## Lecture 11. Antiderivatives. Indefinite Integrals. Riemann Sums. Definite Integrals.

A function $F$ is an **antiderivative** of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *antidifferentiation*. We use capital letters such as $F$ to represent an antiderivative of a function $f$, $G$ to represent an antiderivative of $g$, and so forth.

Obviously, any two antiderivatives of a function differ by a constant. Indeed, the functions $F_1(x) = x^2$ and $F_2(x) = x^2 + 1$ are antiderivatives for the function $f(x) = 2x$.

**Theorem 1.** If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on $I$ is $F(x)+C$ where $C$ is an arbitrary constant.

Thus, the most general antiderivative of $f$ on $I$ is a *family* of functions $F(x)+C$ where $F$ is an antiderivative of $f$ on an interval $I$. We can select a particular antiderivative from this family by assigning a specific value to $C$.

*Example 1*. Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

*Solution*: Since the derivative of $x^3$ is $3x^2$, the general antiderivative $F(x) = x^3 + C$ gives all the antiderivatives of $f(x)$. The condition $F(1) = -1$ determines a specific value for $C$. Substituting $x = 1$ into $F(x) = x^3 + C$ gives $C = -2$. So $F(x) = x^3 - 2$ is the antiderivative satisfying $F(1) = -1$.

**Table. Antiderivative formulas, $k$ a nonzero constant**

| Function | General antiderivative | Function | General antiderivative |
|---|---|---|---|
| 1. $x^n$ | $\frac{1}{n+1}x^{n+1} + C, n \neq -1$ | 8. $e^{kx}$ | $\frac{1}{k}e^{kx} + C$ |
| 2. $\sin kx$ | $-\frac{1}{k}\cos kx + C$ | 9. $\frac{1}{x}$ | $\ln\lvert x\rvert + C,\ x \neq 0$ |
| 3. $\cos kx$ | $\frac{1}{k}\sin kx + C$ | 10. $\frac{1}{\sqrt{1-k^2x^2}}$ | $\frac{1}{k}\sin^{-1} kx + C$ |
| 4. $\sec^2 kx$ | $\frac{1}{k}\tan kx + C$ | 11. $\frac{1}{1+k^2x^2}$ | $\frac{1}{k}\tan^{-1} kx + C$ |
| 5. $\csc^2 kx$ | $-\frac{1}{k}\cot kx + C$ | 12. $\frac{1}{x\sqrt{k^2x^2-1}}$ | $\sec^{-1} kx + C,\ kx > 1$ |
| 6. $\sec kx \tan kx$ | $\frac{1}{k}\sec kx + C$ | 13. $a^{kx}$ | $\left(\frac{1}{k\ln a}\right)a^{kx} + C,\ a > 0, a \neq 1$ |
| 7. $\csc kx \cot kx$ | $-\frac{1}{k}\csc kx + C$ | | |

**Table 2. Antiderivative linearity rules**

| | Function | General antiderivative |
|---|---|---|
| **1.** *Constant Multiple Rule:* | $kf(x)$ | $kF(x)+C$, $k$ a constant |
| **2.** *Negative Rule:* | $-f(x)$ | $-F(x)+C$ |
| **3.** *Sum or Difference Rule:* | $f(x) \pm g(x)$ | $F(x) \pm G(x) + C$ |

### Initial Value problems and Differential Equations

Antiderivatives play several important roles in mathematics and its applications. Methods and techniques for finding them are a major part of calculus. Finding an antiderivative for a function $f(x)$ is the same problem as finding a function $y(x)$ that satisfies the equation $\frac{dy}{dx} = f(x)$. This is called a **differential equation**, since it is an equation involving an unknown function that is being differentiated. To solve it, we need a function $y(x)$ that satisfies the equation. This function is found by taking the antiderivative of $f(x)$. We can fix the arbitrary constant arising in the antidifferentiation process by specifying an initial condition $y(x_0) = y_0$.

This condition means the function $y(x)$ has the value $y_0$ when $x = x_0$. The combination of a differential equation and an initial condition is called an **initial value problem**. Such problems play important roles in all branches of science.

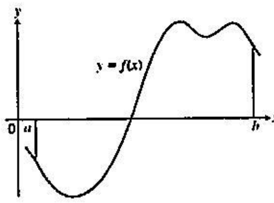
The most general antiderivative $F(x)+C$ of the function $f(x)$ gives the **general solution** $y = F(x)+C$ of the differential equation $dy/dx = f(x)$. The general solution gives all the solutions of the equation. We solve the differential equation by finding its general solution. We then solve the initial value problem by finding the **particular solution** that satisfies the initial condition $y(x_0) = y_0$. In Example 1, the function $y = x^3 - 2$ is the particular solution of the differential equation $dy/dx = 3x^2$ satisfying the initial condition $y(1) = -1$.

**Indefinite Integrals.** The collection of all antiderivatives of $f$ is called the **indefinite integral** of $f$ with respect to $x$, and is denoted by $\int f(x)dx$. The symbol $\int$ is an **integral sign**. The function $f$ is the **integrand** of the integral, and $x$ is the **variable of integration**. Thus, $\int f(x)dx = F(x)+C$, where $F(x)$ is an antiderivative of $f(x)$, and $C$ is an arbitrary constant.

After the integral sign in the notation we just defined, the integrand function is always followed by a differential to indicate the variable of integration.

Thus, we have $\int 2x dx = x^2 + C$, where $C$ is an arbitrary constant.

**Riemann Sums.** We now introduce the notion of a *Riemann sum* which underlines the theory of the definite integral. We begin with an arbitrary bounded function $f$ defined on a closed interval $[a,b]$. The function $f$ may have negative as well positive values.



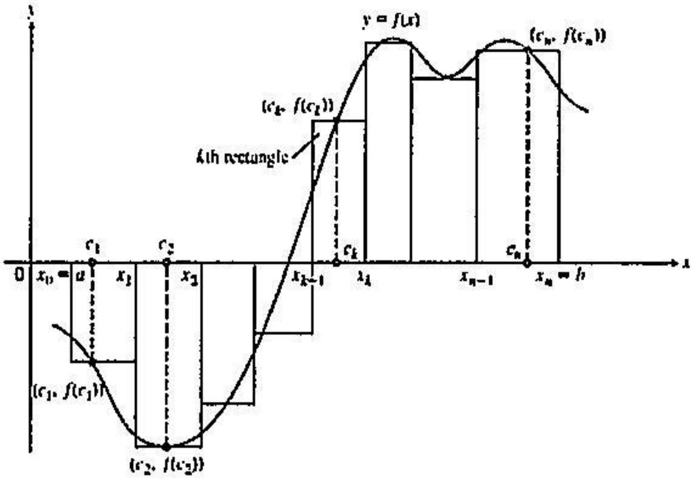

We subdivide the interval $[a,b]$ into subintervals, not necessarily of equal widths (or lengths). To do so, we choose $n-1$ points $\{x_1, x_2, x_3, ..., x_{n-1}\}$ between $a$ and $b$ satisfying $a < x_1 < x_2 < ... < x_{n-1} < b$. To make the notation consistent, we denote $a$ by $x_0$ and $b$ by $x_n$, so that $a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$. The set $P = \{x_0, x_1, x_2, x_3, ..., x_{n-1}, x_n\}$ is called a **partition** of $[a,b]$.

The partition $P$ divides $[a,b]$ into $n$ closed subintervals $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$. The first of these subintervals is $[x_0, x_1]$, the second is $[x_1, x_2]$, and the $k$ **th subinterval** of $P$ is $[x_{k-1}, x_k]$, for $k$ an integer between 1 and $n$.

The width of the first subinterval $[x_0, x_1]$ is denoted by $\Delta x_1$, the width of the second $[x_1, x_2]$ is denoted by $\Delta x_2$, and the width of the $k$th subinterval is $\Delta x_k = x_k - x_{k-1}$. If all $n$ subintervals have equal width, then the common width $\Delta x$ is equal to $(b-a)/n$.

In each subinterval we select some point. The point chosen in the $k$th subinterval $[x_{k-1}, x_k]$ is called $c_k$.

Then on each subinterval we stand a vertical rectangle that stretches from the $x$-axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the $x$-axis, depending on whether $f(c_k)$ is positive or negative, or on the $x$-axis if $f(c_k) = 0$.




On each subinterval we form the product $f(c_k)\cdot\Delta x_k$. This product is positive, negative, or zero, depending on the sign of $f(c_k)$. When $f(c_k)>0,$ the product $f(c_k)\cdot\Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width $\Delta x_k$. When $f(c_k)<0,$ the product $f(c_k)\cdot\Delta x_k$ is a negative number, the negative of the area of rectangle of width $\Delta x_k$ that drops from the $x$-axis to the negative number $f(c_k)$

Finally, we sum all these products to get $S_P=\sum\limits_{k=1}^{n} f(c_k)\Delta x_k$.

The sum $S_P$ is called a **Riemann sum for** $f$ **on the interval** $[a,b]$. There are many such sums, depending on the partition $P$ we choose, and the choices of the points $c_k$ in the subintervals. For instance, we could choose $n$ subintervals all having equal width $\Delta x=(b-a)/n$ to partition of $[a,b]$, and then choose the point $c_k$ to be the right-hand endpoint of each subinterval when forming the Riemann sum. This choice leads to the Riemann sum formula

$$S_n=\sum_{k=1}^{n} f\left(a+k\frac{(b-a)}{n}\right)\cdot\left(\frac{b-a}{n}\right).$$

Similar formulas can be obtained if instead we choose $c_k$ to be the left-hand endpoint, or the midpoint, of each subinterval.

In the cases in which the subintervals all have equal width $\Delta x=(b-a)/n$, we can make them thinner by simply increasing their number $n.$ When a partition has subintervals of varying widths, we can ensure they are all thin by controlling the width of a widest (longest) subinterval. We define the norm of a partition $P,$ written $\|P\|,$ to be the largest of all the subinterval widths. If $\|P\|$ is a small number, then all of the subintervals in the partition $P$ have a small width.

*Example 2*. The set $P=\{0;0{,}2;0{,}6;1;1{,}5;2\}$ is a partition of $[0;2]$. There are five subintervals of $P$: $[0;0{,}2],[0{,}2;0{,}6],[0{,}6;1],[1;1{,}5]$ and $[1{,}5;2]$. The lengths of the subintervals are $\Delta x_1=0{,}2;$ $\Delta x_2=0{,}4;\Delta x_3=0{,}4;\Delta x_4=0{,}5$ and $\Delta x_5=0{,}5.$ The longest subinterval length is 0,5, so the norm of the partition is $\|P\|=0{,}5.$ In this example, there are two subintervals of this length.

Any Riemann sum associated with a partition of a closed interval $[a,b]$ defines rectangles that approximate the region between the graph of a continuous function $f$ and the $x$-axis. Partitions with norm approaching zero lead to collections of rectangles that approximate this region with increasing accuracy.

**Definition of the Definite Integral.** Let $f(x)$ be a function defined on a closed interval $[a,b].$ We say that a number $J$ is the **definite integral of** $f$ **over** $[a,b]$ and that $J$ is the limit of the Riemann sums

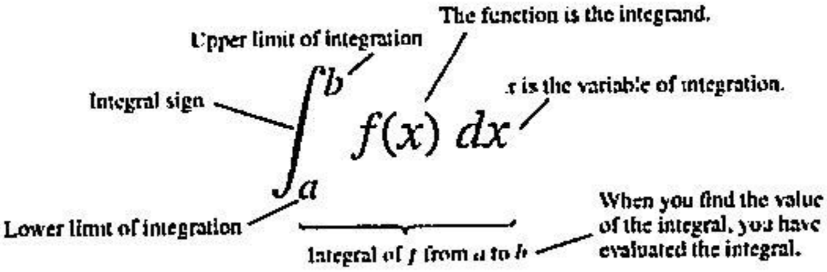
$\sum_{k=1}^{n} f(c_k)\Delta x_k$ if the following condition is satisfied: Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, x_2, ..., x_{n-1}, x_n\}$ of $[a,b]$ with $\| P \| < \delta$ and any choice of $c_k$ in $[x_{k-1}, x_k],$ we have $\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - J \right| < \varepsilon.$

The definition involves a limiting process in which the norm of the partition goes to zero.

We have many choices for a partition $P$ with norm going to zero, and many choices of points $c_k$ for each partition. The definite integral exists when we always get the same limit $J,$ no matter what choices are made. When the limit exists, we write it as the definite integral $J = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k.$

The limit of any Riemann sum is always taken as the norm of the partition approaches zero and the number of subintervals goes to infinity.

The symbol for the number $J$ in the definition of the definite integral is $\int_a^b f(x)dx,$ which is read as ''the integral from $a$ to $b$ of $f$ of $x$ dee $x$'' or sometimes as "the integral from $a$ to $b$ of $f$ of $x$ with respect to $x$". The component parts in the integral symbol also have names:




When the condition in the definition is satisfied, we say that the Riemann sums of $f$ on $[a,b]$ **converge** to the definite integral $J = \int_a^b f(x)dx$ and that $f$ is **integrable** over $[a,b].$

**Theorem 2** (**Integrability of Continuous Functions**) If a function $f$ is continuous over the interval $[a,b],$ or if $f$ has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)dx$ exists and $f$ is integrable over $[a,b].$

*Example 3*. The function $f(x) = \begin{cases} 1, & if\ x\ is\ \ rational \\ 0, & if\ x\ is\ \ irrational \end{cases}$ has no Riemann integral over [0, 1].

We show that upper sum approximations and lower sum approximations converge to different limiting values. If we pick a partition $P$ of [0, 1] and choose $c_k$ to be the point giving the maximum value for $f$ on $[x_{k-1}, x_k]$ then the corresponding Riemann sum is $U = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n}(1)\Delta x_k = 1,$ since each subinterval $[x_{k-1}, x_k]$ contains a rational number where $f(c_k) = 1.$ Note that the lengths of the intervals in the partition sum is equal to 1.

On the other hand, if we pick $c_k$ to be the point giving the minimum value for $f$ on $[x_{k-1}, x_k]$, then the Riemann sum is $L = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n}(0)\Delta x_k = 0$, since each subinterval $[x_{k-1}, x_k]$ contains an irrational number where $f(c_k) = 0$. Since the limit depends on the choices of $c_k$, the function $f$ is not integrable.

**Properties of Definite Integrals.**

**Theorem 3.** When $f$ and $g$ are integrable functions over the interval the definite integral satisfies the following rules:

1. Order of Integration: $\int_a^b f(x)dx = -\int_b^a f(x)dx$. 2. Zero Width Interval: $\int_a^a f(x)dx = 0$

3. Constant Multiple: $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ where $k$ is a constant.

4. Sum and Difference: $\int_a^b \left(f(x) \pm g(x)\right)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

5. Additivity: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$. 6. Max-Min Inequality: If $f$ has maximum value $\max f$ and minimum value $\min f$ on $[a,b]$, then $\min f \cdot (b-a) \le \int_a^b f(x)dx \le \max f \cdot (b-a)$.

7. Domination: $f(x) \ge g(x)$ on $[a,b] \Rightarrow \int_a^b f(x)dx \ge \int_a^b g(x)dx$

$$f(x) \ge 0 \text{ on } [a,b] \Rightarrow \int_a^b f(x)dx \ge 0.$$

*Proof of Rule 6*: Rule 6 says that the integral of $f$ over $[a,b]$ is never smaller than the minimum value of $f$ times the length of the interval and never larger than the maximum value of $f$ times the length of the interval. The reason is that for every partition of $[a,b]$ and for every choice of the points $c_k$,

$\min f \cdot (b-a) = \min f \cdot \sum_{k=1}^{n} \Delta x_k$ because $\sum_{k=1}^{n} \Delta x_k = b-a$.

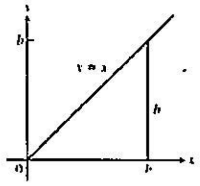
$$= \sum_{k=1}^{n} \min f \cdot \Delta x_k \le \sum_{k=1}^{n} f(c_k)\Delta x_k \le \sum_{k=1}^{n} \max f \cdot \Delta x_k = \max f \cdot \sum_{k=1}^{n} \Delta x_k = \max f \cdot (b-a).$$

Hence, their limit, the integral, does too. □

**Area under the Graph of a Nonnegative Function**

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a,b]$, then the **area under the curve** $y = f(x)$ **over** $[a,b]$ is the integral of $f$ from $a$ to $b$, $A = \int_a^b f(x)dx.$

*Example 4*. Find the area $A$ under $y = x$ over the interval $[0,b]$, $b > 0$.



*Solution*: Since the area equals the definite integral for a nonnegative function, we can quickly derive the definite integral by using the formula for the area of a triangle having base length $b$ and height $y=b.$ The area is $A=(1/2)b\cdot b=b^2/2.$ We conclude that $\int_0^b x dx = b^2/2.$

If $f$ is integrable on $[a,b],$ then its **average value on** $[a,b],$ also called its **mean**, is

$$av(f)=\frac{1}{b-a}\int_a^b f(x)dx.$$

## Glossary

**antiderivative** – первообразная; **indefinite integral** – неопределенный интеграл
**integrand** – подынтегральная функция

## Exercises for Seminar 11

1. Find an antiderivative for each function. Check your answers by differentiation.
a) $-3x^{-4}$; b) $-\pi\, sin\, \pi\, x$; c) $\frac{2}{3}\sec^2\frac{x}{3}$.

2. Find indefinite integrals:
a) $\int(x+1)dx$; b) $\int\left(3t^2+\frac{t}{2}\right)dt$; c) $\int x^{-1/3}dx$; d) $\int 7\sin\frac{\theta}{3}d\theta$; e) $\int(-3\csc^2 x)dx$; f) $\int(e^{3x}+5e^{-x})dx$.

3. Verify the formulas by differentiation:
a) $\int(7x-2)^3dx=\frac{(7x-2)^4}{28}+C$; b) $\int\sec^2(5x-1)dx=\frac{1}{5}\tan(5x-1)+C$.

4. Solve the initial value problems: a) $\frac{dy}{dx}=2x-7, y(2)=0$. b) $\frac{dy}{dx}=\frac{1}{2\sqrt{x}}, y(4)=0$.

5. Evaluate the definite integrals: a) $\int_3^1 7dx$; b) $\int_0^2(2t-3)dt$; c) $\int_2^1\left(1+\frac{z}{2}\right)dz$.

6. Use a definite integral to find the area of the region between the given curve and the $x$-axis on the interval $[0,b].$ a) $y=3x^2$; b) $y=2x$.

7. Graph the function and find its average value over the given interval.
a) $f(x)=x^2-1$ on $[0,\sqrt{3}]$; b) $f(x)=-3x^2-1$ on $[0,1]$.

8. Suppose that $\int_{-3}^0 g(t)dt=\sqrt{2}.$ Find: a) $\int_0^{-3}g(t)dt$ b) $\int_{-3}^0 g(u)du$ c) $\int_{-3}^0[-g(x)]dx$ d) $\int_{-3}^0\frac{g(r)}{\sqrt{2}}dr$

9. Find indefinite integrals:
a) $\int\frac{x+1}{\sqrt{x}}dx$; b) $\int\frac{(1-x)^3}{x\sqrt[3]{x}}dx$; c) $\int\frac{\left(\sqrt{2x}-\sqrt[3]{3x}\right)^2}{x}dx$; d) $\int\frac{x^2dx}{1+x^2}$; e) $\int(2^x+3^x)^2dx$; f) $\int\ (2x-3)^{10}dx$.

## Exercises for Homework 11

1. Find an antiderivative for each function. Check your answers by differentiation.
a) $1/(2x^3)$; b) $cos\frac{\pi\, x}{2}+\pi\, cos\, x$; c) $1-8\csc^2 2x$.

2. Find indefinite integrals: a) $\int(5-6x)dx$; b) $\int\left(\frac{t^2}{2}+4t^3\right)dt$; c) $\int x^{-5/4}dx$;
d) $\int 3\cos 5\theta\, d\theta$; e) $\int\left(-\frac{\sec^2 x}{3}\right)dx$.

3. Verify the formulas by differentiation:

a) $\int (3x+5)^{-2}dx = -\frac{(3x+5)^{-1}}{3} + C$; b) $\int \csc^2\left(\frac{x-1}{3}\right)dx = -3\cot\left(\frac{x-1}{3}\right) + C$.

4. Solve the initial value problems: a) $\frac{dy}{dx} = \frac{1}{x^2} + x, y(0) = -1$. b) $\frac{dr}{d\theta} = \cos \pi\theta, r(0) = 1$.

5. Evaluate the definite integrals: a) $\int_0^2 5x dx$; b) $\int_0^{\sqrt{2}} (t - \sqrt{2})dt$; c) $\int_1^0 (3x^2 + x - 5)dx$.

6. Use a definite integral to find the area of the region between the given curve and the $x$-axis on the interval $[0, b]$. a) $y = \pi x^2$; b) $y = x/2 + 1$.

7. Graph the function and find its average value over the given interval.

a) $f(x) = -x^2/2$ on $[0,3]$; b) $f(x) = 3x^2 - 3$ on $[0,1]$.

8. Find indefinite integrals:

a) $\int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}} dx$; b) $\int \left(1 - \frac{1}{x^2}\right)\sqrt{x\sqrt{x}}dx$; c) $\int \frac{2^{x+1} - 5^{x-1}}{10^x}dx$; d) $\int \sqrt[3]{1 - 3x}dx$; e) $\int \frac{dx}{2+3x^2}$;

f) $\int \frac{dx}{\sqrt{2-3x^2}}$.