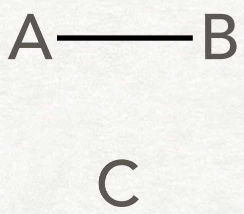
INTRODUCTION TO CATEGORY THEORY

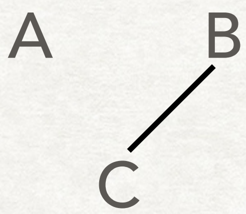
CHAPTER 1 PART 1

LET'S LOOK AT AN EXAMPLE FIRST

Is A connected to C?



NO



NO

LET'S LOOK AT AN EXAMPLE FIRST

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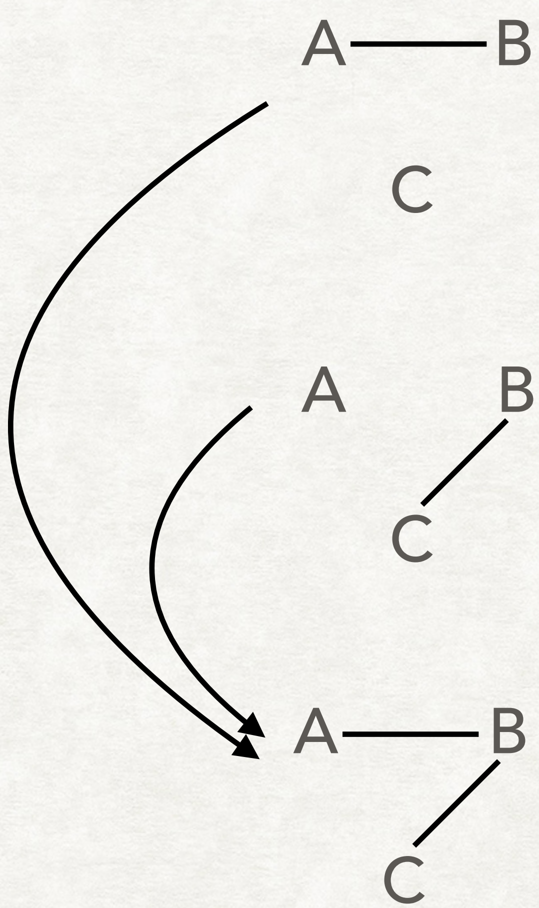


NO

NO

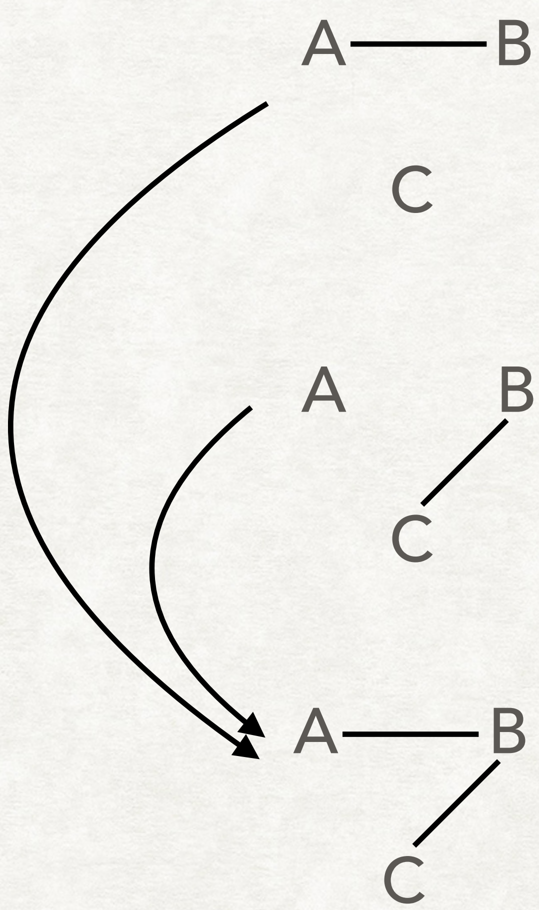
YES

If A ~ B and B ~ C then A ~ C

ANSWER TO COMBINATION OF OBSERVATIONS
!=
COMBINATION OF ANSWERS TO INDIVIDUAL OBSERVATIONS

HOW TO WE FORMULATE IT MATHEMATICALLY?

Is A connected to C?



NO

NO

YES

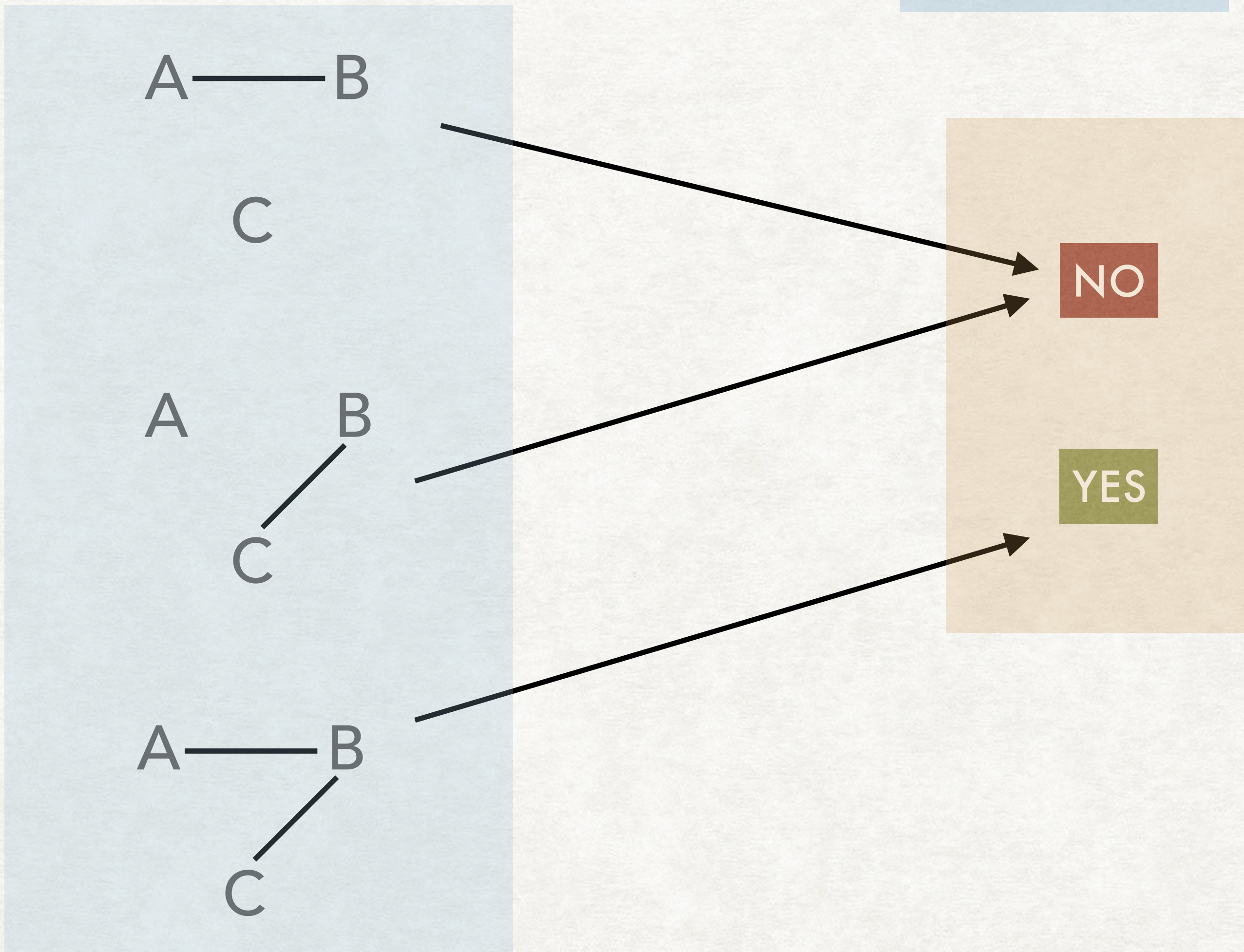
If A ~ B and B ~ C then A ~ C

HOW TO WE FORMULATE IT MATHEMATICALLY?

Is A connected to C?

1. Think of it as a function

Observation → Answer



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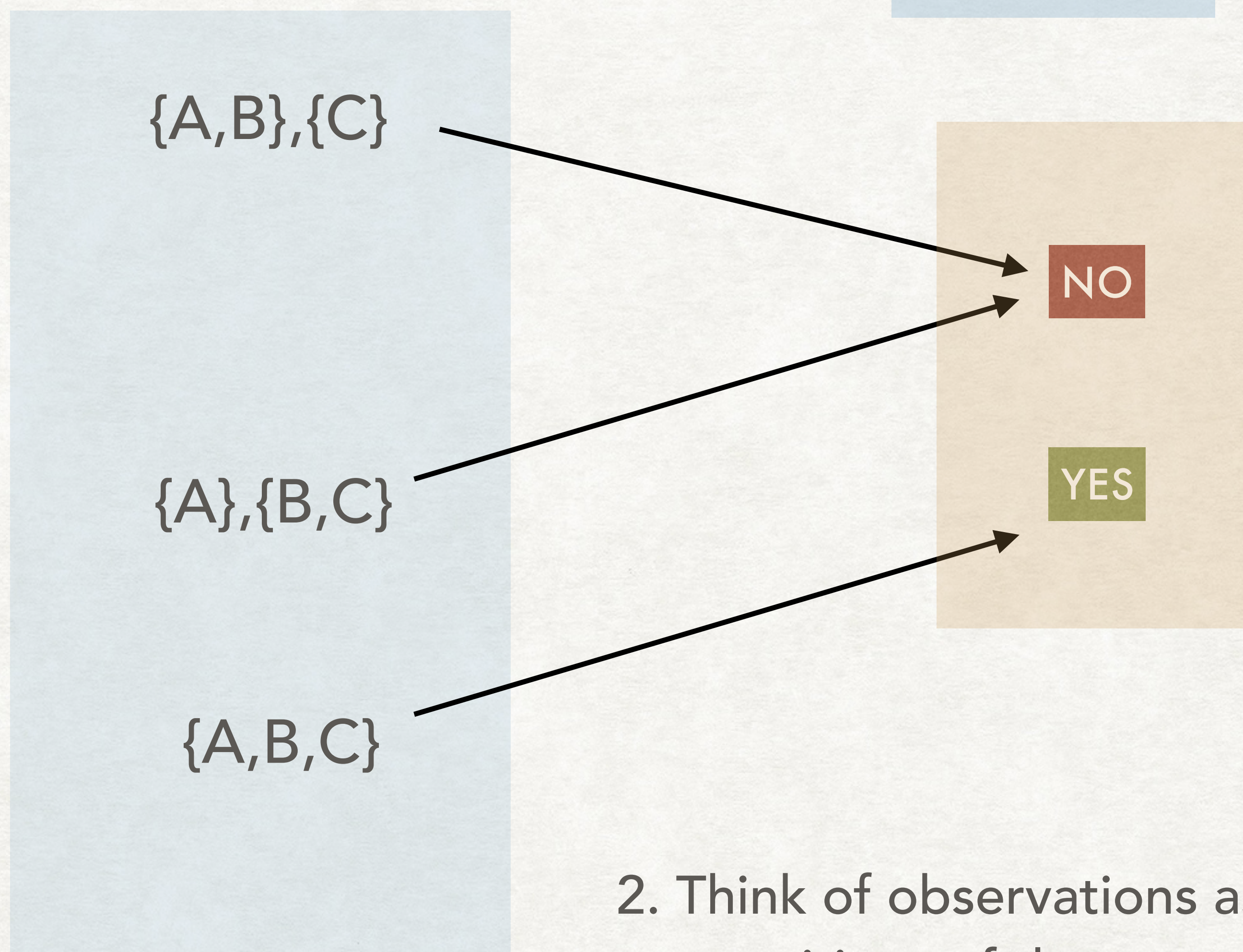


2. Think of observations as partitions of the set {A,B,C}

BUT WHAT IS A PARTITION?

A,B,C,D

Divide a set A into subsets such that each element of A belongs to exactly one subset

Valid partitions

A,C | B,D

A,B,D | C

A,B | C | D

A | B | C | D

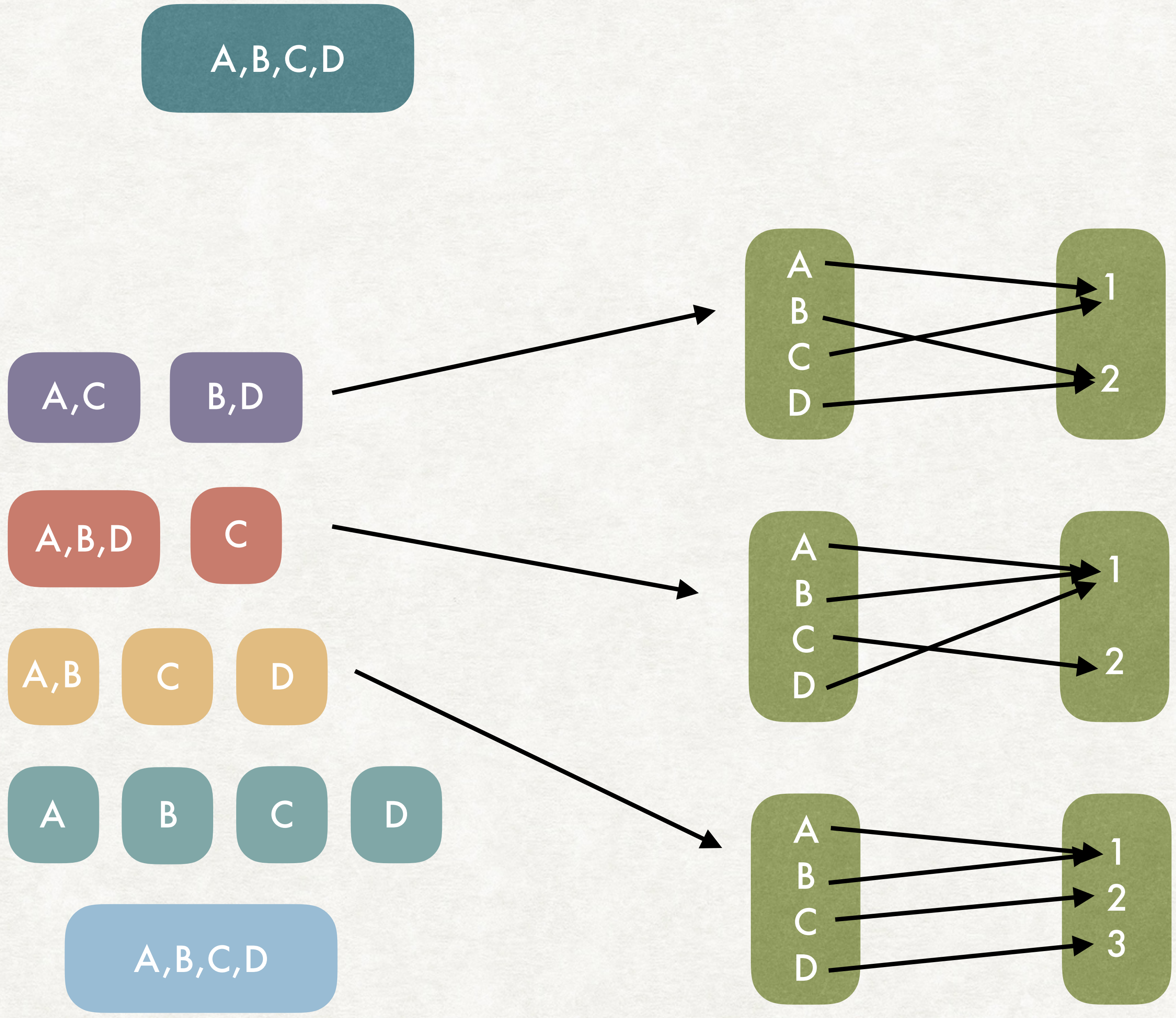
A,B,C,D

Not partitions (why?)

A,B,D | B,C

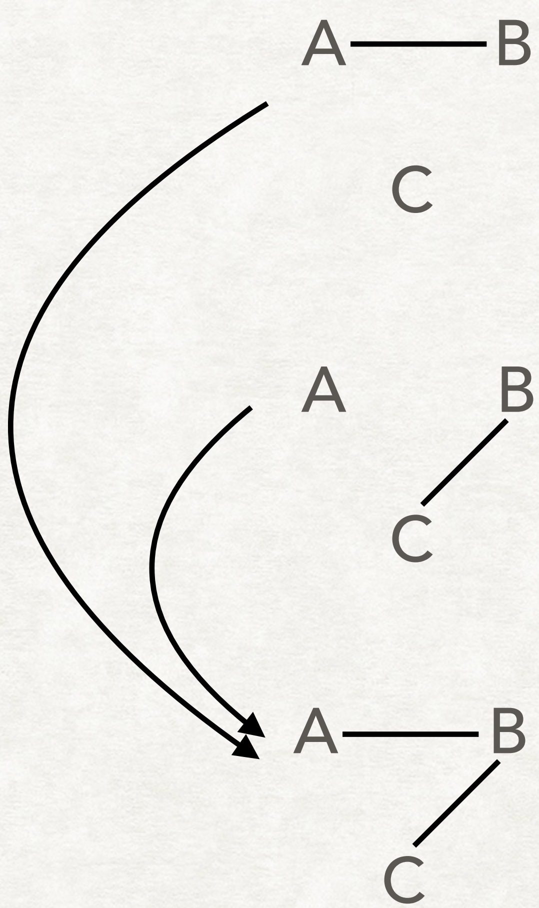
B | C | D

PARTITIONS CAN BE UNDERSTOOD USING SURJECTIVE FUNCTIONS



LET'S COME BACK TO THE ORIGINAL EXAMPLE

Is A connected to C?



NO

NO

YES

If A ~ B and B ~ C then A ~ C

Some are "smaller" than others

"Combining" gives an idea of making a "larger" observation

What do we mean by combining observations?

ANSWER TO COMBINATION OF OBSERVATIONS
!=
COMBINATION OF ANSWERS TO INDIVIDUAL OBSERVATIONS

BINARY RELATIONS AND PREORDERS

Set A = { ∘ , ∘ , ∘ , ∘ , ∘ }

↓

Define a relation R: roughly a condition that involves two elements
Eg: *a* is related to *b* if *b* is divisible by *a*

↓

Pick any two elements *x* and *y*

↓

$x \sim y$ (x is related to y) if R is satisfied by x and y

BINARY RELATIONS AND PREORDERS

Set A = {1,2,3,4}

R: *a* is related to *b* if
sum of a and b is even

$a \sim a \ \forall \ a \in A$

1 ~ 3, 3 ~ 1, 2 ~ 4 and 4 ~ 2

R: *a* is related to *b* if
a + b = 5

2 ~ 3, 3 ~ 2, 1 ~ 4 and 4 ~ 1

BINARY RELATIONS AND PREORDERS

Set A = {1,2,3,4}

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1 ~ 3, 3 ~ 1, 2 ~ 4 and 4 ~ 2

R: *a* is related to *b* if
a + b = 5

2 ~ 3, 3 ~ 2, 1 ~ 4 and 4 ~ 1

Is $a \sim a \; \forall \; a \in A$?	Reflexive property
Does $a \sim b \Rightarrow b \sim a$?	Symmetric property
Does $a \sim b$ and $b \sim c \Rightarrow a \sim c$?	Transitive property

BINARY RELATIONS AND PREORDERS

Set $A = \{1,2,3,4\}$

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R: *a* is related to *b* if
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2 ~ 3, 3 ~ 2, 1 ~ 4 and 4 ~ 1

Is $a \sim a \ \forall a \in A$? YES

Does $a \sim b \Rightarrow b \sim a$? YES

Does $a \sim b$ and $b \sim c \Rightarrow a \sim c$? YES

BINARY RELATIONS AND PREORDERS

Set A = {1,2,3,4}

R: *a* is related to *b* if ***sum of a and b is even***	R: *a* is related to *b* if ***a + b = 5***
$a \sim a\ \forall\ a \in A$ *1 ~ 3, 3 ~ 1, 2 ~ 4 and 4 ~ 2*	*2 ~ 3, 3 ~ 2, 1 ~ 4 and 4 ~ 1*

Is $a \sim a\ \forall\ a \in A$?

Does $a \sim b \Rightarrow b \sim a$?

Does $a \sim b$ and $b \sim c \Rightarrow a \sim c$?

BINARY RELATIONS AND PREORDERS

Set $A = \{1,2,3,4\}$

R: *a* is related to *b* if
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1 ~ 3, 3 ~ 1, 2 ~ 4 and 4 ~ 2

R: *a* is related to *b* if
a + b = 5

2 ~ 3, 3 ~ 2, 1 ~ 4 and 4 ~ 1

Is $a \sim a \ \forall \ a \in A$? NO

Does $a \sim b \Rightarrow b \sim a$? YES

Does $a \sim b$ and $b \sim c \Rightarrow a \sim c$? NO

BINARY RELATIONS AND PREORDERS

Is $a \sim a \ \forall \ a \in A$?	Reflexive property
Does $a \sim b \Rightarrow b \sim a$?	Symmetric property
Does $a \sim b$ and $b \sim c \Rightarrow a \sim c$?	Transitive property

When these two conditions are met,
the binary relation is called a preorder

The relation R is denoted by ≤

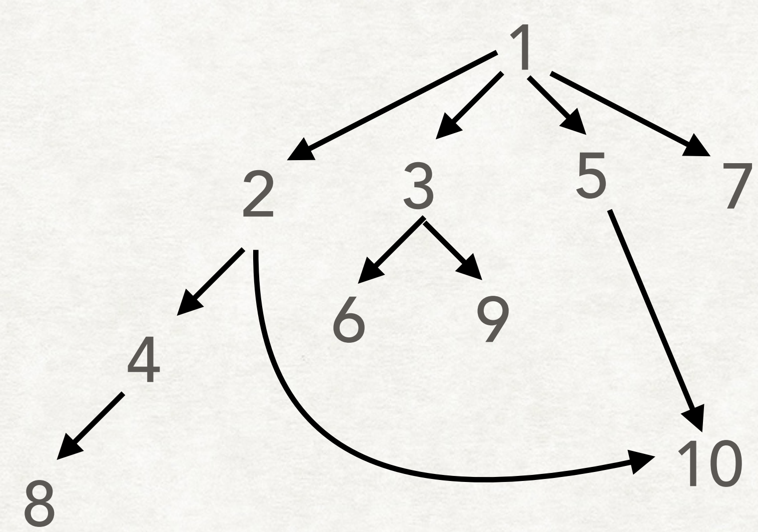
PREORDERS

Define ≤ as:

a ≤ b if a divides b

Thus 3 ≤ 6 but 4 !≤ 6

A = {1,2,3,4,5,6,7,8,9,10}



In a set A, if a is related to b, we can draw an arrow from a to b

1 ≤ 1, 2 ≤ 2 etc

2 ≤ 4 and 4 ≤ 8 ⇒ 2 ≤ 8

PREORDER ON BOOL

Bool is an important preorder:

B = {false, true}

$$\begin{tikzcd} \text{true} \\ \text{false} \arrow[u] \end{tikzcd}$$

false ≤ false
true ≤ true
false ≤ true

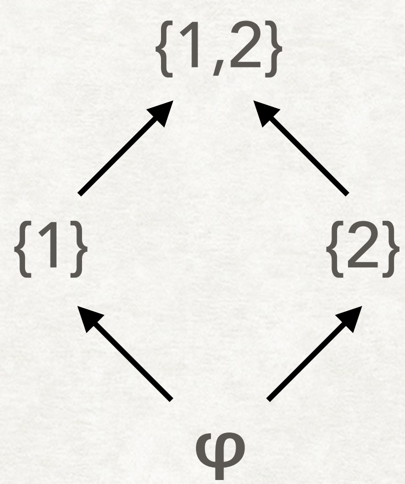
PREORDERS ON POWER SET

Define ≤ as:

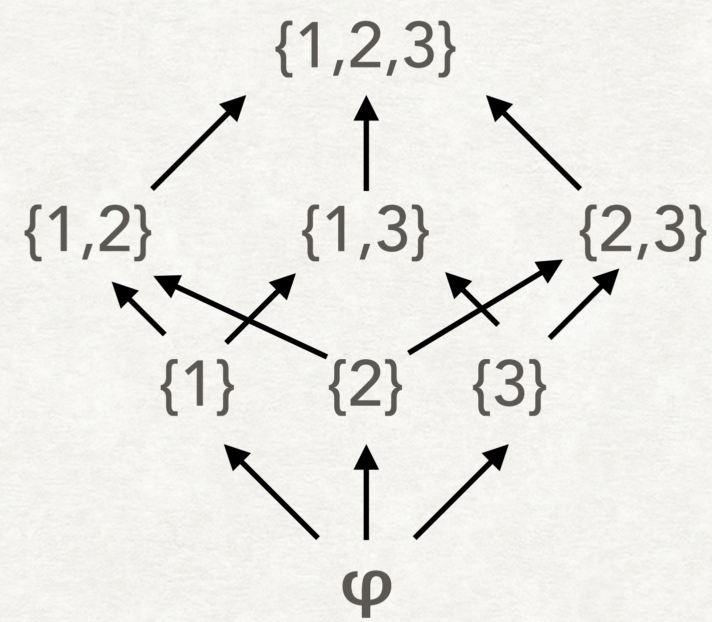
a ≤ b if a ⊆ b

Thus {a,b} ≤ {a,b} but {a} ≤ {a,c}

A = {1,2}



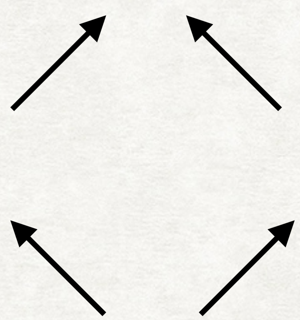
A = {1,2,3}



PREORDERS ON POWER SET

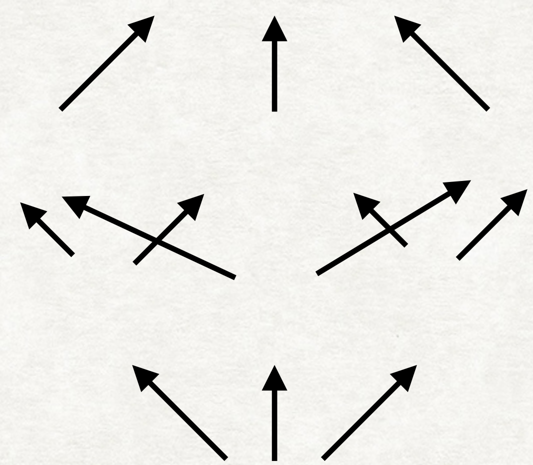
The Hasse diagram for a power set of a finite set with n elements looks like a cube of dimension n

A = {1,2}



Square

A = {1,2,3}



Cube

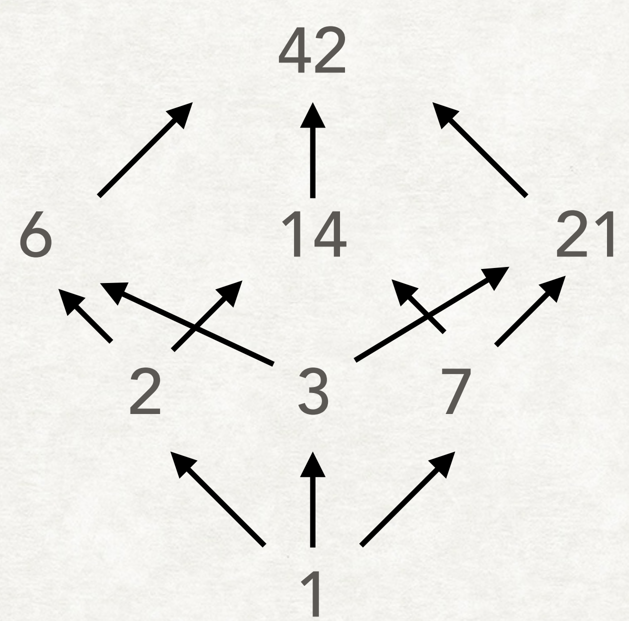
PREORDERS ON DIVISORS

Define ≤ as:

a ≤ b if a divides b

Thus 14 ≤ 42

A = set of all divisors of 42

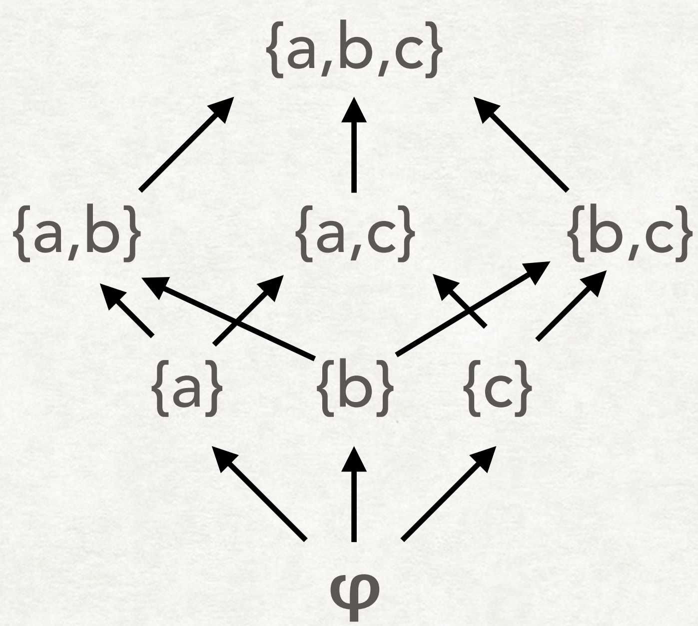


Define ≤ as:

a ≤ b if a ⊆ b

Thus {a,b} ≤ {a,b} but {a} ≤ {a,c}

A = {a,b,c}



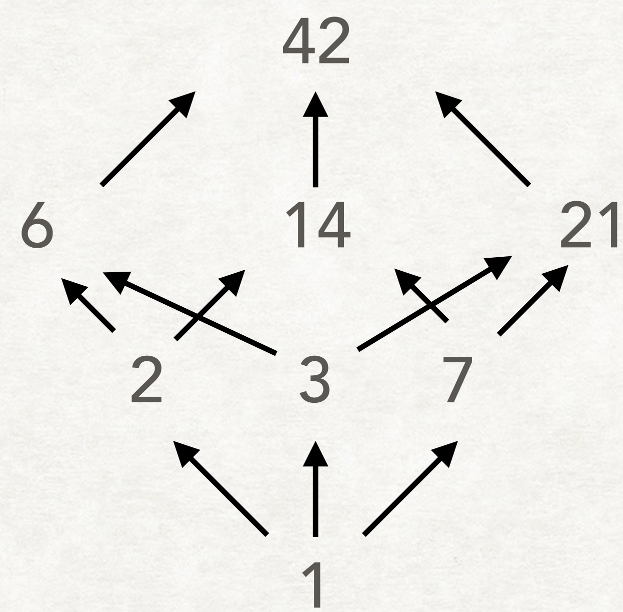
NOTATION FOR PREORDER

If A is a set and the relation ≤ is defined on it, the resulting preorder is denoted by:

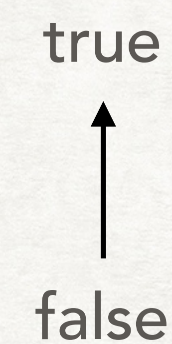
$(A, \leq)$

Eg $(X, \leq_X)$ and $(Y, \leq_Y)$ are two preorders.
The relation $\leq_X$ is defined on X
The relation $\leq_Y$ is defined on Y

MONOTONE MAPS

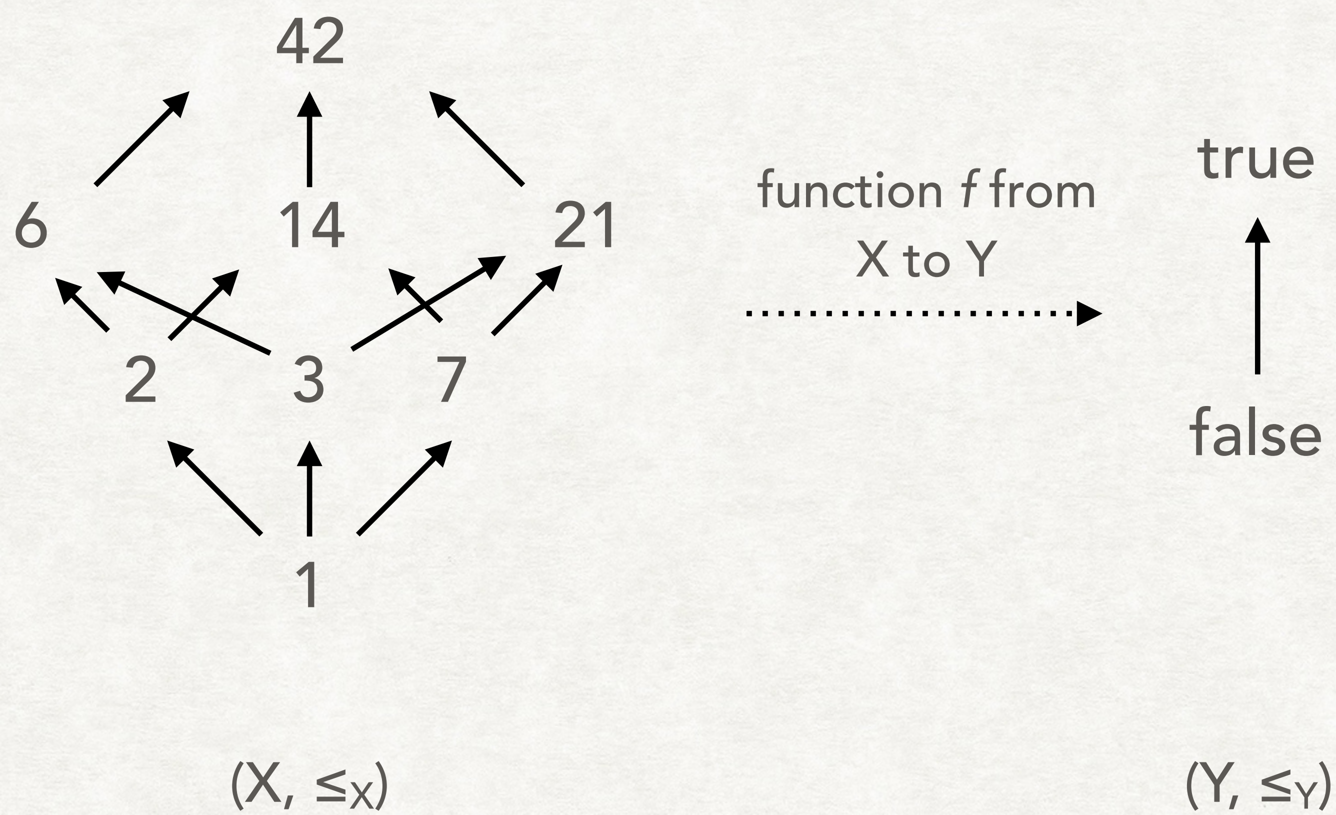


(X, $\leq_X$)



(Y, $\leq_Y$)

MONOTONE MAPS



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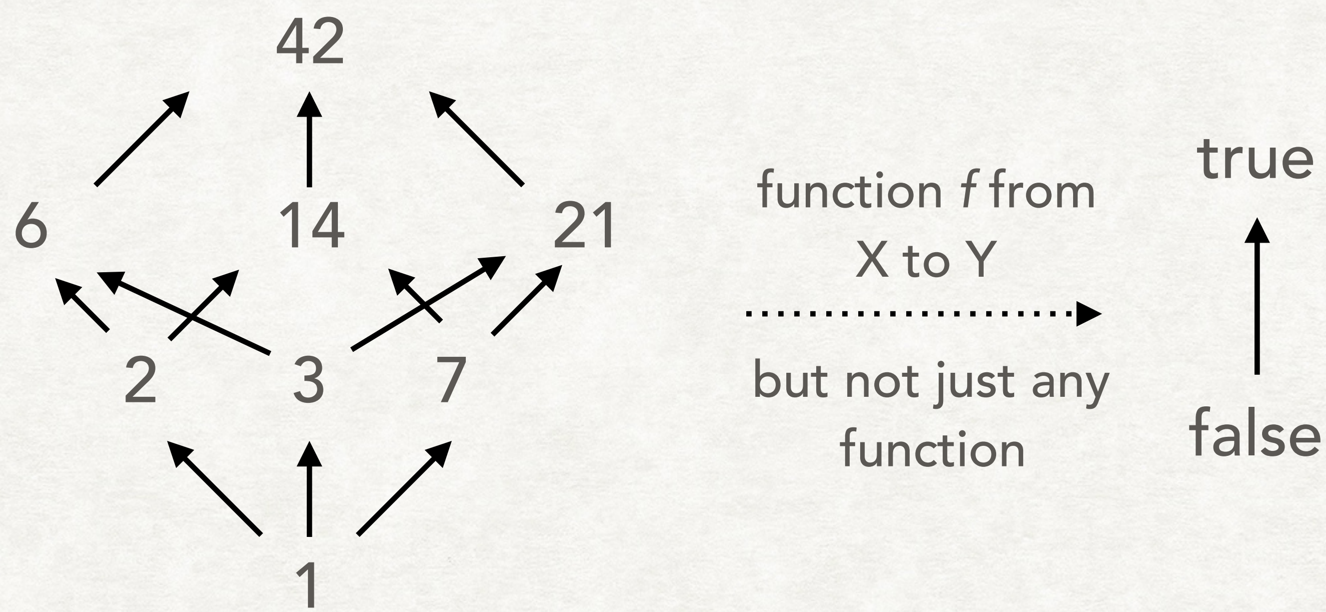
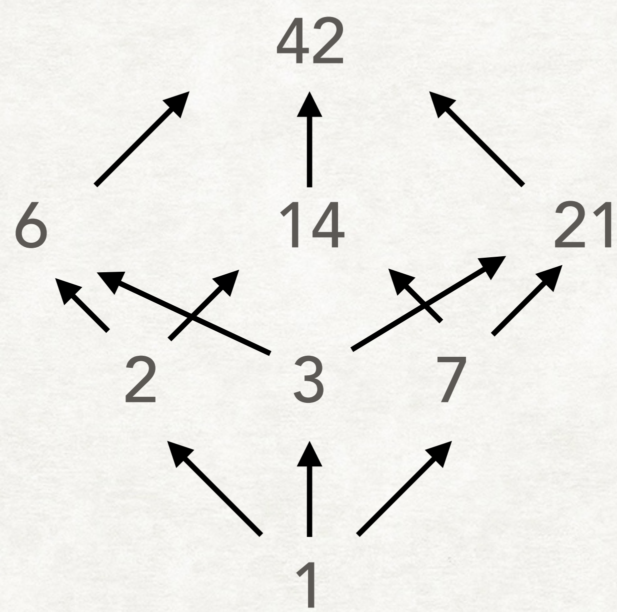


$(X, \leq_X)$ $(Y, \leq_Y)$

MONOTONE MAPS

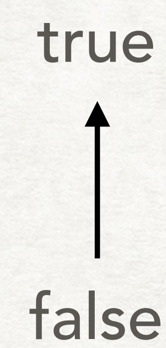


$(X, \leq_X)$

function *f* from X to Y

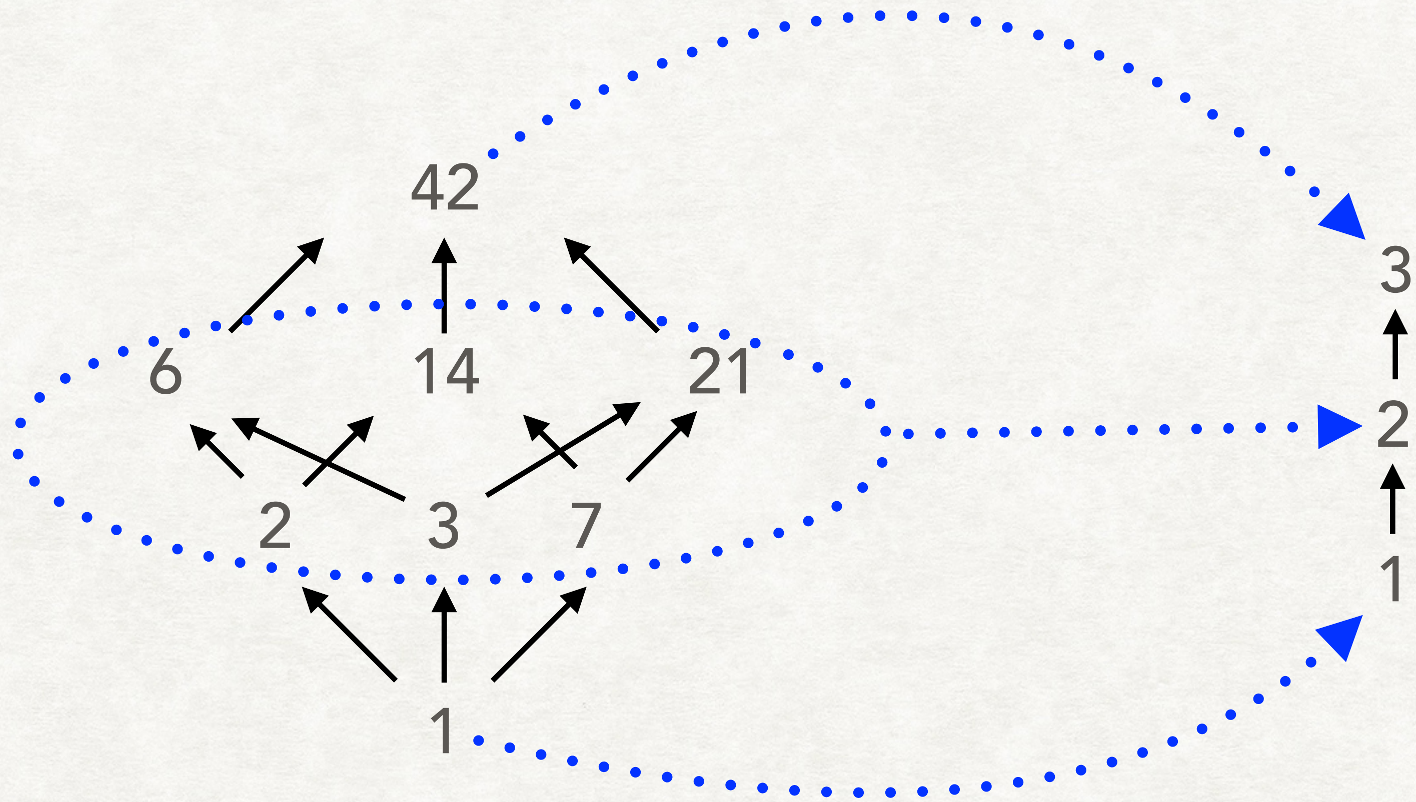
but not just any function

if $x_1 \leq_X x_2$
then
$f(x_1) \leq_Y f(x_2)$



$(Y, \leq_Y)$

MONOTONE MAPS



$(X, \leq_X)$ $(Y, \leq_Y)$

$3 \leq_X 6$ $f(3) \leq_Y f(6)$

$7 \leq_X 21$ $f(7) \leq_Y f(21)$

MONOTONE MAPS

Is this a monotone map?

$$-2 \to -1 \to 0 \to 1 \to 2 \quad \overset{f(x) = x*x}{\dashrightarrow} \quad 0 \to 1 \to 4$$

$(X, \leq_X)$ $(Y, \leq_Y)$

$-2 \leq_X -1$ BUT $f(-2) \; !\leq_Y \; f(-1)$

MEETS AND JOINS

Booleans {true, false}	**Power set of any set A P(a,b,c,d,e,…)**	**Positive integers {1,2,3,4,…}**
AND operation	Intersection of two sets	Highest common factor (最大公因數)
OR operation	Union of two sets	Least common multiple (最小公倍數)

MEETS AND JOINS

These operations share a common structure

Booleans **{true, false}**	**Power set of any set A** **P(a,b,c,d,e,...)**	**Positive integers** **{1,2,3,4,...}**
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MEETS AND JOINS

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a AND b
largest element x such that
$x \leq a$ and $x \leq b$

false AND false = false
false AND true = false
true AND false = false
true AND true = true

true

false

a OR b
smallest element x such that
$a \leq x$ and $b \leq x$

false OR false = false
false OR true = true
true OR false = true
true OR true = true

MEETS AND JOINS

Booleans {true, false}	**Power set of any set A P(a,b,c,d,e,...)**	**Positive integers {1,2,3,4,...}**
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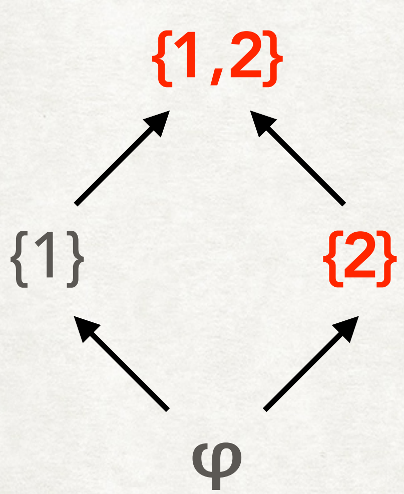
A = {1,2}



$a \cap b$

largest element x such that

$x \leq a$ and $x \leq b$

$\{1,2\} \cap \{2\}$

$\{2\}$

$a \cup b$

smallest element x such that

$a \leq x$ and $b \leq x$

$\{1,2\} \cup \{2\}$

$\{1,2\}$

MEETS AND JOINS

Booleans {true, false}	**Power set of any set A P(a,b,c,d,e,...)**	**Positive integers {1,2,3,4,...}**
AND operation	Intersection of two sets	Highest common factor (最大公因數)
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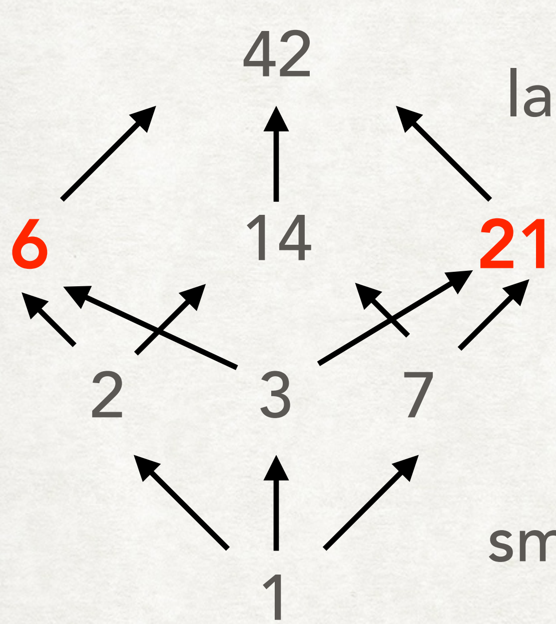


HCF(a,b)
largest element x such that
$x \leq a$ and $x \leq b$

HCF(6,21) = 3
$3 \leq 6$ and $3 \leq 21$

LCM(a,b)
smallest element x such that
$a \leq x$ and $b \leq x$

LCM(6,21) = 42
$6 \leq 42$ and $21 \leq 42$

MEETS AND JOINS

Let $(P, \leq)$ be a preorder, and let $A \subseteq P$ be a subset.

$p \in P$ is a meet of A if

a) $p \leq a \ \forall\ a \in A$

b) $q \leq p \ \forall\ q \in P$ such that $q \leq a$

$p \in P$ is a join of A if

a) $a \leq p \ \forall\ a \in A$

b) $p \leq q \ \forall\ q \in P$ such that $a \leq q$

Example:

$$\begin{tikzcd}
 & d & \\
b \arrow[ur] & & c \arrow[ul] \\
 & a \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$(P, \leq)$

$A = \{b,c\}$

meet of $A = a$

join of $A = d$

MEETS, JOINS & MONOTONE MAPS

Consider a monotone map $f: P \to Q$

$\forall$ p,q $\in$ P if

f(meet of p and q) = meet of f(p) and f(q)

then

f is called meet preserving map

$\forall$ p,q $\in$ P if

f(join of p and q) = join of f(p) and f(q)

then

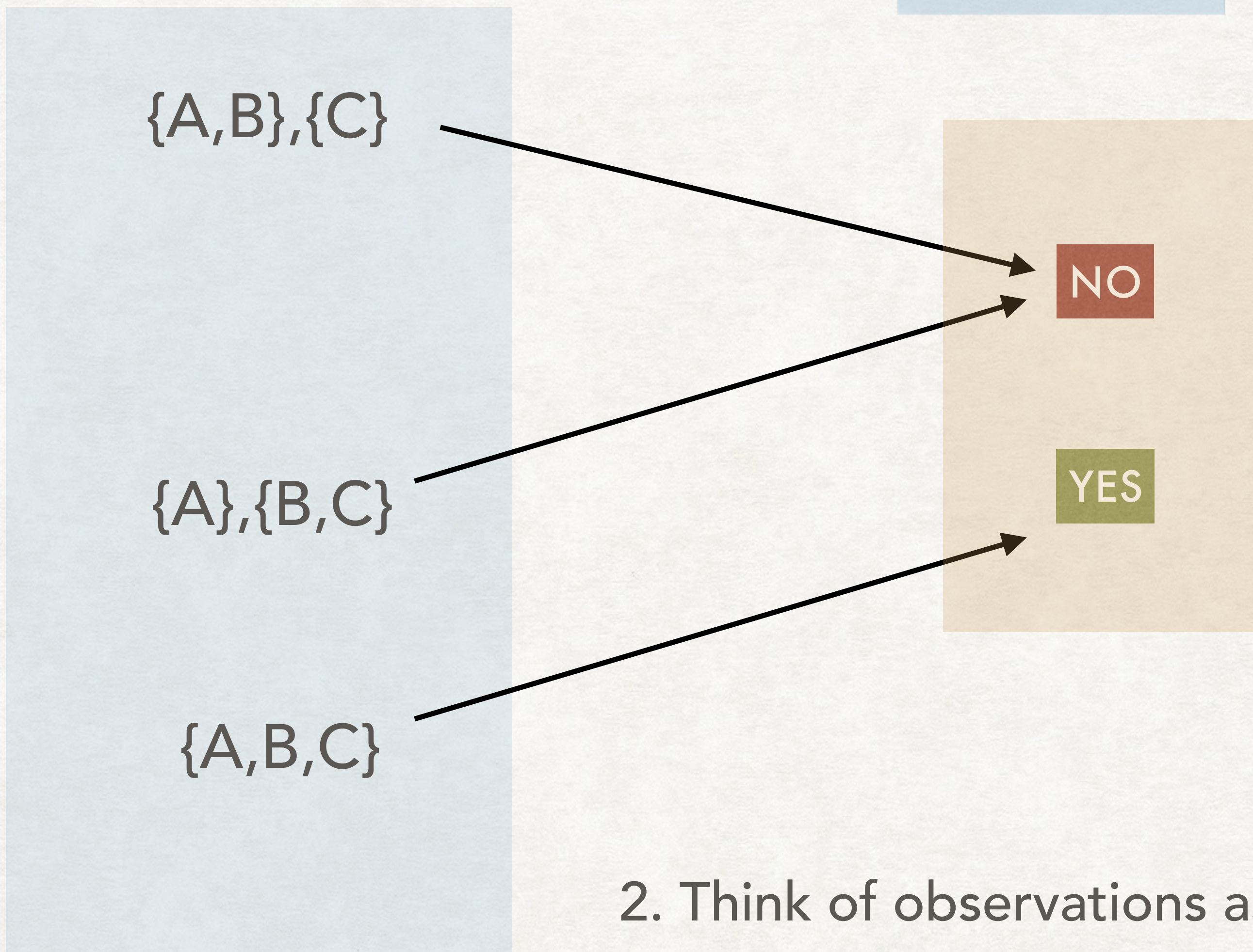
f is called join preserving map

COMING BACK TO THE EXAMPLE

Is A connected to C?

1. Think of it as a function

Observation $\rightarrow$ Answer



2. Think of observations as partitions of the set {A,B,C}

COMING BACK TO THE EXAMPLE

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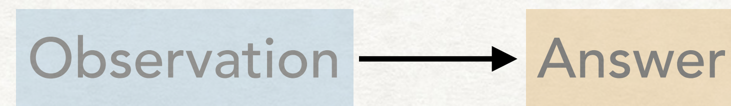

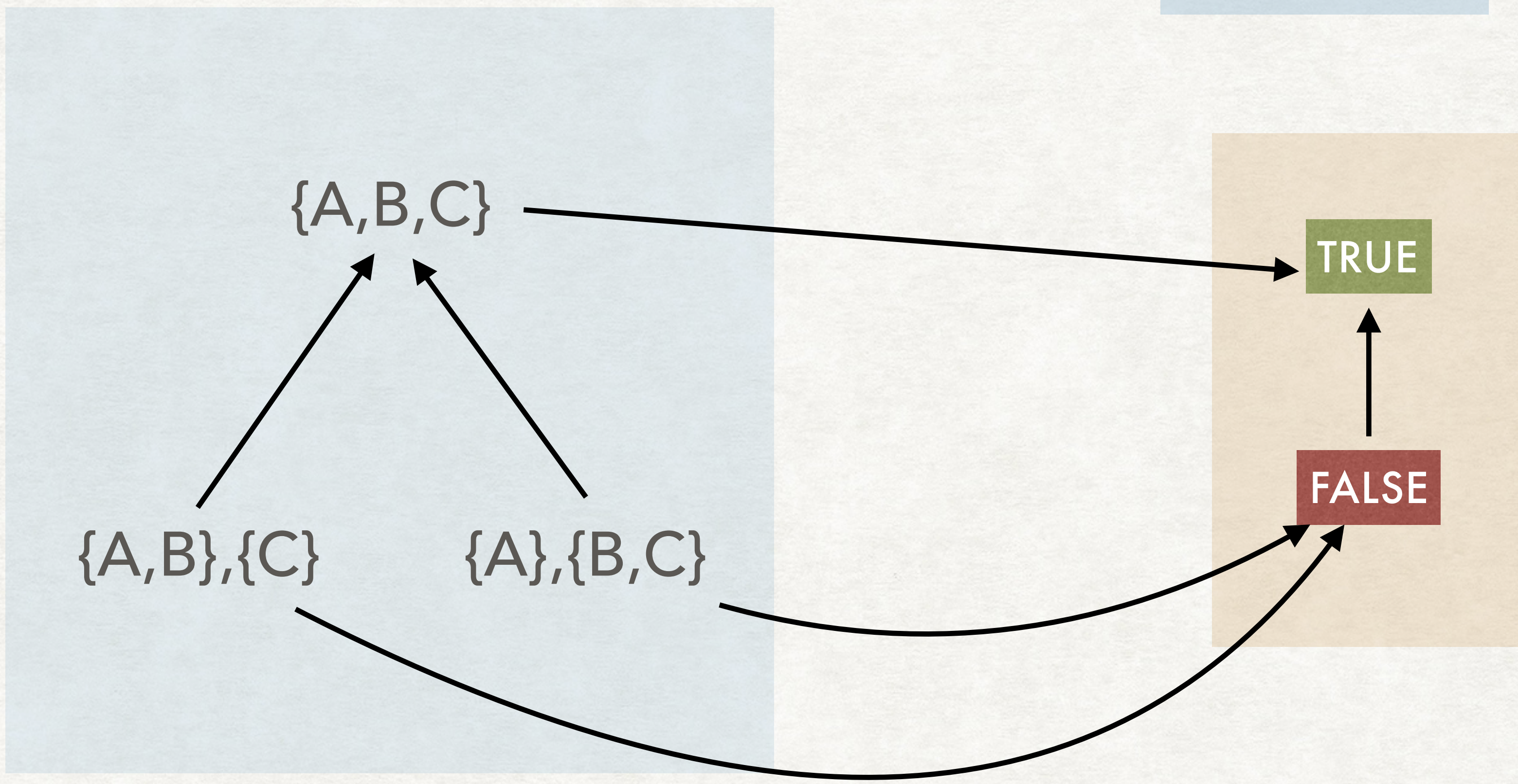

3. View the sets as preorders

2. Think of observations as partitions of the set {A,B,C}

COMING BACK TO THE EXAMPLE

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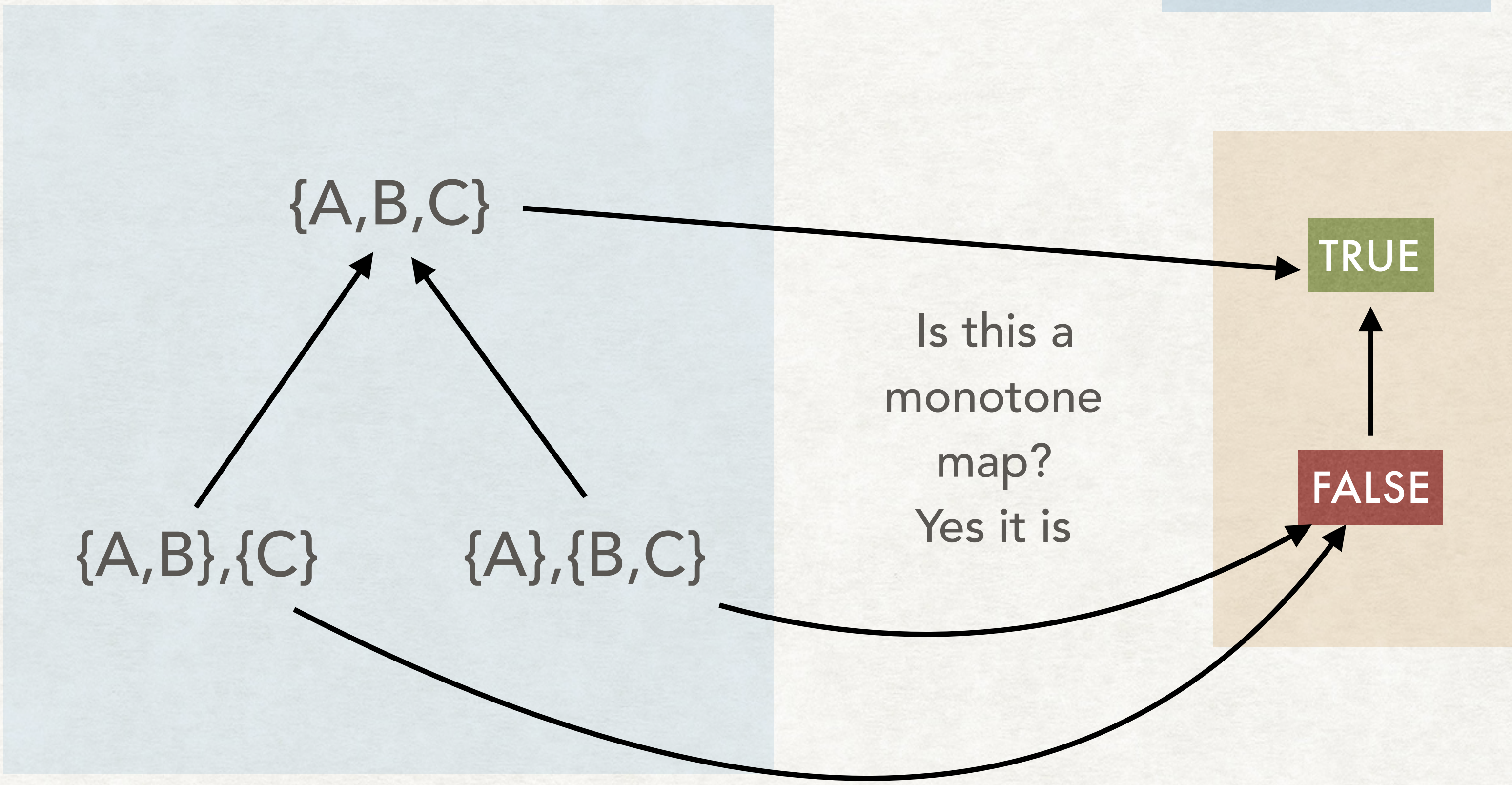
Observation → Answer



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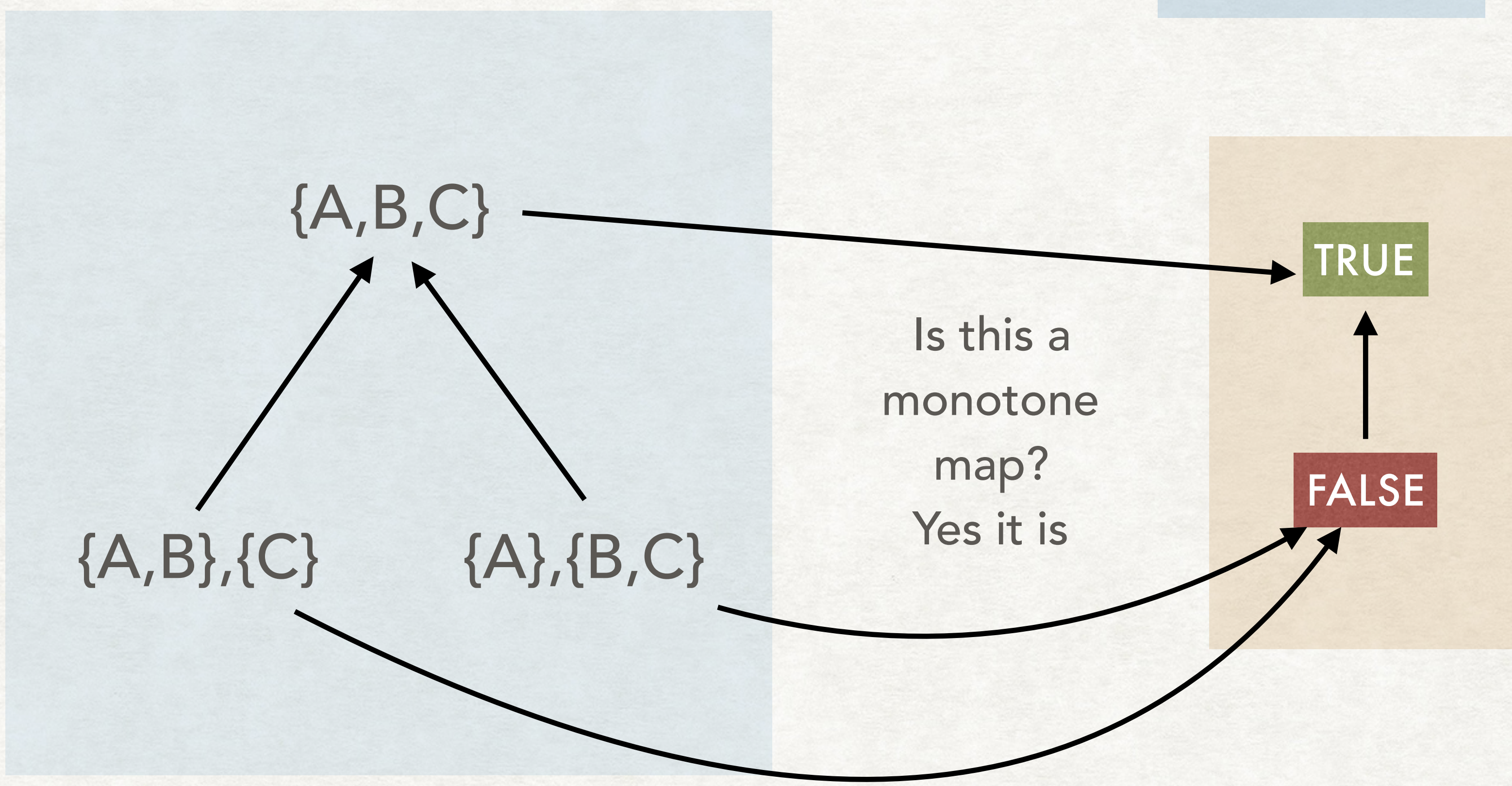

Does it preserve meets? Yes! Eg

f (meet of {A,B},{C} and {A,B,C})
= f({A,B},{C})
= false

meet of f({A,B},{C}) and f({A,B,C})
= meet of (false and true)
= false

3. View the sets as preorders

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COMING BACK TO THE EXAMPLE

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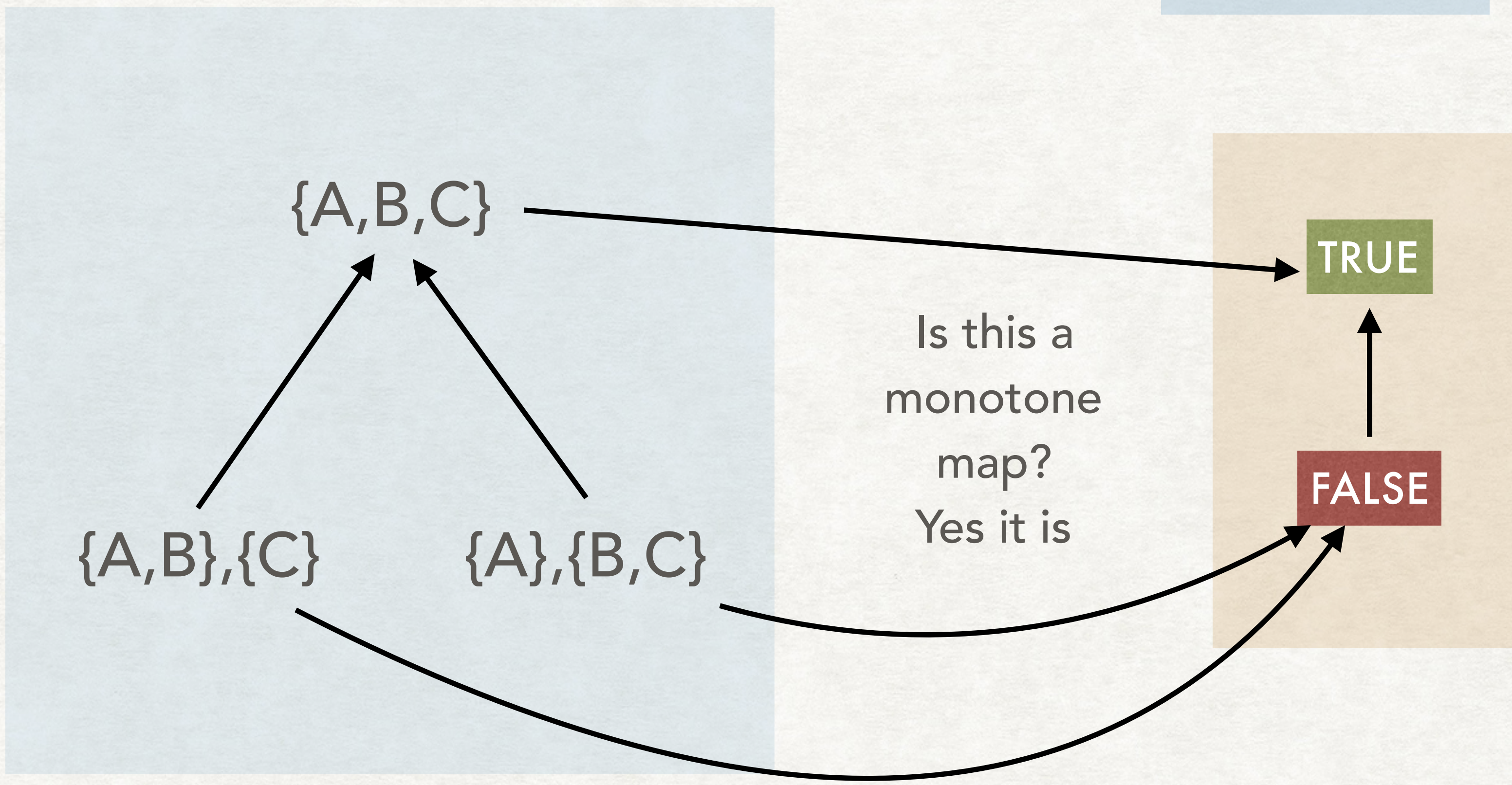
Observation → Answer



Does it preserve joins? No

f (join of {A,B},{C} and {A},{B,C})
= f({A,B,C})
= true

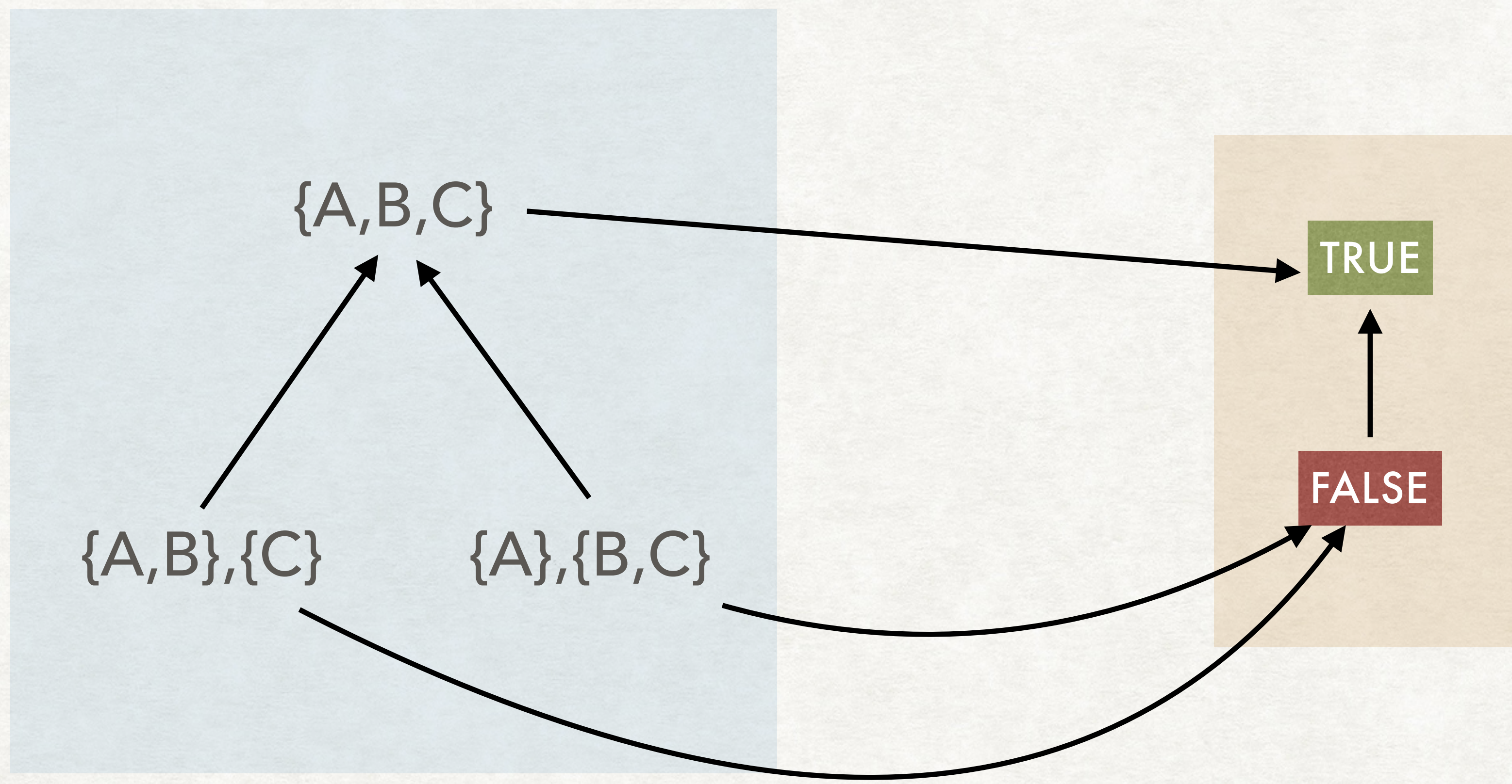
join of f({A,B},{C}) and f({A},{B,C})
= meet of (false and false)
= false

3. View the sets as preorders

2. Think of observations as partitions of the set {A,B,C}

GENERATIVE EFFECTS

Is A connected to C?



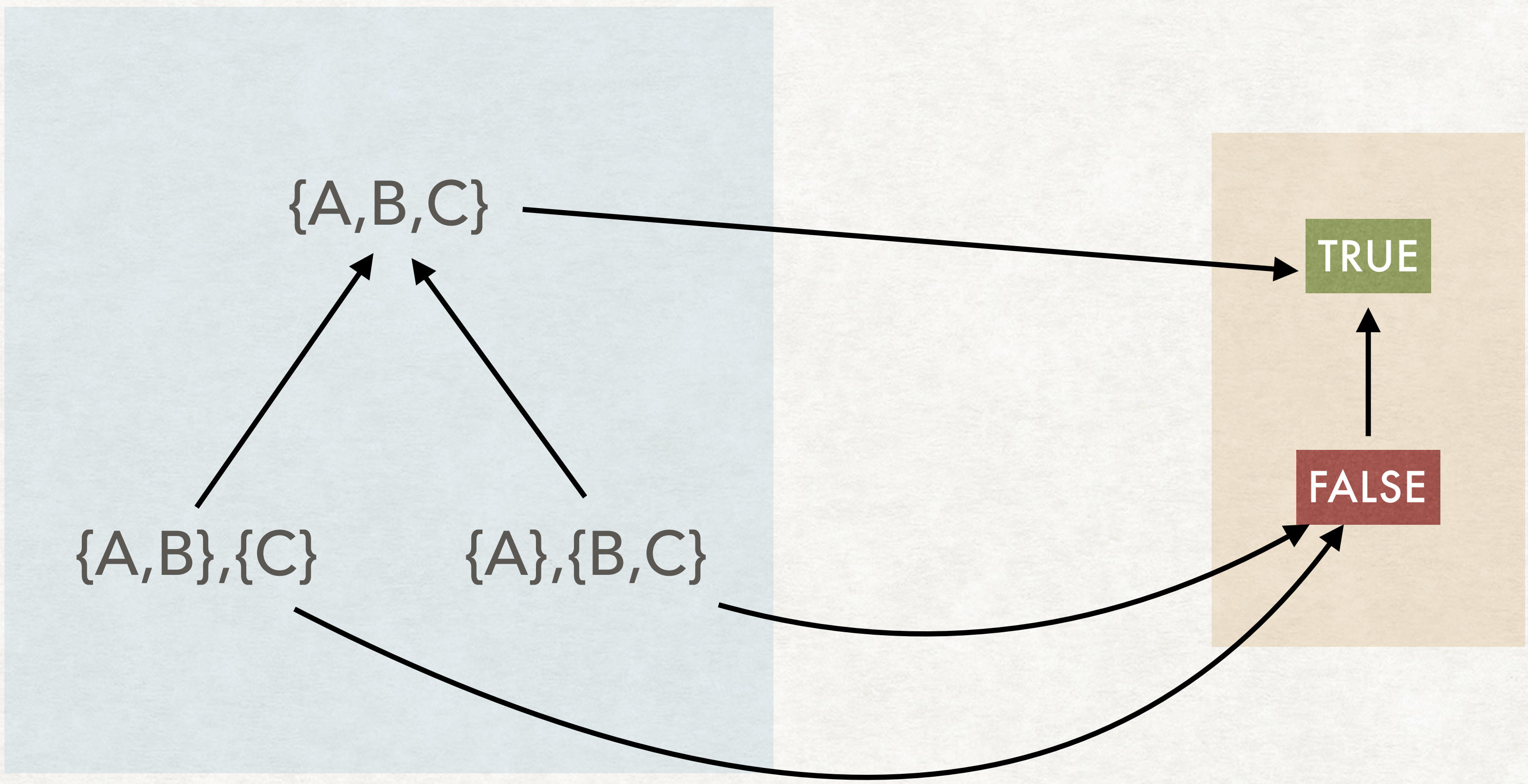
a monotone map $f : P \to Q$ has a generative effect if there exist elements $a, b \in P$ such that

$$f(a) \vee f(b) \neq f(a \vee b).$$

GENERATIVE EFFECTS

Is A connected to C?

P = features



Q = observations of features

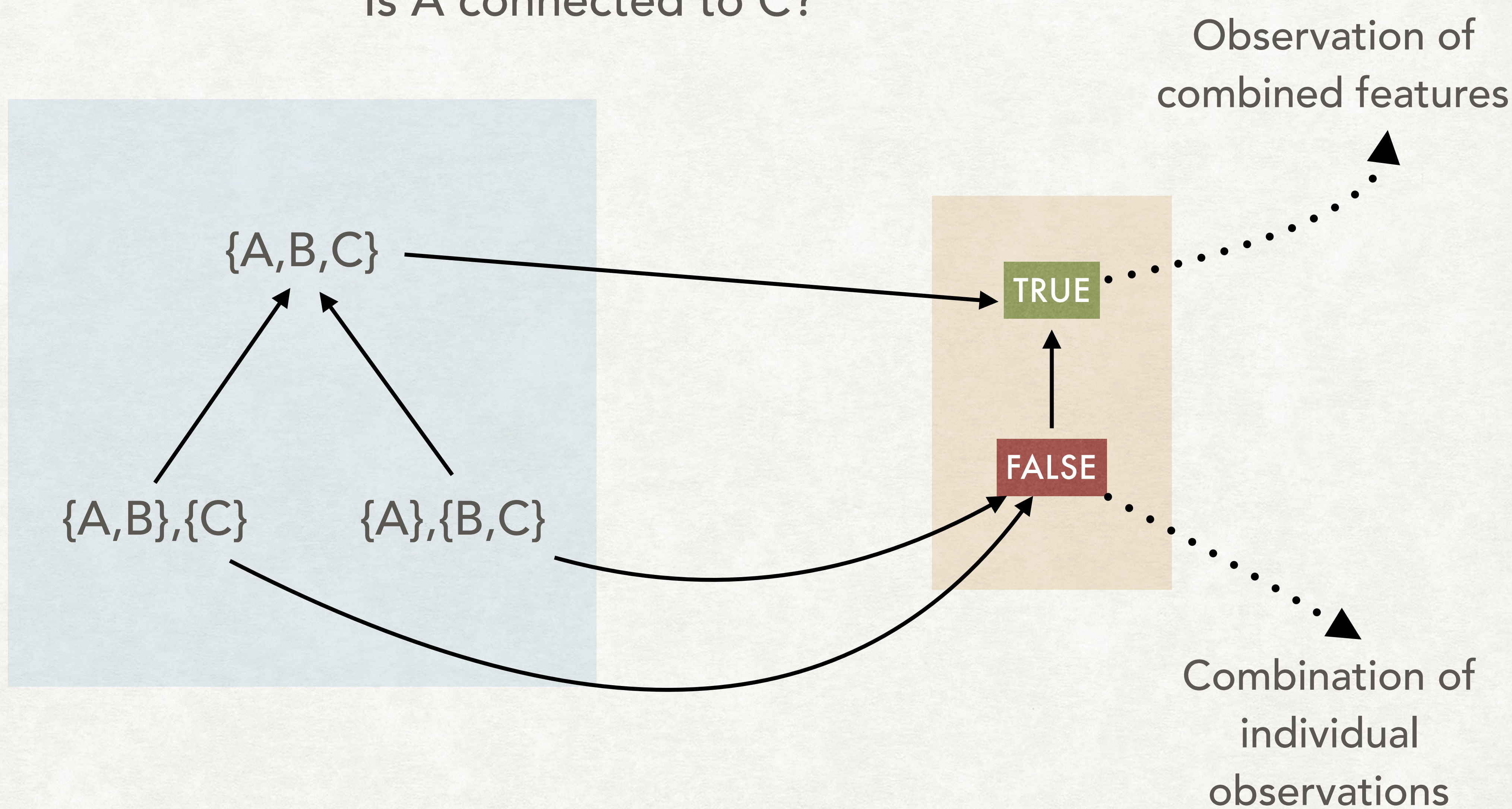
a monotone map f : P → Q has a generative effect if there exist elements a, b ∈ P such that

$f(a) \vee f(b) \neq f(a \vee b)$.

GENERATIVE EFFECTS

Is A connected to C?

P = features



a monotone map $f : P \to Q$ has a generative effect if there exist elements $a, b \in P$ such that $f(a) \vee f(b) \neq f(a \vee b)$.

THANK YOU