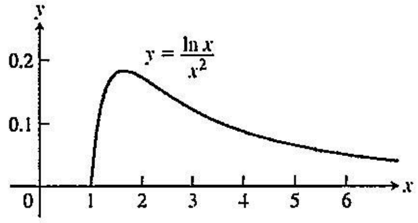
Lecture 15. Improper Integrals

Up to now, we have required definite integrals to have two properties. First, the domain of integration $[a,b]$ must be finite. Second, the range of the integrand must be finite on this domain. In practice, we may encounter problems that fail to meet one or both of these conditions. The integral for the area under the curve $y=(\ln x)/x^2$ from x = 1 to $x=\infty$ is an example for which the domain is infinite. The integral for the area under the curve of $y=1/\sqrt{x}$ between x = 0 and $x=1$ is an example for which the range of the integrand is infinite.



In either case, the integrals are said to be *improper* and are calculated as limits.

Infinite Limits of Integration

Consider the infinite region (unbounded on the right) that lies under the curve $y=e^{-x/2}$ in the first quadrant. You might think this region has infinite area, but we will see that the value is finite. We assign a value to the area in the following way. First find the area $A(b)$ of the portion of the region that is bounded on the right by $x = b$: $A(b)=\int\limits_0^b e^{-x/2}dx=-2e^{-x/2}\Big|_0^b=-2e^{-b/2}+2.$

Then find the limit of $A(b)$ as $b\to\infty$: $\lim\limits_{b\to\infty}A(b)=\lim\limits_{b\to\infty}(-2e^{-b/2}+2)=2.$

The value we assign to the area under the curve from 0 to ∞ is $\int\limits_0^\infty e^{-x/2}dx=\lim\limits_{b\to\infty}\int\limits_0^b e^{-x/2}dx=2.$

Definition. Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a,\infty)$, then $\int\limits_a^\infty f(x)dx=\lim\limits_{b\to\infty}\int\limits_a^b f(x)dx.$

2. If $f(x)$ is continuous on $(-\infty,b]$, then $\int\limits_{-\infty}^b f(x)dx=\lim\limits_{a\to-\infty}\int\limits_a^b f(x)dx.$

3. If $f(x)$ is continuous on $(-\infty,\infty)$, then $\int\limits_{-\infty}^\infty f(x)dx=\int\limits_{-\infty}^c f(x)dx+\int\limits_c^\infty f(x)dx,$ where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Example. Is the area under the curve $y=(\ln x)/x^2$ from $x=1$ to $x=\infty$ finite? If so what is its value?

Solution: $\int\limits_1^\infty \frac{\ln x}{x^2}dx=\lim\limits_{b\to\infty}\int\limits_1^b \frac{\ln x}{x^2}dx=\Big|\ u=\ln x,\ du=dx/x,\ dv=dx/x^2,\ v=-1/x\ \Big|=$

$$=\lim_{b\to\infty}\left(\left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b-\int_1^b\left(-\frac{1}{x}\right)\frac{1}{x}dx\right)=\lim_{b\to\infty}\left(-\frac{\ln b}{b}-\left[\frac{1}{x}\right]_1^b\right)=\lim_{b\to\infty}\left(-\frac{\ln b}{b}-\frac{1}{b}+1\right)$$
$$=1.$$

Thus, the improper integral converges and the area has finite value 1.

The Integral $\int_1^{\infty} \frac{dx}{x^p}$. The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if $p > 1$ and diverges if $p \le 1$.

Example. For what values of *p* does the integral $\int_1^{\infty} \frac{dx}{x^p}$ converge? When the integral does converge, what is its value?

*Solution***:** If $p \neq 1$, $\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\Big|_1^b = \frac{1}{1-p}(b^{-p+1} - 1) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)$.

Thus, $\int_1^{\infty} \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^p} = \lim_{b\to\infty} \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right) = \begin{cases} 1/(p-1), & p > 1 \\ \infty, & p < 1 \end{cases}$ because $\lim_{b\to\infty} \frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1 \end{cases}$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$.

If *p* = 1, the integral also diverges: $\int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln x\Big|_1^b = \lim_{b\to\infty}(\ln b - \ln 1) = \infty$.

Integrands with Vertical Asymptotes. Another type of improper integral arises when the integrand has a vertical asymptote – an infinite discontinuity – at a limit of integration or at some point between the limits of integration. If the integrand *f* is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of *f* and above the *x*-axis between the limits of integration.

Definition. Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If *f*(*x*) is continuous on $(a,b]$ and discontinuous at *a*, then $\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx$.

2. If *f*(*x*) is continuous on $[a,b)$ and discontinuous at *b*, then $\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx$.

3. If *f*(*x*) is discontinuous at *c*, where $a < c < b$, and continuous on $[a,c) \cup (c,b]$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

Example. Investigate the convergence of $\int_0^1 \frac{1}{1-x}dx$.

Solution: The integrand $f(x) = 1/(1-x)$ is continuous on $[0, 1)$ but is discontinuous at *x* = 1 and becomes infinite as $x \to 1^-$. We evaluate the integral as

$$\lim_{b\to 1^-} \int_0^b \frac{1}{1-x}dx = \lim_{b\to 1^-}\left[-\ln|1-x|\right]_0^b = \lim_{b\to 1^-}\left[-\ln(1-b) + 0\right] = \infty.$$

The limit is infinite, so the integral diverges.

Example. Evaluate $\int_0^3 \frac{dx}{(x-1)^{2/3}}$.

Solution: The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$. Thus, by Part 3 of the definition above, $\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$.

Next, we evaluate each improper integral on the right-hand side of this equation.

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} 3(x-1)^{1/3}\Big|_0^b = \lim_{b\to 1^-}\left[3(b-1)^{1/3} + 3\right] = 3.$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+} 3(x-1)^{1/3}\Big|_c^3 = \lim_{c\to 1^+}\left[3(3-1)^{1/3} - 3(c-1)^{1/3}\right] = 3\sqrt[3]{2}.$$

We conclude $\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$

Tests for Convergence and Divergence.

When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges. If the integral diverges, that's the end of the story. If it converges, we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

Example. Does the integral $\int_1^\infty e^{-x^2}\,dx$ converge?

Solution: By definition, $\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx.$

We cannot evaluate this integral directly because it is non-elementary. But we *can* show that its limit as $b \to \infty$ is finite. We know that $\int_1^b e^{-x^2}\,dx$ is an increasing function of b. Therefore either it becomes infinite as $b \to \infty$ or it has a finite limit as $b \to \infty$. It does not become infinite: For every value of $x \ge 1$, we have $e^{-x^2} \le e^{-x}$ so that $\int_1^b e^{-x^2}\,dx \le \int_1^b e^{-x}\,dx = -e^{-b} + e^{-1} < e^{-1} = 0{,}36788.$

Hence, $\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx$ converges to some definite finite value. We do not know exactly what the value is except that it is something positive and less than 0.37.

Theorem 1 (Direct Comparison Test) Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1) $\int_a^\infty f(x)dx$ converges if $\int_a^\infty g(x)dx$ converges.

2) $\int_a^\infty g(x)dx$ diverges if $\int_a^\infty f(x)dx$ diverges.

Proof. The reasoning behind the argument establishing Theorem 1 is similar to that in Example above. If $0 \le f(x) \le g(x)$ for $x \ge a$, then we have $\int_a^b f(x)dx \le \int_a^b g(x)dx,\ b > a.$

From this it can be argued, as in Example above, that $\int_a^\infty f(x)dx$ converges if $\int_a^\infty g(x)dx$ converges.

Turning this around says that $\int_a^\infty g(x)dx$ diverges if $\int_a^\infty f(x)dx$ diverges. □

Theorem 2 (Limit Comparison Test) If positive functions f and g are continuous on $[a, \infty)$, and if $\lim_{x\to\infty}\frac{f(x)}{g(x)} = L,\ 0 < L < \infty,$ then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge or both diverge.

Although the improper integrals of two functions from a to ∞ may both converge, this does not mean that their integrals necessarily have the same value, as the next example shows.

Example. Show that $\int_1^\infty \frac{dx}{1+x^2}$ converges by comparison with $\int_1^\infty \frac{dx}{x^2}$. Find and compare the two integral values.

Solution: The functions $f(x) = 1/x^2$ and $g(x) = 1/(1+x^2)$ are positive and continuous on $[1, \infty)$. Also, $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{1/x^2}{1/(1+x^2)} = \lim_{x\to\infty}\frac{1+x^2}{x^2} = \lim_{x\to\infty}\left(\frac{1}{x^2}+1\right) = 0+1=1,$ a positive finite limit. Therefore, $\int_1^\infty \frac{dx}{1+x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges. The integrals converge to different values, however: $\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^2} = \lim_{b\to\infty}\left(-\frac{1}{x}\right)_1^b = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1$ and

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_1^b \frac{dx}{1+x^2} = \lim_{b\to\infty}[\arctan b - \arctan 1] = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}.$$

Exercises for Seminar 15

1. The following integrals converge. Evaluate the integrals:

a) $\int_0^\infty \frac{dx}{x^2+1}$ b) $\int_0^1 \frac{dx}{\sqrt{x}}$ c) $\int_{-1}^1 \frac{dx}{x^{2/3}}$ d) $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$ e) $\int_{-1}^\infty \frac{d\theta}{\theta^2+5\theta+6}$

2. Using Tests for Convergence test the following integrals for convergence:

a) $\int_0^{\pi/2} \tan\theta\, d\theta$ b) $\int_0^1 \frac{\ln x}{x^2}dx$ c) $\int_0^{\ln 2} x^{-2}e^{-1/x}dx$ d) $\int_\pi^\infty \frac{2+\cos x}{x}dx$ e) $\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}}$

Exercises for Homework 15

1. The following integrals converge. Evaluate the integrals:

a) $\int_1^\infty \frac{dx}{x^{1,001}}$ b) $\int_0^4 \frac{dx}{\sqrt{4-x}}$ c) $\int_{-8}^1 \frac{dx}{x^{1/3}}$ d) $\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$ e) $\int_0^\infty \frac{dx}{(1+x)(x^2+1)}$

2. Using Tests for Convergence test the following integrals for convergence:

a) $\int_0^{\pi/2} \cot\theta\, d\theta$ b) $\int_1^2 \frac{dx}{x\ln x}$ c) $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$ d) $\int_\pi^\infty \frac{1+\cos x}{x^2}dx$ e) $\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$