Lecture 14. Trigonometric integrals. Integration of rational functions.

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral $\int sec^2 x\, dx = tan\, x + C$.

The general idea is to use identities to transform the integrals we have to find into integrals that are easier to work with.

Products of Powers of Sines and Cosines: We begin with integrals of the form $\int \sin^m x \cos^n x dx$, where *m* and *n* are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to *m* and *n* being odd or even.

Case 1. If ***m* is odd**, we write *m* as $2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1-\cos^2 x)^k \sin x. \quad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x\, dx$ equal to $-d(\cos x)$.

Case 2. If ***m* is even and *n* is odd** in $\int \sin^m x \cos^n x dx$, we write *n* as $2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain $\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1-\sin^2 x)^k \cos x$.

We then combine the single $\cos x$ with dx and set $\cos x\, dx$ equal to $d(\sin x)$.

Case 3. If **both *m* and *n* are even** in $\int \sin^m x \cos^n x dx$, we substitute

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2} \quad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Here are some examples illustrating each case.

Example 1. Evaluate $\int \sin^3 x \cos^2 x dx$.

Solution: This is an example of Case 1. $\int \sin^3 x \cos^2 x dx = \int \sin^2 x \cos^2 x \sin x dx =$

$= \int (1-\cos^2 x)\cos^2 x(-d(\cos x)) = |u = \cos x| = \int (1-u^2)u^2(-du) = \int (u^4 - u^2)du =$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{cos^5 x}{5} - \frac{cos^3 x}{3} + C.$$

Example 2. Evaluate $\int \sin^2 x \cos^4 x dx$.

Solution: This is an example of Case 3. $\int \sin^2 x \cos^4 x dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx =$

$= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x + \cos^2 2x)dx = \frac{1}{8}\int (1+\cos 2x - \cos^2 2x - \cos^3 2x)dx =$

$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int \left(\cos^2 2x + \cos^3 2x\right)dx\right].$

For the term involving $\cos^2 2x$, we use $\int \cos^2 2x dx = \frac{1}{2}\int (1+\cos 4x)dx = \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + C_1$.

For the $\cos^3 2x$ term, we have $\int \cos^3 2x dx = \int (1-\sin^2 2x)\cos 2x dx = |u = \sin 2x| =$

$= \frac{1}{2}\int (1-u^2)du = \frac{1}{2}(\sin 2x - \frac{1}{3}\sin^3 2x) + C_2$.

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

Eliminating Square Roots. In the next example, we use the identity $\cos^2\theta = (1+\cos 2\theta)/2$ to eliminate a square root.

Example. Evaluate $\int_0^{\pi/4} \sqrt{1+\cos 4x}dx.$

Solution: To eliminate the square root, we use the identity $\cos^2\theta = (1+\cos 2\theta)/2$ or $1+\cos 2\theta = 2\cos^2\theta.$ With $\theta = 2x,$ this becomes $1+\cos 4x = 2\cos^2 2x.$ Therefore,

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}dx = \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx =$$

$$= |\cos 2x \geq 0 \ on\ [0,\pi/4]| = \sqrt{2}\int_0^{\pi/4} \cos 2x dx = \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1-0] = \frac{\sqrt{2}}{2}.$$

Integrals of Powers of $\tan x$ **and** $\sec x$: We know how to integrate the tangent and secant and their squares. To integrate higher powers, we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1,$ and integrate by parts when necessary to reduce the higher powers to lower powers.

Example 3. Evaluate $\int \tan^4 x dx.$

Solution:

$$\int \tan^4 x dx = \int \tan^2 x \cdot \tan^2 x dx = \int \tan^2 x \cdot (\sec^2 x - 1)dx = \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx =$$

$$= \int \tan^2 x \sec^2 x dx - \int (\sec^2 x - 1)dx = \int \tan^2 x \sec^2 x dx - \int \sec^2 x dx + \int dx.$$

In the first integral, we let $u = \tan x,$ $du = \sec^2 x dx$ and have $\int u^2 du = \frac{1}{3}u^3 + C_1.$

The remaining integrals are standard forms, so $\int \tan^4 x dx = \frac{1}{3}\tan^3 x - \tan x + x + C.$

Example 4. Evaluate $\int \sec^3 x dx.$

Solution: We integrate by parts using $u = \sec x,$ $dv = \sec^2 x dx,$ $v = \tan x,$ $du = \sec x \tan x dx.$

Then $\int \sec^3 x dx = \sec x \tan x - \int (\tan x)(\sec x \tan x dx) = \sec x \tan x - \int (\sec^2 x - 1)\sec x dx =$

$$= \sec x \tan x + \int \sec x dx - \int \sec^3 x dx.$$

Combining the two secant-cubed integrals gives $2\int \sec^3 x dx = \sec x \tan x + \int \sec x dx$ and

$$\int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Example 5. Evaluate $\int \tan^4 x \sec^4 x dx.$

Solution: $\int \tan^4 x \sec^4 x dx = \int (\tan^4 x)(1+\tan^2 x)(\sec^2 x)dx = \int (\tan^4 x + \tan^6 x)(\sec^2 x)dx =$

$$= \int \tan^4 x \sec^2 x dx + \int \tan^6 x \sec^2 x dx = |u = \tan x| = \int u^4 du + \int u^6 du = \frac{u^5}{5} + \frac{u^7}{7} + C =$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C.$$

Products of Sines and Cosines: The integrals $\int \sin mx \sin nx dx,$ $\int \sin mx \cos nx dx$ and $\int \cos mx \cos nx dx$ arise in many applications involving periodic functions. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities: $\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$ (3)

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x] \quad (4)$$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]. \qquad (5)$$

These identities come from the angle sum formulas for the sine and cosine functions. They give functions whose antiderivatives are easily found.

Example 6. Evaluate $\int \sin 3x \cos 5x dx$.

Solution: From Equation (4) with m = 3 and n = 5, we get

$$\int \sin 3x \cos 5x dx = \frac{1}{2}\int \left[\sin(-2x) + \sin 8x\right]dx = \frac{1}{2}\int (\sin 8x - \sin 2x)dx = -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.$$

Trigonometric Substitutions.

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. The most common substitutions are $x = a\tan\theta, x = a\sin\theta, x = a\sec\theta.$ These substitutions are effective in transforming integrals involving $\sqrt{a^2+x^2}, \sqrt{a^2-x^2}$ and $\sqrt{x^2-a^2}$ into integrals we can evaluate directly.

With $x = a\tan\theta,\ a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta.$

With $x = a\sin\theta,\ a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta.$

With $x = a\sec\theta,\ x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$

Example 7. Evaluate $\int \frac{dx}{\sqrt{4+x^2}}$.

Solution: We set $x = 2\tan\theta,\ dx = 2\sec^2\theta\, d\theta,\ 4 + x^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta.$

Then $\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C =$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C.$$

Example 8. Evaluate $\int \frac{x^2 dx}{\sqrt{9-x^2}}$.

Solution: We set $x = 3\sin\theta, dx = 3\cos\theta\, d\theta, 9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta.$

Then $\int \frac{x^2 dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{|3\cos\theta|} = 9\int \sin^2\theta\, d\theta = 9\int \frac{1-\cos 2\theta}{2} d\theta = \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C =$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C = \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C = \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.$$

Example 9. Evaluate $\int \frac{dx}{\sqrt{25x^2-4}},\ x > \frac{2}{5}.$

Solution: We first rewrite the radical as $\sqrt{25x^2-4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)} = 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}$ to put the radicand in the form $x^2 - a^2$. We then substitute $x = \frac{2}{5}\sec\theta,\ dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta,$

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25} = \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta, \qquad \sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta|$$

With these substitutions, we have $\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{dx}{5\sqrt{x^2-(4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\, d\theta}{5\cdot(2/5)\tan\theta} =$

$$= \frac{1}{5}\int \sec\theta\, d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C = \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C.$$

Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are easily integrated. For instance, the rational function $(5x-3)/(x^2-2x-3)$ can be rewritten as $\dfrac{5x-3}{x^2-2x-3} = \dfrac{2}{x+1} + \dfrac{3}{x-3}$.

You can verify this equation algebraically by placing the fractions on the right side over a common denominator $(x + 1)(x - 3)$. The skill acquired in writing rational functions as such a sum is useful in other settings as well (for instance, when using certain transform methods to solve differential equations). To integrate the rational function $(5x-3)/(x^2-2x-3)$ on the left side of our previous expression, we simply sum the integrals of the fractions on the right side:

$$\int \frac{5x-3}{(x+1)(x-3)}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx = 2\ln|x+1| + 3\ln|x-3| + C.$$

The method for rewriting rational functions as a sum of simpler fractions is called **the method of partial fractions**. In the case of the preceding example, it consists of finding constants A and B such that $\dfrac{5x-3}{x^2-2x-3} = \dfrac{A}{x+1} + \dfrac{B}{x-3}$. (1)

We call the fractions $A/(x + 1)$ and $B/(x - 3)$ **partial fractions** because their denominators are only part of the original denominator $x^2 - 2x - 3$. We call A and B **undetermined coefficients** until suitable values for them have been found.

To find A and B, we first clear Equation (1) of fractions and regroup in powers of x, obtaining

$$5x - 3 = A(x - 3) + B(x + 1) = (A + B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal: $A + B = 5$, $-3A + B = -3$. Solving these equations simultaneously gives $A = 2$ and $B = 3$.

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

● *The degree of $f(x)$ must be less than the degree of $g(x)$*. That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term.

● *We must know the factors of $g(x)$*. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Here is how we find the partial fractions of a proper fraction $f(x)/g(x)$ when the factors of g are known. A quadratic polynomial (or factor) is **irreducible** if it cannot be written as the product of two linear factors with real coefficients. That is, the polynomial has no real roots.

Method of Partial Fractions when $f(x)/g(x)$ is Proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions: $\dfrac{A_1}{x-r} + \dfrac{A_2}{(x-r)^2} + \dots + \dfrac{A_m}{(x-r)^m}$.

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \dots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example 1. Use partial fractions to evaluate $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx.$

Solution: The partial fraction decomposition has the form

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}.$$

To find the values of the undetermined coefficients A, B, and C, we clear fractions and get
$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1) = (A+B+C)x^2 + (4A+2B)x +$
$+(3A-3B-C)$.

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x, obtaining: Coefficient of $x^2: A+B+C=1$

Coefficient of $x^1: 4A+2B=4$

Coefficient of $x^0: 3A-3B-C=1$

Thus, the solution is $A=3/4,\ B=1/2$ and $C=-1/4.$ Hence we have

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx = \frac{3}{4}\int\frac{dx}{x-1} + \frac{1}{2}\int\frac{dx}{x+1} - \frac{1}{4}\int\frac{dx}{x+3} = \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K,$$

where K is the arbitrary constant of integration (to avoid confusion with the undetermined coefficient we labeled as C).

The next example shows how to handle the case when $f(x)/g(x)$ is an improper fraction. It is a case where the degree of f is larger than the degree of g.

Example 2. Use partial fractions to evaluate $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx.$

Solution: First we divide the denominator into the numerator to get a polynomial plus a proper fraction: $2x^3-4x^2-x-3 = 2x(x^2-2x-3)+5x-3$.

Then we write the improper fraction as a polynomial plus a proper fraction, and we have

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx = \int 2x dx + \int \frac{5x-3}{x^2-2x-3}dx = \int 2x dx + \int \frac{2dx}{x+1} + \int \frac{3dx}{x-3} =$$
$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C.$$

Other Ways to Determine the Coefficients

Another way to determine the constants that appear in partial fractions is to differentiate, as in the next example. Still another is to assign selected numerical values to x.

Example 3. Find A, B, and C in the equation $\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$ by clearing fractions, differentiating the result, and substituting $x=-1$.

Solution: We first clear fractions: $x-1 = A(x+1)^2 + B(x+1) + C.$ Substituting $x=-1$ shows $C=-2.$ We then differentiate both sides with respect to x, obtaining $1 = 2A(x+1) + B.$ Substituting $x=-1$ shows $B = 1$. We differentiate again to get $0 = 2A,$ which shows $A = 0.$

Hence, $\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}$.

In some problems, assigning small values to x, such as $x = 0, \pm 1, \pm 2,$ to get equations in A, B, and C provides a fast alternative to other methods.

Example 4. Find A, B, and C in the expression $\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$ by assigning numerical values to x.

Solution: Clear fractions to get $x^2+1 = A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$.
Then let x = 1, 2, 3 successively to find A, B, and C:

$x = 1$: $1^2+1 = A\cdot(-1)\cdot(-2)+B\cdot 0+C\cdot 0 \Rightarrow A=1$

$x = 2$: $2^2+1 = A\cdot 0+B\cdot 1\cdot(-1)+C\cdot 0 \Rightarrow B=-5$

$x = 3$: $3^2+1 = A\cdot 0+B\cdot 0+C\cdot 2\cdot 1 \Rightarrow C=5$.

Conclusion: $\dfrac{x^2+1}{(x-1)(x-2)(x-3)} = \dfrac{1}{x-1}-\dfrac{5}{x-2}+\dfrac{5}{x-3}$.

Glossary

a proper fraction – правильная дробь

Exercises for Seminar 14

1. (Power of sines and cosines). Evaluate the integrals:

a) $\int \cos^3 x \sin x dx$ b) $\int \sin^3 x dx$ c) $\int \sin^3 x \cos^3 x dx$ d) $\int 16 \sin^2 x \cos^2 x dx$

2. (Integrating Square Roots). Evaluate the integrals:

a) $\int\limits_0^{2\pi} \sqrt{\dfrac{1-\cos x}{2}} dx$ b) $\int\limits_0^{\pi} \sqrt{1-\sin^2 t}\, dt$

3. (Powers of tangents and secants). Evaluate the integrals:

a) $\int \sec^2 x \tan x dx$ b) $\int \sec^3 x \tan x dx$ c) $\int \sec^4 \theta\, d\theta$ d) $\int \tan^5 x dx$

4. (Products of sines and cosines). Evaluate the integrals:

a) $\int \sin 3x \cos 2x dx$ b) $\int \cos 3x \cos 4x dx$

5. Use various trigonometric identities before you evaluate the integrals:

a) $\int \sin^2 \theta \cos 3\theta\, d\theta$ b) $\int \cos^3 \theta \sin 2\theta\, d\theta$

6. Using trigonometric substitutions evaluate the integrals:

a) $\int \dfrac{dx}{\sqrt{9+x^2}}$ b) $\int \sqrt{25-t^2}\, dt$ c) $\int \dfrac{dx}{\sqrt{4x^2-49}},\ x > \dfrac{7}{2}$ d) $\int \dfrac{\sqrt{y^2-49}}{y} dy,\ y > 7$

7. Expand quotients by partial fractions:

a) $\dfrac{5x-13}{(x-3)(x-2)}$ b) $\dfrac{x+4}{(x+1)^2}$ c) $\dfrac{z}{z^3-z^2-6z}$

8. Evaluate the integrals:

a) $\int \dfrac{dx}{1-x^2}$ b) $\int \dfrac{x+4}{x^2+5x-6} dx$ c) $\int \dfrac{dx}{(x^2-1)^2}$ d) $\int \dfrac{y^2+2y+1}{(y^2+1)^2} dy$ e) $\int \dfrac{x^2-x+2}{x^3-1} dx$

9. (Improper Fractions). Evaluate the integrals:

a) $\int \dfrac{2x^3-2x^2+1}{x^2-x} dx$ b) $\int \dfrac{9x^3-3x+1}{x^3-x^2} dx$ c) $\int \dfrac{y^4+y^2-1}{y^3+y} dy$

Exercises for Homework 14

1. (Power of sines and cosines). Evaluate the integrals:

a) $\int \sin^4 2x \cos 2x dx$ b) $\int \cos^3 4x dx$ c) $\int \cos^3 2x \sin^5 2x dx$ d) $\int 8\cos^4 2\pi x dx$

2. (Integrating Square Roots). Evaluate the integrals:

a) $\int_0^{\pi} \sqrt{1-\cos 2x} dx$ b) $\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1-\cos x}} dx$

3. (Powers of tangents and secants). Evaluate the integrals:

a) $\int \sec x \tan^2 x dx$ b) $\int \sec^3 x \tan^3 x dx$ c) $\int 3\sec^4 3x dx$ d) $\int \cot^6 2x dx$

4. (Products of sines and cosines). Evaluate the integrals:

a) $\int \sin 2x \cos 3x dx$ b) $\int \sin x \sin 7x dx$

5. Use various trigonometric identities before you evaluate the integrals:

a) $\int \cos^2 2\theta \sin\theta \, d\theta$ b) $\int \sin\theta \cos\theta \cos 3\theta \, d\theta$

6. Using trigonometric substitutions evaluate the integrals:

a) $\int \frac{3dx}{\sqrt{1+9x^2}}$ b) $\int \sqrt{1-9t^2} dt$ c) $\int \frac{5dx}{\sqrt{25x^2-9}}, \ x > \frac{3}{5}$ d) $\int \frac{\sqrt{y^2-25}}{y^3} dy, \ y > 5$

7. Expand quotients by partial fractions:

a) $\frac{5x-7}{x^2-3x+2}$ b) $\frac{z+1}{z^2(z-1)}$

8. Evaluate the integrals:

a) $\int \frac{dx}{x^2+2x}$ b) $\int \frac{2x+1}{x^2-7x+12} dx$ c) $\int \frac{x^2 dx}{(x-1)(x^2+2x+1)}$ d) $\int \frac{8x^2+8x+2}{(4x^2+1)^2} dx$ e) $\int \frac{dx}{x^4+x}$

9. (Improper Fractions). Evaluate the integrals:

a) $\int \frac{x^4 dx}{x^2-1}$ b) $\int \frac{16x^3 dx}{4x^2-4x+1}$