Lecture 13. Definite Integral Substitutions. Techniques of Integration.

There are two methods for evaluating a definite integral by substitution. One method is to find an antiderivative using substitution and then to evaluate the definite integral by applying the Evaluation Theorem. The other method extends the process of substitution directly to definite integrals by changing the limits of integration.

Theorem 1 (**Substitution in Definite Integrals**) If g' is continuous on the interval $[a,b]$ and f is continuous on the range of $g(x)=u,$ then $\int_a^b f(g(x))\cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$

Proof. Let F denote any antiderivative of f. Then, $\int_a^b f(g(x))\cdot g'(x)dx = F(g(x))\Big|_{x=a}^{x=b} =$

$$= F(g(b)) - F(g(a)) = F(u)\Big|_{u=g(a)}^{u=g(b)} = \int_{g(a)}^{g(b)} f(u)du. \quad \square$$

Example 1. Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx.$

Solution: **Method 1**: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 1.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx = \left| u = x^3+1, du = 3x^2dx;\ x=-1 \Rightarrow u=(-1)^3+1=0,\ x=1 \Rightarrow u=1^3+1=2 \right| =$$

$$= \int_0^2 \sqrt{u}du = \frac{2}{3}u^{3/2}\Big|_0^2 = \frac{2}{3}\left[2^{3/2}-0^{3/2}\right] = \frac{4\sqrt{2}}{3}.$$

Method 2: Transform the integral as an indefinite integral, integrate, change back to $x,$ and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}dx = \left| u = x^3+1, du = 3x^2dx \right| = \int \sqrt{u}du = \frac{2}{3}u^{3/2}+C = \frac{2}{3}(x^3+1)^{3/2}+C.$$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx = \frac{2}{3}(x^3+1)^{3/2}\Big|_{-1}^{1} = \frac{4\sqrt{2}}{3}.$$

Definite Integrals of Symmetric Functions.

Theorem 2. Let f be continuous on the symmetric interval $[-a,a].$

(a) If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx.$ (b) If f is odd, then $\int_{-a}^{a} f(x)dx = 0.$

Proof of Part (a). $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx = -\int_0^{-a} f(x)dx + \int_0^a f(x)dx = -\int_0^a f(-u)(-du) +$

$$+\int_0^a f(x)dx = \int_0^a f(-u)du + \int_0^a f(x)dx = \int_0^a f(u)du + \int_0^a f(x)dx = 2\int_0^a f(x)dx.$$

The proof of part (b) is entirely similar. $\square$

Areas Between Curves.

If f and g are continuous with $f(x)\ge g(x)$ throughout $[a,b],$ then the area of the region between the curves $y=f(x)$ and $y=g(x)$ from a to b is the integral of $(f-g)$ from a to b:

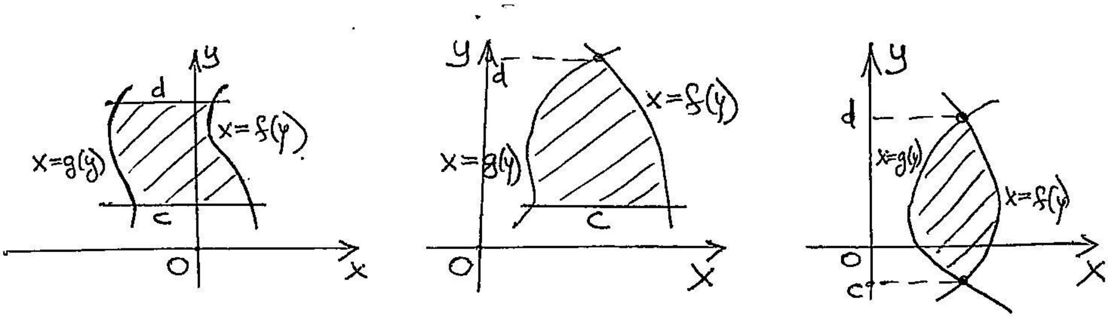
$$A = \int_a^b [f(x) - g(x)]dx.$$

Example 2. Find the area of the region bounded above by the curve $y = 2e^{-x} + x$, below by the curve $y = e^x/2$, on the left by $x = 0$, and on the right by $x = 1$.

Solution: $A = \int_0^1 [(2e^{-x} + x) - (1/2)e^x]dx = [-2e^{-x} + (1/2)x^2 - (1/2)e^x]_0^1 = 3 - 2/e - e/2 \approx 0{,}9051.$

Integration with Respect to y

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x. For regions like these:



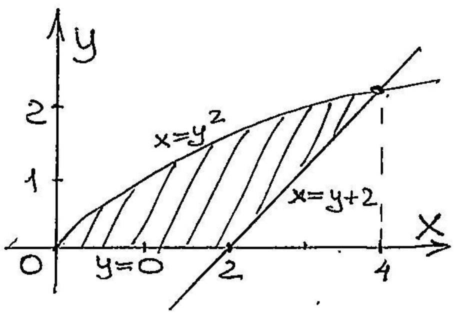
use the formula $A = \int_c^d [f(y) - g(y)]dy.$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

Example 3. Find the area of the region in the first quadrant that is bounded from the left by $x = y^2$ and from the right by $x = y + 2$.

Solution: The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$. The lower limit of integration is $y = 0$. We find the upper limit by solving $x = y + 2$ and $x = y^2$ simultaneously for y: $y + 2 = y^2$.

$y^2 - y - 2 = 0 \Rightarrow (y+1)(y-2) = 0 \Rightarrow y = -1, y = 2.$ The upper limit of integration is $d = 2$.



The area of the region is $A = \int_c^d [f(y) - g(y)]dy = \int_0^2 [y + 2 - y^2]dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^2 = \frac{10}{3}.$

Integration by parts. Integration by parts is a technique for simplifying integrals of the form $\int f(x)g(x)dx$. It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integrals $\int x\cos x dx$ and $\int x^2 e^x dx$ are such integrals because *f*(*x*) = *x* or $f(x) = x^2$ can be differentiated repeatedly to become zero, and *g*(*x*) = cos *x* or $g(x) = e^x$ can be integrated repeatedly without difficulty.

Integration by parts also applies to integrals like $\int \ln x dx$ and $\int e^x \cos x dx$. In the first case, $f(x) = \ln x$ is easy to differentiate and *g*(*x*) = 1 easily integrates to *x*. In the second case, each part of the integrand appears again after repeated differentiation or integration.

Product Rule in Integral Form

If *f* and *g* are differentiable functions of *x*, the Product Rule says that

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x)g(x)]\, dx = \int [f'(x)g(x) + f(x)g'(x)]\, dx \quad \text{or}$$

$$\int \frac{d}{dx}[f(x)g(x)]\, dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

Rearranging the terms of this last equation, we get

$$\int f(x)g'(x)dx = \int \frac{d}{dx}[f(x)g(x)]dx - \int f'(x)g(x)dx,$$

leading to the **integration by parts** formula

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \qquad (1)$$

Sometimes it is easier to remember the formula if we write it in differential form. Let *u* = *f*(*x*) and *v* = *g*(*x*). Then *du* = *f′*(*x*) *dx* and *dv* = *g′*(*x*) *dx*. Using the Substitution Rule, the integration by parts formula becomes **Integration by Parts Formula:** $\int udv = uv - \int vdu$ (2)

This formula expresses one integral, $\int udv$, in terms of a second integral, $\int vdu$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for u and dv. The next examples illustrate the technique. To avoid mistakes, we always list our choices for u and dv, then we add to the list our calculated new terms du and v, and finally we apply the formula in Equation (2).

Example 1. Find $\int x \cos x dx$.

Solution: We use the formula $\int udv = uv - \int vdu$ with $u = x$, $dv = \cos x dx$: then we have $du = dx$, $v = \sin x$. Then $\int x\cos x dx = x\sin x - \int \sin x dx = x\sin x + \cos x + C$.

There are four apparent choices available for u and dv in Example 1:

1. Let *u* = 1 and $dv = x\cos x dx$. 2. Let *u* = *x* and $dv = \cos x dx$.

3. Let *u* = *x* cos *x* and $dv = dx$. 4. Let *u* = cos *x* and $dv = x dx$.

Choice 2 was used in Example 1. The other three choices lead to integrals we don’t know how to integrate. For instance, Choice 3, with *du* = (cos *x* – *x* sin *x*) *dx*, leads to the integral

$$\int (x\cos x - x^2 \sin x)dx.$$

The goal of integration by parts is to go from an integral $\int udv$ that we don’t see how to evaluate to an integral $\int vdu$ that we can evaluate. Generally, you choose dv first to be as much of the integrand, including *dx*, as you can readily integrate; *u* is the leftover part. When finding v from dv, any antiderivative will work and we usually pick the simplest one; no arbitrary constant of integration is needed in *y* because it would simply cancel out of the right-hand side of (2).

Example 2. Find $\int \ln x dx.$

Solution: Since $\int \ln x dx$ can be written as $\int \ln x \cdot 1\, dx,$ we use the formula $\int u dv = uv - \int v du$ with $u = \ln x,\ dv = dx$, and consequently $du = \frac{1}{x} dx,\ v = x.$ Then from Equation (2),

$$\int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + C.$$

Sometimes we have to use integration by parts more than once.

Example 3. Evaluate $\int x^2 e^x dx.$

Solution: With $u = x^2, dv = e^x dx$ we have $du = 2x dx,\ v = e^x$. Then $\int x^2 e^x dx = x^2 e^x - 2\int x e^x dx.$ The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x, dv = e^x dx$. Then $du = dx,\ v = e^x,$ and $\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C.$

Using this last evaluation, we then obtain $\int x^2 e^x dx = x^2 e^x - 2\int x e^x dx = x^2 e^x - x e^x + 2e^x + C,$ where the constant of integration is renamed after substituting for the integral on the right.

The technique of Example 3 works for any integral $\int x^n e^x dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy.

Example 4. Evaluate $\int e^x \cos x dx.$

Solution: Let $u = e^x$ and $dv = \cos x dx.$ Then $du = e^x dx,\ v = \sin x,$ and

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx.$$

The second integral is like the first except that it has sin x in place of cos x. To evaluate it, we use integration by parts with $u = e^x, dv = \sin x dx, du = e^x dx, v = -\cos x.$

Then $\int e^x \cos x dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x dx)\right) = e^x \sin x + e^x \cos x - \int e^x \cos x dx.$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give $2\int e^x \cos x dx = e^x \sin x + e^x \cos x + C_1.$ Dividing by 2 and renaming the constant of integration give $\int e^x \cos x dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$

Example 5. Obtain a formula that expresses the integral $\int \cos^n x dx$ in terms of an integral of a lower power of cos x.

Solution: We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x.$ Then we let $u = \cos^{n-1} x$ and $dv = \cos x dx,$ so that $du = (n-1)\cos^{n-2} x\ (-\sin x dx)$ and $v = \sin x.$ Integration by parts then gives

$\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x dx = \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x dx =$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x dx.$

If we add $(n-1)\int \cos^n x dx$ to both sides of this equation, we obtain

$$n\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x dx.$$

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate.

Evaluating Definite Integrals by Parts

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)dx$$

Example 6. Find $\int_0^4 xe^{-x}dx$.

Solution:

$$\int_0^4 xe^{-x}dx = |u = x, dv = e^{-x}dx \Rightarrow du = dx, v = -e^{-x}|$$

$$= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})dx = 1 - 5e^{-4}.$$

Glossary

integration by parts – интегрирование по частям

Exercises for Seminar 13

1. Use the Substitution Formula in Theorem 1 to evaluate the integrals:

a) $\int_0^3 \sqrt{y+1}dy$; b) $\int_0^{\pi/4} \tan x \sec^2 x dx$.

2. Find the areas of the regions enclosed by the lines and curves:

a) $y = x^2 - 2$ and $y = 2$. b) $y = x^4$ and $y = 8x$.

3. Find the areas of the regions enclosed by the lines and curves:

a) $x = 2y^2, x = 0,$ and $y = 3$. b) $x - y^2 = 0$ and $x + 2y^2 = 3$.

4. Evaluate the integrals using integration by parts.

a) $\int x \sin \frac{x}{2} dx$; b) $\int t^2 \cos t\, dt$; c) $\int x \ln x dx$;

d) $\int xe^x dx$; e) $\int e^\theta \sin\theta\, d\theta$.

5. Evaluate the integrals by using a substitution prior to integration by parts.

a) $\int e^{\sqrt{3s+9}} ds$; b) $\int_0^{\pi/3} x \tan^2 x dx$; c) $\int \sin(\ln x) dx$.

Exercises for Homework 13

1. Use the Substitution Formula in Theorem 1 to evaluate the integrals:

a) $\int_0^1 r\sqrt{1-r^2}dr$; b) $\int_0^\pi 3\cos^2 x \sin x dx$.

2. Find the areas of the regions enclosed by the lines and curves:

a) $y = 2x - x^2$ and $y = -3$. b) $y = x^2 - 2x$ and $y = x$.

3. Find the areas of the regions enclosed by the lines and curves:
a) $y^2 - 4x = 4$ and $4x - y = 16$. b) $x + y^2 = 0$ and $x + 3y^2 = 2$.

4. Evaluate the integrals using integration by parts.
a) $\int \theta \cos \pi\theta \, d\theta$; b) $\int x^2 \sin x \, dx$; c) $\int x^3 \ln x dx$;
d) $\int x e^{3x} dx$; e) $\int e^{-y} \cos y \, dy$.

5. Evaluate the integrals by using a substitution prior to integration by parts.
a) $\int_0^1 x\sqrt{1-x} dx$; b) $\int \ln(x + x^2) dx$; c) $\int z(\ln z)^2 dz$.