\limits_{x\to -1^+} f(x)$ and $\lim\limits_{x\to -1^-} f(x)$. d) Does $\lim\limits_{x\to -1} f(x)$ exist? If so, what is it? If not, why not?

6. Find the following limits:

a) $\lim\limits_{x\to 1^+}\sqrt{\frac{x-1}{x+2}}$; b) $\lim\limits_{x\to 1^-}\left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right)$.

7. Using $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ find the limits: a) $\lim\limits_{t\to 0}\frac{\sin kt}{t}$ (k constant); b) $\lim\limits_{x\to 0}\frac{\tan 2x}{x}$;
c) $\lim\limits_{x\to 0}\frac{x\csc 2x}{\cos 5x}$; d) $\lim\limits_{h\to 0^-}\frac{h}{\sin 3h}$.