Lecture 6. Differential Calculus: chain rule, implicit differentiation, rule for inverses.

The Chain Rule. How do we differentiate $F(x) = sin(x^2 - 4)$? This function is the composite $f \circ g$ of two functions $y = f(u) = sin\, u$ and $u = g(x) = x^2 - 4$ that we know how to differentiate. The answer, given by the *Chain Rule*, says that the derivative is the product of the derivatives of f and g.

Theorem 1 (The Chain Rule) If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) * g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$, where $\frac{dy}{du}$ is evaluated at $u = g(x)$.

A Proof of One Case of the Chain Rule: Let Δu be the change in u when x changes by Δx, so that $\Delta u = g(x + \Delta x) - g(x)$. Then the corresponding change in y is $\Delta y = f(u + \Delta u) - f(u)$.

If $\Delta u \neq 0$, we can write the fraction $\Delta y/\Delta x$ as the product $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} * \frac{\Delta u}{\Delta x}$ and take the limit as $\Delta x \to 0$: $\frac{dy}{dx} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} = \lim\limits_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \lim\limits_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{dy}{du}\frac{du}{dx}$.

Example 1. Find the derivative of $x(t) = cos(t^2 + 1)$.

Solution: In this instance, x is a composite function: $x = cos\, u$ and $u = t^2 + 1$. We have $\frac{dx}{du} = -sin\, u, \frac{du}{dt} = 2t$. By the Chain Rule, $\frac{dx}{dt} = \frac{dx}{du} * \frac{du}{dx} = -sin\, u * 2t = -2t\, sin(t^2 + 1)$.

"Outside-Inside" Rule

A difficulty with the Leibniz notation is that it doesn't state specifically where the derivatives in the Chain Rule are supposed to be evaluated. So it sometimes helps to think about the Chain Rule using functional notation. If $y = f(g(x))$, then $\frac{dy}{dx} = f'(g(x)) * g'(x)$. In words, differentiate the "outside" function f and evaluate it at the "inside" function $g(x)$ left alone; then multiply by the derivative of the "inside function."

Example 2. Differentiate $sin(x^2 + e^x)$ with respect to x.

Solution: We apply the Chain Rule directly and find $\frac{d}{dx} sin(x^2 + e^x) = cos(x^2 + e^x) * (2x + e^x)$.

Some useful formulas: $\frac{d}{dx} e^u = e^u \frac{du}{dx}, \frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$.

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative.

Example 3. Find the derivative of $g(t) = tan(5 - sin\, 2\, t)$.

Solution: $g'(t) = \frac{d}{dt}(tan(5 - sin\, 2\, t)) = sec^2(5 - sin\, 2\, t) * \frac{d}{dt}(5 - sin\, 2\, t) =$

$$sec^2(5 - sin\, 2\, t) * (0 - cos\, 2\, t) * \frac{d}{dt}(2t)) = sec^2(5 - sin\, 2\, t) * (-cos\, 2\, t) * 2$$
$$= -2\, cos\, 2\, t\, sec^2(5 - sin\, 2\, t).$$

Implicit Differentiation: Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y explicitly in terms of the variable x. We have learned rules for differentiating functions defined in this way. Another situation occurs when we encounter equations like $x^3 + y^3 - 9xy = 0, y^2 - x = 0$ or $x^2 + y^2 - 25 = 0$.

These equations define an *implicit* relation between the variables x and y. In some cases we may be able to solve such an equation for y as an explicit function (or even several functions) of x. When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find dy/dx by *implicit differentiation*.

Example 4. Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.
Solution: We can solve this problem by differentiating the given equation of the circle implicitly with respect to x: $\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25) \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$.
The slope at $(3, -4)$ is $-x/y|_{(3,-4)} = 3/4$.

Implicit Differentiation Rule: 1. Differentiate both sides of the equation with respect to *x*, treating *y* as a differentiable function of *x*.
2. Collect the terms with *dy/dx* on one side of the equation and solve for *dy/dx*.
Example 5. Find *dy/dx* if $y^2 = x^2 + \sin x\, y$.
Solution: We differentiate the equation implicitly.
$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin x\, y) \Rightarrow 2y\frac{dy}{dx} = 2x + (\cos x\, y)\frac{d}{dx}(xy)$
$2y\frac{dy}{dx} = 2x + (\cos x\, y)\left(y + x\frac{dy}{dx}\right) \Rightarrow 2y\frac{dy}{dx} - (\cos x\, y)\left(x\frac{dy}{dx}\right) = 2x + (\cos x\, y)y$
$(2y - x\cos x\, y)\frac{dy}{dx} = 2x + y\cos x\, y \Rightarrow \frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$.

Derivatives of Higher Order
Implicit differentiation can also be used to find higher derivatives.
Example 6. Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.
Solution: To start, we differentiate both sides of the equation with respect to *x* in order to find $y' = dy/dx$.
$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8) \Rightarrow 6x^2 - 6yy' = 0 \Rightarrow y' = \frac{x^2}{y}$, when $y \neq 0$.
We now apply the Quotient Rule to find y'': $y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2}y'$
Finally, we substitute $y' = x^2/y$ to express y'' in terms of *x* and *y*.
$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}$, when $y \neq 0$.

Tangents and Normal Lines
Example 7. Show that the point (2,4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there.
Solution: The point (2,4) lies on the curve because its coordinates satisfy the equation given for the curve: $2^3 + 4^3 - 9 \cdot 2 \cdot 4 = 8 + 64 - 72 = 0$. To find the slope of the curve at (2,4), we first use implicit differentiation to find a formula for $\frac{dy}{dx}$: $3x^2 + 3y^2\frac{dy}{dx} - 9y - 9x\frac{dy}{dx} = 0$.
We have $\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}$. We then evaluate the derivative at (2,4): $\frac{dy}{dx}\Big|(2,4) = \frac{4}{5}$.
The tangent at (2,4) is the line through (2,4) with slope $4/5$: $y - 4 = (4/5)(x - 2)$.

$$y = (4/5)x + 12/5.$$

The normal to the curve at (2,4) is the line perpendicular to the tangent there, the line through (2,4) with slope $-5/4$: $y - 4 = (-5/4)(x - 2) \Rightarrow y = (-5/4)x + 13/2$.

Derivatives of Inverse Functions and Logarithms
Early we defined the natural logarithm function $f^{-1}(x) = \ln x$ as the inverse of the natural exponential function $f(x) = e^x$. This is one of the most important function-inverse pairs in mathematics and science. Here we learn a rule for differentiating the inverse of a differentiable function and we apply the rule to find the derivative of the natural logarithm function.
Theorem 2 (The Derivative Rule for Inverses) If *f* has an interval *I* as domain and $f'(x)$ exists and is never zero on *I*, then f^{-1} is differentiable at every point in its domain (the range of *f*). The value of $(f^{-1})'$ at a point *b* in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$: $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$ or $\frac{df^{-1}}{dx}\Big|x = b = \frac{1}{\frac{df}{dx}|x=f^{-1}(b)}$.

Theorem 2 makes two assertions. The first of these has to do with the conditions under which f^{-1} is differentiable; the second assertion is a formula for the derivative of f^{-1} when it exists. While we omit the proof of the first assertion, the second one is proved in the following way:

$$f(f^{-1}(x)) = x \Rightarrow \frac{d}{dx} f(f^{-1}(x)) = 1 \Rightarrow f'(f^{-1}(x)) \frac{d}{dx} f^{-1}(x) = 1 \Rightarrow \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

Example 8. The function $f(x) = x^2, x > 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/(2\sqrt{x})$. Let's verify that Theorem 2 gives the same formula for the derivative of $f^{-1}(x)$: $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2(f^{-1}(x))} = \frac{1}{2\sqrt{x}}$.

Equation in Theorem 2 sometimes enables us to find specific values of df^{-1}/dx without knowing a formula for f^{-1}.

Example 9. Let $f(x) = x^3 - 2, x > 0$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution: We apply Theorem 2 to obtain the value of the derivative of f^{-1} at $x = 6$:

$$\frac{df}{dx}|x = 2 = 3x^2|x = 2 = 12 \Rightarrow \left.\frac{df^{-1}}{dx}\right|x = f(2) = \frac{1}{\frac{df}{dx}|_{x=2}} = \frac{1}{12}.$$

Derivative of the Natural Logarithm Function

Since we know the exponential function $f(x) = e^x$ is differentiable everywhere, we can apply Theorem 2 to find the derivative of its inverse $f^{-1}(x) = \ln x$:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{f^{-1}(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Thus, $\frac{d}{dx}(\ln x) = \frac{1}{x}, x > 0$.

The Chain Rule extends this formula to positive functions $u(x)$: $\frac{d}{dx} \ln u = \frac{1}{u}\frac{du}{dx}, u > 0$.

Example 10. $\frac{d}{dx} \ln(x^2 + 3) = \frac{1}{x^2+3}\frac{d}{dx}(x^2 + 3) = \frac{2x}{x^2+3}$.

The Derivatives of a^u and $\log_a u$: We start with the equation $a^x = e^{\ln(a^x)} = e^{x \ln a}, a > 0$:

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \frac{d}{dx}(x \ln a) = a^x \ln a.$$

With the Chain Rule, we get a more general form for the derivative of a general exponential function a^u: If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and $\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}$.

Example 11. $\frac{d}{dx} 3^{\sin x} = 3^{\sin x}(\ln 3)\frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3) \cos x$.

To find the derivative of $\log_a u$ for an arbitrary base ($a > 0, a \neq 1$), we start with the change-of-base formula for logarithms and express $\log_a u$ in terms of natural logarithms, $\log_a x = \frac{\ln x}{\ln a}$.

$$\frac{d}{dx} \log_a x = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx} \ln x = \frac{1}{\ln a} * \frac{1}{x} = \frac{1}{x \ln a}.$$

If u is a differentiable function of x and $u > 0$, the Chain Rule gives a more general formula: For $a > 0$ and $a \neq 1$, $\frac{d}{dx} \log_a u = \frac{1}{u \ln a}\frac{du}{dx}$.

Logarithmic Differentiation: The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**, is illustrated in the next example.

Example 12. Find *dy/dx* if $y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, x > 1$.

We take the natural logarithm of both sides and simplify the result with the algebraic properties of logarithms:

$$\ln y = \ln \frac{(x^2+1)(x+3)^{1/2}}{x-1} = \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1).$$

We then take derivatives of both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1} \cdot 2x + \frac{1}{2} \cdot \frac{1}{x+3} - \frac{1}{x-1} \Rightarrow \frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Finally, we substitute for y: $\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$.

Irrational Exponents and the Power Rule (General Version)

The definition of the general exponential function enables us to raise any positive number to any real power *n*, rational or irrational. That is, we can define the power function $y = x^n$ for any exponent *n*.

For any $x > 0$ and for any real number $n, x^n = e^{n \ln x}$. Because the logarithm and exponential functions are inverses of each other, the definition gives $\ln x^n = n \ln x$, for all real numbers *n*. That is, the rule for taking the natural logarithm of any power holds for *all* real exponents *n*, not just for rational exponents.

General Power Rule for Derivatives: For $x > 0$ and for any real number $n, \frac{d}{dx}x^n = nx^{n-1}$. If $x \leq 0$, then the formula holds whenever the derivative, x^n, and x^{n-1} all exist.

Proof. Differentiating x^n with respect to *x* gives

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{n \ln x} = e^{n \ln x}\frac{d}{dx}(n \ln x) = x^n \cdot \frac{n}{x} = nx^{n-1}.$$

Example 13. Differentiate $f(x) = x^x, x > 0$.

Solution: We note that $f(x) = x^x = e^{x \ln x}$, so differentiation gives

$$f'(x) = \frac{d}{dx}(e^{x \ln x}) = e^{x \ln x}\frac{d}{dx}(x \ln x) = e^{x \ln x}\left(\ln x + x \cdot \frac{1}{x}\right) = x^x(\ln x + 1).$$

Theorem 3 (The Number *e* as a Limit). The number *e* can be calculated as the limit

$$e = \lim_{x \to 0}(1+x)^{1/x}.$$

Proof. If $f(x) = \ln x$, then $f'(x) = 1/x$, so $f'(1) = 1$. But, by the definition of derivative,

$$f'(1) = \lim_{h \to 0}\frac{f(1+h) - f(1)}{h} = \lim_{x \to 0}\frac{f(1+x) - f(1)}{x} = \lim_{x \to 0}\frac{\ln(1+x) - \ln(1)}{x}$$

$$= \lim_{x \to 0}\frac{1}{x}\ln(1+x) =$$

$$= \lim_{x \to 0}\ln(1+x)^{1/x} = \ln\left[\lim_{x \to 0}(1+x)^{1/x}\right].$$

Because $f'(1) = 1$, we have $\ln\left[\lim_{x \to 0}(1+x)^{1/x}\right] = 1$. Therefore, exponentiating both sides we get $\lim_{x \to 0}(1+x)^{1/x} = e$.

Approximating the limit in Theorem 3 by taking *x* very small gives approximations to *e*. Its value is *e* ≈ 2.718281828459045 to 15 decimal places.

Glossary

chain rule – цепное правило; **implicit** – неявный

Exercises for Seminar 6

1. For given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

a) $y = 6u - 9, u = \left(\frac{1}{2}\right)x^4$; b) $y = \sin u, u = 3x + 1$; c) $y = \sqrt{u}, u = \sin x$.

2. Find the derivatives of the functions:

a) $y = (2x + 1)^5$; b) $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$; c) $s = \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t$;
d) $r = (\csc\theta + \cot\theta)^{-1}$; e) $y = \tan^2(\sin^3 t)$.

3. Use implicit differentiation to find dy/dx:

a) $x^2y + xy^2 = 6$; b) $2xy + y^2 = x + y$; c) $y^2 = \frac{x-1}{x+1}$; d) $x + \tan(xy) = 0$.

4. Use implicit differentiation to find dy/dx and then d^2y/dx^2:

a) $x^2 + y^2 = 1$; b) $y^2 = e^x + 2x$; c) $2\sqrt{y} = x - y$.

5. Verify that the given point is on the curve and find the tangent and normal line to the curve at the given point:

a) $x^2 + xy - y^2 = 1, (2,3)$; b) $6x^2 + 3xy + 2y^2 + 17y - 6 = 0, (-1,0)$.

6. Find the derivative of y with respect to x, t, or θ, as appropriate:

a) $y = \ln 3x + x$; b) $y = \ln(t^2)$; c) $y = \ln(\ln x)$; d) $y = \ln\frac{1}{x\sqrt{x+1}}$.

7. Use logarithmic differentiation to find the derivative of y with respect to the given independent variable: a) $y = \sqrt{x(x+1)}$; b) $y = \sqrt{\frac{t}{t+1}}$; c) $y = (\sin\theta)\sqrt{\theta + 3}$; d) $y = \sqrt[3]{\frac{x(x-2)}{x^2+1}}$.

8. Find the derivative of y with respect to the given independent variable:

a) $y = 2^x$; b) $y = x^\pi$; c) $y = \log_4 x + \log_4 x^2$.

9 (Logarithmic differentiation with Exponentials) Use logarithmic differentiation to find dy/dx:

a) $y = (x+1)^x$; b) $y = \left(\sqrt{x}\right)^x$; c) $y = (\sin x)^x$.

10. Find $f^{-1}(x)$. Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that at these points $df^{-1}/dx = 1/(df/dx)$.

a) $f(x) = 2x + 3, a = -1$; b) $f(x) = 5 - 4x, a = 1/2$.

11. Let $f(x) = x^3 - 3x^2 - 1, x \geq 2$. Find the value of df^{-1}/dx at the point $x = -1 = f(3)$.

Exercises for Homework 6

1. For given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

a) $y = 2u^3, u = 8x - 1$; b) $y = \cos u, u = e^{-x}$; c) $y = -\sec u, u = \frac{1}{x} + 7x$.

2. Find the derivatives of the functions:

a) $y = \left(\frac{\sqrt{x}}{2} - 1\right)^{-10}$; b) $y = \sqrt{3x^2 - 4x + 6}$; c) $s = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$;
d) $r = 6(\sec\theta - \tan\theta)^{\frac{3}{2}}$; e) $y = \cos^4(\sec^2 3t)$.

3. Use implicit differentiation to find dy/dx:

a) $x^3 + y^3 = 18xy$; b) $x^3 - xy + y^3 = 1$;
c) $x^3 = \frac{2x-y}{x+3y}$; d) $x^4 + \sin y = x^3y^2$.

4. Use implicit differentiation to find dy/dx and then d^2y/dx^2:

a) $x^{2/3} + y^{2/3} = 1$; b) $y^2 - 2x = 1 - 2y$; c) $xy + y^2 = 1$.

5. Verify that the given point is on the curve and find the tangent and normal line to the curve at the given point:

a) $y^2 - 2x - 4y - 1 = 0, (-2,1)$; b) $x^2 - \sqrt{3}xy + 2y^2 = 5, \left(\sqrt{3}, 2\right)$.

6. Find the derivative of y with respect to x, t, or θ, as appropriate:

a) $y = \frac{1}{\ln 3x}$; b) $y = \ln\left(t^{3/2}\right) + \sqrt{t}$; c) $y = \ln(\ln(\ln x))$; d) $y = \frac{1}{2}\ln\frac{1+x}{1-x}$.

7. Use logarithmic differentiation to find the derivative of y with respect to the given independent variable: a) $y = \sqrt{(x^2+1)(x-1)^2}$; b) $y = \sqrt{\frac{1}{t(t+1)}}$;

c) $y = (\tan\theta)\sqrt{2\theta + 1}$; d) $y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$.

8. Find the derivative of y with respect to the given independent variable:

a) $y = 5^{\sqrt{x}}$; b) $y = \log_2 5\,\theta$; c) $y = \theta\,\sin(\log_7 \theta)$.

9 (Logarithmic differentiation with Exponentials) Use logarithmic differentiation to find dy/dx:

a) $y = x^{(x+1)}$; b) $y = x^{\sqrt{x}}$; c) $y = x^{\sin x}$.

10. Find $f^{-1}(x)$. Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that at these points $df^{-1}/dx = 1/(df/dx)$.

a) $f(x) = (1/5)x + 7, a = -1$; b) $f(x) = 2x^2, x \geq 0, a = 5$.

11. Let $f(x) = x^2 - 4x - 5, x > 2$. Find the value of df^{-1}/dx at the point $x = 0 = f(5)$.