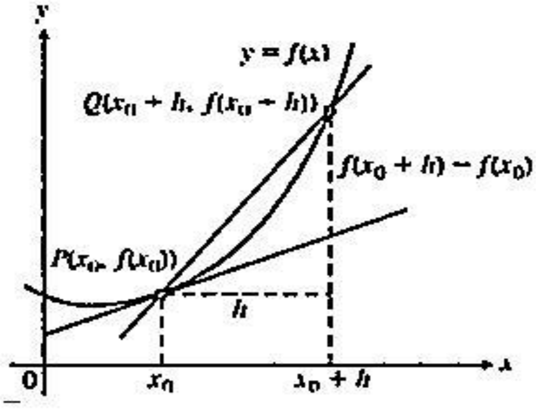
Lecture 5. Derivatives: Tangents. Derivative at a point.

Finding a tangent to the graph of a function: To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we calculate the slope of the secant through P and a nearby point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$. If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.



The slope of the tangent line at P is $\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$.

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number $m = \lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ (provided the limit exists). The **tangent line** to the curve at P is the line through P with this slope.

Example 1. (a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$? (b) Where does the slope equal $-1/4$? (c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution: (a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is $\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0}\frac{1/(a+h)-1/(a)}{h} = \lim\limits_{h\to 0}\frac{1}{h}\frac{a-(a+h)}{a(a+h)} = \lim\limits_{h\to 0}\frac{-1}{a(a+h)} = -\frac{1}{a^2}$.

The number a may be positive or negative, but not 0. When $a = -1$, the slope is -1.

(b) The slope of $y = 1/x$ at the point $x = a$ is $-\frac{1}{a^2}$. It will be -1/4 provided that $-\frac{1}{a^2} = -\frac{1}{4}$.

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$.

(c) The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep. We see this situation again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent levels off becoming more and more horizontal.

Rates of Change: Derivative at a Point. The expression $\frac{f(x_0+h)-f(x_0)}{h}, h \neq 0$ is called the **difference quotient of f at x_0 with increment** h. If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation:

The **derivative of a function f at a point x_0**, denoted $f'(x_0)$ (read "f prime of x_0"), is $f'(x_0) = \lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ provided this limit exists.

The following are all interpretations for the limit of the difference quotient, $\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$.

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

The Derivative as a Function: We now investigate the derivative as a *function* derived from f by considering the limit at each point x in the domain of f.
The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, provided the limit exists.
We use the notation $f(x)$ in the definition to emphasize the independent variable x with respect to which the derivative function $f'(x)$ is being defined. The domain of f' is the set of points in the domain of f for which the limit exists, which means that the domain may be the same as or smaller than the domain of f. If f' exists at a particular x, we say that f is **differentiable (has a derivative) at *x***. If f' exists at every point in the domain of f, we call f **differentiable**.
If we write $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x. Therefore, we have an equivalent definition of the derivative (***alternative formula for the derivative***):

$$f'(x) = \lim_{z\to x}\frac{f(z)-f(x)}{z-x}.$$

This formula is sometimes more convenient to use when finding a derivative function, and focuses on the point z that approaches x.
The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation $\frac{d}{dx}f(x)$ as another way to denote the derivative $f'(x)$.
Example 2. Differentiate $f(x) = \frac{x}{x-1}$.
Solution. $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{x+h}{x+h-1}-\frac{x}{x-1}}{h} = \lim_{h\to 0}\frac{(x+h)(x-1)-x(x+h-1)}{h(x+h-1)(x-1)} =$

$$= \lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)} = -\frac{1}{(x-1)^2}.$$

Example 3. Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.
Solution. We use the alternative formula to calculate f': $f'(x) = \lim_{z\to x}\frac{f(z)-f(x)}{z-x} = \lim_{z\to x}\frac{\sqrt{z}-\sqrt{x}}{z-x} =$ $\lim_{z\to x}\frac{\sqrt{z}-\sqrt{x}}{(\sqrt{z}-\sqrt{x})(\sqrt{z}+\sqrt{x})} = \lim_{z\to x}\frac{1}{\sqrt{z}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$.
There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f(x)) = D_x f(x).$$

The symbols d/dx and D indicate the operation of differentiation. We read dy/dx as "the derivative of y with respect to x," and df/dx and $(d/dx)f(x)$ as "the derivative of f with respect to x."
To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \frac{dy}{dx}|x = a = \frac{df}{dx}|x = a = \frac{d}{dx}f(x)|x = a.$$

Differentiable on an Interval. One-Sided Derivatives. A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[\boldsymbol{a}, \boldsymbol{b}]$ if it is differentiable on the interior (a, b) and if the limits $\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}$ (**right-hand derivative at** a) and $\lim_{h\to 0^-}\frac{f(b+h)-f(b)}{h}$ (**left-hand derivative at** b) exist at the endpoints.
Right-hand and left-hand derivatives may be defined at any point of a function's domain.
Example 4. Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.
Solution. The derivative of $y = mx + b$ is the slope m. Thus, to the right of the origin,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 * x) = 1.$$

To the left, $\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 * x) = -1$.

There is no derivative at the origin because the one-sided derivatives differ there:
Right-hand derivative of $|x|$ at zero $= \lim\limits_{h\to 0^+}\frac{|0+h|-|0|}{h} = \lim\limits_{h\to 0^+}\frac{|h|}{h} = \lim\limits_{h\to 0^+}\frac{h}{h} = 1.$
Left-hand derivative of $|x|$ at zero $= \lim\limits_{h\to 0^-}\frac{|0+h|-|0|}{h} = \lim\limits_{h\to 0^-}\frac{|h|}{h} = \lim\limits_{h\to 0^-}\frac{-h}{h} = -1.$

Theorem 1 (**Differentiability Implies Continuity**) If f has a derivative at $x = c$, then f is continuous at $x = c$.
Proof. Given that $f'(c)$ exists, we must show that $\lim\limits_{x\to c} f(x) = f(c)$, or equivalently, that $\lim\limits_{h\to 0} f(c+h) = f(c)$. If $h \neq 0$, then $f(c + h) = f(c) + (f(c + h) - f(c)) = f(c) + \frac{f(c+h)-f(c)}{h} * h.$
Now take limits as $h \to 0$. $\lim\limits_{h\to 0} f(c+h) = \lim\limits_{h\to 0} f(c) + \lim\limits_{h\to 0}\frac{f(c+h)-f(c)}{h} * \lim\limits_{h\to 0} h =$
$= f(c) + f'(c) * 0 = f(c).$ □
Theorem 1 says that if a function has a discontinuity at a point (for instance, a jump discontinuity), then it cannot be differentiable there.
Caution: The converse of Theorem 1 is false (see Example 4).

Differentiation Rules: This section introduces several rules that allow us to differentiate constant functions, power functions, polynomials, exponential functions, rational functions, and certain combinations of them, simply and directly, without having to take limits each time.

Powers, Multiples, Sums, and Differences
A simple rule of differentiation is that the derivative of every constant function is zero.

Derivative of a Constant Function: If f has the constant value $f(x) = c$, then $\frac{df}{dx} = \frac{d}{dx}(c) = 0.$
Proof. We apply the definition of the derivative to $f(x) = c$, the function whose outputs have the constant value c. At every value of x, we find that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0. \ \square$$

Derivative of a Positive Integer Power: If n is a positive integer, then $\frac{d}{dx}x^n = nx^{n-1}$.
Proof. The formula $z^n - x^n = (z-x)(z^{n-1} + z^{n-2}x + \ldots + zx^{n-2} + x^{n-1})$ can be verified by multiplying out the right-hand side. Then from the alternative formula for the definition of the derivative, $f'(x) = \lim\limits_{z\to x}\frac{f(z)-f(x)}{z-x} = \lim\limits_{z\to x}\frac{z^n-x^n}{z-x} = \lim\limits_{z\to x}(z^{n-1} + z^{n-2}x + \ldots + zx^{n-2} + x^{n-1}) = nx^{n-1}$. □
The last rule is actually valid for all real numbers n.

Derivative Constant Multiple Rule: If u is a differentiable function of x, and c is a constant, then $\frac{d}{dx}(cu) = c\frac{du}{dx}$.
Proof. $\frac{d}{dx}(cu) = \lim\limits_{h\to 0}\frac{cu(x+h)-cu(x)}{h} = c\lim\limits_{h\to 0}\frac{u(x+h)-u(x)}{h} = c\frac{du}{dx}$. □

Derivative Sum Rule: If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Proof. We apply the definition of the derivative to $f(x) = u(x) + v(x)$:

$$\frac{d}{dx}[u(x)+v(x)] = \lim_{h\to 0}\frac{[u(x+h)+v(x+h)]-[u(x)+v(x)]}{h}$$
$$= \lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h} + \frac{v(x+h)-v(x)}{h}\right]$$

$= \lim\limits_{h\to 0}\frac{u(x+h)-u(x)}{h} + \lim\limits_{h\to 0}\frac{v(x+h)-v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}$. □

Derivative of the Natural Exponential Function: $\frac{d}{dx}(e^x) = e^x$.

Derivative Product Rule: If u and v are differentiable functions of x, then so is their product uv, and $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ or in prime notation, $(uv)' = u'v + uv'$.

Proof. $\frac{d}{dx}(uv) = \lim\limits_{h\to 0}\frac{u(x+h)v(x+h)-u(x)v(x)}{h} = \lim\limits_{h\to 0}\frac{u(x+h)v(x+h)-u(x+h)v(x)+u(x+h)v(x)}{h}$

$$\frac{-u(x)v(x)}{h} = \lim_{h\to 0}\left[u(x+h)\frac{v(x+h)-v(x)}{h} + v(x)\frac{u(x+h)-u(x)}{h}\right] =$$

$= \lim\limits_{h\to 0}u(x+h)\cdot\lim\limits_{h\to 0}\frac{v(x+h)-v(x)}{h} + v(x)\cdot\lim\limits_{h\to 0}\frac{u(x+h)-u(x)}{h} = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}$. □

Derivative Quotient Rule: If u and v are differentiable at x, and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

Second- and Higher-Order Derivatives: If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$. The function $f''$is called the **second derivative** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$, is the **third derivative** of $y$with respect to x. The names continue as you imagine, with $y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^n y$ denoting the n**th derivative** of y with respect to x for any positive integer n.

Derivatives of Trigonometric Functions.

$$(\sin x)' = \cos x\,, (\cos x)' = -\sin x,$$

$$(\tan x)' = \sec^2 x = \frac{1}{\cos^2 x}, (\cot x)' = -\csc^2 x = -\frac{1}{\sin^2 x}.$$

Proof (Derivative of the Sine Function). To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we use the sum identity for the sine function:

$$\sin(x+h) = \sin x\cos h + \cos x\sin h.$$

Then $f'(x) = \lim\limits_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim\limits_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} = \lim\limits_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}$

$= \lim\limits_{h\to 0}\sin\ x\frac{\cos h-1}{h} + \lim\limits_{h\to 0}\cos x\frac{\sin h}{h} = \sin x\lim\limits_{h\to 0}\frac{\cos h-1}{h} + \cos x\lim\limits_{h\to 0}\frac{\sin h}{h}$ =sin x *0 + cos x*1= cos x. □

Glossary

tangent – касательный; **slope** – наклон; **secant** – секущий
a nearby point – близлежащая точка; **increment** – приращение
steep – крутой, отвесный (вертикальный)

Exercises for Seminar 5

1. Find the slope of the function’s graph at the given point. Then find an equation for the line tangent to the graph there.
a) $f(x) = x^2 + 1, (2, 5)$; b) $g(x) = \frac{x}{x-2}, (3, 3)$.

2. Using the definition, calculate the derivatives of the functions. Then find the values of the derivatives as specified.

a) $f(x) = 4 - x^2; f'(-3), f'(0), f'(1)$. b) $g(t) = 1/t^2; g'(-1), g'(2), g'(\sqrt{3})$.

3. Using the alternative formula for derivatives $f'(x) = \lim_{z \to x} \frac{f(z)-f(x)}{z-x}$, find the derivative:

a) $f(x) = \frac{1}{x+2}$; b) $g(x) = \frac{x}{x-1}$.

4. Find the first and second derivatives:

a) $y = \frac{4x^3}{3} - x + 2e^x$. b) $y = 6x^2 - 10x - 5x^{-2}$.

5. Find the derivatives of the functions:

a) $v = \frac{1+x-4\sqrt{x}}{x}$; b) $y = \frac{(x+1)(x+2)}{(x-1)(x-2)}$.

6. Find the derivatives of all orders of the functions:

a) $y = \frac{x^4}{2} - \frac{3}{2}x^2 - x$; b) $y = (x-1)(x+2)(x+3)$.

7. Find the derivatives of the following functions:

a) $y = -10x + 3\cos x$; b) $y = x^2 \cos x$;

c) $y = \frac{\cot x}{1+\cot x}$; d) $y = (\sec x + \tan x)(\sec x - \tan x)$.

Exercises for Homework 5

1. Find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

a) $f(x) = x - 2x^2, (1, -1)$; b) $g(x) = \frac{8}{x^2}, (2, 2)$.

2. Using the definition, calculate the derivatives of the functions. Then find the values of the derivatives as specified.

a) $F(x) = (x-1)^2 + 1; F'(-1), F'(0), F'(2)$. b) $k(z) = \frac{1-z}{2z}; k'(-1), k'(1), k'(\sqrt{2})$.

3. Using the alternative formula for derivatives $f'(x) = \lim_{z \to x} \frac{f(z)-f(x)}{z-x}$, find the derivative:

a) $f(x) = x^2 - 3x + 4$; b) $g(x) = 1 + \sqrt{x}$.

4. Find the first and second derivatives:

a) $y = \frac{x^3}{3} + \frac{x^2}{2} + e^{-x}$; b) $r = \frac{1}{3s^2} - \frac{5}{2s}$.

5. Find the derivatives of the functions:

a) $r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$; b) $y = \frac{x^2+3e^x}{2e^x-x}$.

6. Find the derivatives of all orders of the functions:

a) $y = \frac{x^5}{120}$; b) $y = (4x^2+3)(2-x)x$.

7. Find the derivatives of the following functions:

a) $y = \frac{3}{x} + 5\sin x$; b) $y = \sqrt{x}\sec x + 3$;

c) $y = \frac{\cos x}{1+\sin x}$; d) $y = x^3 \sin x \cos x$.