Lecture 4. Limits Involving Infinity. Asymptotes of Graphs.

Here we investigate the behavior of a function when the magnitude of the independent variable x becomes increasingly large, or $x \to \pm\infty$. We further extend the concept of limit to *infinite limits*, which are not limits as before, but rather a new use of the term limit. Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large in magnitude. We use these limit ideas to analyze the graphs of functions having *horizontal* or *vertical asymptotes*.

Finite Limits as $x \to \pm\infty$. The symbol for infinite (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds.

Definition: 1. We say that $f(x)$ has the **limit L as x approaches infinity** and write $\lim\limits_{x\to\infty} f(x) = L$ if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x with $x > M$ we have $|f(x) - L| < \varepsilon$.

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write $\lim\limits_{x\to-\infty} f(x) = L$ if, for every number $\varepsilon > 0$, there exists a corresponding number N such that for all x with $x < N$ we have $|f(x) - L| < \varepsilon$.

Example 1. Show that (a) $\lim\limits_{x\to\infty} \frac{1}{x} = 0$ and (b) $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$.

Solution: (a) Let $\varepsilon > 0$ be given. We must find a number M such that for all x with $x > M$ we have $\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon$. The implication will hold if $M = 1/\varepsilon$ or any larger positive number. This proves $\lim\limits_{x\to\infty}(1/x) = 0$.

(b) Let $\varepsilon > 0$ be given. We must find a number N such that for all x with $x < N$ we have $\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon$. The implication will hold if $N = -1/\varepsilon$ or any number less than $-1/\varepsilon$. This proves $\lim\limits_{x\to-\infty}(1/x) = 0$.

Theorem 1 All the Limit Laws in Theorem 1 (Lecture 2) are true when we replace $\lim\limits_{x\to c}$ by $\lim\limits_{x\to\infty}$ or $\lim\limits_{x\to-\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

Example 2. The properties of Theorem 1 are used to calculate limits in the same way as when x approaches a finite number c.

(a) $\lim\limits_{x\to\infty}(5 + 1/x) = \lim\limits_{x\to\infty} 5 + \lim\limits_{x\to\infty}(1/x) = 5 + 0 = 5.$

(b) $\lim\limits_{x\to\infty} \frac{\pi\sqrt{3}}{x^2} = \lim\limits_{x\to\infty}(\pi\sqrt{3} \cdot (1/x) \cdot (1/x) = \lim\limits_{x\to\infty} \pi\sqrt{3} \cdot \lim\limits_{x\to\infty}(1/x) \cdot \lim\limits_{x\to\infty}(1/x) = \pi\sqrt{3} \cdot 0 \cdot 0 = 0.$

Limits at Infinity of Rational Functions. To determine the limit of a rational function as $x \to \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator. The result that depends on the degrees of the polynomials involved.

Example 3. (a) $\lim\limits_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim\limits_{x\to\infty} \frac{5 + 8/x - 3/x^2}{3 + 2/x^2} = \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3}.$

(b) $\lim\limits_{x\to\infty} \frac{11x + 2}{2x^3 - 1} = \lim\limits_{x\to\infty} \frac{11/x^2 + 2/x^3}{2 - 1/x^3} = \frac{0 + 0}{2 - 0} = 0.$

Horizontal Asymptotes. If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an *asymptote* of the graph.

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either $\lim\limits_{x\to\infty} f(x) = b$ or $\lim\limits_{x\to-\infty} f(x) = b$.

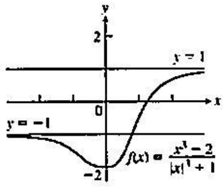
Example 4. Find the horizontal asymptotes of the graph of $f(x) = \dfrac{x^3-2}{|x|^3+1}$.

Solution: We calculate the limits as $x \to \pm\infty$.

For $x \ge 0$: $\lim\limits_{x\to\infty} \dfrac{x^3-2}{|x|^3+1} = \lim\limits_{x\to\infty} \dfrac{x^3-2}{x^3+1} = \lim\limits_{x\to\infty} \dfrac{1-2/x^3}{1+1/x^3} = 1.$

For $x < 0$: $\lim\limits_{x\to-\infty} \dfrac{x^3-2}{|x|^3+1} = \lim\limits_{x\to-\infty} \dfrac{x^3-2}{(-x)^3+1} = \lim\limits_{x\to-\infty} \dfrac{1-2/x^3}{-1+1/x^3} = -1.$

The horizontal asymptotes are $y = -1$ and $y = 1$.



Oblique Asymptotes. If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \pm\infty$.

Example 5. Find the oblique asymptote of the graph of $f(x) = \dfrac{x^2-3}{2x-4}$.

Solution: We are interested in the behavior as $x \to \pm\infty$. We divide $(2x-4)$ into (x^2-3):
$x^2 - 3 = (2x-4)(x/2+1)+1.$

This tells us that $f(x) = \dfrac{x^2-3}{2x-4} = \left(\dfrac{x}{2}+1\right) + \left(\dfrac{1}{2x-4}\right).$

As $x \to \pm\infty$, the remainder $r(x) = 1/(2x-4)$, whose magnitude gives the vertical distance between the graphs of f and $g(x) = x/2+1$, goes to zero, making the slanted line $g(x) = x/2+1$ as asymptote of the graph of f.

We also have an equivalent definition of an oblique asymptote: A line $y = kx + b$ is an **oblique asymptote** of a curve $y = f(x)$ if there are limits $k = \lim\limits_{x\to+\infty} \dfrac{f(x)}{x}$, $b = \lim\limits_{x\to+\infty} [f(x) - kx]$ or $k = \lim\limits_{x\to-\infty} \dfrac{f(x)}{x}$, $b = \lim\limits_{x\to-\infty} [f(x) - kx]$.

Infinite Limits. Let us look again at the function $f(x) = 1/x$. As $x \to 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B, however large, the values of f become larger still. Thus, f has no limit as $x \to 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$. We write $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} 1/x = \infty.$

In writing this equation, we are *not* saying that the limit exists. Nor are we saying that there is a real number ∞, for there is no such number. Rather, we are saying that $\lim\limits_{x\to0^+} 1/x$ *does not exist because* $1/x$ *becomes arbitrarily large and positive as* $x \to 0^+$.

As $x \to 0^-$, the values of f become arbitrarily large and negative. Given any negative real number $-B$, the values of f eventually lie below $-B$. We write $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 1/x = -\infty$. Again, we are *not* saying that the limit exists and equals the number $-\infty$. There is no real number $-\infty$. We are describing the behavior of a function whose limit as $x \to 0^-$ *does not exist because* $1/x$ *becomes arbitrarily large and negative.*

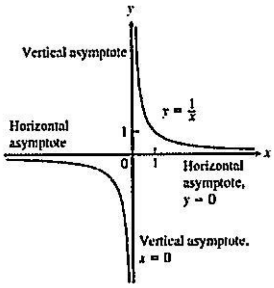
Definition: 1. We say that $f(x)$ **approaches infinity as** x **approaches** c, and write $\lim_{x \to c} f(x) = \infty$, if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x with $0 < |x - c| < \delta$ we have $f(x) > B$.

2. We say that $f(x)$ **approaches minus infinity as** x **approaches** c, and write $\lim_{x \to c} f(x) = -\infty$, if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x with $0 < |x - c| < \delta$ we have $f(x) < -B$.

Example 6. Prove that $\lim_{x \to 0} \dfrac{1}{x^2} = \infty$.

Solution: Given $B > 0$, we want to find $\delta > 0$ such that $0 < |x - 0| < \delta$ implies $\dfrac{1}{x^2} > B$. Now, $\dfrac{1}{x^2} > B$ if and only if $x^2 < \dfrac{1}{B}$ or, equivalently, $|x| < \dfrac{1}{\sqrt{B}}$. Thus, choosing $\delta = 1/\sqrt{B}$ (or any smaller positive number), we see that $|x| < \delta$ implies $\dfrac{1}{x^2} > \dfrac{1}{\delta^2} \geq B$. Therefore, by definition, $\lim_{x \to 0} \dfrac{1}{x^2} = \infty$.

Vertical Asymptotes. Notice that the distance between a point on the graph of $f(x) = 1/x$ and the y-axis approaches zero as the point moves vertically along the graph and away from the origin.



The function $f(x) = 1/x$ is unbounded as x approaches 0 because $\lim_{x \to 0^+} \dfrac{1}{x} = \infty$ and $\lim_{x \to 0^-} \dfrac{1}{x} = -\infty$. We say that the line $x = 0$ (the y-axis) is a *vertical asymptote* of the graph of $f(x) = 1/x$. Observe that the denominator is zero at $x = 0$ and the function is undefined there.

Definition. A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$.

Example 7. Find the horizontal and vertical asymptotes of the curve $y = \dfrac{x+3}{x+2}$.

Solution: We are interested in the behavior as $x \to \pm\infty$ and the behavior as $x \to -2$, where the denominator is zero. The asymptotes are quickly revealed if we recast the rational function as a polynomial with a remainder, by dividing $(x+2)$ into $(x+3)$: $x + 3 = (x+2) + 1$. This result enables us to rewrite y as: $y = 1 + \dfrac{1}{x+2}$.

As $x \to \pm\infty$, the curve approaches the horizontal asymptote $y = 1$; as $x \to -2$, the curve approaches the vertical asymptote $x = -2$.

Glossary

magnitude – величина, размер; **to outgrow** – перерастать
to surpass – превосходить, превышать; **to reveal** – раскрыть
to recast – перестроить, переделать

Exercises for Seminar 4

1. Find the limit of each function as $x \to \infty$ and as $x \to -\infty$:
a) $f(x) = 2/x - 3$; b) $g(x) = 1/(2 + 1/x^2)$.

2. Find the limits: a) $\lim\limits_{x\to\infty} \dfrac{\sin 2x}{x}$; b) $\lim\limits_{t\to-\infty} \dfrac{2 - t + \sin t}{t + \cos t}$.

3. Find the limit of each rational function as $x \to \infty$ and as $x \to -\infty$:
a) $f(x) = \dfrac{2x+3}{5x+7}$; b) $f(x) = \dfrac{x+1}{x^2+3}$.

4. Find the limits: a) $\lim\limits_{x\to\infty} \sqrt{\dfrac{8x^2 - 3}{2x^2 + x}}$; b) $\lim\limits_{x\to-\infty} \dfrac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$.

5 (Infinite limits). Find the limits: a) $\lim\limits_{x\to 0^+} \dfrac{1}{3x}$; b) $\lim\limits_{x\to 2^-} \dfrac{3}{x-2}$; c) $\lim\limits_{x\to 8^+} \dfrac{2x}{x+8}$.

6. Find the horizontal and vertical asymptotes: a) $y = \dfrac{1}{x-1}$; b) $y = \dfrac{x+5}{x+4}$.

7. Finding limits of differences when $x \to \pm\infty$: a) $\lim\limits_{x\to\infty}\left(\sqrt{x+9} - \sqrt{x+4}\right)$; b) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+3} + x\right)$.

8. Find the oblique asymptotes: a) $y = \dfrac{x^2}{x-1}$; b) $y = \dfrac{x^2-1}{x}$

Exercises for Homework 4

1. Find the limit of each function as $x \to \infty$ and as $x \to -\infty$:
a) $f(x) = \pi - 2/x^2$; b) $g(x) = 1/(8 - 5/x^2)$.

2. Find the limits: a) $\lim\limits_{\theta\to-\infty} \dfrac{\cos\theta}{3\theta}$; b) $\lim\limits_{r\to\infty} \dfrac{r + \sin r}{2r + 7 - 5\sin r}$.

3. Find the limit of each rational function as $x \to \infty$ and as $x \to -\infty$:
a) $f(x) = \dfrac{2x^3 + 7}{x^3 - x^2 + x + 7}$; b) $f(x) = \dfrac{5x^8 - 2x^3 + 9}{3 + x - 4x^5}$.

4. Find the limits: a) $\lim\limits_{x\to-\infty}\left(\dfrac{x^2 + x - 1}{8x^2 - 3}\right)^{1/3}$; b) $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+1}}{x+1}$.

5 (Infinite limits). Find the limits: a) $\lim\limits_{x\to 0^-} \dfrac{5}{2x}$; b) $\lim\limits_{x\to 3^+} \dfrac{1}{x-3}$; c) $\lim\limits_{x\to 5^-} \dfrac{3x}{2x+10}$.

6. Find the horizontal and vertical asymptotes: a) $y = \dfrac{-3}{x-3}$; b) $y = \dfrac{2x}{x+1}$.

7. Finding limits of differences when $x \to \pm\infty$:
a) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+25} - \sqrt{x^2-1}\right)$; b) $\lim\limits_{x\to-\infty}\left(\sqrt{9x^2 - x} + 3x\right)$.

8. Find the oblique asymptotes: a) $y = \dfrac{x^2+1}{x-1}$; b) $y = \dfrac{x^3+1}{x^2}$.