Lecture 8. Application of Derivatives: Extreme values of functions, Monotonicity.

Extreme Values of Functions. Let f be a function with domain D. Then f has an **absolute maximum (absolute minimum)** value on D at a point c if $f(x) \le f(c)$ ($f(x) \ge f(c)$) for all x in D. Maximum and minimum values are called **extreme values** of the function f. Absolute maxima or minima are also referred as **global** maxima or minima.

Theorem 1 (Extreme Value Theorem) If f is continuous on a closed interval $[a,b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a,b]$. That is, there are numbers x_1 and x_2 in $[a,b]$ with $f(x_1) = m, f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a,b]$.

Let's consider the function $y = x^2$. Its domain is $(-\infty, \infty)$. This example shows that an absolute maximum value may not exist if the interval fails to be both closed and finite.

A function f has a **local maximum (local minimum)** value at a point c within its domain D if $f(x) \le f(c)$ ($f(x) \ge f(c)$) for all $x \in D$ lying in some open interval containing c.

A list of all local maxima will automatically include the absolute maximum if there is one. Similarly, a list of all local minima will include the absolute minimum if there is one.

Theorem 2 (**The First Derivative Theorem for Local Extreme Values**) If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

Proof. To prove that $f'(c)$ is zero at a local extremum, we show first that $f'(c)$ cannot be positive and second that $f'(c)$ cannot be negative. The only number that is neither positive nor negative is zero, so that is what $f'(c)$ must be.

To begin, suppose that f has a local maximum value at $x = c$ so that $f(x) - f(c) \le 0$ for all values of x near enough to c. Since c is an interior point of f's domain, $f'(c)$ is defined by the two-sided limit $\lim\limits_{x\to c} \dfrac{f(x) - f(c)}{x - c}$. This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f'(c)$. When we examine these limits separately, we find that

$$f'(c) = \lim_{x\to c^+} \frac{f(x) - f(c)}{x - c} \le 0. \qquad (1)$$

Similarly,
$$f'(c) = \lim_{x\to c^-} \frac{f(x) - f(c)}{x - c} \ge 0. \qquad (2)$$

Together, Equations (1) and (2) imply $f'(c) = 0$. This proves the theorem for local maximum values. To prove it for local minimum values, we simply use $f(x) \ge f(c)$, which reverses the inequalities in Equations (1) and (2). □

Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined.

If we recall that all the domains we consider are intervals or unions of separate intervals, the only places where a function f can possibly have an extreme value (local or global) are

1. interior points where $f' = 0$.

2. interior points where f' is undefined.

3. endpoints of the domain of f.

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

Thus the only domain points where a function can assume extreme values are critical points and endpoints.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

Example 1. Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2,1]$.

Solution: The function is differentiable over the entire domain, so the only critical point is where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$: $f(0) = 0, f(-2) = 4, f(1) = 1$.

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$.

Theorem 3 (**Rolle's Theorem**) Suppose that $y = f(x)$ is continuous over the closed interval $[a,b]$ and differentiable at every point of its interior (a,b). If $f(a) = f(b)$, then there is at least one number c in (a,b) at which $f'(c) = 0$.

Proof. Being continuous, f assumes absolute maximum and minimum values on $[a,b]$ by Theorem 1. These can occur only

1. at interior points where f' is zero,

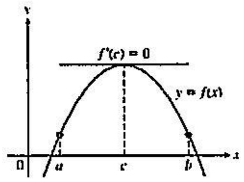
2. at interior points where f' does not exist,

3. at endpoints of the function's domain, in this case a and b.

By the hypothesis, f has a derivative at every interior point. That rules out possibility (2), leaving us with interior points where $f' = 0$ and with the two endpoints a and b.

If either the maximum or the minimum occurs at a point c between a and b, then $f'(c) = 0$ by Theorem 2, and we have found a point for Rolle's Theorem.

If both the absolute maximum and the absolute minimum occur at the endpoints, then because $f(a) = f(b)$ it must be the case that f is a constant with $f(x) = f(a) = f(b)$ for every $x \in [a,b]$. Therefore $f'(x) = 0$ and the point c can be taken anywhere in the interior (a,b). □



Theorem 4 (**The Mean Value Theorem**) Suppose $y = f(x)$ is continuous over a closed interval $[a,b]$ and differentiable on the interval's interior (a,b). Then there is at least one point c in (a,b) at which $\dfrac{f(b) - f(a)}{b - a} = f'(c)$. (1)

Proof. Consider the following functions: $g(x) = f(a) + \dfrac{f(b) - f(a)}{b - a}(x - a)$ (2)

and $$h(x) = f(x) - g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a). \quad (3)$$

The function h satisfies the hypotheses of Rolle's Theorem on $[a,b]$. It is continuous on $[a,b]$ and differentiable on (a,b) because both f and g are. Also, $h(a) = h(b) = 0$. Therefore $h'(c) = 0$ at some point $c \in (a,b)$. This is the point we want for Equation (1) in the theorem.

To verify Equation (1), we differentiate both sides of Equation (3) with respect to x and then set $x = c$: $h'(x) = f'(x) - \dfrac{f(b) - f(a)}{b - a} \Rightarrow 0 = h'(c) = f'(c) - \dfrac{f(b) - f(a)}{b - a} \Rightarrow f'(c) = \dfrac{f(b) - f(a)}{b - a}$,

which is what we set out to prove. □

Mathematical Consequences.

Corollary 1 If $f'(x)=0$ at each point x of an open interval (a,b), then $f(x)=C$ for all $x\in(a,b)$, where C is a constant.

Proof. We want to show that f has a constant value on the interval (a,b). We do so by showing that if x_1 and x_2 are any two points in (a,b) with $x_1<x_2$, then $f(x_1)=f(x_2)$. Now f satisfies the hypotheses of the Mean Value Theorem on $[x_1,x_2]$: It is differentiable at every point of $[x_1,x_2]$ and hence continuous at every point as well. Therefore, $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}=f'(c)$ at some point c between x_1 and x_2. Since $f'=0$ throughout (a,b), this equation implies successively that $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}=0$, $f(x_2)-f(x_1)=0$, and $f(x_1)=f(x_2)$. □

Corollary 2 If $f'(x)=g'(x)$ at each point x in an open interval (a,b), then there exists a constant C such that $f(x)=g(x)+C$ for all $x\in(a,b)$. That is, $f-g$ is a constant function on (a,b).

Proof. At each point $x\in(a,b)$ the derivative of the difference function $h=f-g$ is $h'(x)=f'(x)-g'(x)=0$. Thus, $h(x)=C$ on (a,b) by Corollary 1. That is, $f(x)-g(x)=C$ on (a,b), so $f(x)=g(x)+C$. □

Monotonic Functions and the First Derivative Test: In sketching the graph of a differentiable function, it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval. This paragraph gives a test to determine where it increases and where it decreases. We also show how to test the critical points of a function to identify whether local extreme values are present.

A function that is increasing or decreasing on an interval is said to be **monotonic** on the interval.

Corollary 3. Suppose that *f* is continuous on $[a,b]$ and differentiable on (a, b).

If $f'(x)>0$ at each point $x\in(a,b)$, then *f* is increasing on $[a,b]$.

If $f'(x)<0$ at each point $x\in(a,b)$, then *f* is decreasing on $[a,b]$.

Proof. Let x_1 and x_2 be any two points in $[a,b]$ with $x_1<x_2$. The Mean Value Theorem (Lagrange's theorem) applied to *f* on $[x_1,x_2]$ says that $f(x_2)-f(x_1)=f'(c)(x_2-x_1)$ for some *c* between x_1 and x_2. The sign of the right-hand side of this equation is the same as the sign of $f'(c)$ because x_2-x_1 is positive. Therefore, $f(x_2)>f(x_1)$ if f' is positive on (a, b) and $f(x_2)<f(x_1)$ if f' is negative on (a,b). □

Corollary 3 tells us that $f(x)=\sqrt{x}$ is increasing on the interval $[0,b]$ for any $b>0$ because $f'(x)=1/(2\sqrt{x})$ is positive on $(0, b)$. The derivative does not exist at $x = 0$, but Corollary 3 still applies. The corollary is valid for infinite as well as finite intervals, so $f(x)=\sqrt{x}$ is increasing on $[0,\infty)$.

To find the intervals where a function *f* is increasing or decreasing, we first find all of the critical points of *f*.

Example 2. Find the critical points of $f(x)=x^3-12x-5$ and identify the open intervals on which *f* is increasing and on which *f* is decreasing.

Solution: The function f is everywhere continuous and differentiable. The first derivative $f'(x)=3x^2-12=3(x+2)(x-2)$ is zero at $x=-2$ and $x = 2$. These critical points subdivide the domain of *f* to create non-overlapping open intervals $(-\infty,-2),(-2,2)$, and $(2,\infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f' at a convenient point in each subinterval. The behavior of *f* is determined by then applying Lemma 1 to each subinterval. Thus, f is increasing on $(-\infty,-2)$ and $(2,\infty)$, and f is decreasing on $(-2, 2)$.

First Derivative Test for Local Extrema: Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

The test for local extrema at endpoints is similar, but there is only one side to consider in determining whether f is increasing or decreasing, based on the sign of f'.

Proof of the First Derivative Test. Part (1). Since the sign of f' changes from negative to positive at c, there are numbers a and b such that $a < c < b$, $f' < 0$ on (a, c), and $f' > 0$ on (c, b). If $x \in (a,c)$, then $f(c) < f(x)$ because $f' < 0$ implies that f is decreasing on $[a,c]$. If $x \in (c,b)$, then $f(c) < f(x)$ because $f' > 0$ implies that f is increasing on $[c,b]$. Therefore, $f(x) \geq f(c)$ for every $x \in (a,b)$. By definition, f has a local minimum at c. Parts (2) and (3) are proved similarly. □

Example 3. Find the critical points of $f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$. Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution. The first derivative $f'(x) = \dfrac{4(x-1)}{3x^{2/3}}$ is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain, so the critical points $x = 0$ and $x = 1$ are the only places where f might have an extreme value. Corollary 3 implies that f decreases on $(-\infty, 0)$, decreases on (0, 1), and increases on $(1, \infty)$. The First Derivative Test for Local Extrema tells us that f does not have an extreme value at $x = 0$ (f' does not change sign) and that f has a local minimum at $x = 1$ (f' changes from negative to positive). The value of the local minimum is $f(1) = -3$. This is also an absolute minimum since f is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$.

Glossary

to attain – достигать

Exercises for Seminar 8

1. Find the absolute maximum and minimum values of each function on the given interval.

a) $f(x) = (2/3)x - 5,\ -2 \leq x \leq 3$; b) $f(x) = x^2 - 1,\ -1 \leq x \leq 2$.

2. Determine all critical points for each function.

a) $y = x^2 - 6x + 7$; b) $y = x^2 - 32\sqrt{x}$.

3. Find the critical points, domain endpoints, and extreme values (absolute and local) for each function:

a) $y = x^{2/3}(x+2)$; b) $y = \begin{cases} 4 - 2x, & x \leq 1 \\ x + 1, & x > 1 \end{cases}$.

4. Find the value or values of c that satisfy the equation $\dfrac{f(b) - f(a)}{b - a} = f'(c)$ in the conclusion of the Mean Value Theorem for the following functions and intervals:

a) $f(x) = x^2 + 2x - 1,\ [0,1]$; b) $f(x) = x + 1/x,\ [1/2, 2]$.

5. Which of the following functions satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

a) $f(x) = x^{\frac{2}{3}}, [-1,\ 8]$; b) $f(x) = \begin{cases} (\sin x)/x, & -\pi \leq x < 0 \\ 0, & x = 0 \end{cases}$.

6. (Analyzing Functions from Derivatives). Answer the following questions about the functions whose derivatives are given: (1) What are the critical points of f ? (2) On what open intervals is f increasing or decreasing? (3) At what points, if any, does f assume local maximum and minimum values.

a) $f'(x) = x(x-1)$; b) $f'(x) = (x-1)^2(x+2)$; c) $f'(x) = 1-\dfrac{4}{x^2}$, $x \neq 0$;

d) $f'(x) = (\sin x - 1)(2\cos x + 1),\ 0 \le x \le 2\pi$.

7. (Identifying Extrema). Find the open intervals on which the function is increasing and decreasing. Identify the function's local and absolute extreme values, if any, saying where they occur.

a) $g(t) = -t^2 - 3t + 3$; b) $g(x) = x\sqrt{8-x^2}$; c) $f(x) = x^{1/3}(x+8)$; d) $f(x) = x\ln x$.

Exercises for Homework 8

1. Find the absolute maximum and minimum values of each function on the given interval.

a) $f(x) = -x-4,\ -4 \le x \le 1$; b) $f(x) = 4 - x^2,\ -2 \le x \le 1$.

2. Determine all critical points for each function.

a) $f(x) = 6x^2 - x^3$; b) $y = \sqrt{2x - x^2}$.

3. Find the critical points, domain endpoints, and extreme values (absolute and local) for each function:

a) $y = x^{2/3}(x^2-4)$; b) $y = \begin{cases} 3-x, & x < 0 \\ 3+2x-x^2, & x \ge 0 \end{cases}$.

4. Find the value or values of c that satisfy the equation $\dfrac{f(b)-f(a)}{b-a} = f'(c)$ in the conclusion of the Mean Value Theorem for the following functions and intervals:

a) $f(x) = x^{2/3},\ [0,1]$; b) $f(x) = \sin^{-1} x,\ [-1,1]$.

5. Which of the following functions satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

a) $f(x) = x^{\frac{4}{5}}, [0, 1]$; b) $f(x) = \begin{cases} x^2 - x, & -2 \le x \le -1 \\ 2x^2 - 3x - 3, & -1 < x \le 0 \end{cases}$.

6. (Analyzing Functions from Derivatives). Answer the following questions about the functions whose derivatives are given: (1) What are the critical points of f ? (2) On what open intervals is f increasing or decreasing? (3) At what points, if any, does f assume local maximum and minimum values.

a) $f'(x) = (x-1)(x+2)$; b) $f'(x) = (x-1)^2(x+2)^2$; c) $f'(x) = 3 - \dfrac{6}{\sqrt{x}}$, $x \neq 0$.

7. (Identifying Extrema). Find the open intervals on which the function is increasing and decreasing. Identify the function's local and absolute extreme values, if any, saying where they occur.

a) $g(t) = -3t^2 + 9t + 5$; b) $g(x) = x^2\sqrt{5-x}$;

c) $f(x) = x^{2/3}(x+5)$; d) $f(x) = x^2 \ln x$.