Lecture 12. Fundamental Theorem of Calculus. Substitution Method.

Here we present the Fundamental Theorem of Calculus, which is the central theorem of integral calculus. It connects integration and differentiation, enabling us to compute integrals using an antiderivative of the integrand function rather than by taking limits of Riemann sums.

Theorem 1 (Mean Value Theorem for Definite Integrals) If f is continuous on $[a,b]$, then at some point c in $[a,b]$, $f(c)=\frac{1}{b-a}\int_a^b f(x)dx.$

Proof. If we divide both sides of the Max-Min Inequality by $(b-a)$, we obtain

$$\min f \le \frac{1}{b-a}\int_a^b f(x)dx \le \max f.$$

Since f is continuous, the Intermediate Value Theorem for Continuous Functions say that f must assume every value between $\min f$ and $\max f$. It must therefore assume the value $\frac{1}{b-a}\int_a^b f(x)dx$ at some point c in $[a,b]$. □

Example 1. Show that if f is continuous on $[a,b], a \neq b,$ and if $\int_a^b f(x)dx = 0,$ then $f(x)=0$ at least once in $[a,b]$.

Solution: The average value of f on $[a,b]$ is $av(f)=\frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{b-a}\cdot 0 = 0.$

By the Mean Value Theorem, f assumes this value at some point $c \in [a,b]$.

Theorem 2 (The Fundamental Theorem of Calculus, Part 1) If f is continuous on $[a,b]$, then $F(x)=\int_a^x f(t)dt$ is continuous on $[a,b]$ and differentiable on (a,b) and its derivative is $f(x)$:

$$F'(x)=\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Proof. We prove by applying the definition of the derivative directly to the function $F(x)$, when x and $x+h$ are in (a,b). This means writing out the difference quotient $\frac{F(x+h)-F(x)}{h}$ and showing that its limit as $h \to 0$ is the number $f(x)$ for each x in (a,b). Doing so, we find

$$F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)dt-\int_a^x f(t)dt\right]=\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt.$$

According the Mean Value Theorem for Definite Integrals, the value before taking the limit in the last expression is one of the values taken on by f in the interval between x and $x+h$. That is, for some number c in this interval, $\frac{1}{h}\int_x^{x+h} f(t)dt = f(c)$.

As $h \to 0, x+h$ approaches x, forcing c to approach x also (because c is trapped between x and $x+h$). Since f is continuous at x, $f(c)$ approaches $f(x)$: $\lim_{h\to 0} f(c) = f(x)$.

In conclusion, we have $F'(x)=\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt = \lim_{h\to 0} f(c) = f(x).$

If $x=a$ or $b,$ then the limit of $\dfrac{F(x+h)-F(x)}{h}$ is interpreted as a one-sided limit with $h\to 0^+$ or $h\to 0^-$, respectively. Then F is continuous over $[a,b]$. This concludes the proof. □

Theorem 2 (The Fundamental Theorem of Calculus, Part 2) If f is continuous over $[a,b]$ and F is any antiderivative of f on $[a,b]$, then $\int_a^b f(x)dx = F(b)-F(a).$

Proof. Part 1 tell us that an antiderivative of f exists, namely $G(x)=\int_a^x f(t)dt.$ Thus, if F is any antiderivative of f, then $F(x)=G(x)+C$ for some constant C for $a<x<b.$
Since both F and G are continuous on $[a,b]$, we see that $F(x)=G(x)+C$ also holds when $x=a$ and $x=b$ by taking one-sided limits (as $x\to a^+$ and $x\to b^-$).
Evaluating $F(b)-F(a)$, we have

$$F(b)-F(a)=[G(b)+C]-[G(a)+C]=G(b)-G(a)=\int_a^b f(t)dt-\int_a^a f(t)dt=\int_a^b f(t)dt. \ \square$$

Example 2. (a) $\int_0^\pi \cos x dx = \sin x\Big]_0^\pi = \sin\pi - \sin 0 = 0-0=0$

(b) $\int_{-\pi/4}^0 \sec x\tan x dx = \sec x\Big]_{-\pi/4}^0 = \sec 0 - \sec(-\pi/4) = 1-\sqrt{2}$.

The Integral of a Rate. We can interpret Part 2 of the Fundamental Theorem in another way. If F is any antiderivative of f, then $F'=f$. The equation in the theorem can then be rewritten as

$$\int_a^b F'(x)dx = F(b)-F(a).$$

Now $F'(x)$ represents the rate of change of the function $F(x)$ with respect to x, so the last equation asserts that the integral of F' is just the net change in F as x changes from a to b.

Theorem 3 (**The Net Change Theorem**) The net change in a differentiable function $F(x)$ over an interval $a\le x\le b$ is the integral of its rate of change: $F(b)-F(a)=\int_a^b F'(x)dx.$

Total Area. To find the area between the graph of $y=f(x)$ and the x-axis over the interval $[a,b]$:
1. Subdivide $[a,b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

Example. Find the total area of the region between the x-axis and the graph of $f(x)=x^3-x^2-2x,\ -1\le x\le 2.$

Solution: First find the zeros of f. Since $f(x)=x^3-x^2-2x=x(x+1)(x-2),$ the zeros are $x=0, -1$ and 2. The zeros subdivide $[-1, 2]$ into two subintervals: $[-1, 0]$, on which $f\ge 0$, and $[0, 2]$, on which $f\le 0$. We integrate f over each subinterval and add the absolute values of the calculated integrals:

$\int_{-1}^{0}(x^3 - x^2 - 2x)dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = \frac{5}{12}$; $\int_{0}^{2}(x^3 - x^2 - 2x)dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = -\frac{8}{3}$.

Total enclosed area $= \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$.

We must distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)dx$ is a *number*. An indefinite integral $\int f(x)dx$ is a *function* plus an arbitrary constant C.

Here we begin to develop more general techniques for finding antiderivatives of functions we can't easily recognize as a derivative.

Substitution method: If u is a differentiable function of x and n is any number different from -1, the Chain Rule tells us that $\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}$. From another point of view, this same equation says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n(du/dx)$. Therefore,

$$\int u^n \frac{du}{dx}dx = \frac{u^{n+1}}{n+1} + C.$$

The last integral is equal to the simpler integral $\int u^n du = \frac{u^{n+1}}{n+1} + C$, which suggests that the simpler expression du can be substituted for $(du/dx)dx$ when computing an integral. Leibniz, one of the founders of calculus, had the insight that indeed this substitution could be done, leading to the *substitution method* for computing integrals.

Example 1. Find the integral $\int (x^3 + x)^5(3x^2 + 1)dx$.

Solution: We set $u = x^3 + x$. Then $du = \frac{du}{dx}dx = (3x^2 + 1)dx$, so that by substitution we have

$$\int (x^3 + x)^5(3x^2 + 1)dx = \int u^5 du = \frac{u^6}{6} + C = \frac{(x^3 + x)^6}{6} + C.$$

Example 2. Find $\int \sqrt{2x+1}dx$.

Solution: The integral does not fit the formula $\int u^n du$, with $u = 2x + 1$ and $n = 1/2$, because $du = \frac{du}{dx}dx = 2dx$ is not precisely dx. The constant factor 2 is missing from the integral. However, we can introduce this factor after the integral sign if we compensate for it by a factor of $1/2$ in front of the integral sign. So we write

$$\int \sqrt{2x+1}dx = \frac{1}{2}\int \sqrt{2x+1} \cdot 2dx = \frac{1}{2}\int u^{1/2}du = \frac{1}{2}\frac{u^{3/2}}{3/2} + C = \frac{1}{3}(2x+1)^{3/2} + C.$$

Theorem 1 (The Substitution Rule). If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then $\int f(g(x))g'(x)dx = \int f(u)du$.

Proof. By the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ whenever F is an antiderivative of f: $\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$.

If we make the substitution $u = g(x)$, then $\int f(g(x))g'(x)dx = \int \frac{d}{dx}F(g(x))dx = F(g(x)) + C =$

$= F(u) + C = \int F'(u)du = \int f(u)du.$ □

The substitution method to evaluate $\int f(g(x))\, g'(x)dx$:

1. Substitute $u = g(x)$ and $du = (du/dx)dx = g'(x)dx$ to obtain $\int f(u)du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

Example 3. Find $\int \sec^2(5x+1)\cdot 5dx$.

Solution: We substitute $u = 5x+1$ and $du = 5dx$. Then,

$$\int \sec^2(5x+1)\cdot 5dx = \int \sec^2 u du = \tan u + C = \tan(5x+1) + C.$$

Example 4. $\int x^2 e^{x^3} dx = \int e^{x^3}\cdot x^2 dx = \left| u = x^3 \Rightarrow du = 3x^2 dx \Rightarrow (1/3)du = x^2 dx \right| = \int e^u \cdot \frac{1}{3} du =$

$$= \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C.$$

Example 5. Evaluate $\int x\sqrt{2x+1}\, dx$.

Solution: We suggest the substitution $u = 2x+1$ with $du = 2dx$. Then $\sqrt{2x+1}dx = \frac{1}{2}\sqrt{u}du$.

However, in this case the integrand contains an extra factor of x multiplying the term $\sqrt{2x+1}$. To adjust for this, we solve the substitution equation $u = 2x+1$ to obtain $x = (u-1)/2$, and find that $x\sqrt{2x+1}dx = \frac{1}{2}(u-1)\cdot\frac{1}{2}\sqrt{u}du$. The integration now becomes

$$\int x\sqrt{2x+1}\, dx = \frac{1}{4}\int (u-1)\sqrt{u}du = \frac{1}{4}\int (u-1)u^{1/2}du = \frac{1}{4}\int (u^{3/2} - u^{1/2})du = \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C =$$

$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C.$$

Example 6. Sometimes we can use trigonometric identities to transform integrals we do not know to evaluate into ones we can evaluate using the Substitution Rule.

a) $\int \sin^2 x dx = \left| \sin^2 x = \frac{1-\cos 2x}{2} \right| = \int \frac{1-\cos 2x}{2} dx = \frac{1}{2}\int (1-\cos 2x)dx = \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C =$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

b) $\int \cos^2 x dx = \left| \cos^2 x = \frac{1+\cos 2x}{2} \right| = \int \frac{1+\cos 2x}{2} dx = \frac{x}{2} + \frac{\sin 2x}{4} + C.$

c) $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = \left| u = \cos x \Rightarrow du = -\sin x dx \right| = \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C =$

$$= \ln\frac{1}{|\cos x|} + C = \ln|\sec x| + C.$$

Example 7. An integrand may require some algebraic manipulations before the substitution method can be applied.

a) $\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x dx}{e^{2x}+1} = \left| u = e^x \Rightarrow du = e^x dx \right| = \int \frac{du}{u^2+1} = \tan^{-1} u + C = \tan^{-1}(e^x) + C.$

b) $\int \sec x dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx = \left| u = \tan x + \sec x \Rightarrow \right.$

$$\left. \Rightarrow du = (\sec^2 x + \sec x \tan x)dx \right| = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C.$$

The integrals of $\cot x$ and $\csc x$ are found in a way similar to those used for finding the integrals of $\tan x$ and $\sec x$.

Integrals of the tangent, cotangent, secant and cosecant functions

$$\int \tan x dx = \ln|\sec x| + C \qquad \int \sec x dx = \ln|\sec x + \tan x| + C$$
$$\int \cot x dx = \ln|\sin x| + C \qquad \int \csc x dx = -\ln|\csc x + \cot x| + C$$

Trying Different Substitutions: The success of the substitution method depends on finding a substitution that changes an integral we cannot evaluate directly into one that we can. Finding the right substitution gets easier with practice and experience. If the first substitution fails, try another substitution, possibly coupled with other algebraic or trigonometric simplifications to the integrand.

Example 8. Evaluate $\int \frac{2zdz}{\sqrt[3]{z^2+1}}$.

Solution: **Method 1**. $\int \frac{2zdz}{\sqrt[3]{z^2+1}} = \left| u = z^2 + 1 \Rightarrow du = 2zdz \right| = \int \frac{du}{u^{1/3}} = \int u^{-1/3} du = \frac{u^{2/3}}{2/3} + C =$

$= \frac{3}{2} u^{2/3} + C = \frac{3}{2}(z^2+1)^{2/3} + C.$

Method 2. $\int \frac{2zdz}{\sqrt[3]{z^2+1}} = \left| u = \sqrt[3]{z^2+1} \Rightarrow u^3 = z^2 + 1 \Rightarrow 3u^2 du = 2zdz \right| = 3\int u du = 3 \cdot \frac{u^2}{2} + C =$

$= \frac{3}{2}(z^2+1)^{2/3} + C.$

Exercises for Seminar 12

1. Calculate the integrals: a) $\int_1^4 (5x + x^2) dx$; b) $\int_1^2 x^3(x^2 - 1) dx$.

2. Find the derivatives: (a) by evaluating the integral and differentiating the result; (b) by differentiating the integral directly. a) $\frac{d}{dx}\int_0^{\sqrt{x}} \cos t \, dt$; b) $\frac{d}{dt}\int_0^{t^4} \sqrt{u}\, du$.

3. Find dy/dx.

a) $y = \int_0^x \sqrt{1+t^2}\, dt$; b) $y = \int_{\sqrt{x}}^0 \sin(t^2)\, dt$.

4. Find the total area between the region and the x-axis:

a) $y = -x^2 - 2x,\ -3 \le x \le 2.$ b) $y = x^3 - 3x^2 + 2x,\ 0 \le x \le 2.$

5. Evaluate the indefinite integrals by using the given substitutions:

a) $\int 2(2x+4)^4 dx,\ u = 2x + 4$; b) $\int 2x(x^2+5)^{-4} dx,\ u = x^2 + 5$;

c) $\int (3x+2)(3x^2+4x)^4 dx,\ u = 3x^2 + 4x$.

6. Evaluate the integrals:

a) $\int \sqrt{3-2s}\, ds$; b) $\int \theta \sqrt[4]{1-\theta^2}\, d\theta$; c) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx$; d) $\int \frac{x}{\sqrt{1+x}} dx$; e) $\int \frac{1}{x^2}\sqrt{2 - \frac{1}{x}}\, dx$.

f) $\int x e^{-x^2} dx$; g) $\int \frac{sinx}{\sqrt{cos^3 x}} dx$; h) $\int \frac{sinx}{\sqrt{cos2x}} dx$.

Exercises for Homework 12

1. Calculate the integrals: a) $\int_{-3}^{4}(4-x^2)dx$; b) $\int_{-1}^{0}x(2x^2-1)^4\,dx.$

2. Find the derivatives: (a) by evaluating the integral and differentiating the result; (b) by differentiating the integral directly.

a) $\frac{d}{dx}\int_{1}^{\sin x}3t^2\,dt$; b) $\frac{d}{d\theta}\int_{0}^{\tan\theta}\sec^2 y\,dy$.

3. Find dy/dx.

a) $y=\int_{1}^{x}\frac{1}{t}dt,\ x>0$. b) $y=x\int_{2}^{x^2}\sin(t^3)\,dt$.

4. Find the total area between the region and the x-axis:

a) $y=3x^2-3,\ -2\le x\le 2.$ b) $y=x^{1/3}-x,\ -1\le x\le 8.$

5. Evaluate the indefinite integrals by using the given substitutions:

a) $\int 7\sqrt{7x-1}dx,\ u=7x-1$. b) $\int\frac{4x^3}{(x^4+1)^2}dx,\ u=x^4+1$.

6. Evaluate the integrals:

a) $\int\frac{1}{\sqrt{5s+4}}ds$; b) $\int 3y\sqrt{7-3y^2}dy$; c) $\int\sqrt{\sin x}\cos^3 x dx$; d) $\int\sqrt{\frac{x-1}{x^5}}dx$; e) $\int\frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}dx$.

f) $\int\frac{e^x}{2+e^x}dx$; g) $\int tanxdx$; h) $\int\frac{sinx+cosx}{\sqrt[3]{sinx-cosx}}dx$; l) $\int\frac{cosx}{\sqrt{cos2x}}dx.$