Divisibilidade

Divisão Euclidiana

Há algo errado: o resultado coincide com o do sage, com o resto negativo se $b < 0$. Verificar!

In [1]:

```
def div_euc(a,b):
  k = 0
  if a == 0:
    return (0,0)
  if a > 0 and b > 0:
    while a-k*b >= 0:
      k += 1
    return (k-1, a-(k-1)*b)
  elif a < 0 and b > 0:
    while a-k*b < 0:
      k -= 1
    return (k, a-k*b)
  elif a < 0 and b < 0:
    while a-k*b <= 0:
      k += 1
    return (k-1, a-(k-1)*b)
  else:
  #a > 0 e b < 0
    while a-k*b > 0:
      k -= 1
    return (k, a-k*b)
div_euc(10,-3)
```

Out[1]:

```
(-4, -2)
```

MDC e Bézout-Bachet

In [2]:

```
a = 342
b = 101
(q,r) = (b,a.quo_rem(b)[1])
while r != 0:
  (q,r) = (r,q.quo_rem(r)[1])

print("mdc({},{})=".format(a,b), q)
```

```
mdc(342,101)= 1
```

Algoritmo de Euclides estendido

Não está funcionando quando $b \mid a$. Tem que testar se o primeiro `r0` é igual a 0

In [3]:

```
a = 101
b = 342
(q0,r0) = a.quo_rem(b)
(q1,r1) = b.quo_rem(r0)
(x0,y0,x1,y1) = (1,-q0,-q1,1+q0*q1)

while r1 > 0:
    raux = r0
    (q0,r0,q1,r1) = (q1,r1,r0.quo_rem(r1)[0],r0.quo_rem(r1)[1])
#     (q1,r1) = raux.quo_rem(r1)
    print(raux,'=',q1,'*',r0,'+',r1)
    (x0,y0,x1,y1) = (x1,y1,x0-q1*x1,y0-q1*y1)

print(x0,y0,r0)
print(x0*a+y0*b)
```

```
101 = 2 * 39 + 23
39 = 1 * 23 + 16
23 = 1 * 16 + 7
16 = 2 * 7 + 2
7 = 3 * 2 + 1
2 = 2 * 1 + 0
149 -44 1
1
```

XGCD com matrizes

In [ ]:

```
a = 1001
b = 109
M = Matrix([[1,0],[0,1],[a,b]])
while M[2,1] != 0:
    q = floor(M[2,0]/M[2,1])
    M = M*matrix([[0,1],[1,-q]])
    print(M)
    input()
print(M[2,0],M[0,0],M[1,0])
print(xgcd(a,b))
print(M[2,0]==M[0,0]*a+b*M[1,0])
```

```
[  0   1]
[  1  -9]
[109  20]
```

Frações contínuas

A célula a seguir calcula a n-ésima convergente p_n/q_n de uma fração contínua usando a recorrência

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2} \\ q_n = a_n q_{n-1} + q_{n-2} \end{cases}$$

In [ ]:

```
def f(a,n):
    if n == 0:
        return (a[0],1)
    if n == 1:
        return (a[0]*a[1]+1,a[1])
    pn = a[n]*f(a,n-1)[0] + f(a,n-2)[0]
    qn = a[n]*f(a,n-1)[1] + f(a,n-2)[1]
    return (pn,qn)

a = continued_fraction(pi^2)
print(a)
print(f(a,n)[0]/f(a,n)[1])
n = 10
print(a.convergent(n) == f(a,n)[0]/f(a,n)[1])
```

Conjectura de Collatz

In [ ]:

```
def f(n):
  if n % 2 == 0:
    return n/2
  else:
    return 3*n+1

def collatz(k):
  i = 0
  if k == 1:
      return 1
  k = floor(k)
  while f(k) != 1:
      k = f(k)
      i = i+1
      #print("f^i(k) = ", k, "\t i = ", i)
  return i
list_plot([(k,collatz(k)) for k in srange(1,1000)])
```

Fatoração

Algoritmo para fatoração de um dado n, com e sem recursividade. Note que o algoritmo não é eficiente.

In [ ]:

```
def fat(n, fatores = []):
    if n == 1:
        return fatores
    p = 2
    while  not p.divides(n):
        p = p.next_prime()

    return fat(n/p,fatores+[p])

# print(fat(1729))

# fat99 = fat(99)
# print(fat99)
# print(prod(fat(99)) == 99)

def fat_sem_recursao(n):
    fatores = []
    p = 2
    i = 0
    while n != 1:
        while (not p.divides(n)):
            i+=1
            print(p,n)
            p = p.next_prime()
        fatores.append(p)
        n = n/p
        if p^2 > n:
            p = n
    print("Iter.:", i)
    return fatores

n = 7*11*13*19*29*31^3*41
ff = fat_sem_recursao(n)
print(ff)
print('fat ok', prod(ff)== n)
```

Fibonacci e Cálculo simbólico (Explore!)

In [ ]:

```
# def fib(n):
#   if n == 0 or n == 1:
#     return 1
#   return fib(n-1) + fib(n-2)
# [fib(n) for n in [0..10]]

def fib_g(n):
    phi = (1+sqrt(5))/2
    psi = (1-sqrt(5))/2
    return (phi^n - psi^n)/(phi-psi)
print(fib_g(6))
show(fib_g(6))
print(expand(fib_g(6)))
```

Plotando o gráfico de curvas implicitas

In [1]:

```
var('y')
c1 = implicit_plot(y^2 == x^3 + x,(-1,1),(-1,1),color='green')
c2 = implicit_plot(y == 3*x,(-1,1),(-1,1),color='red')
p = point((0,0),size=40,color='black')
# show(c1+c2+p)
```

Resolvendo uma recorrência linear

Encontramos a fórmula geral da recorrência $x_n = 4x_{n-1} + 3x_{n-2} - 14x_{n-3} - 6x_{n-4}$ com termos iniciais $x_0 = x_1 = 4, x_2 = 22$ e $x_3 = 58$.

In [2]:

```
a = (1+sqrt(2))
b = (1-sqrt(2))
c = (1+sqrt(7))
d = (1-sqrt(7))

f = (x-a)*(x-b)*(x-c)*(x-d)

expand(f)
x_inicial = [4,a+b+c+d,a^2+b^2+c^2+d^2,a^3+b^3+c^3+d^3]

# por recorrência
def v(n):
    global x_inicial
    if n<=3:
        return x_inicial[n]
    else:
        return 4*v(n-1)+3*v(n-2)-14*v(n-3)-6*v(n-4)
[expand(v(n)) for n in [0..10]]

# pela fórmula geral
def vg(n):
    global a,b,c,d,x_inicial
    return a^n + b^n + c^n + d^n
print([(expand(v(n)),expand(vg(n))) for n in [0..15]])

var('n')
show(a^n+b^n+c^n+d^n)
```

```
[(4, 4), (4, 4), (22, 22), (58, 58), (218, 218), (714, 714), (2566, 
2566), (9006, 9006), (32418, 32418), (116482, 116482), (421702, 421
702), (1528366, 1528366), (5553314, 5553314), (20195634, 20195634), 
(73515142, 73515142), (267730878, 267730878)]
```

$$\left(\sqrt{7}+1\right)^{n} + \left(-\sqrt{7}+1\right)^{n} + \left(\sqrt{2}+1\right)^{n} + \left(-\sqrt{2}+1\right)^{n}$$

In [3]:

```
a = (1+sqrt(2))
b = (1-sqrt(2))
c = (1+sqrt(7))
d = (1-sqrt(7))
f = (x-a)*(x-b)*(x-c)*(x-d)

"""
CASO GERAL:
construir o polinômio através dos coeficientes
de recorrência da sequência e chamar de a,b,c
e d suas raízes
"""

x_inicial = [1,-1,1,-1]
var('a1,a2,a3,a4')
eqs = [
    a1+a2+a3+a4 == x_inicial[0],
    a1*a + a2*b+a3*c+a4*d ==  x_inicial[1],
    a1*a^2 + a2*b^2+a3*c^2+a4*d^2 ==  x_inicial[2],
    a1*a^3 + a2*b^3+a3*c^3+a4*d^3 ==  x_inicial[3],
]
show(eqs)
s = solve(eqs,a1,a2,a3,a4)
coef = [s[0][i].rhs() for i in [0..3]]

# por recorrência
def v(n):
    global x_inicial
    if n<=3:
        return x_inicial[n]
    else:
        return 4*v(n-1)+3*v(n-2)-14*v(n-3)-6*v(n-4)

# pela fórmula geral
def vg(n):
    global a,b,c,d,coef
    return coef[0]*a^n + coef[1]*b^n + coef[2]*c^n + coef[3]*d^n


print([(expand(v(n)),expand(vg(n))) for n in [0..15]])
```

$$\Big[a_1 + a_2 + a_3 + a_4 = 1, a_3(\sqrt{7}+1) - a_4(\sqrt{7}-1) + a_1(\sqrt{2}+1) - a_2(\sqrt{2}-1) =$$

$$+ a_4(\sqrt{7}-1)^2 + a_1(\sqrt{2}+1)^2 + a_2(\sqrt{2}-1)^2 = 1, a_3(\sqrt{7}+1)^3 - a_4(\sqrt{7}-1$$

$$- a_2(\sqrt{2}-1)^3 = (-1)\Big]$$

```
[(1, 1), (-1, -1), (1, 1), (-1, -1), (7, 7), (17, 17), (97, 97), (3
47, 347), (1399, 1399), (5177, 5177), (19465, 19465), (71723, 7172
3), (264415, 264415), (969257, 969257), (3549361, 3549361), (129730
67, 12973067)]
```

Primos

In [4]:

```
#l = [[10^i,prime_pi(10^i),RR(10^i/log(10^i)),prime_pi(10^i)/RR(10^i/log(10^i))]
for i in range(1,14)]
for i in range(8):
    print('10^{}  & {:d} & {:.2f} & {:.5f} \\\\'.format(i+1,l[i][1],float(l[i][2
]),float(l[i][3])))
```

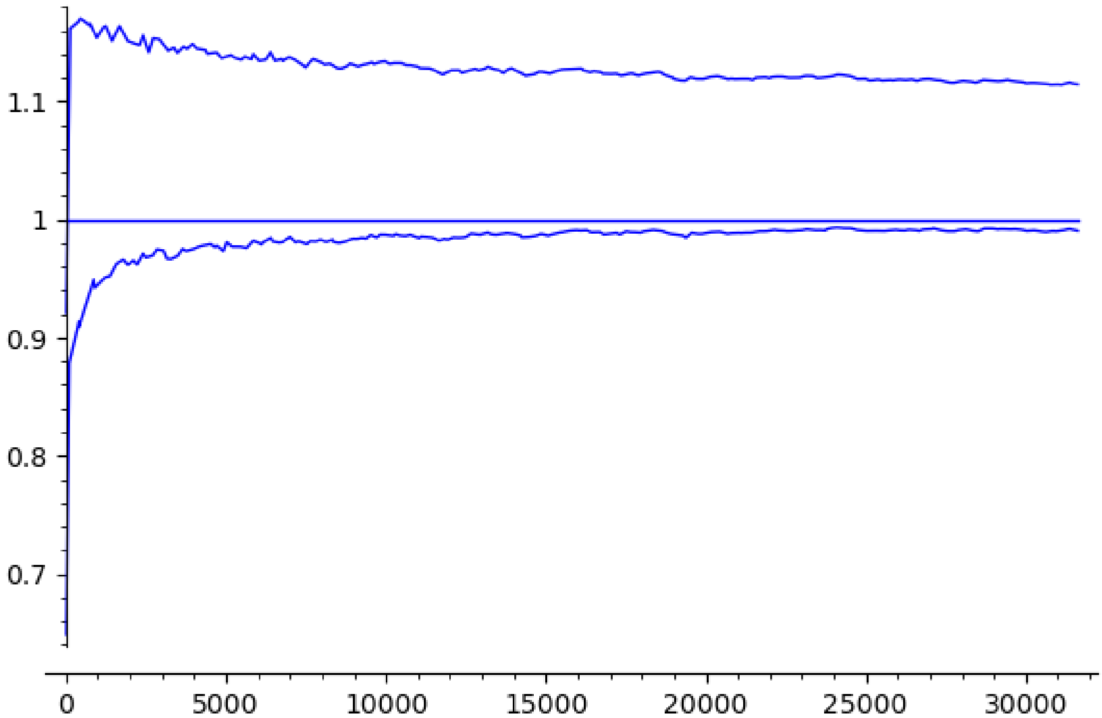
```
---------------------------------------------------------------------------
NameError                                 Traceback (most recent call last)
<ipython-input-4-460c10afb783> in <module>
      1 #l = [[10^i,prime_pi(10^i),RR(10^i/log(10^i)),prime_pi(10^i)/RR(10^i/log(10^i))] for i in range(1,14)]
      2 for i in range(Integer(8)):
----> 3     print('10^{}  & {:d} & {:.2f} & {:.5f} \\\\'.format(i+Integer(1),l[i][Integer(1)],float(l[i][Integer(2)]),float(l[i][Integer(3)])))

NameError: name 'l' is not defined
```

In [5]:

```
def tnp(x):
    return RR(x/log(x))
def erro(x):
    return prime_pi(x)/tnp(x)
def erroli(x):
    return prime_pi(x)/li(x)
# plot(tnp,10,2000)+plot(prime_pi,10,2000)
K = 10^4.5
plot(erro,10,K) + plot(lambda x:1, 10, K)+plot(erroli,10,K)
```

Out[5]:



In [6]:

```
# # n = 100
# @interact(x = (50,20000,100))
# def _(x=50):
#     def bertrand(n):
#         return prime_pi(2*n) - prime_pi(n)
#     P = plot(bertrand,1,x)
#     show(P)
# # IntSlider(min=-10,max=30,step=1,value=10)
@interact(x=(10,100))
def pp(x):
    P = plot(prime_pi,1,x)
    show(P)
```

Algoritmo AKS

Implementamos aqui o algoritmo AKS que verifica primalidade em tempo polinomial. Definimos primeiramente a função `is_perfect_power` que verifica se o inteiro em questão é uma potência perfeita. Usamos o fato que, se $n = a^b$, como $a \geq 2$, $b \leq \log_2(n)$, caso contrário $a^b > 2^{\log_2(n)} = n$. Assim, definindo $m = \lfloor \log_2(n) \rfloor$, basta verificar se $(\sqrt[k]{n})^k = n$ para algum $k = 2, \ldots, m$.

`Exception (FLINT memory_manager). Unable to allocate memory.` na linha 48 (funciona para números pequenos). verificar forma mais eficiente de efetuar esse cálculo.

In [9]:

```
def is_perfect_power(n):
#     m = 0
#     while N(n/2^m) > 1:
#         m+=1
#     print(m,floor(log(n,base=2)))
    m = floor(log(n,base=2))
#     print(m)
    for i in [2..(m)]:
        if floor(N(n^(1/i)))^i == n:
#             print("n = ", (n^(1/i)), "^", i)
            return True
    return False

def o(a,r):
    if gcd(a,r) != 1:
        return "Erro: r e n nao coprimos"
    Zr = Zmod(r)
    return a.multiplicative_order()

is_perfect_power(2^123*3^124)

def aks(n):
    if is_perfect_power(n):
        print("teste 1")
        return True
    r = 2
    if r.divides(n):
        print("teste 1.1 (par)")
        return False
    while o(r,n) < (log(n,base=2))^2: #só quero que essa linha seja executada se gcd(r.n) = 1
        while gcd(r,n) > 1:
            r += 1

    for a in [2..r]:
        if gcd(a,n)<n and gcd(a,n)>1:
            print("teste 2")
#             print(a,n)
#             print(gcd(a,n))
            return False
    if n <= r:
        print("teste 3")
        return True
    Zn = Zmod(n)
    Pn.<t> = Zn[]
    P = QuotientRing(Pn,Pn.ideal(t^r - 1))
    for a in [1..(floor(sqrt(euler_phi(r))*log(n,base=2)))]:
#         print(n,r,a)
        if P((t+a))^n != P(t^n+a):
            print("teste 4")
            return False
    print("teste final")
    return True

cand = [random_prime(10^5,lbound=10^4)+2 for i in [1..50]]

print(all([aks(n)==is_prime(n) for n in cand]))
```

```
teste 4
teste 4
teste final
teste 4
teste 4
teste 4
teste 4
teste final
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste final
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste 4
teste final
teste 4
teste 4
teste 4
teste 4
teste final
teste 4
teste 4
teste 4
teste 4
teste 4
True
```

Funções Aritméticas

In [10]:

```
def psi_pp(n):
    fatoracao = factor(n)
    if len(fatoracao) != 1:
        return "Erro: {} não é potência de primo".format(n)
    p = fatoracao[0][0]
    e = fatoracao[0][1]
    return p^e + p^(e-1)

print("psi(16) = ", psi_pp(16))
print("psi(22) = ", psi_pp(22))
```

```
psi(16) =  24
psi(22) =  Erro: 22 não é potência de primo
```

In [ ]:

```
def psi(n):
    if n <= 0:
        return "Erro: n deve ser positivo"
    if n == 1:
        return 1
    fatoracao = factor(n)
    total = prod([psi_pp(p^e) for (p,e) in fatoracao])
    return total

# A = list_plot([(i,psi(i)) for i in [1..100]])
plot(lambda x:euler_phi(floor(x))/x,1,100,color='red')
```

Sequência de alíquota

In [11]:

```
n = 25312

K = 1000
k = 0
aliquot = [n]

while k < K:
    sk = aliquot[-1]
    if sk == 0:
        break
    if sigma(sk)-sk in aliquot:
        break
    aliquot.append(sigma(sk)-sk)

    k += 1
show(aliquot)
show(len(aliquot))
```

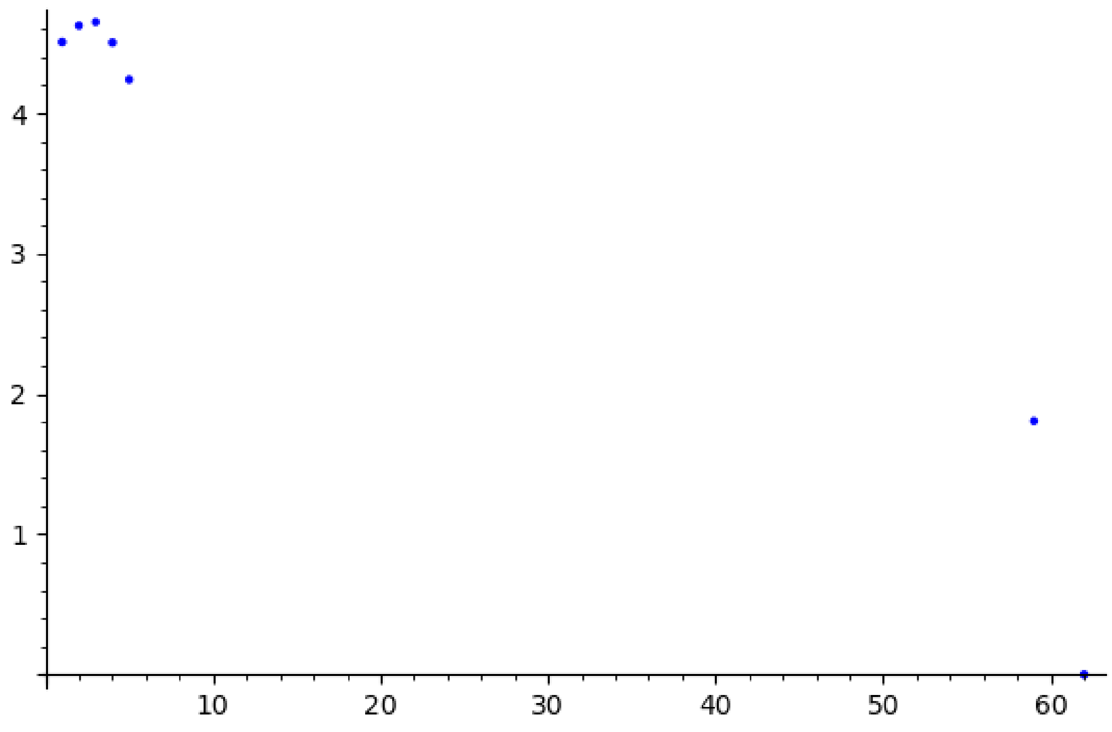
[25312, 32144, 42070, 44618, 31894, 17354, 8680, 14360, 18040, 27320, 34240, 48
49064, 42946, 22394, 11200, 20296, 19304, 19096, 26984, 23626, 11816, 13624, 140
10840, 13640, 20920, 26240, 38020, 41864, 36646, 19298, 9652, 8268, 12900, 25292
3514, 2534, 1834, 1334, 826, 614, 310, 266, 214, 110, 106, 56, 64, 6

64

In [12]:

```
list_plot([(i,log(aliquot[i],10)) for i in [1..5]+[len(aliquot)-5,len(aliquot)-2
]])
```

Out[12]:

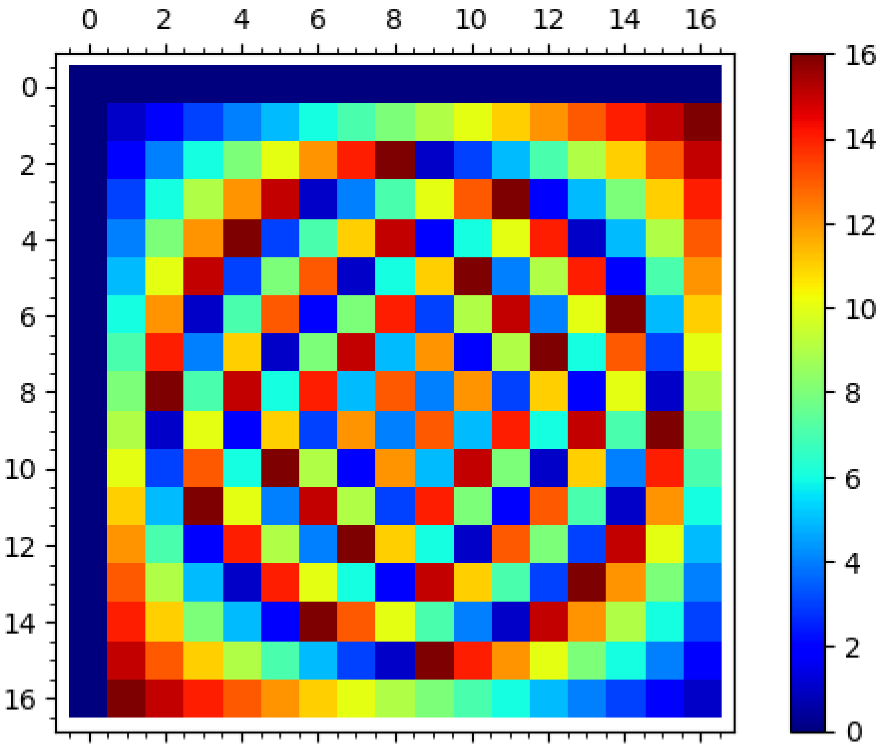


Congruências

In [13]:

```
Z10 = Zmod(14.next_prime())
a = Z10(3)
a.multiplicative_order()
matrix_plot([[i*j for i in Z10] for j in Z10],cmap='jet',colorbar=True)
```

Out[13]:



In [14]:

```
print(Z10.addition_table(names='digits'))
print(Z10.multiplication_table(names='digits'))
```

```
 +  00 01 02 03 04 05 06 07 08 09 10 11 12 13 14 15 16
  +---------------------------------------------------
00| 00 01 02 03 04 05 06 07 08 09 10 11 12 13 14 15 16
01| 01 02 03 04 05 06 07 08 09 10 11 12 13 14 15 16 00
02| 02 03 04 05 06 07 08 09 10 11 12 13 14 15 16 00 01
03| 03 04 05 06 07 08 09 10 11 12 13 14 15 16 00 01 02
04| 04 05 06 07 08 09 10 11 12 13 14 15 16 00 01 02 03
05| 05 06 07 08 09 10 11 12 13 14 15 16 00 01 02 03 04
06| 06 07 08 09 10 11 12 13 14 15 16 00 01 02 03 04 05
07| 07 08 09 10 11 12 13 14 15 16 00 01 02 03 04 05 06
08| 08 09 10 11 12 13 14 15 16 00 01 02 03 04 05 06 07
09| 09 10 11 12 13 14 15 16 00 01 02 03 04 05 06 07 08
10| 10 11 12 13 14 15 16 00 01 02 03 04 05 06 07 08 09
11| 11 12 13 14 15 16 00 01 02 03 04 05 06 07 08 09 10
12| 12 13 14 15 16 00 01 02 03 04 05 06 07 08 09 10 11
13| 13 14 15 16 00 01 02 03 04 05 06 07 08 09 10 11 12
14| 14 15 16 00 01 02 03 04 05 06 07 08 09 10 11 12 13
15| 15 16 00 01 02 03 04 05 06 07 08 09 10 11 12 13 14
16| 16 00 01 02 03 04 05 06 07 08 09 10 11 12 13 14 15

 *  00 01 02 03 04 05 06 07 08 09 10 11 12 13 14 15 16
  +---------------------------------------------------
00| 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
01| 00 01 02 03 04 05 06 07 08 09 10 11 12 13 14 15 16
02| 00 02 04 06 08 10 12 14 16 01 03 05 07 09 11 13 15
03| 00 03 06 09 12 15 01 04 07 10 13 16 02 05 08 11 14
04| 00 04 08 12 16 03 07 11 15 02 06 10 14 01 05 09 13
05| 00 05 10 15 03 08 13 01 06 11 16 04 09 14 02 07 12
06| 00 06 12 01 07 13 02 08 14 03 09 15 04 10 16 05 11
07| 00 07 14 04 11 01 08 15 05 12 02 09 16 06 13 03 10
08| 00 08 16 07 15 06 14 05 13 04 12 03 11 02 10 01 09
09| 00 09 01 10 02 11 03 12 04 13 05 14 06 15 07 16 08
10| 00 10 03 13 06 16 09 02 12 05 15 08 01 11 04 14 07
11| 00 11 05 16 10 04 15 09 03 14 08 02 13 07 01 12 06
12| 00 12 07 02 14 09 04 16 11 06 01 13 08 03 15 10 05
13| 00 13 09 05 01 14 10 06 02 15 11 07 03 16 12 08 04
14| 00 14 11 08 05 02 16 13 10 07 04 01 15 12 09 06 03
15| 00 15 13 11 09 07 05 03 01 16 14 12 10 08 06 04 02
16| 00 16 15 14 13 12 11 10 09 08 07 06 05 04 03 02 01
```

Equações diofantinas

A equação $aX + bY = c$

In [15]:

```
def dioflin(a,b,c):
    if a == 0 and b == 0:
        return "a e b ambos nulos"
    (d,xt,yt) = xgcd(a,b)
    if not d.divides(c):
        return "mdc(a,b) não divide c. A eq. não possui solução"
    sol = (c/d*xt,c/d*yt)
    return sol


(a,b,c) = (6841684,5168416,8484)
(x,y) = dioflin(a,b,c)
print(x,y)
print(x*a+y*b, c)
```

```
408209781 -540367170
8484 8484
```

Lixo

In [ ]:

```
# x = var('x')
# eqn =  x^3 + sqrt(2)*x + 5 == 0
# a = solve(eqn, x)[0].rhs()
# pretty_print(a)
# show(continued_fraction(pi))
# A = matrix(3,3,{(0,0):1})
# show(A)
print(RR(e))
RR100 = RealField(prec=100)
print(RR100)
print(RR100(e))
p = next_prime(10)
matrix_plot([[mod(i^j,p) for i in range(p)] for j in range(p)],cmap='jet')
```

In [ ]:

```

```