Lecture 10. Indeterminate Forms. L’Hopital’s Rule. Cauchy Mean Value Theorem

John Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerators and denominators both approach zero or $+\infty$. The rule is known today as l'Hopital's Rule, after Guillaume de l’Hopital. He wrote the first introductory differential calculus text, where the rule first appeared in print. Limit involving transcendental functions often require some use of the rule for their calculation.

Indeterminate Form $0/0$. If we want to know how the function $F(x)=\dfrac{x-\sin x}{x^3}$ behaves near $x=0$ (where it is undefined), we can examine the limit of $F(x)$ as $x\to 0$. We cannot apply the Quotient Rule for limits because the limit of the denominator is 0. Moreover, in this case, both the numerator and denominator approach 0, and $0/0$ is undefined. Such limits may or may not exist in general, but the limit does exist for the function $F(x)$ under discussion by applying l’Hopital’s Rule.

If continuous functions $f(x)$ and $g(x)$ are both zero at $x=a,$ then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ cannot be found by substituting $x=a.$ The substitution produces $0/0,$ a meaningless expression, which we cannot evaluate. We use $0/0$ as a notation for an expression known as an **indeterminate form**. Other meaningless expressions often occur, such as $\infty/\infty,$ $\infty\cdot 0,$ $\infty-\infty,$ 0^0 and $1^\infty,$ which cannot be evaluated in a consistent way; these are called indeterminate forms as well. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancellation, rearrangement of terms, or other algebraic manipulations. L’Hopital’s Rule enables us to draw on our success with derivatives to evaluate limits that otherwise lead to indeterminate forms.

Theorem 1 (**L'Hopital's rule**). Suppose that $f(a)=g(a)=0,$ that f and g are differentiable on an open interval I containing $a,$ and that $g'(x)\neq 0$ on I if $x\neq a.$ Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Example 1. The following limits involve $0/0$ indeterminate forms, so we apply l’Hopital’s Rule. In some cases, it must be applies repeatedly.

a) $\lim\limits_{x\to 0}\dfrac{3x-\sin x}{x}=\lim\limits_{x\to 0}\dfrac{3-\cos x}{1}=\dfrac{3-\cos x}{1}\Big|_{x=0}=2.$

b) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}=\lim\limits_{x\to 0}\dfrac{\frac{1}{2\sqrt{1+x}}}{1}=\dfrac{1}{2}.$

c) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1-x/2}{x^2}=\lim\limits_{x\to 0}\dfrac{(1/2)(1+x)^{-1/2}-1/2}{2x}=\left|\dfrac{0}{0}\right|=\lim\limits_{x\to 0}\dfrac{-(1/4)(1+x)^{-3/2}}{2}=-\dfrac{1}{8}.$

Using L’Hopital’s Rule. To find $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ by l’Hopital’s Rule, we continue to differentiate f and $g,$ so long as we still get the form $0/0$ at $x=a.$ But as soon as one or the other of these derivatives is different from zero at $x=a$ we stop differentiating. L’Hopital’s Rule does not apply when either the numerator or denominator has a finite nonzero limit.

L’Hopital’s Rule applies to one-sided limits as well.

Example 2. In this example, the one-sided limits are different:

a) $\lim\limits_{x\to 0^+}\dfrac{\sin x}{x^2}=\lim\limits_{x\to 0^+}\dfrac{\cos x}{2x}=\infty.$ b) $\lim\limits_{x\to 0^-}\dfrac{\sin x}{x^2}=\lim\limits_{x\to 0^-}\dfrac{\cos x}{2x}=-\infty.$

Indeterminate Forms $\infty/\infty, \infty\cdot 0, \infty-\infty$. Sometimes when we try to evaluate a limit as $x\to a$ by substituting $x=a$ we get an indeterminate form like ∞/∞, $\infty\cdot 0$, or $\infty-\infty$, instead of $0/0$. We first consider the form ∞/∞.
More advanced treatments of calculus prove that l'Hopital's Rule applies to the indeterminate form ∞/∞ as to $0/0$. If $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$ as $x\to a$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ provided the limit on the right exists. In the notation $x\to a$, a may be either finite or infinite. Moreover, $x\to a$ may be replaced by the one-sided limits $x\to a^+$ or $x\to a^-$.
Example 3. Find the limits of these ∞/∞ forms:
a) $\lim\limits_{x\to\pi/2}\dfrac{\sec x}{1+\tan x}$. The numerator and denominator are discontinuous at $x=\pi/2$, so we investigate the one-sided limits here. To apply l'Hopital's Rule, we can choose I to be any open interval with $x=\pi/2$ as an endpoint.

$$\lim_{x\to(\pi/2)^-}\frac{\sec x}{1+\tan x}=\left|\frac{\infty}{\infty}\right|=\lim_{x\to(\pi/2)^-}\frac{\sec x\tan x}{\sec^2 x}=\lim_{x\to(\pi/2)^-}\sin x=1.$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.
b) $\lim\limits_{x\to\infty}\dfrac{\ln x}{2\sqrt{x}}=\lim\limits_{x\to\infty}\dfrac{1/x}{1/\sqrt{x}}=\lim\limits_{x\to\infty}\dfrac{1}{\sqrt{x}}=0.$ c) $\lim\limits_{x\to\infty}\dfrac{e^x}{x^2}=\lim\limits_{x\to\infty}\dfrac{e^x}{2x}=\lim\limits_{x\to\infty}\dfrac{e^x}{2}=\infty.$
Next we turn our attention to the indeterminate forms $\infty\cdot 0$ and $\infty-\infty$. Sometimes these forms can be handled by using algebra to convert them to a $0/0$ or ∞/∞ form. Here again we do not mean to suggest that $\infty\cdot 0$ and $\infty-\infty$ is a number. They are only notations for functional behaviors when considering limits. Here are examples of how we might work with these indeterminate forms.
Example 4. Find the limit of these $\infty\cdot 0$ forms:
a) $\lim\limits_{x\to\infty}\left(x\sin\dfrac{1}{x}\right)=\left|h=1/x\right|=\lim\limits_{h\to 0^+}\left(\dfrac{1}{h}\sin h\right)=\lim\limits_{h\to 0^+}\dfrac{\sin h}{h}=1.$
b) $\lim\limits_{x\to 0^+}\sqrt{x}\ln x=\lim\limits_{x\to 0^+}\dfrac{\ln x}{1/\sqrt{x}}=\lim\limits_{x\to 0^+}\dfrac{1/x}{-1/2x^{3/2}}=\lim\limits_{x\to 0^+}\left(-2\sqrt{x}\right)=0.$
Example 5. Find the limits of this $\infty-\infty$ form:

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\sin x}{x\sin x}=\lim_{x\to 0}\frac{1-\cos x}{\sin x+x\cos x}=\lim_{x\to 0}\frac{\sin x}{2\cos x-x\sin x}=\frac{0}{2}=0.$$

Indeterminate Powers. Limits that lead to the indeterminate forms 1^∞, 0^0, and ∞^0 can sometimes be handled by first taking the logarithm of the function. We use l'Hopital's Rule to find the limit of the logarithm expression and then exponentiate the result to find the original function limit. This procedure is justified by the continuity of the exponential function and it is formulated as follows (the formula is also valid for one-sided limits):
If $\lim\limits_{x\to a}\ln f(x)=L$, **then** $\lim\limits_{x\to a}f(x)=\lim\limits_{x\to a}e^{\ln f(x)}=e^L$ (**here** a **may be either finite or infinite**).
Example 6. Apply l'Hopital's Rule to show that $\lim\limits_{x\to 0^+}(1+x)^{1/x}=e.$
Solution: The limit leads to the indeterminate form 1^∞. We let $f(x)=(1+x)^{1/x}$ and find $\lim\limits_{x\to 0^+}\ln f(x)$. Since $\ln f(x)=\ln(1+x)^{1/x}=\dfrac{1}{x}\ln(1+x)$, l'Hopital's Rule now applies to give

$$\lim_{x\to 0^+}\ln f(x)=\lim_{x\to 0^+}\frac{\ln(1+x)}{x}=\lim_{x\to 0^+}\frac{\frac{1}{1+x}}{1}=\frac{1}{1}=1.$$

Therefore, $\lim_{x\to 0^+}(1+x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^1 = e.$

Example 7. Find $\lim_{x\to\infty} x^{1/x}$.

Solution: The limit leads to the indeterminate form ∞^0. We let $f(x) = x^{1/x}$ and find $\lim_{x\to\infty}\ln f(x)$.

Since $\ln f(x) = \ln x^{1/x} = \dfrac{\ln x}{x}$, l'Hopital's Rule gives

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty}\frac{\ln x}{x} = \left|\frac{\infty}{\infty}\right| = \lim_{x\to\infty}\frac{1/x}{1} = \lim_{x\to\infty}\frac{1}{x} = 0.$$

Therefore $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1.$

Proof of L'Hopital's Rule

The proof of l'Hopital's Rule is based on Cauchy's Mean Value Theorem, an extension of the Mean Value Theorem that involves two functions instead of one. We prove Cauchy's Theorem first and then show how it leads l'Hopital's Rule.

Theorem 2 (**Cauchy's Mean Value Theorem**) Suppose functions f and g are continuous on $[a,b]$ and differentiable throughout (a,b) and also suppose $g'(x) \neq 0$ throughout (a,b). Then there exists a number c in (a,b) at which $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$.

Proof. We apply the Mean Value Theorem twice. First, we use it to show that $g(a) \neq g(b)$. For if $g(b)$ did equal $g(a)$, then the Mean Value Theorem would give $g'(c) = \dfrac{g(b)-g(a)}{b-a} = 0$ for some c between a and b, which cannot happen because $g'(x) \neq 0$ in (a,b).

We next apply the Mean Value Theorem to the function

$$F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}\left[g(x)-g(a)\right].$$

This function is continuous and differentiable where f and g are, and $F(b) = F(a) = 0$. Therefore, there is a number c between a and b for which $F'(c) = 0$. When expressed in terms of f and g, this equation becomes $F'(c) = f'(c) - \dfrac{f(b)-f(a)}{g(b)-g(a)}\left[g'(c)\right] = 0$ so that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}. \quad \square$$

Proof of l'Hopital's Rule. We first establish the limit equation for the case $x \to a^+$. The method needs almost no change to apply to $x \to a^-$, and the combination of these two cases establishes the result.

Suppose that x lies to the right of a. Then $g'(x) \neq 0$, and we can apply Cauchy's Theorem to the closed interval from a to x. This step produces a number c between a and x such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x)-f(a)}{g(x)-g(a)}.$$

But $f(a) = g(a) = 0$, so $\dfrac{f'(c)}{g'(c)} = \dfrac{f(x)}{g(x)}$. As x approaches a, c approaches a because it always lies between a and x. Therefore, $\lim_{x\to a^+}\dfrac{f(x)}{g(x)} = \lim_{c\to a^+}\dfrac{f'(c)}{g'(c)} = \lim_{x\to a^+}\dfrac{f'(x)}{g'(x)}$, which establish l'Hopital's Rule for the case where x approaches a from above. The case where x approaches a from below is proved by applying Cauchy's Theorem to the closed interval $[x, a]$, $x < a$. $\square$

Glossary

indeterminate form – неопределенность

Exercises for Seminar 10

1. (Finding Limit in Two Ways). Use l’Hopital’s Rule to evaluate the limit. Then evaluate the limit using a method studied before.

a) $\lim\limits_{x\to -2}\dfrac{x+2}{x^2-4}$; b) $\lim\limits_{x\to\infty}\dfrac{5x^2-3x}{7x^2+1}$; c) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$.

2. Using l’Hopital’s Rule to find the limits:

a) $\lim\limits_{x\to 0}\dfrac{8x^2}{\cos x-1}$; b) $\lim\limits_{\theta\to\pi/2}\dfrac{2\theta-\pi}{\cos(2\pi-\theta)}$; c) $\lim\limits_{x\to 0}\dfrac{x2^x}{2^x-1}$;

d) $\lim\limits_{x\to\infty}\dfrac{\ln(x+1)}{\log_2 x}$; e) $\lim\limits_{x\to\infty}(\ln 2x-\ln(x+1))$.

3. (Indeterminate Powers and Products). Find the limits:

a) $\lim\limits_{x\to 1^+} x^{1/(1-x)}$; b) $\lim\limits_{x\to 0^+}\left(\ln\dfrac{1}{x}\right)^x$; c) $\lim\limits_{x\to 0^+} x^x$; d) $\lim\limits_{x\to\infty}\left(\dfrac{x+2}{x-1}\right)^x$.

Exercises for Homework 10

1. (Finding Limit in Two Ways). Use l’Hopital’s Rule to evaluate the limit. Then evaluate the limit using a method studied before.

a) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{x}$; b) $\lim\limits_{x\to 1}\dfrac{x^3-1}{4x^3-x-3}$; c) $\lim\limits_{x\to\infty}\dfrac{2x^2+3x}{x^3+x+1}$.

2. Using l’Hopital’s Rule to find the limits:

a) $\lim\limits_{x\to 0}\dfrac{\sin x-x}{x^3}$; b) $\lim\limits_{\theta\to\pi/3}\dfrac{3\theta+\pi}{\sin(\theta+(\pi/3))}$;

c) $\lim\limits_{x\to 0}\dfrac{3^x-1}{2^x-1}$; d) $\lim\limits_{x\to\infty}\dfrac{\log_2 x}{\log_3(x+3)}$; e) $\lim\limits_{x\to 0^+}(\ln x-\ln\sin x)$.

3. (Indeterminate Powers and Products). Find the limits:

a) $\lim\limits_{x\to 1^+} x^{1/(x-1)}$; b) $\lim\limits_{x\to e^-}(\ln x)^{1/(x-e)}$; c) $\lim\limits_{x\to 0^+}\left(1+\dfrac{1}{x}\right)^x$; d) $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x+2}\right)^{1/x}$.