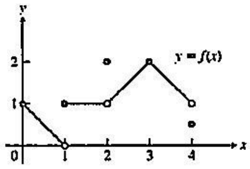
Lecture 3. Continuity. Points of discontinuity. Continuous functions.

Continuity: Intuitively, any function $y = f(x)$ whose graph can be sketched over its domain in one unbroken motion is an example of a continuous function. Such functions play an important role in the study of calculus and its applications.



Example. At which numbers does the function f in Figure appear to be not continuous? Explain why. What occurs at other numbers in the domain?

Solution: First, we observe that the domain of the function is the closed interval [0, 4], so we will be considering the numbers x within that interval. From the figure, we notice right away that there are breaks in the graph at the numbers $x = 1, x = 2$, and $x = 4$. The breaks appear as jumps, which we identify later as “jump discontinuities”. These are numbers for which the function is not continuous, and we discuss each in turn.

Numbers at which the graph of f has breaks:

At $x = 1$, the function fails to have a limit. It does have both a left-hand limit, $\lim_{x\to 1^-} f(x) = 0$, as well as a right-hand limit, $\lim_{x\to 1^+} f(x) = 1$, but the limit values are different, resulting in a jump in the graph. The function is not continuous at $x = 1$.

At $x = 2$, the function does have a limit, $\lim_{x\to 2} f(x) = 1$, but the value of the function is $f(2) = 2$. The limit and function values are not the same, so there is a break in the graph and f is not continuous at $x = 2$.

At $x = 4$, the function does have a left-hand limit at this right endpoint, $\lim_{x\to 4^-} f(x) = 1$, but again the value of the function $f(4) = 1/2$ differs from the value of the limit. We see again a break in the graph of the function at this endpoint and the function is not continuous from the left.

Numbers at which the graph of f has no breaks:

At $x = 0$, the function has a right-hand limit at this left endpoint, $\lim_{x\to 0^+} f(x) = 1$, and the value of the function is the same, $f(0) = 1$. So, no break occurs in the graph of the function at this endpoint, and the function is continuous from the right at $x = 0$.

At all other numbers $x = c$ in the domain, which we have not considered, the function has a limit equal to the value of the function at the point, so $\lim_{x\to c} f(x) = f(c)$. For example, $\lim_{x\to 5/2} f(x) = f(5/2) = 3/2$. No breaks appear in the graph of the function at any of these remaining numbers and the function is continuous at each of them.

Continuity at a Point. Let c be a real number on the x-axis.

The function f is **continuous at c** if $\lim_{x\to c} f(x) = f(c)$.

The function f is **right-continuous at c (or continuous from the right)** if $\lim_{x\to c^+} f(x) = f(c)$.

The function f is **left-continuous at c (or continuous from the left)** if $\lim_{x\to c^-} f(x) = f(c)$.

Obviously, a function f is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c. We say that a function is **continuous over a closed interval $[a, b]$** if it is right-continuous at a, left-continuous at b, and continuous at all interior points of the interval.

If a function is not continuous at an interior point c of its domain, we say that f is **discontinuous at c**, and that c is a point of discontinuity of f. Note that a function f can be continuous, right-continuous, or left-continuous only at a point c for which $f(c)$ is defined.

Example 1. The function $f(x) = \sqrt{4 - x^2}$ is continuous over its domain $[-2, 2]$. It is right-continuous at $x = -2$, and it is left-continuous at $x = 2$.

We summarize continuity at an interior point in the form of a test.

Continuity Test for an interior point: A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.
1. $f(c)$ exists (c lies in the domain of f).
2. $\lim\limits_{x\to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim\limits_{x\to c} f(x) = f(c)$ (the limit equals the function value).
For one-sided continuity and continuity at an endpoint of an interval, the limit in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.
Points of discontinuity.
Example 2. The function $f(x) = \frac{x^2-1}{x-1}$ has a discontinuity at $x = 1$, but this discontinuity is ***removable***: the function has a limit as $x \to 1$, and we can remove the discontinuity by setting $f(1)$ equal to this limit (the one-sided limits exist and have the same value).
If a point of discontinuity $a \in E$ of a function $f: E \to R$ is such that there exist two one-sided limits $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$, and $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} f(x)$, then a is called a ***removable discontinuity*** of the function f.
Example 3. The unit step function $U(x) = \begin{cases} 0, x < 0 \\ 1, x \geq 0 \end{cases}$ is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a ***jump discontinuity*** at $x = 0$: the one-sided limits exist but have different values.
A point $a \in E$ is called a ***discontinuity of first kind*** for a function $f: E \to R$ if two one-sided limits exist $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$, but at least one of them is not equal to the value $f(a)$ that the function assumes at a.
Obviously, both a jump discontinuity and a removal discontinuity are discontinuities of first kind.
The function $f(x) = 1/x^2$ has an ***infinite discontinuity*** at $x = 0$. The function $f(x) = sin(1/x)$ has an ***oscillating discontinuity*** at $x = 0$: it oscillates too much to have a limit as $x \to 0$.
If $a \in E$ is a point of discontinuity of a function $f: E \to R$ and at least one of one-sided limits does not exist, then a is called a ***discontinuity of second kind***.
Obviously, both an infinite discontinuity and an oscillating discontinuity are discontinuities of second kind.
Continuous functions. Generally, we want to describe the continuity behavior of a function throughout its entire domain, not only at a single point. We know how to do that if the domain is a closed interval. In the same way, we define a **continuous function** as one that is continuous at every point in its domain. This is a property of the *function*. A function always has a specified domain, so if we change the domain, we change the function, and this may change its continuity property as well. If a function is discontinuous at one or more points of its domain, we say it is a **discontinuous** function.
The function $y = 1/x$ is a continuous function because it is continuous at every point of its domain. It has a point of discontinuity at $x = 0$, however, because it is not defined there; that is, it is discontinuous on any interval containing $x = 0$.
The identity function $f(x) = x$ and constant functions are continuous everywhere.
Algebraic combinations of continuous functions are continuous wherever they are defined.
Theorem 1 (Properties of continuous functions) If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$:
1. Sums: $f + g$ 2. Differences: $f - g$ 3. Constant multiplies: $k \cdot f$, for any number k
4. Products: $f \cdot g$ 5. Quotients: f/g, provided $g(c) \neq 0$ 6. Powers: f^n, n a positive integer
7. Roots: $\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer.
For instance, to prove the sum property we have
$$\lim_{x\to c}(f+g)(x) = \lim_{x\to c}\big(f(x)+g(x)\big) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c) + g(c) = (f+g)(c).$$
This shows that $f + g$ is continuous.

Example 4. An algebraic polynomial $P(x) = a_0x^n + a_1x^{n-1}+\ldots+a_n$ is a continuous function on R.
Indeed, it follows by induction from 1 and 3 of Theorem 1 that the sum and product of any finite number of functions that are continuous at a point are themselves continuous at that point. Since both a constant function and the function *f(x)* = *x* are continuous on R, then follows that the function ax^m is continuous for any a and m, and consequently the polynomial *P(x)* is also.
Example 5. A rational function $R(x) = P(x)/Q(x)$ – a quotient of polynomials – is continuous wherever it is defined, that is, where $Q(x) \neq 0$.
Inverse functions and Continuity. The inverse function of any function continuous on an interval is continuous over its domain. Thus, the result is suggested by the observation that the graph of f^{-1}, being the reflection of the graph of f across the line $y = x$, cannot have any breaks in it when the graph of f has no breaks.
Composites.
Theorem 2 (**Composite of continuous functions**) If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.
Example 6. Show that the function $y = \sqrt{x^2 - 2x - 5}$ is continuous on its natural domain.
Solution: The square root function is continuous on $[0, \infty)$ because it is a root of the continuous identity function $h(x) = x$. The given function is then the composite of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$, and is continuous on its natural domain.
Theorem 2 is actually a consequence of a more general result, which we now state and prove.
Theorem 3 (**Limits of continuous functions**) If g is continuous at the point b and $\lim\limits_{x\to c} f(x) = b$, then $\lim\limits_{x\to c} g(f(x)) = g(b) = g(\lim\limits_{x\to c} f(x))$.
Proof. Let $\varepsilon > 0$ be given. Since g is continuous at b, there exists a number $\delta_1 > 0$ such that $|g(y) - g(b)| < \varepsilon$ whenever $0 < |y - b| < \delta_1$. Since $\lim\limits_{x\to c} f(x) = b$, there exists a $\delta > 0$ such that $|f(x) - b| < \delta_1$ whenever $0 < |x - c| < \delta$. If we let $y = f(x)$, we then have that $|y - b| < \delta_1$ whenever $0 < |x - c| < \delta$, which implies from the first statement that $|g(y) - g(b)| = |g(f(x)) - g(b)| < \varepsilon$ whenever $0 < |x - c| < \delta$. From the definition of limit, this proves that $\lim\limits_{x\to c} g(f(x)) = g(b)$. □
Example 7. As an application of Theorem 3, we have the following calculations:
$$\lim_{x\to\pi/2} cos\left(2x + sin\left(\frac{3\pi}{2} + x\right)\right) = cos\left(\lim_{x\to\pi/2} 2x + \lim_{x\to\pi/2} sin\left(\frac{3\pi}{2} + x\right)\right) = cos(\pi + sin\,2\,\pi)$$
$$= cos\,\pi = -1.$$
Intermediate Value Theorem for Continuous Functions
Functions that are continuous on intervals have properties that make them particularly useful in mathematics and its applications. One of these is the *Intermediate Value Property*. A function is said to have the **Intermediate Value property** if whenever it takes on two values, it also takes on the values in between.
Theorem 4 (**The Intermediate Value Theorem for continuous functions**) If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.
Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.
Continuous Extension to a Point. Sometimes the formula that describes a function f does not make sense at a point $x = c$. It might nevertheless be possible to extend the domain of f, to include $x = c$, creating a new function that is continuous at $x = c$. For example, the function $y = f(x) = (sin\,x)/x$ is continuous at every point except $x = 0$, since the origin is not in its domain. Since $y = (sin\,x)/x$ has a finite limit as $x \to 0$, we can extend the function's domain to include the point

$x = 0$ in such a way that the extended function is continuous at $x = 0$. We define the new function $F(x) = \begin{cases} (\sin x)/x, x \neq 0 \\ 1, \quad x = 0. \end{cases}$

The function $F(x)$ is continuous at $x = 0$, because $\lim_{x\to 0} \frac{\sin x}{x} = F(0)$, so it meets the requirements for continuity.

More generally, a function (such as a rational function) may have a limit at a point where it is not defined. If $f(c)$ is not defined, but $\lim_{x\to c} f(x) = L$ exists, we can define a new function $F(x)$ by the rule $F(x) = \begin{cases} f(x), if x \text{ is in the domain of } f \\ \quad L, if\ x = c. \end{cases}$

The function $F$is continuous at $x = c$. It is called the **continuous extension of** f to $x = c$. For rational functions f, continuous extensions are often found by cancelling common factors in the numerator and denominator.

Example 8. Show that $f(x) = \frac{x^2+x-6}{x^2-4}, x \neq 2$ has a continuous extension to $x = 2$, and find that extension.

Solution: Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}.$$

The new function $F(x) = \frac{x+3}{x+2}$ is equal to $f(x)$ for $x \neq 2$, but it is continuous at $x = 2$, having there the value of 5/4. Thus, F is the continuous extension of f to $x = 2$.

Uniform continuity. A function $f: E \to R$ is *uniformly continuous* on a set $E \subseteq R$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ for all points $x_1, x_2 \in E$ such that $|x_1 - x_2| < \delta$.

More briefly, ($f: E \to R$ is uniformly continuous):

$$(\forall\varepsilon > 0\ \exists\delta > 0\ \forall x_1, x_2 \in E\ (|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon)).$$

Let us now discuss the concept of uniform continuity.

1° If a function is uniformly continuous on a set, it is continuous at each point of that set. Indeed, in the definition just given it suffices to set $x_1 = x$ and $x_2 = a$, and we see that the definition of continuity of a function $f: E \to R$ at a point $a \in E$ is satisfied.

2° Generally speaking, the continuity of a function does not imply its uniform continuity.

Theorem 5. A function that is continuous on a closed interval is uniformly continuous on that interval.

Example 9. The function $f(x) = \sin(1/x)$, which we have encountered many times, is continuous on the open interval]0, 1[= E. However, in every neighborhood of 0 in the set E the function assumes both values –1 and 1. Hence, for $\varepsilon < 2$, the condition $|f(x_1) - f(x_2)| < \varepsilon$ does not hold.

In this connection it is useful to write out explicitly the negation of the property of uniform continuity for a function: ($f: E \to R$ is not uniformly continuous):

$$(\exists\varepsilon > 0\ \forall\delta > 0\ \exists x_1, x_2 \in E(|x_1 - x_2| < \delta \wedge |f(x_1) - f(x_2)| \geq \varepsilon)).$$

Example 10. If the function $f: E \to R$ is unbounded in every neighborhood of a fixed point $x_0 \in E$, then it is not uniformly continuous.

Indeed, in that case for any $\delta > 0$ there are points x_1 and x_2 in every δ-neighborhood of x_0 such that $|f(x_1) - f(x_2)| > 1$ although $|x_1 - x_2| < \delta$.

Such is the situation with the function $f(x) = \sin(1/x)$ on the set $R\backslash 0$. In this case $x_0 = 0$. The same situation holds in regard to $\log_a x$, which is defined on the set of positive numbers and unbounded in a neighborhood of $x_0 = 0$.

Example 11. The function $f(x) = x^2$, which is continuous on R, is not uniformly continuous on R.

In fact, at the points $x_n' = \sqrt{n+1}$ and $x_n'' = \sqrt{n}$, where $n \in N$, we have $f(x_n') = n+1$ and $f(x_n'') = n$, so that $f(x_n') - f(x_n'') = 1$. But $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$, so that for any $\delta > 0$ there are points x_n' and x_n'' such that $|x_n' - x_n''| < \delta$, yet $f(x_n') - f(x_n'') = 1$.
Example 12. The function $f(x) = sin(x^2)$, which is continuous and bounded on R, is not uniformly continuous on R. Indeed, at the points $x_n' = \sqrt{\frac{\pi}{2}(n+1)}$ and $x_n'' = \sqrt{\frac{\pi}{2}n}$, where $n \in N$, we have $|f(x_n') - f(x_n'')| = 1$, while $\lim_{n\to\infty}|x_n' - x_n''| = 0$.

Glossary

uniform continuity – равномерная непрерывность

Exercises for Seminar 3

1. At what points are the functions continuous?
a) $y = \frac{1}{x-2} - 3x$; b) $y = \frac{x+1}{x^2-4x+3}$; c) $y = |x-1| + sin\,x$; d) $y = \frac{cos\,x}{x}$;
e) $y = tan\frac{\pi x}{2}$; f) $y = \sqrt{2x+3}$.
2. Find the limits. Are the functions continuous at the point being approached?
a) $\lim_{x\to\pi} sin(x - sin\,x)$; c) $\lim_{x\to 0} tan\left(\frac{\pi}{4}cos(sin\,x^{1/3})\right)$.
3. For what value of a is the following function continuous at every x?
a) $f(x) = \begin{cases} x^2-1, x<3 \\ 2ax, x \geq 3 \end{cases}$; b) $f(x) = \begin{cases} ax^2 - 2a, x \geq 2 \\ 12, x < 2 \end{cases}$.
4. Find points of discontinuity of functions and determine their kind:
a) $f(x) = \frac{x}{(1+x)^2}$; b) $f(x) = \frac{x}{sin\,x}$; c) $f(x) = cos^2\frac{1}{x}$
5. Show that the function $f(x) = 1/x$ is continuous on the interval (0, 1), but it is not uniformly continuous on this interval.
6. Investigate the following functions on uniform continuity:
a) $f(x) = \frac{x}{4-x^2}, -1 \leq x \leq 1$. c) $f(x) = \frac{sin\,x}{x}, 0 < x < \pi$.
7. For an arbitrary number $\varepsilon > 0$ find $\delta = \delta(\varepsilon)$ satisfying the conditions of uniform continuity for the function $f(x)$ on a given interval if:
a) $f(x) = 5x - 3, -\infty < x < +\infty$. b) $f(x) = \frac{1}{x}, 0{,}1 \leq x \leq 1$.
8. Define $g(3)$ in a way that extends $g(x) = (x^2-9)/(x-3)$ to be continuous at $x = 3$.

Exercises for Homework 3

1. At what points are the functions continuous?
a) $y = \frac{1}{(x+2)^2} + 4$; b) $y = \frac{x+3}{x^2-3x-10}$; c) $y = \frac{1}{|x|+1} - \frac{x^2}{2}$; d) $y = \frac{x+2}{cos\,x}$;
e) $y = \frac{x\,tan\,x}{x^2+1}$; f) $y = \sqrt[4]{3x-1}$.
2. Find the limits. Are the functions continuous at the point being approached?
a) $\lim_{t\to 0} sin\left(\frac{\pi}{2}cos(tan\,t)\right)$; b) $\lim_{y\to 1} sec(y\,sec^2 y - tan^2 y - 1)$.
3. For what value of a is the following function continuous at every x?
a) $f(x) = \begin{cases} x, x < -2 \\ ax^2, x \geq -2 \end{cases}$; b) $f(x) = \begin{cases} \frac{x-a}{a+1}, x < 0 \\ x^2 + a, x > 0 \end{cases}$.
4. Find points of discontinuity of functions and determine their kind:
a) $f(x) = \frac{1+x}{1+x^3}$; b) $f(x) = \sqrt{\frac{1-cos\,\pi x}{4-x^2}}$; c) $f(x) = \frac{x^2-1}{x^3-3x+2}$.
5. Show that the function $f(x) = sin(\pi/x)$ is continuous and bounded on the interval (0, 1), but it is not uniformly continuous on this interval.

6. Investigate the following functions on uniform continuity:

a) $f(x) = ln\, x\,, 0 < x < 1$. b) $f(x) = e^x \cos\frac{1}{x}, 0 < x < 1$.

7. For an arbitrary number $\varepsilon > 0$ find $\delta = \delta(\varepsilon)$ satisfying the conditions of uniform continuity for the function $f(x)$ on a given interval if:

a) $f(x) = x^2 - 2x - 1, -2 \leq x \leq 5$. b) $f(x) = \sqrt{x},\ \ 1 \leq x < +\infty$.

8. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.