GALOIS CONNECTIONS

INTRODUCTION TO CATEGORY THEORY

QUICK REVIEW: PREORDERS, MEET AND JOIN

A = {1, 2}

A and B are sets (collections of unique objects)

Items in a set are called *elements*

B = {3, 4}

You might ask: what can I do with elements of a set?

QUICK REVIEW: PREORDERS, MEET AND JOIN

A = {1, 2}

B = {3, 4}

A and B are sets (collections of unique objects)

Items in a set are called *elements*

You might ask: what can I do with elements of a set?

I CAN COMPARE TWO ELEMENTS

CONCEPT OF A IS "LESS" THAN B

I CAN MAKE TWO ELEMENTS INTERACT TO CREATE A NEW ELEMENT

ANY OTHER IDEAS?

CONCEPT OF A IS "LESS" THAN B

WE CAN DEFINE THIS OURSELVES

Can ask questions like:

What is the largest element smaller than {x,y,z}? → MEET

What is the smallest element larger than {a,b,c}? → JOIN

QUICK REVIEW: PREORDERS, MEET AND JOIN

A = {1, 2}

B = {1.4, 2.2}

I CAN COMPARE TWO ELEMENTS

CONCEPT OF A IS “LESS” THAN B

WE CAN DEFINE THIS OURSELVES

A = 1 ≤ 2

B = 1.4 ≤ 2.2

… also remember that a ≤ a for every element a of a set

Thus 1 ≤ 1, 2 ≤ 2 and so on

REPLACE ≤ WITH →

A = $1 \rightarrow 2$

B = $1.4 \rightarrow 2.2$

Interpret this as:

WE HAVE TWO SETS A AND B.

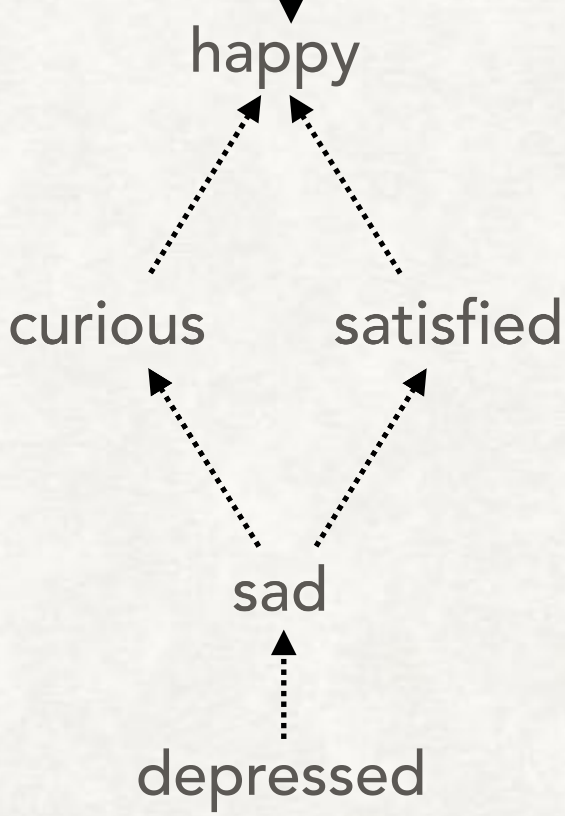
WE KNOW WHICH ELEMENTS ARE SMALLER THAN OR EQUAL TO WHICH OTHER ELEMENTS

QUICK REVIEW: PREORDERS, MEET AND JOIN

Emotions =
{ depressed, sad, curious, satisfied, happy}

I CAN COMPARE TWO ELEMENTS

A≤B IF I PREFER B TO A
A≤B 如果 B 比 A 好



Can ask questions like:

What is the largest element smaller than {x,y,z}?
What is the smallest element larger than {a,b,c}?

MEET OF {CURIOUS, SATISFIED} = SAD
JOIN OF {CURIOUS, SATISFIED} = HAPPY

QUICK REVIEW: PREORDERS, MEET AND JOIN

Emotions =

{ depressed, sad, curious, satisfied, happy}

I CAN COMPARE TWO ELEMENTS

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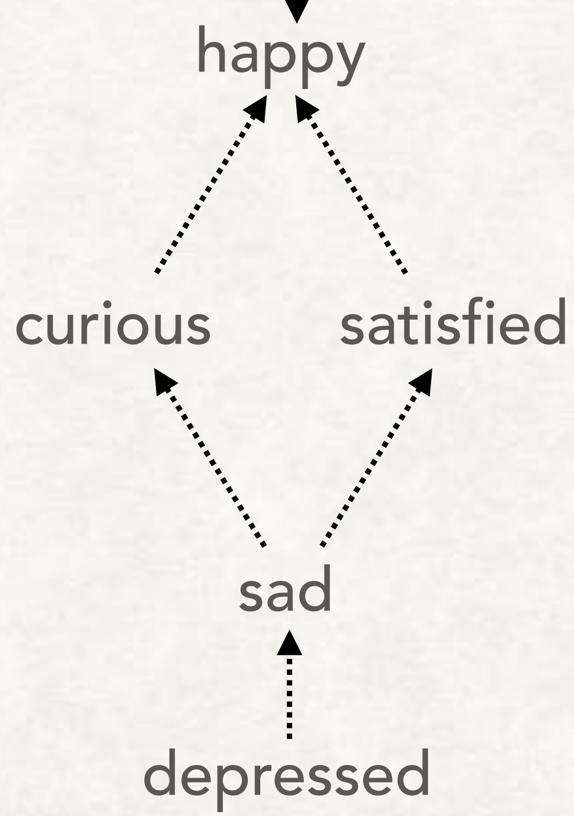

Remember:

sad ≤ sad, curious ≤ curious and so on

Can ask questions like:

What is the largest element smaller than {x,y,z}?

What is the smallest element larger than {a,b,c}?

MEET OF {CURIOUS, SATISFIED} = SAD

JOIN OF {CURIOUS, SATISFIED} = HAPPY

EXAMPLE #1

Preorder A $1 \longrightarrow 2$ Find nearest neighbor in preorder B

Preorder B $1.4 \longrightarrow 2.2$ Find nearest neighbor in preorder A

LET'S LOOK AT AN EXAMPLE FIRST

$$\begin{tikzcd} \text{Preorder A} & 1 \arrow[r] \arrow[d, bend right] & 2 \arrow[d, bend right] \\ \text{Preorder B} & 1.4 \arrow[r] \arrow[u, bend right] & 2.2 \arrow[u, bend right] \end{tikzcd}$$

Find nearest neighbor
in preorder B

Find nearest neighbor
in preorder A

$$l(1) = 1.4 \qquad r(1.4) = 1$$

$$l(2) = 2.2 \qquad r(2.2) = 2$$

LET'S LOOK AT AN EXAMPLE FIRST

Preorder A

Preorder B

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, bend right] & 2 \arrow[d, bend right] \\ 1.4 \arrow[r] \arrow[u, bend right] & 2.2 \arrow[u, bend right] \end{tikzcd}$$

Find nearest neighbor in preorder B

Find nearest neighbor in preorder A

$l(1) = 1.4$

$l(2) = 2.2$

$r(1.4) = 1$

$r(2.2) = 2$

l and r are inverses of each other

$$r(\,l(x)\,) = x$$

$$l(\,r(y)\,) = y$$

LET'S LOOK AT AN EXAMPLE FIRST

Preorder A $1 \longrightarrow 2$

Preorder B $1.4 \longrightarrow 2.2$

Preorder A $1 \longrightarrow 2$

Find nearest neighbor in preorder B

Preorder B $1.8 \longrightarrow 2.2$

Find nearest neighbor in preorder A

LET'S LOOK AT AN EXAMPLE FIRST

Preorder A $1 \longrightarrow 2$

Preorder B $1.6 \longrightarrow 2.2$

Preorder A $1 \longrightarrow 2$

Find nearest neighbor in preorder B

Preorder B $1.8 \longrightarrow 2.2$

Find nearest neighbor in preorder A

$l(1) = 1.8$

$l(2) = ?$

$r(1.8) = 2$

$r(2.2) = 2$

Note that the function r is not invertible here

LET'S LOOK AT AN EXAMPLE FIRST

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, bend right, color=magenta, "l"'] & 2 \arrow[dl, bend right, color=magenta] \\ 1.8 \arrow[r] \arrow[ur, bend right, color=blue] & 2.2 \arrow[u, bend right, color=blue, "r"'] \end{tikzcd}$$

$l(1) = 1.8$ $\qquad$ $r(1.8) = 2$

$\underline{l(2) = 1.8}$ $\qquad$ $r(2.2) = 2$

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, bend right, color=magenta, "l"'] & 2 \arrow[d, bend right, color=magenta] \\ 1.8 \arrow[r] \arrow[ur, bend right, color=blue] & 2.2 \arrow[u, bend right, color=blue, "r"'] \end{tikzcd}$$

$l(1) = 1.8$ $\qquad$ $r(1.8) = 2$

$\underline{l(2) = 2.2}$ $\qquad$ $r(2.2) = 2$

WHICH ONE IS A BETTER CHOICE?

LET'S LOOK AT AN EXAMPLE FIRST

$$\downarrow l \qquad \begin{tikzcd} 1 \arrow[r] \arrow[d, bend right] & 2 \arrow[dl] \\ 1.8 \arrow[r] \arrow[ur] & 2.2 \arrow[u, bend right] \end{tikzcd} \qquad r \uparrow$$

$l(1) = 1.8$ $\qquad$ $r(1.8) = 2$

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$$\downarrow l \qquad \begin{tikzcd} 1 \arrow[r] \arrow[d, bend right] & 2 \arrow[d] \\ 1.8 \arrow[r] \arrow[ur] & 2.2 \arrow[u, bend right] \end{tikzcd} \qquad r \uparrow$$

$l(1) = 1.8$ $\qquad$ $r(1.8) = 2$

$\underline{l(2) = 2.2}$ $\qquad$ $r(2.2) = 2$

WHICH ONE IS A BETTER CHOICE?

FUNCTIONS L AND R SHOULD PRESERVE SOMETHING FROM ISOMORPHIC MAPS

LET'S LOOK AT AN EXAMPLE FIRST

$$\downarrow l \qquad \begin{tikzcd} 1 \arrow[r] \arrow[d, bend right] & 2 \arrow[dl] \\ 1.8 \arrow[r] \arrow[ur, bend right] & 2.2 \arrow[u, bend right] \end{tikzcd} \qquad r \uparrow$$

$l(1) = 1.8$ $\qquad$ $r(1.8) = 2$

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$l(1) = 1.8$ $\qquad$ $r(1.8) = 2$

$\underline{l(2) = 2.2}$ $\qquad$ $r(2.2) = 2$

WHICH ONE IS A BETTER CHOICE?

FUNCTIONS L AND R SHOULD PRESERVE SOMETHING FROM ISOMORPHIC MAPS

$L(X) \leq Y$
$\Leftrightarrow$
$X \leq R(Y)$

LET'S LOOK AT AN EXAMPLE FIRST

$$
\begin{tikzcd}
1 \arrow[r] \arrow[d, bend right, "l"'] & 2 \arrow[dl, bend right, "l"'] \\
1.8 \arrow[r] \arrow[ur, bend right, "r"'] & 2.2 \arrow[u, bend right, "r"']
\end{tikzcd}
$$

$l(1) = 1.8$ $\quad$ $r(1.8) = 2$

$\underline{l(2) = 1.8}$ $\quad$ $r(2.2) = 2$

$2 \leq g(1.8) \Leftrightarrow f(2) \leq 1.8$

$$
\begin{tikzcd}
1 \arrow[r] \arrow[d, bend right, "l"'] & 2 \arrow[d, bend right, "l"'] \\
1.8 \arrow[r] \arrow[ur, bend right, "r"'] & 2.2 \arrow[u, bend right, "r"']
\end{tikzcd}
$$

$l(1) = 1.8$ $\quad$ $r(1.8) = 2$

$\underline{l(2) = 2.2}$ $\quad$ $r(2.2) = 2$

$2 \leq g(1.8)$ but $f(2) \;! \!\leq 1.8$?

WHICH ONE IS A BETTER CHOICE?

FUNCTIONS L AND R SHOULD PRESERVE SOMETHING FROM ISOMORPHIC MAPS

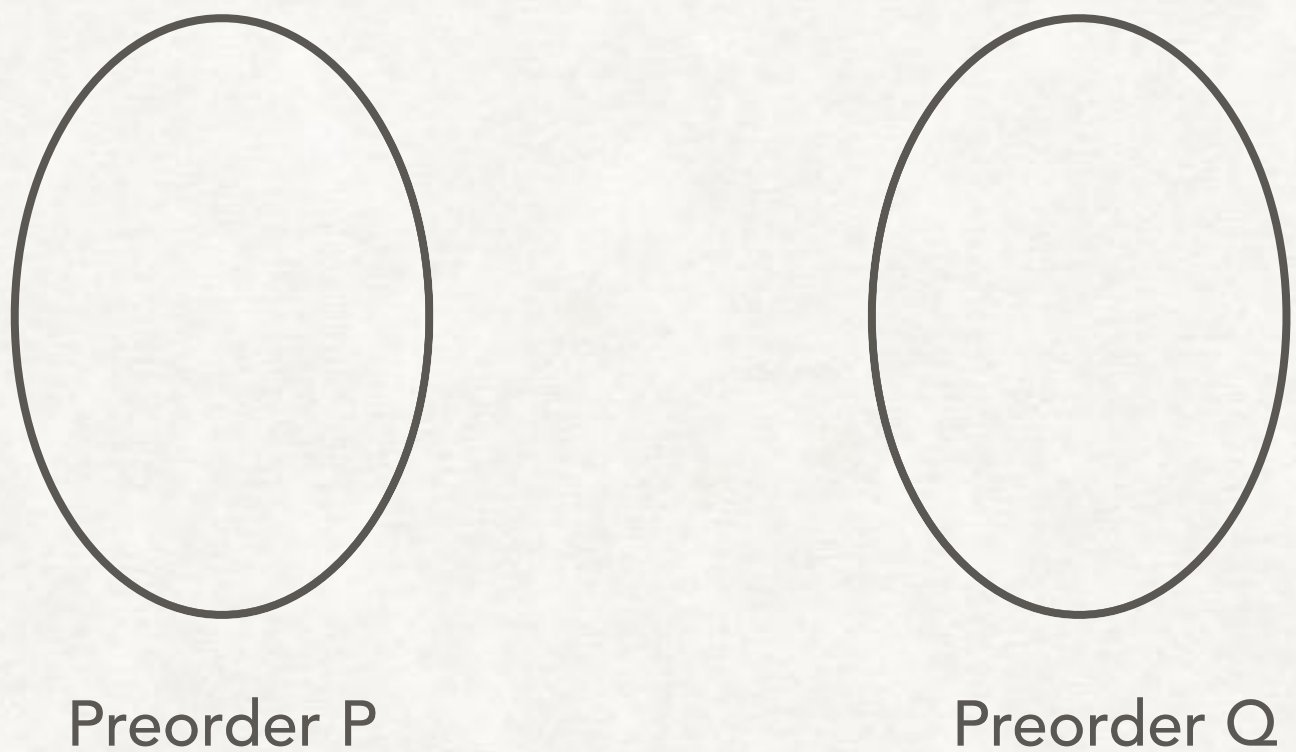
$$l(x) \leq y \\ \Leftrightarrow \\ x \leq r(y)$$

Aims to minimize disagreement

GALOIS CONNECTION

DEFINITION

Given preorders P and Q



GALOIS CONNECTION

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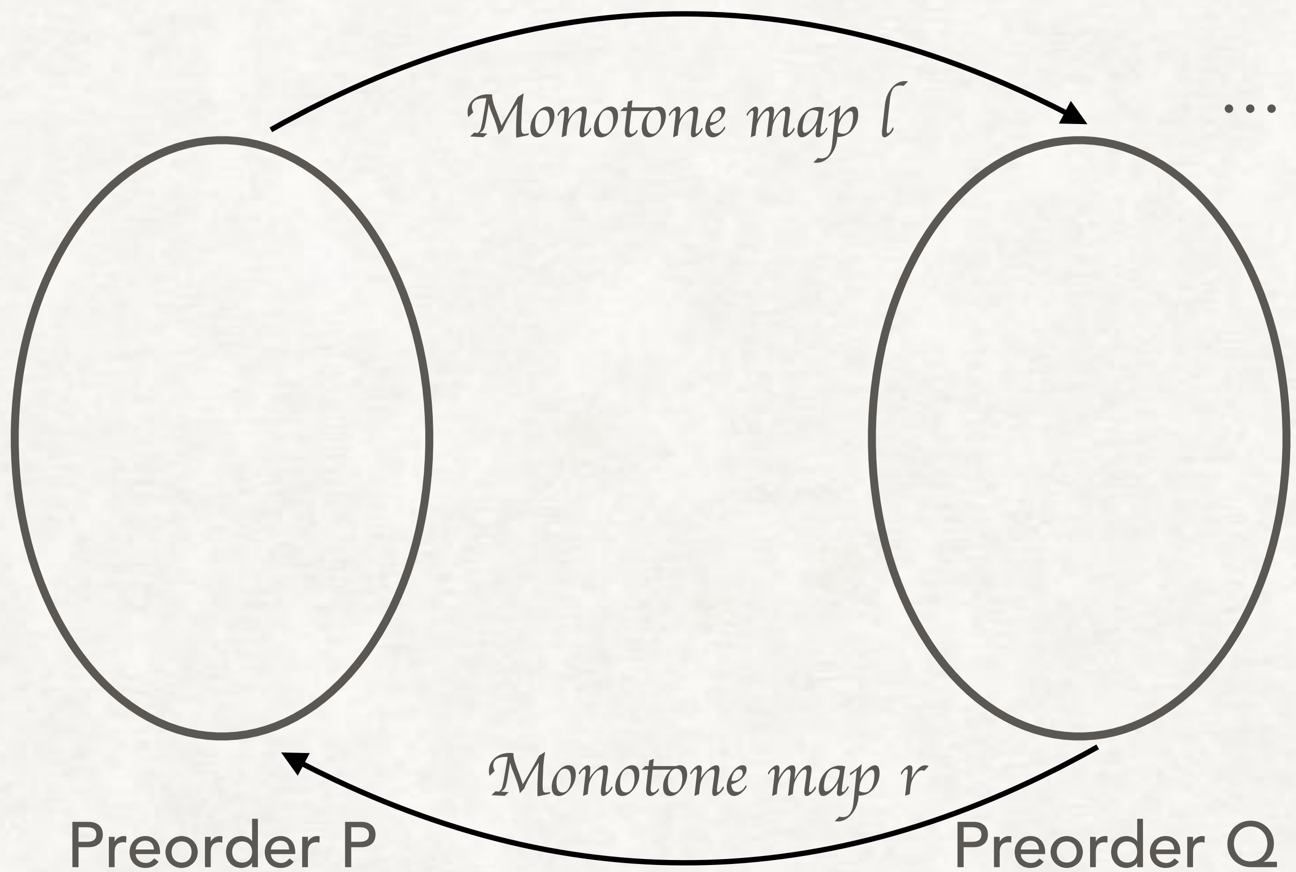

Given preorders P and Q

… and a pair of monotone maps
$l: P \to Q$ and $r: Q \to P$

GALOIS CONNECTION

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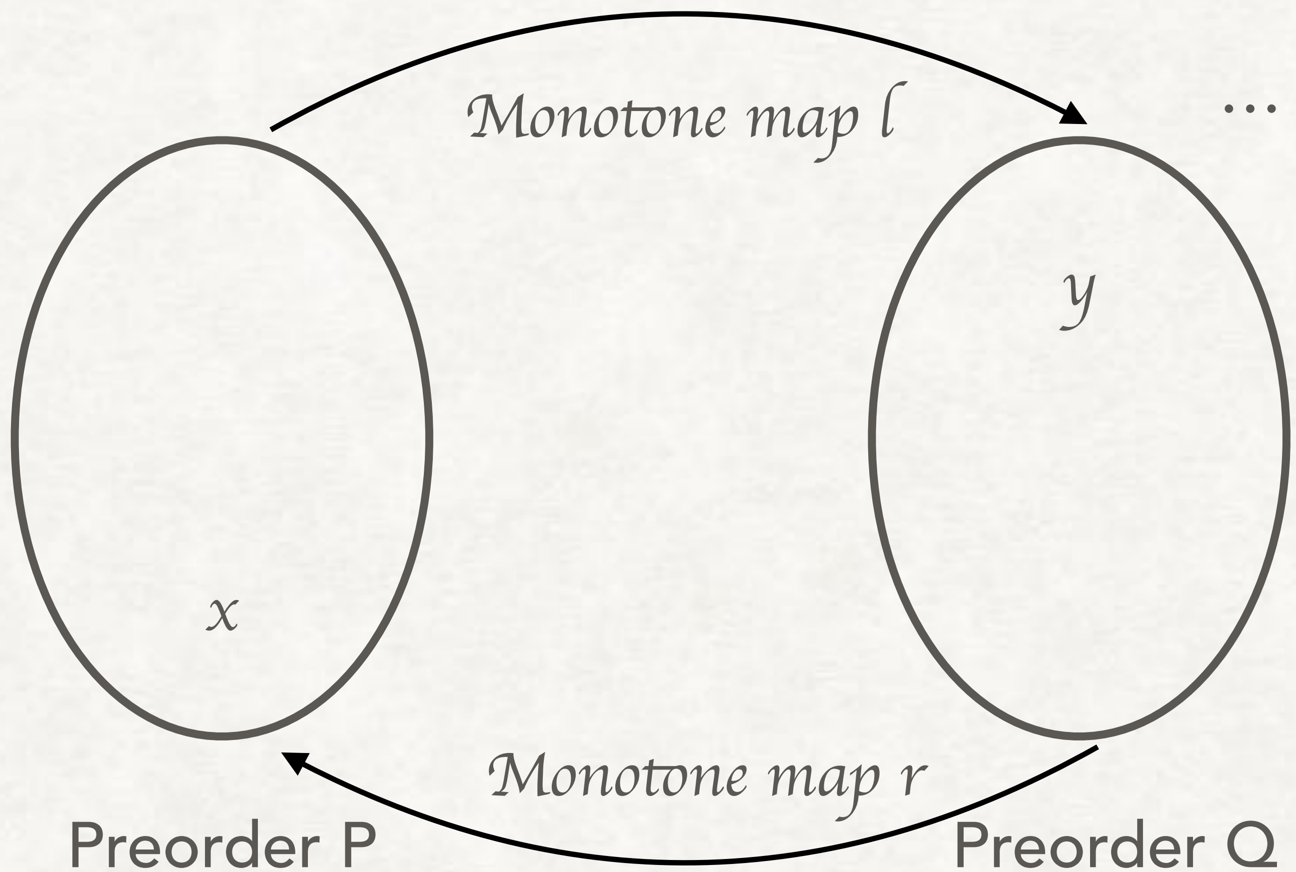

$$x \leq r(y) \Leftrightarrow l(x) \leq y$$

$$\text{for all } x \in P,\ y \in Q$$

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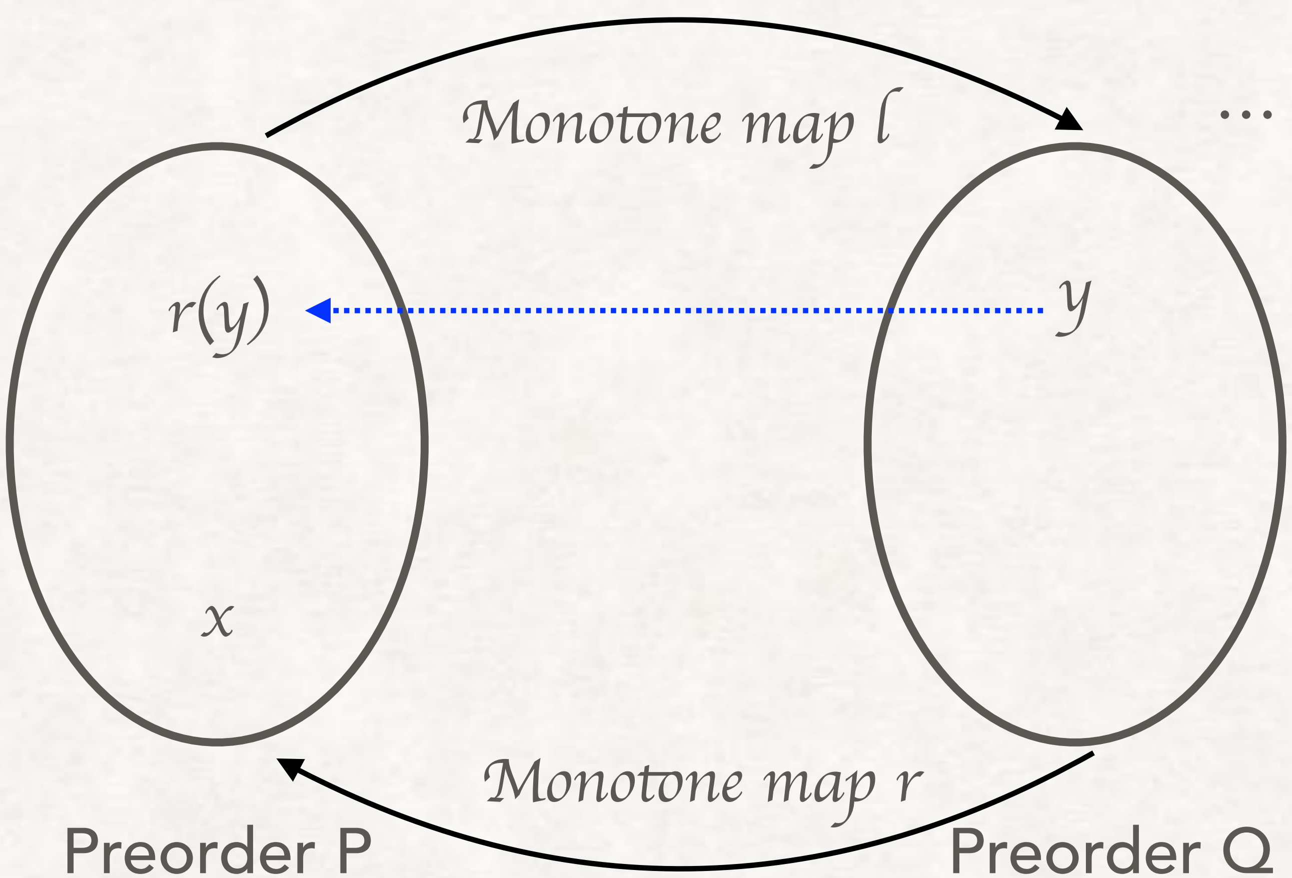

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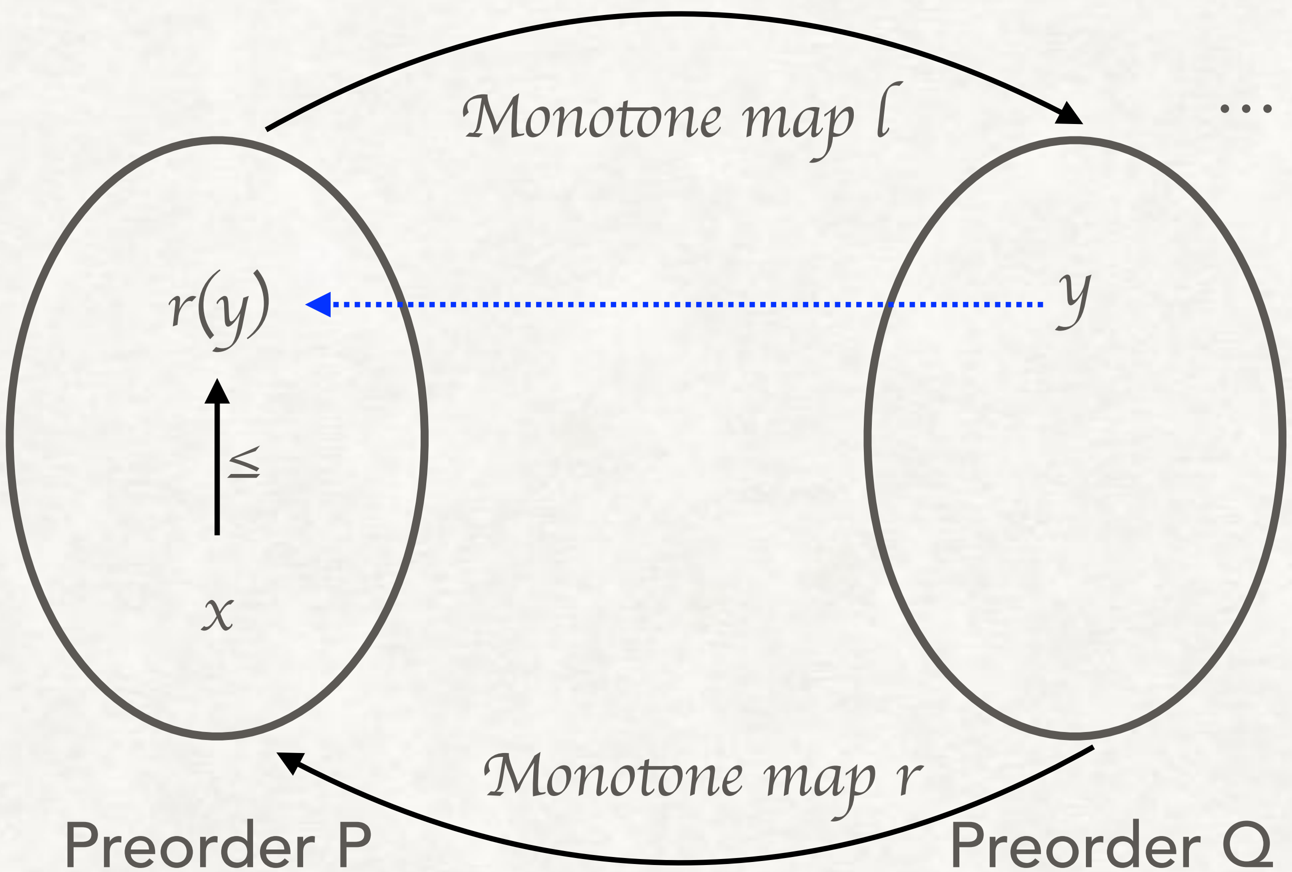
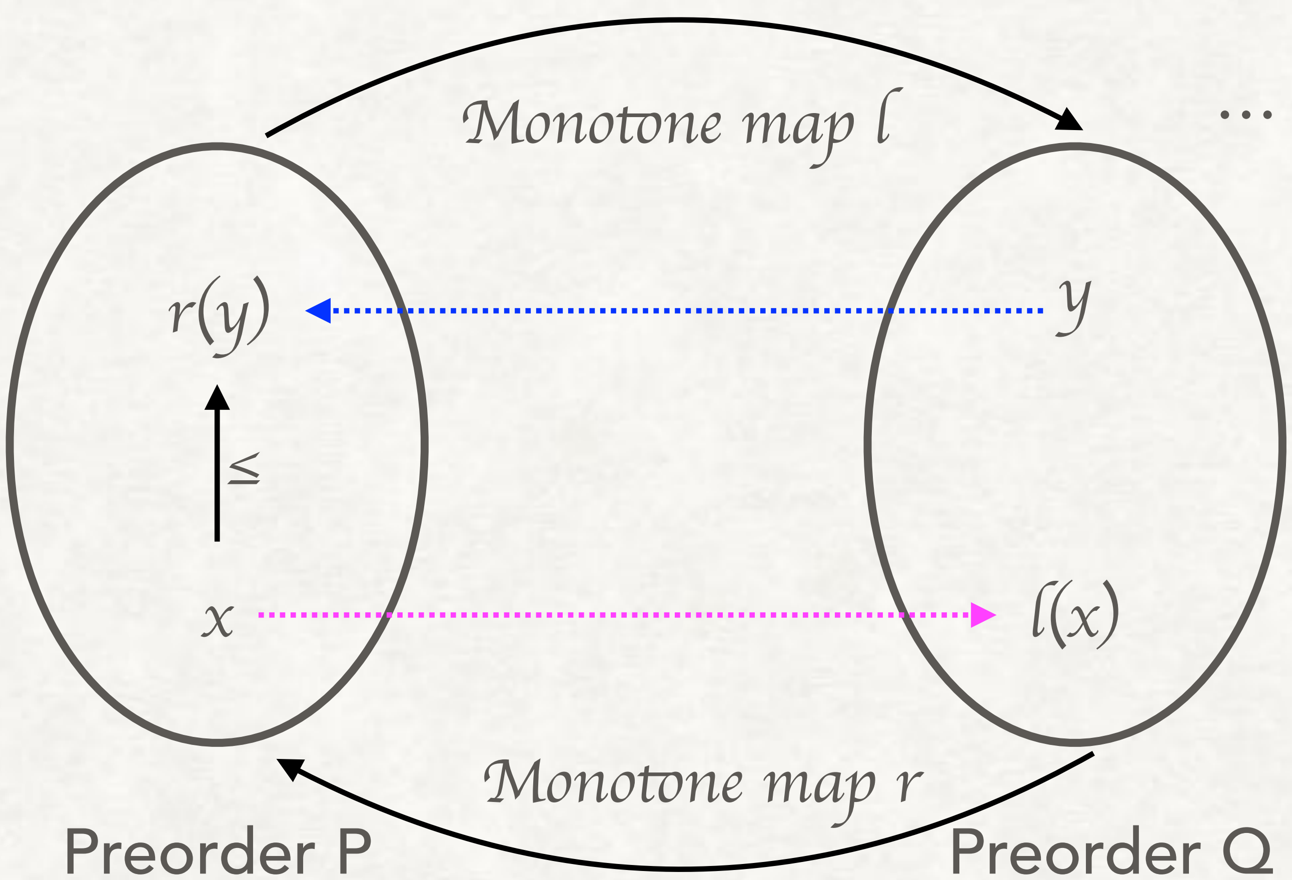


$$x \leq r(y) \Leftrightarrow l(x) \leq y$$

for all $x \in P, y \in Q$

GALOIS CONNECTION

DEFINITION



Given preorders P and Q

… and a pair of monotone maps
$l : P \to Q$ and $r : Q \to P$

$$x \leq r(y) \Leftrightarrow l(x) \leq y$$

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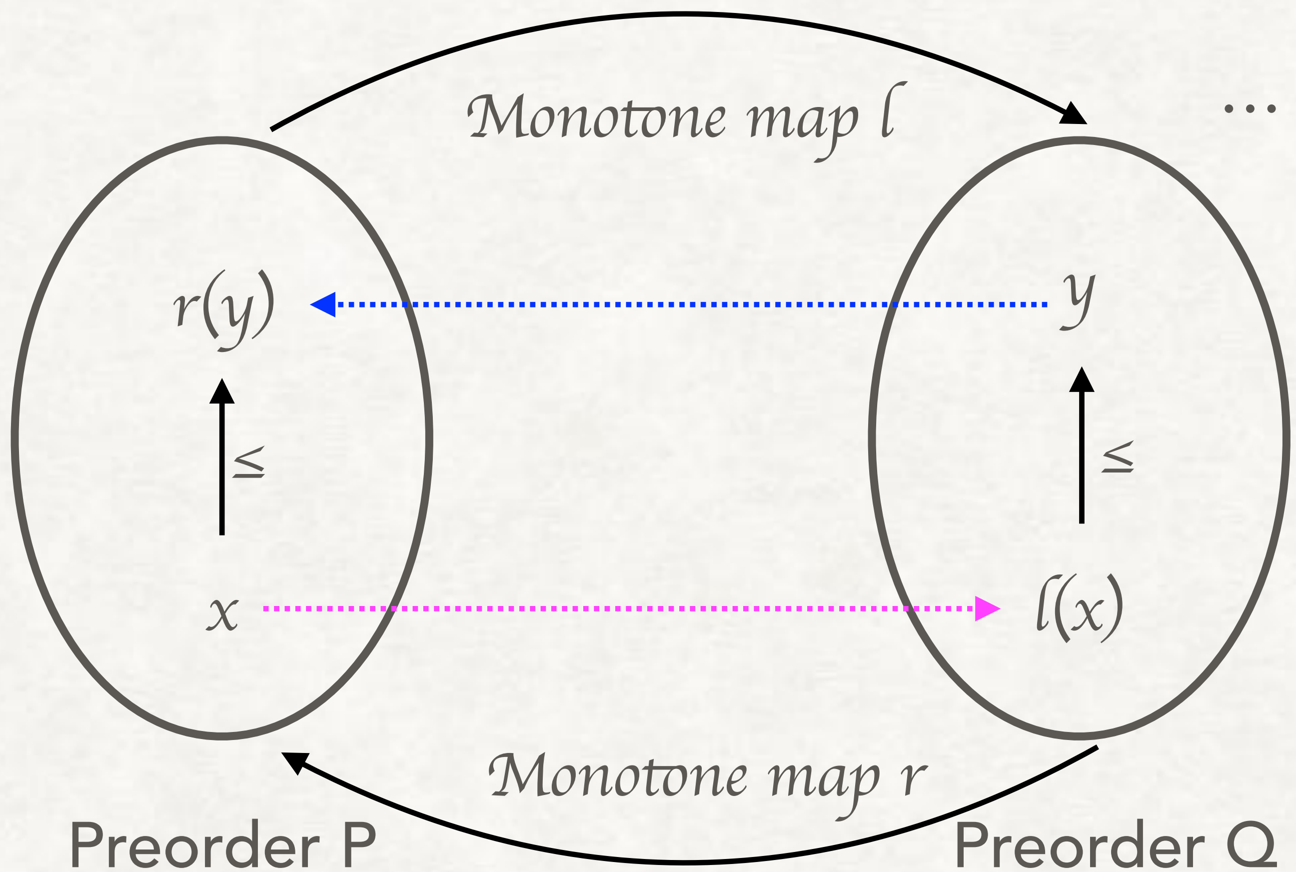

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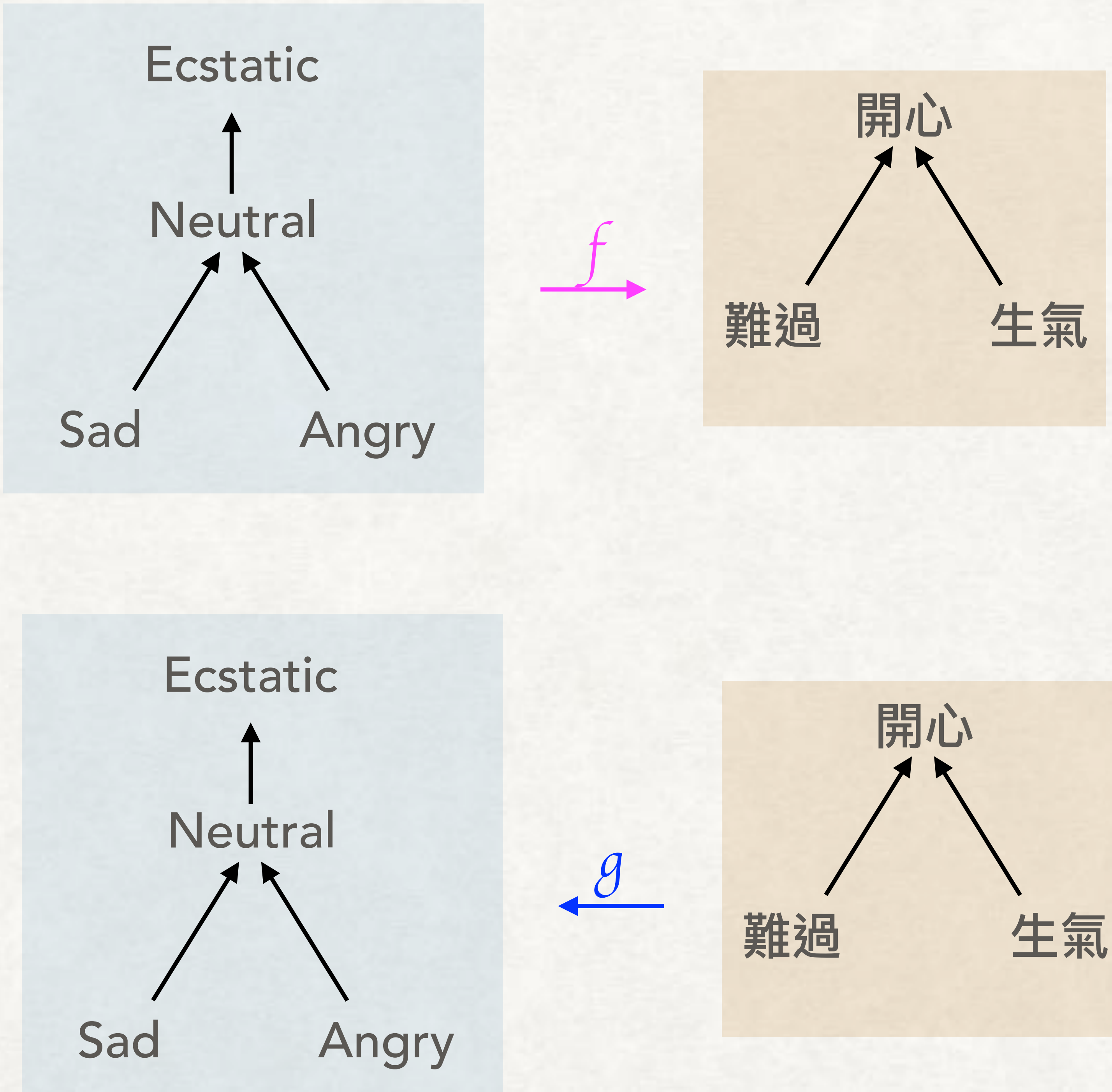
EXAMPLE #2: TRANSLATION



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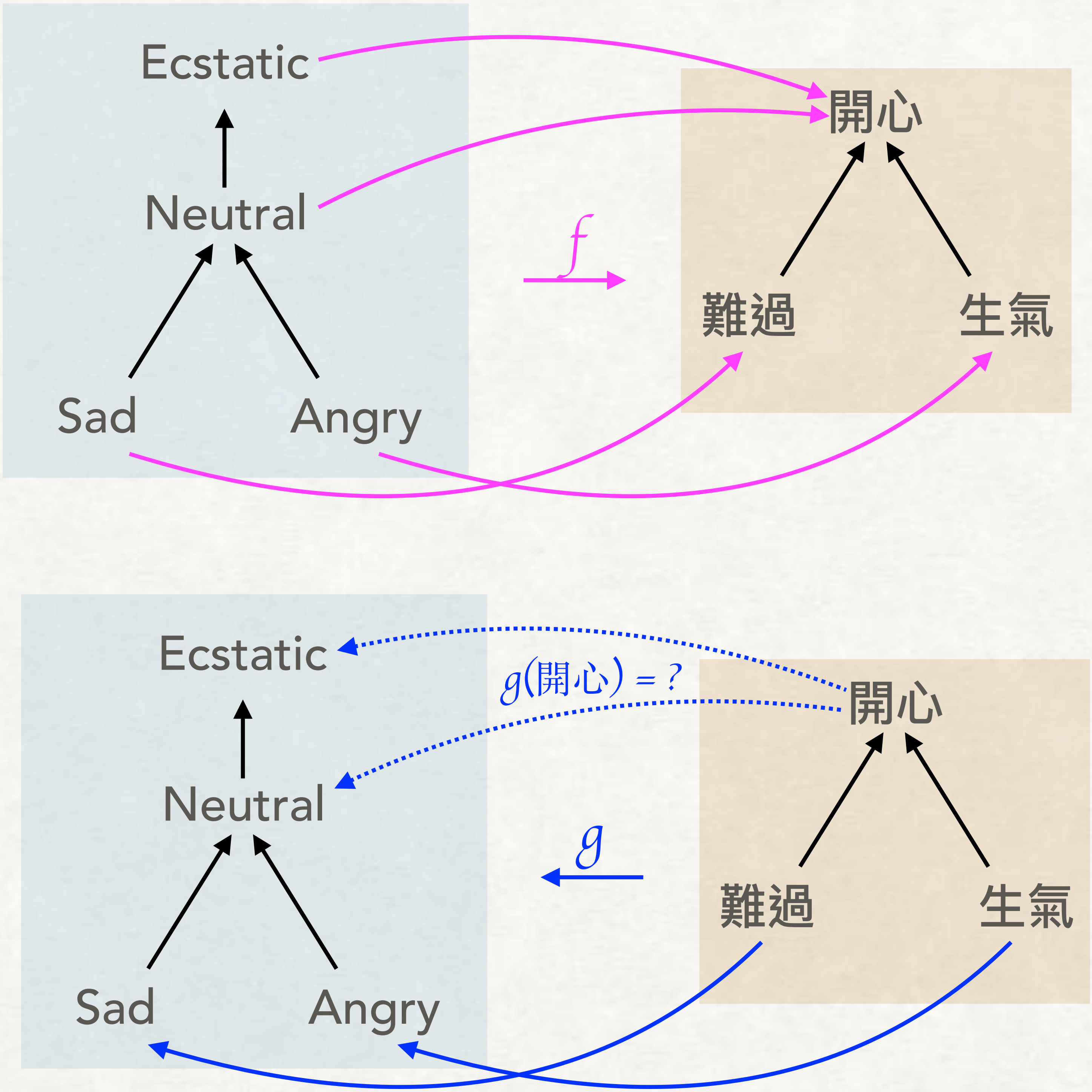

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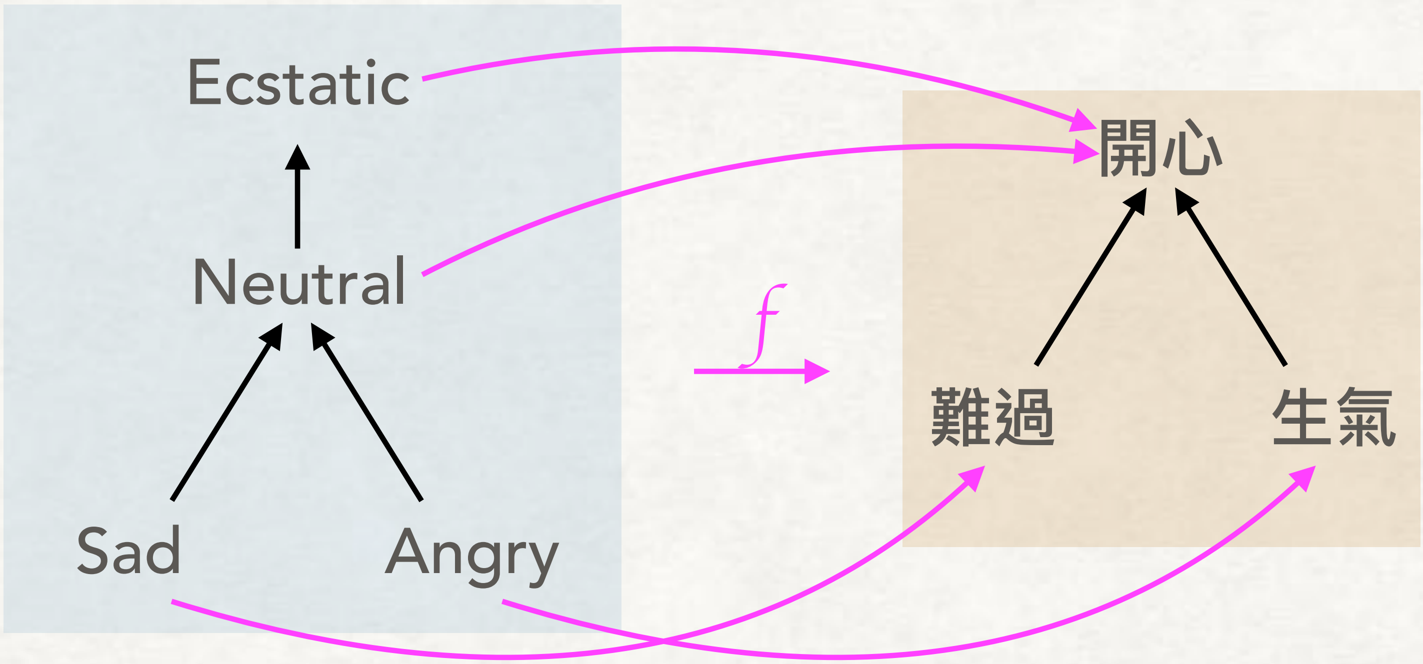

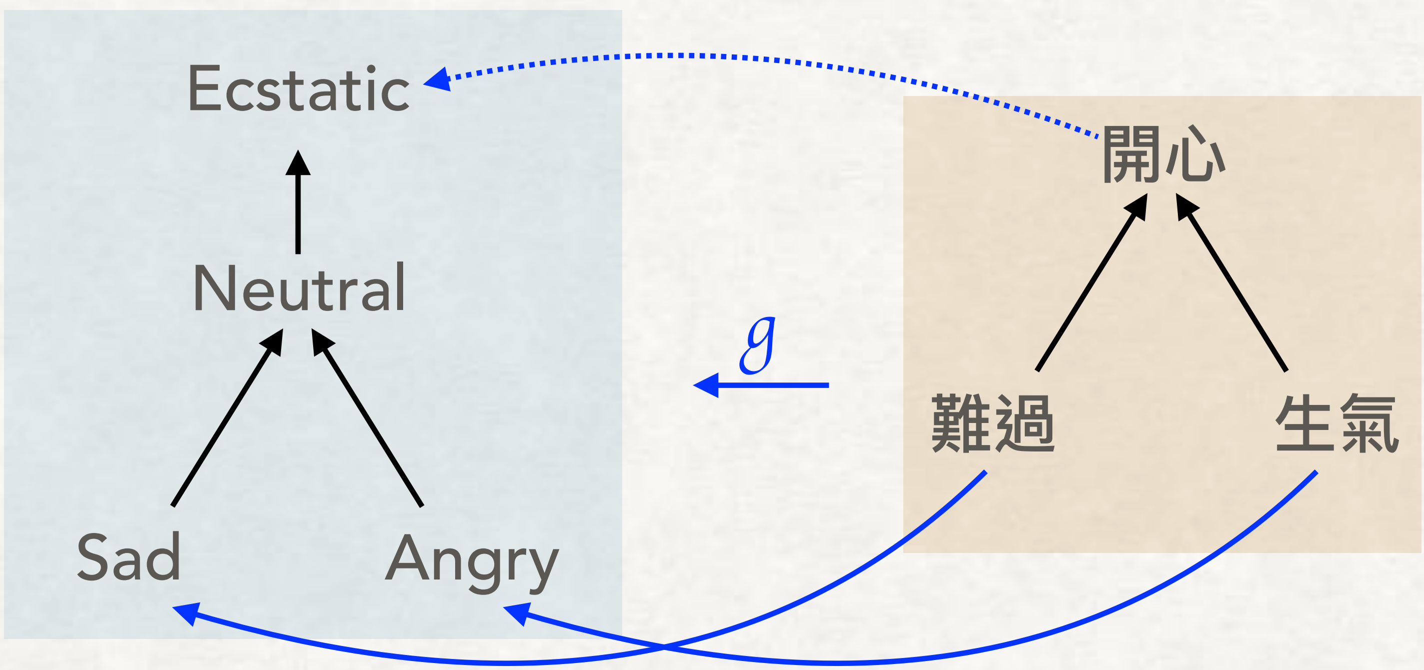

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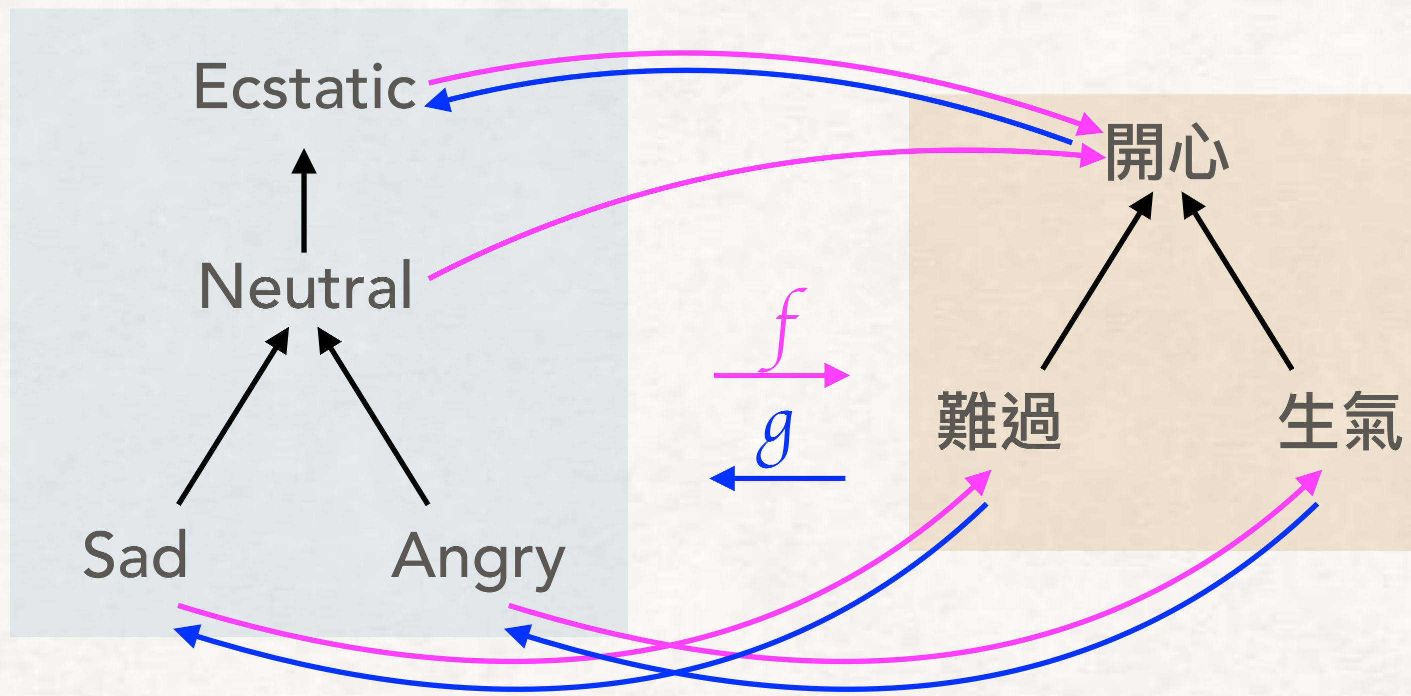


neutral ≤ *g*(開心) ⇔ *f*(*neutral*) ≤ 開心

Obvious for all other terms:

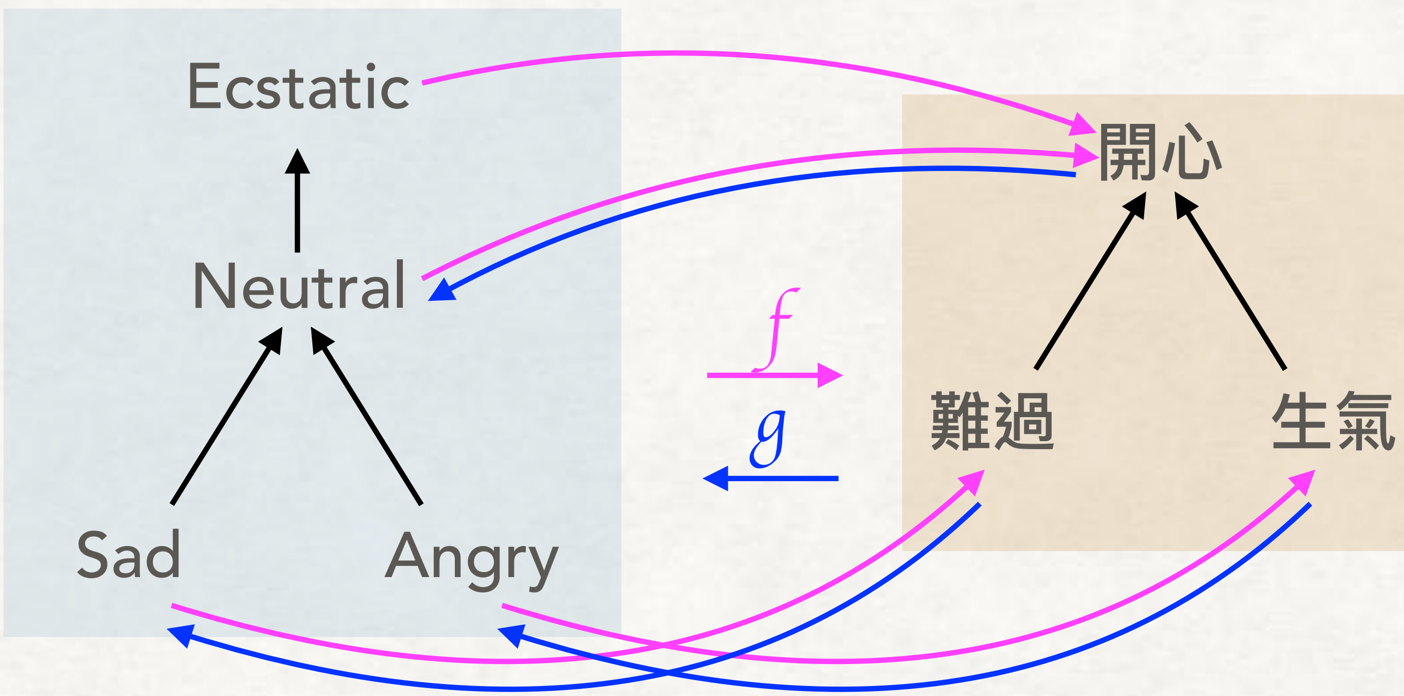
sad ≤ *g*(難過) ⇔ *f*(*sad*) ≤ 難過

… and so on

f: left adjoint

g: right adjoint

EXAMPLE #2: TRANSLATION



$g(\text{開心}) \leq ecstatic \Leftrightarrow \text{開心} \leq f(ecstatic)$

Obvious for all other terms:

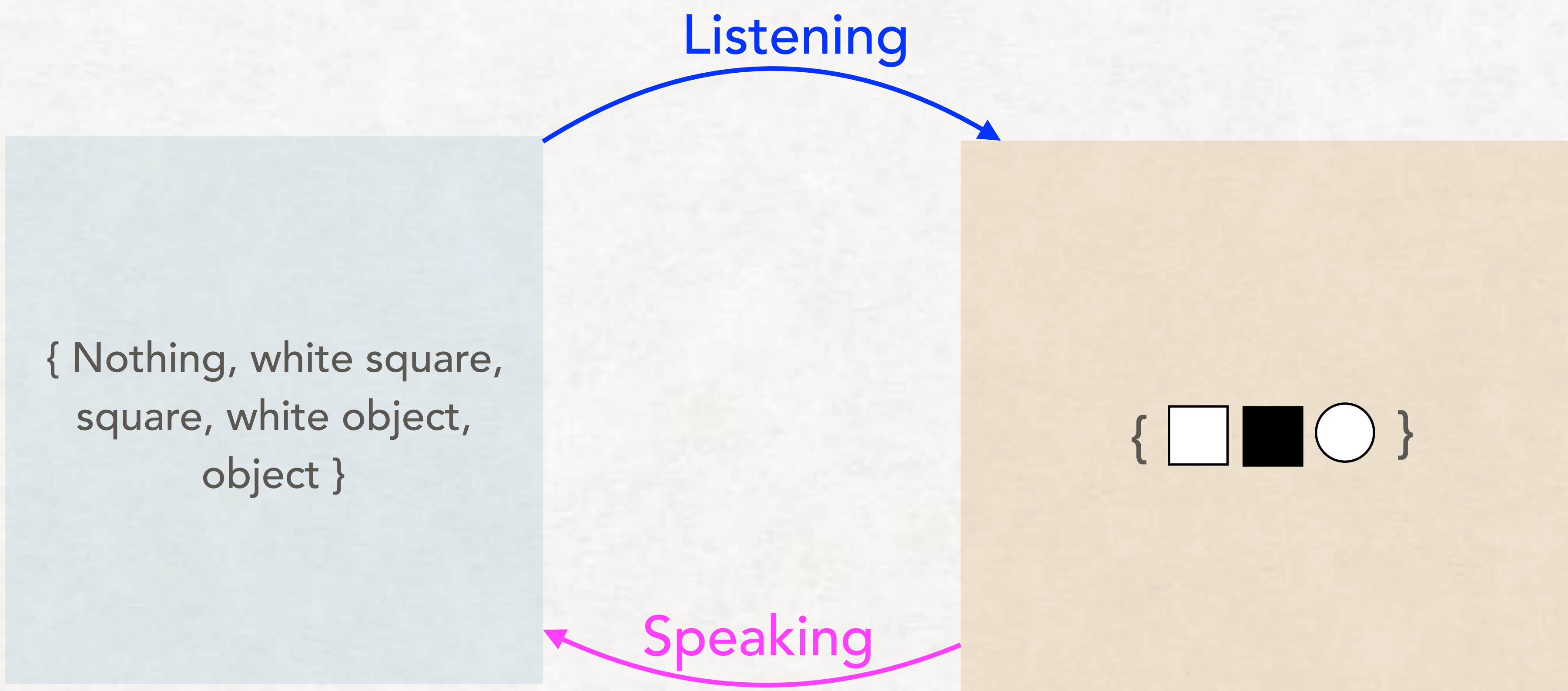
$g(\text{難過}) \leq sad \Leftrightarrow \text{難過} \leq f(sad)$

... and so on

f: right adjoint

g: left adjoint

EXAMPLE #3: COMMUNICATION



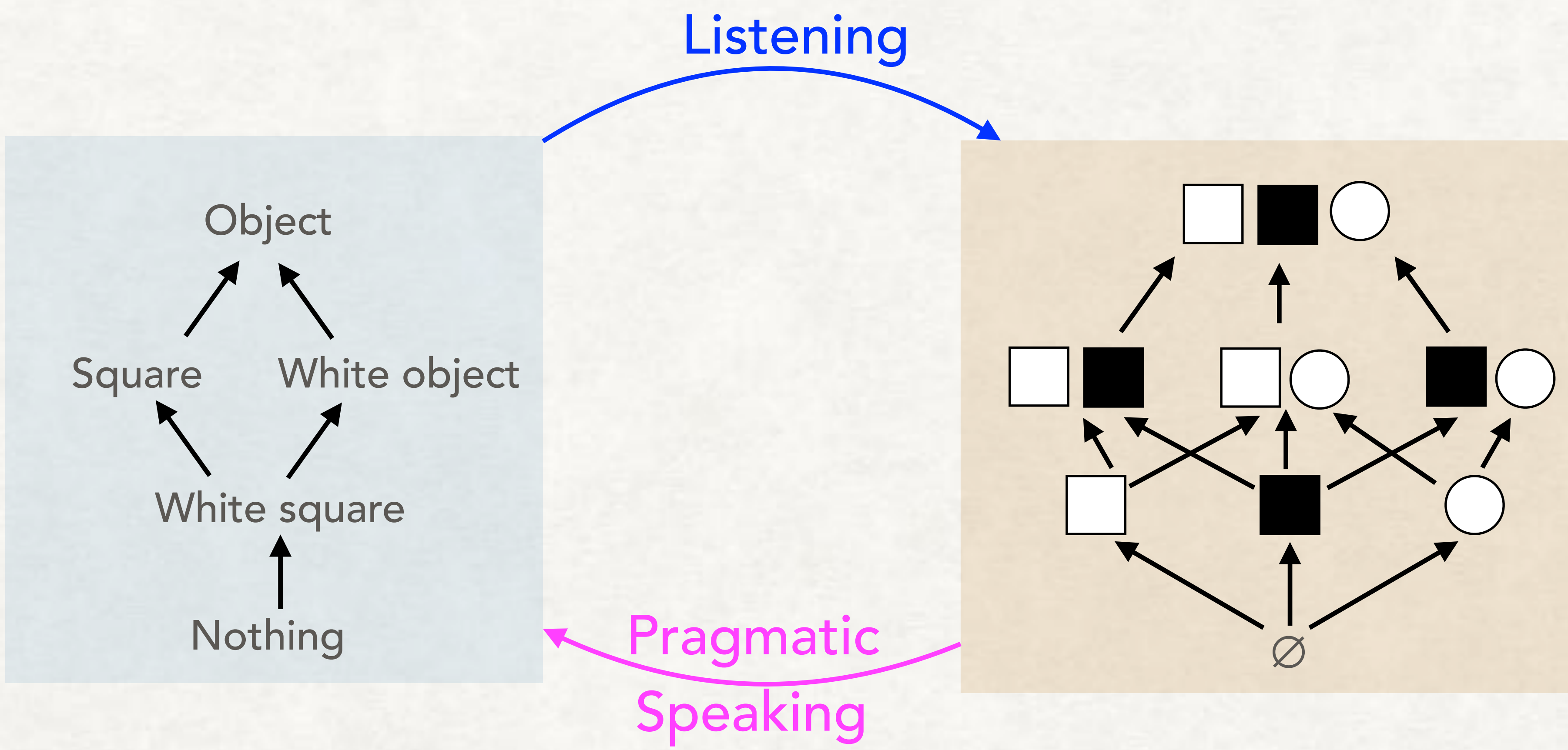
Language

Abstract concept: all possible subsets of □ ■ ○

Define ≤: $a \leq b$ if a implies b

Define ≤: $a \leq b$ if a is subset of b

EXAMPLE #3: COMMUNICATION



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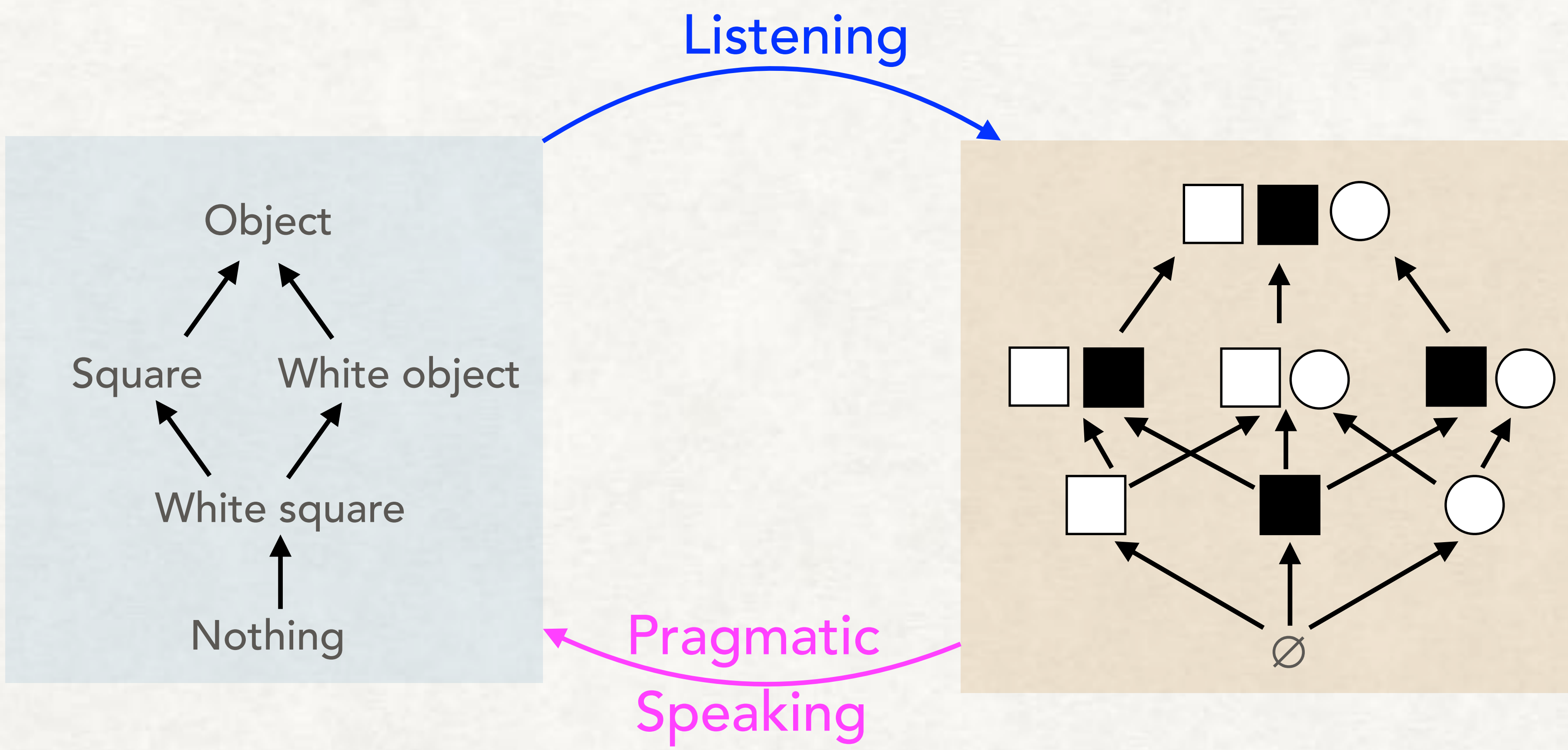


Language

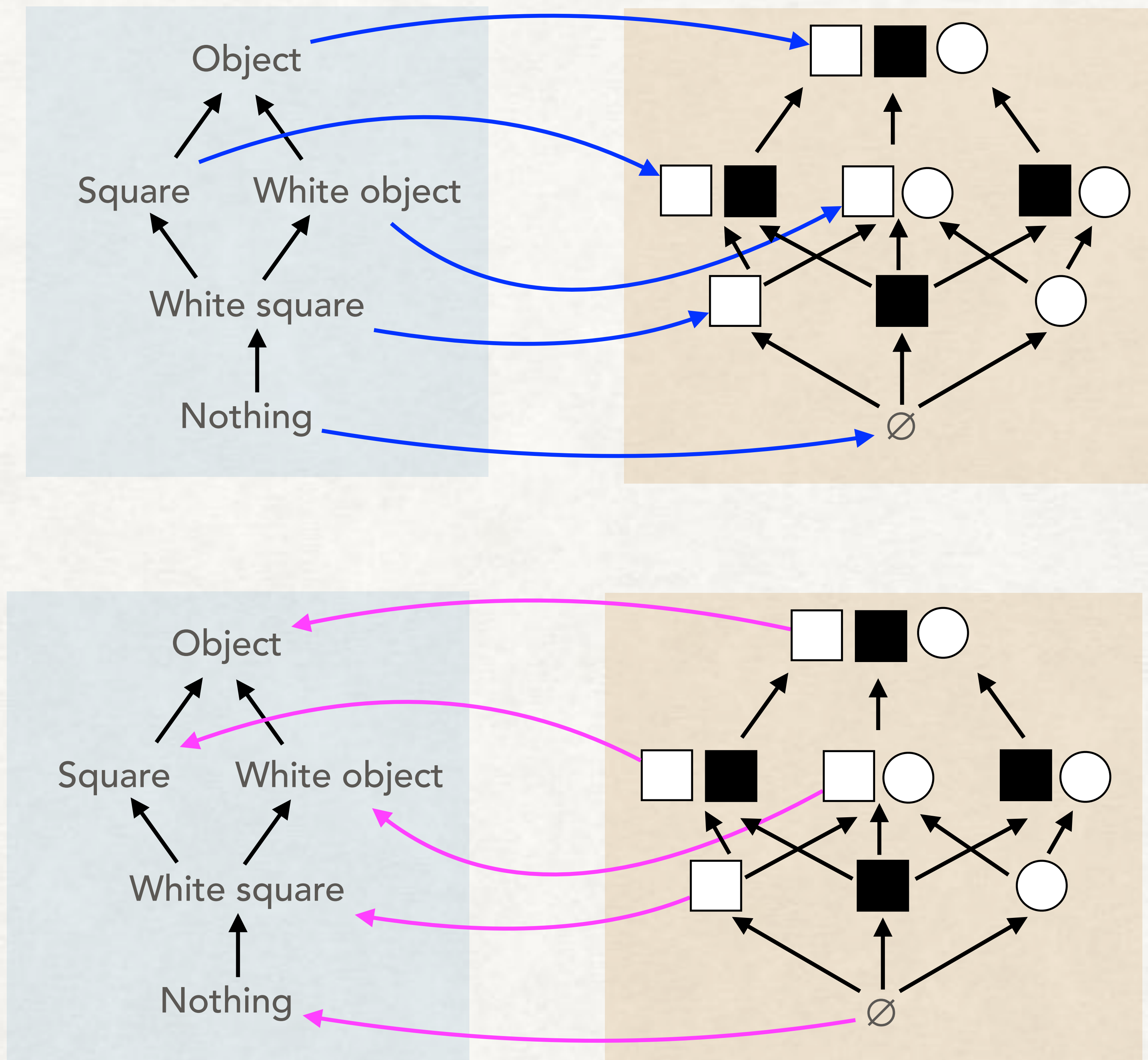
Abstract concept: all possible subsets of

Pragmatic means the speaker wants to be ***correct*** and ***as specific*** as possible

Quantity

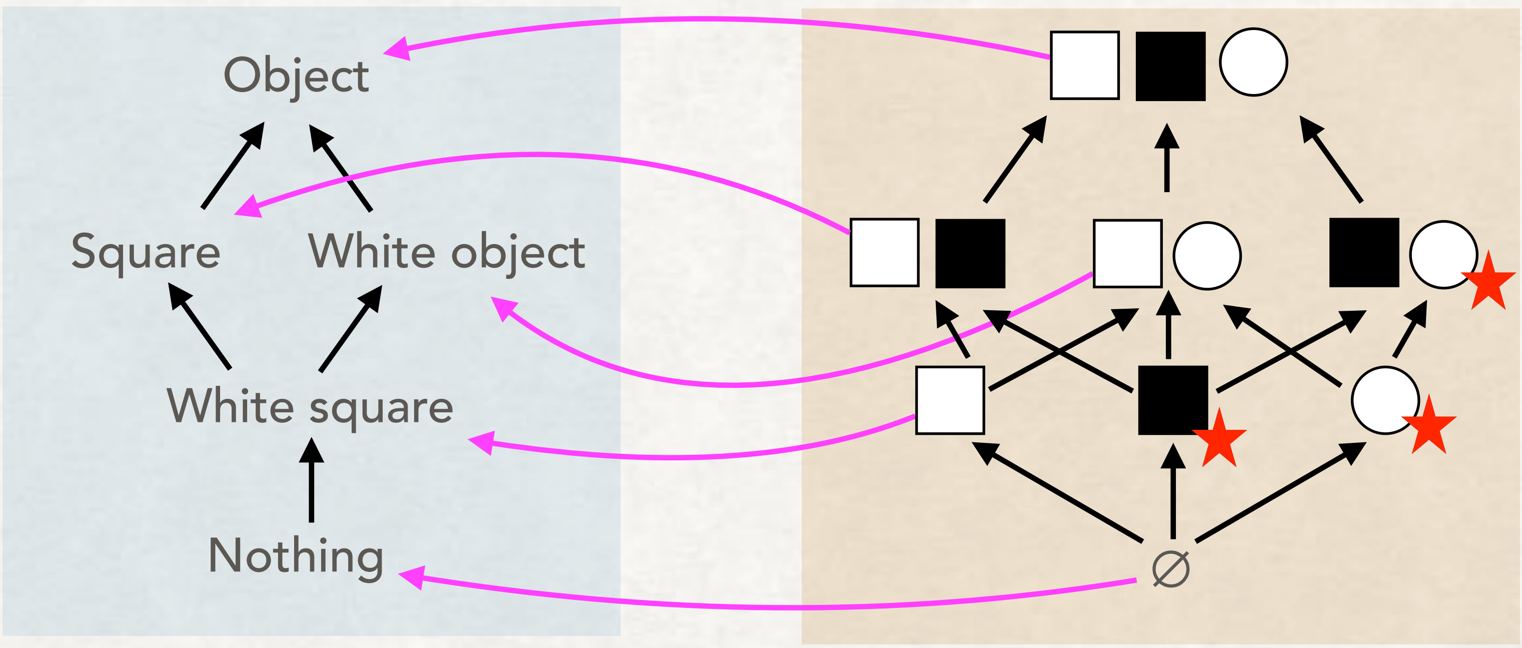
Quality

Listening



Speaking

EXAMPLE #3: COMMUNICATION



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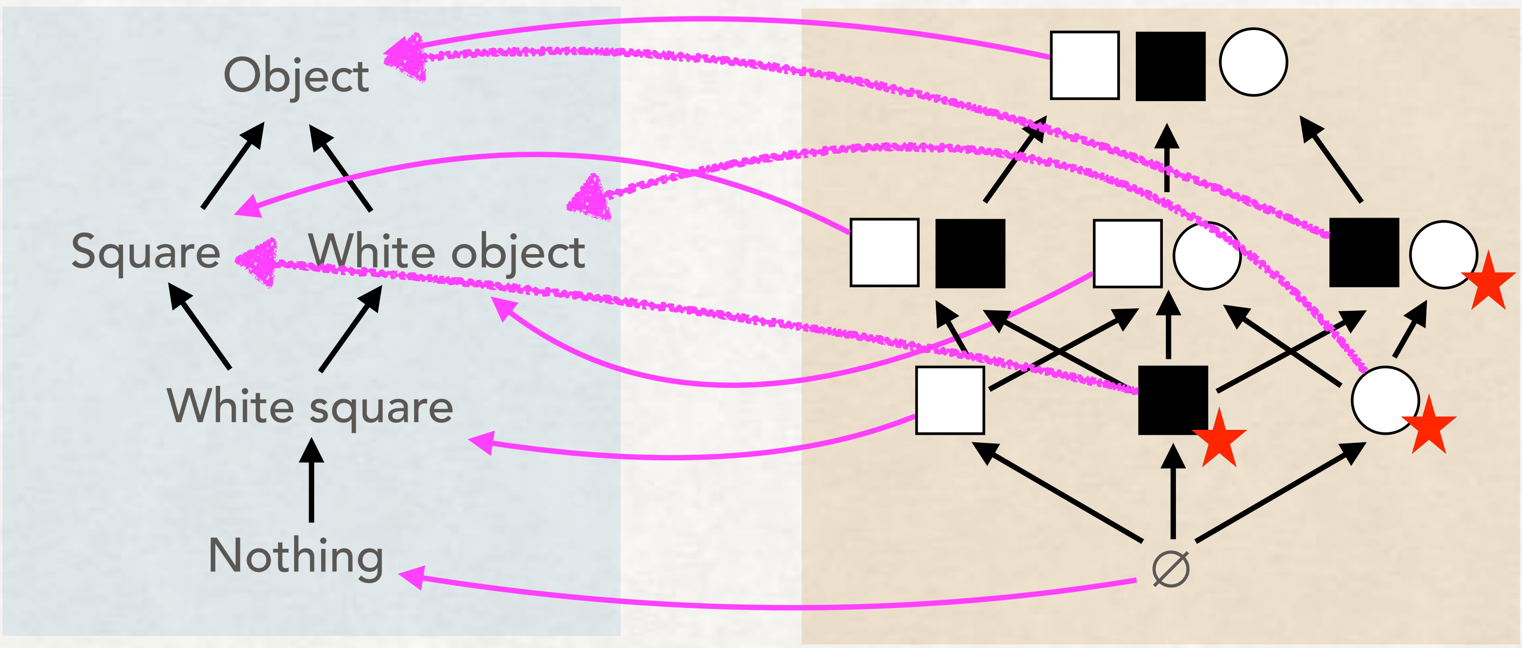
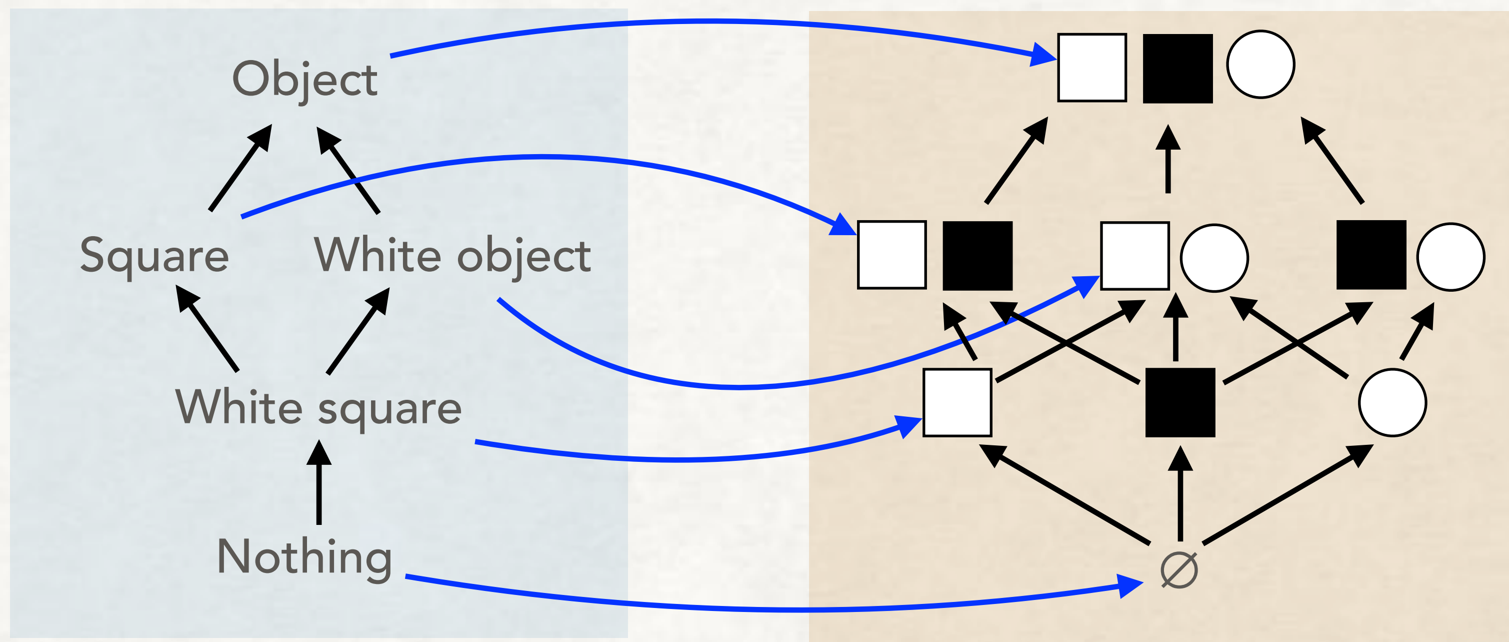
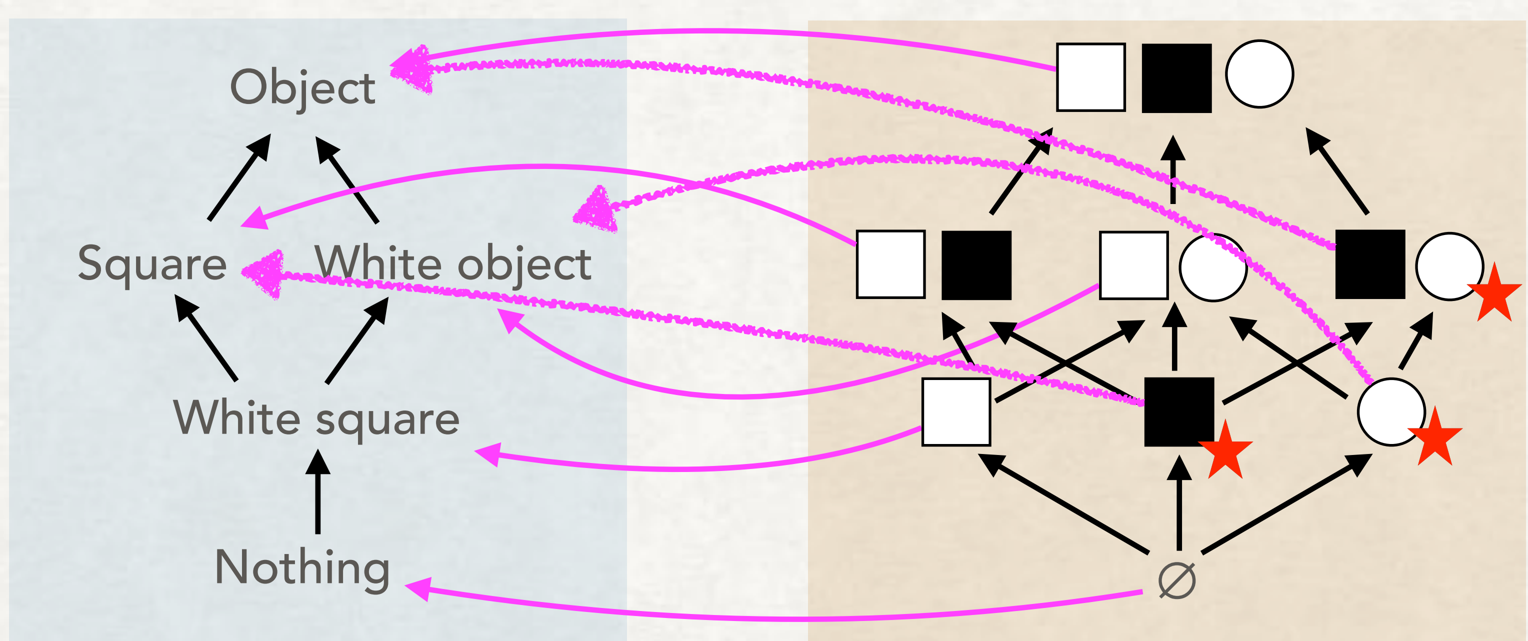


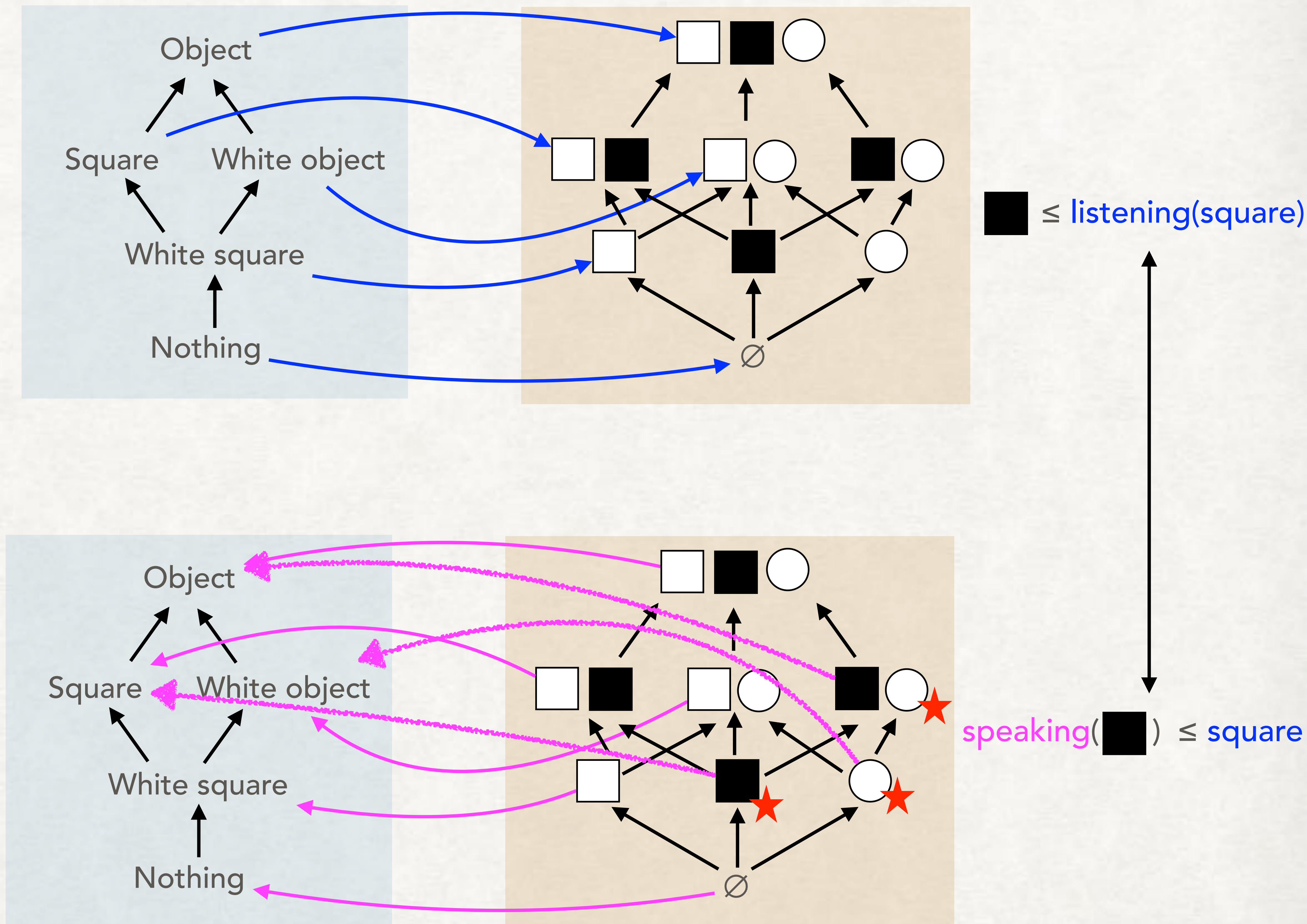
Listening





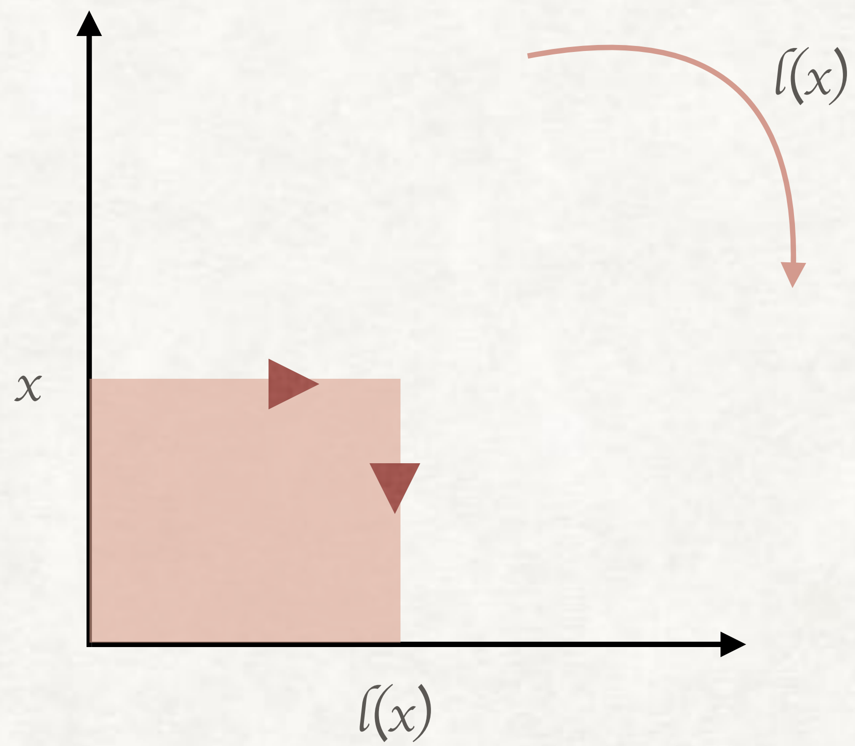
Speaking

Listening ⟶ Right adjoint

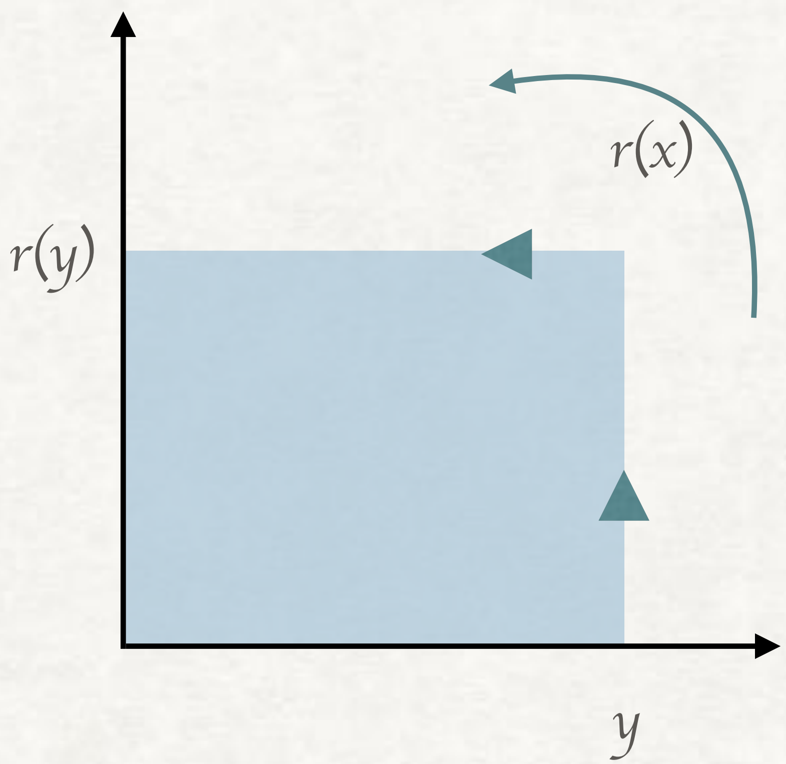


Speaking ⟶ Left adjoint

VISUAL EXPLANATION OF GALOIS CONNECTION



Left adjoint



Right adjoint

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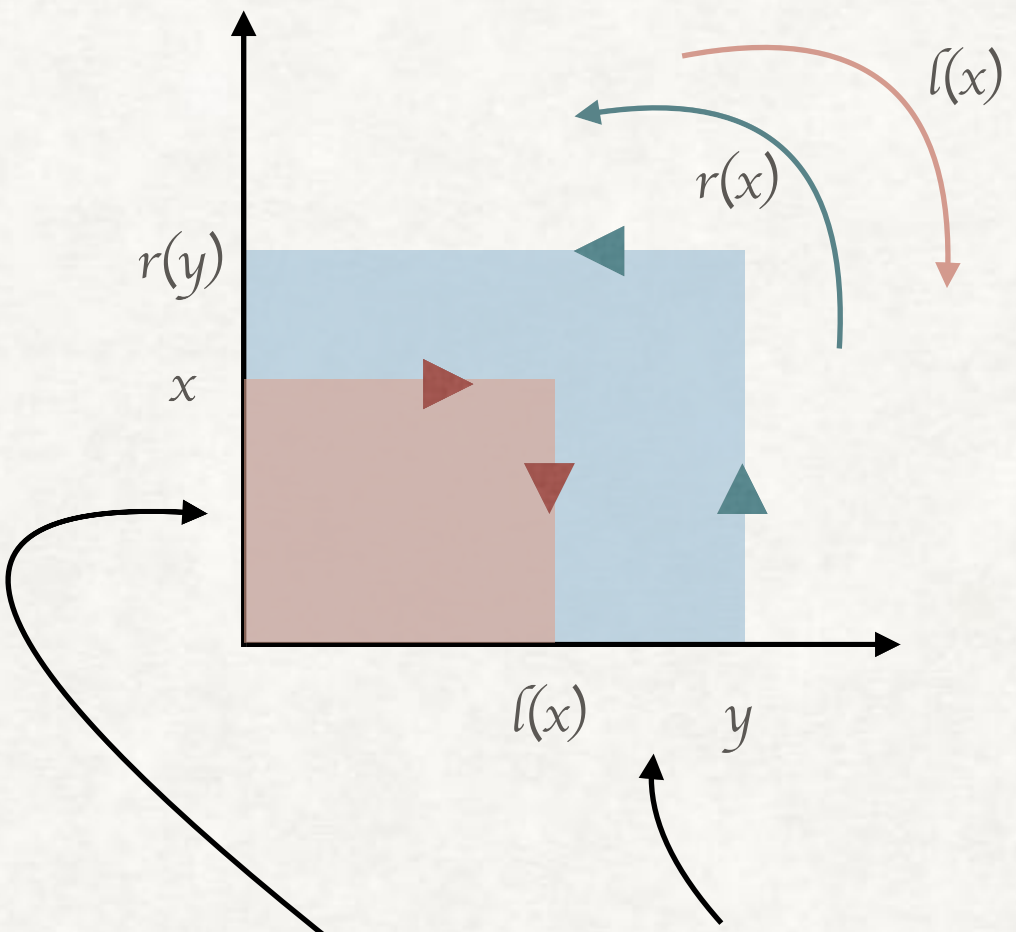
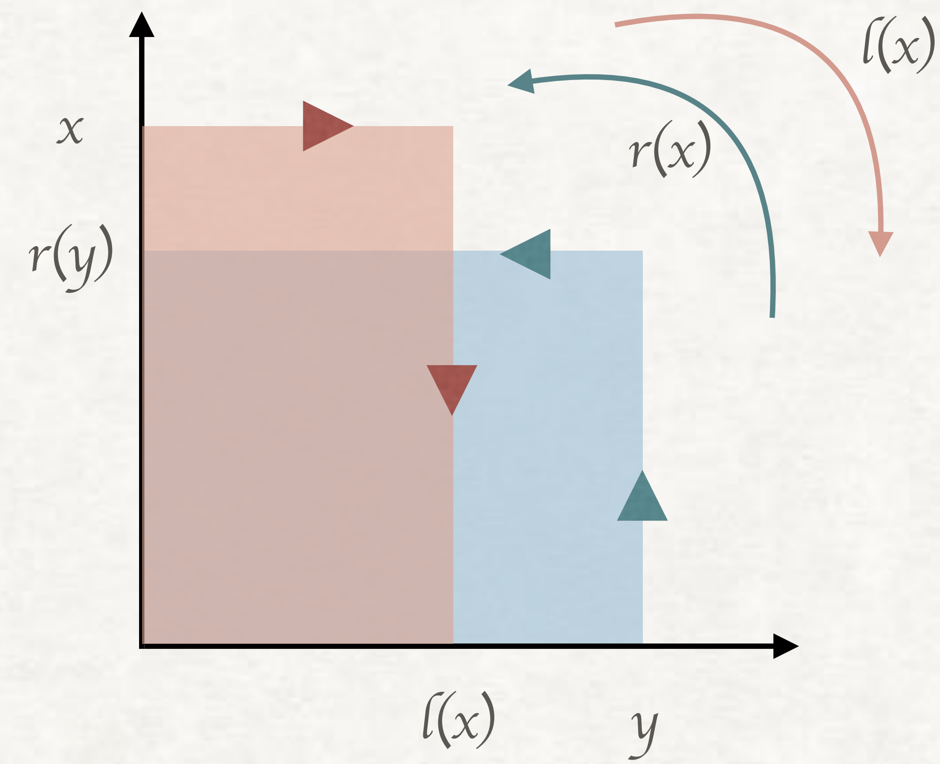


$x \leq r(y) \Leftrightarrow l(x) \leq y$

RECTANGLES DO NOT INTERSECT

RECTANGLE$_{\text{LEFT-ADJOINT}}$ CONTAINED BY RECTANGLE$_{\text{RIGHT-ADJOINT}}$

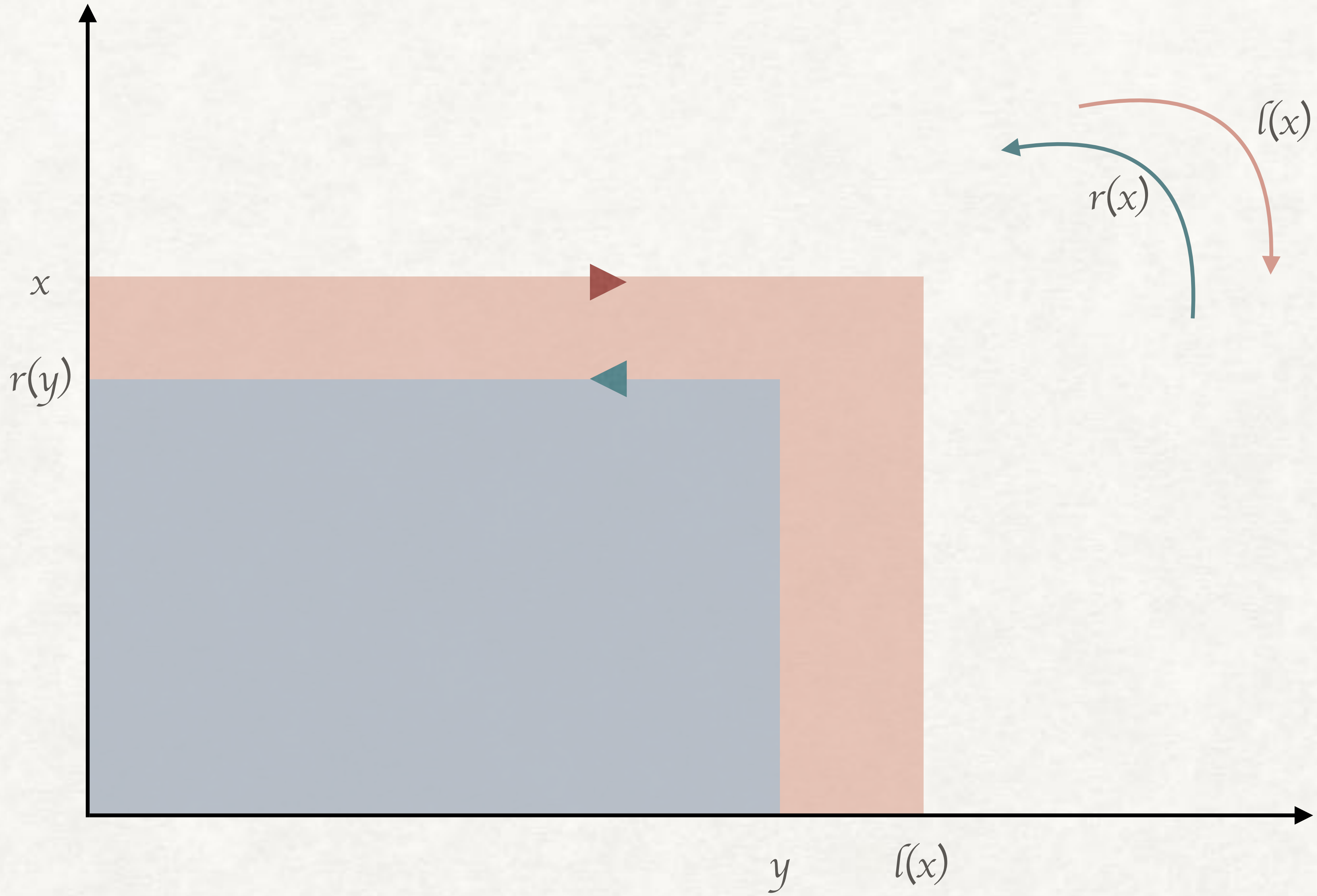


No longer true:

$x \leq r(y) \Leftrightarrow l(x) \leq y$

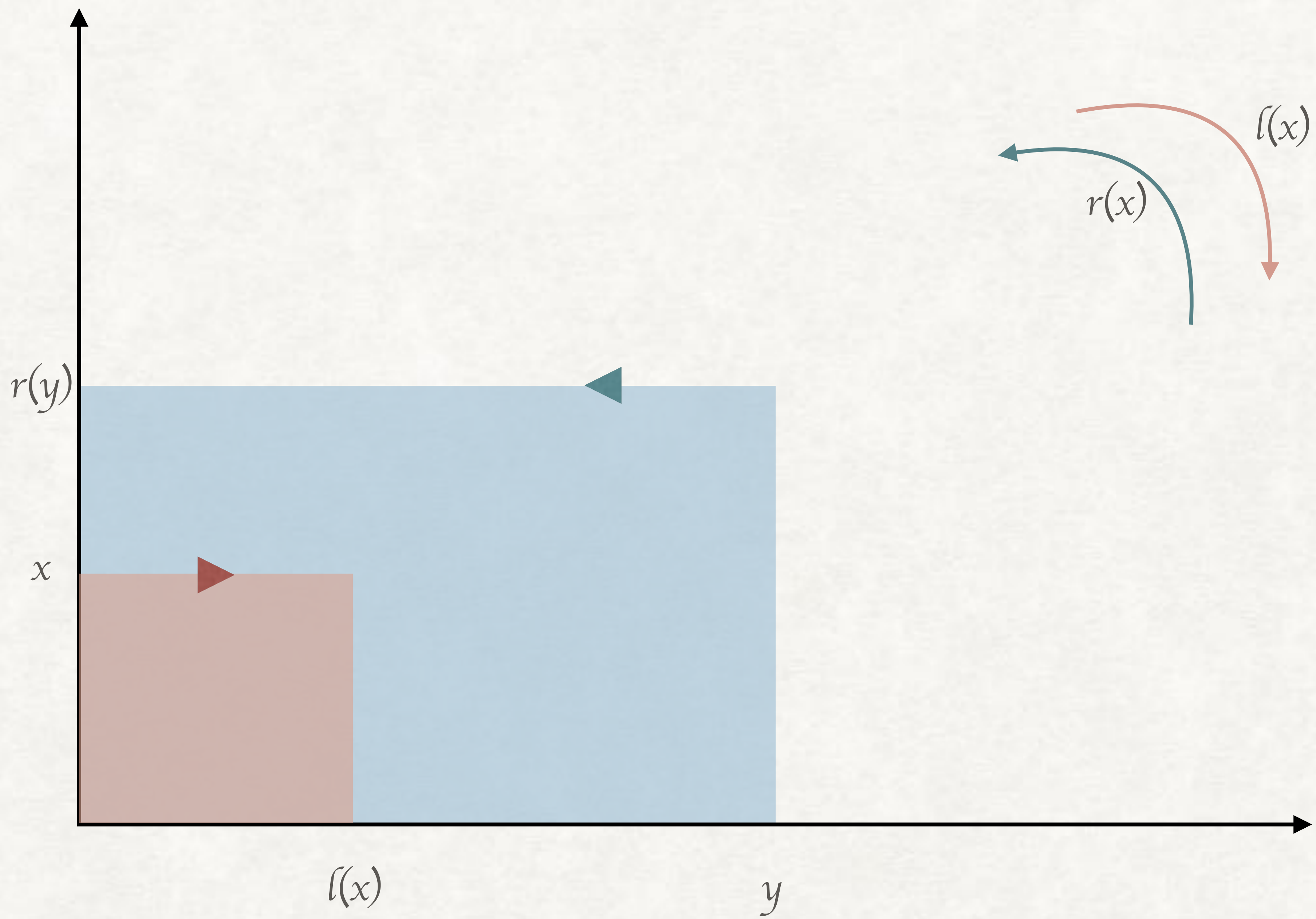
RECTANGLES INTERSECT

VISUAL EXPLANATION OF GALOIS CONNECTION



Is this a Galois connection?

VISUAL EXPLANATION OF GALOIS CONNECTION



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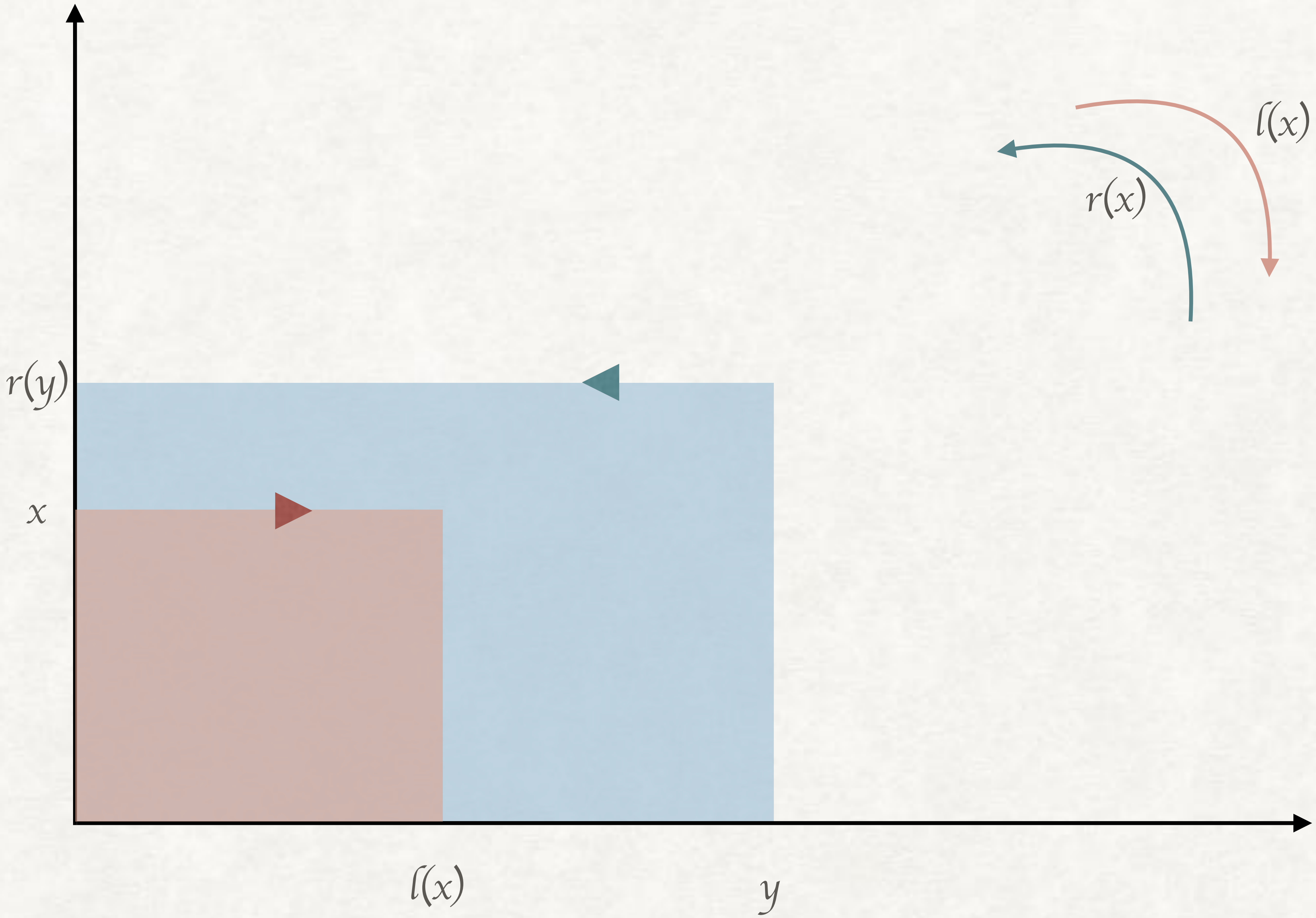

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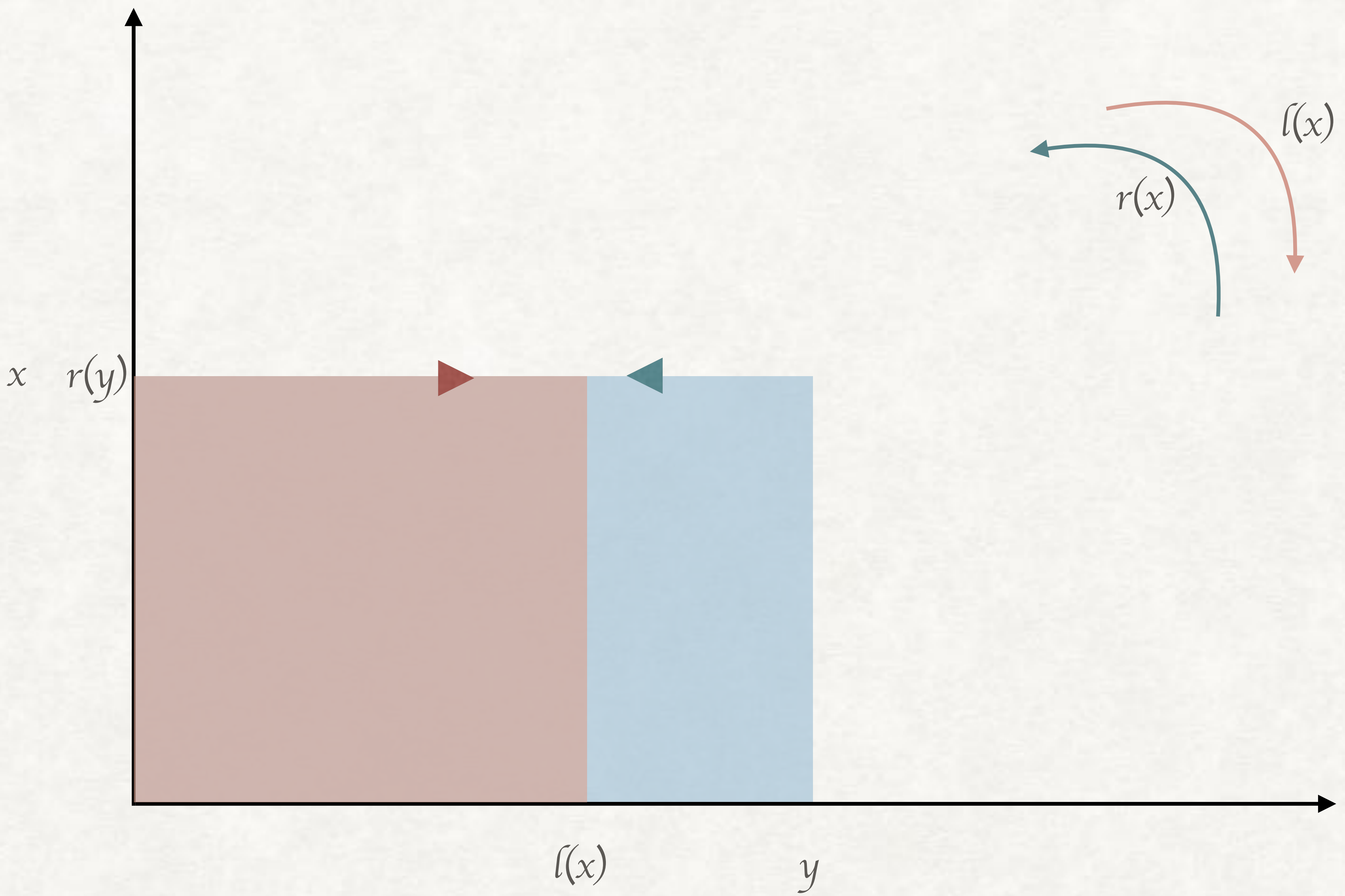
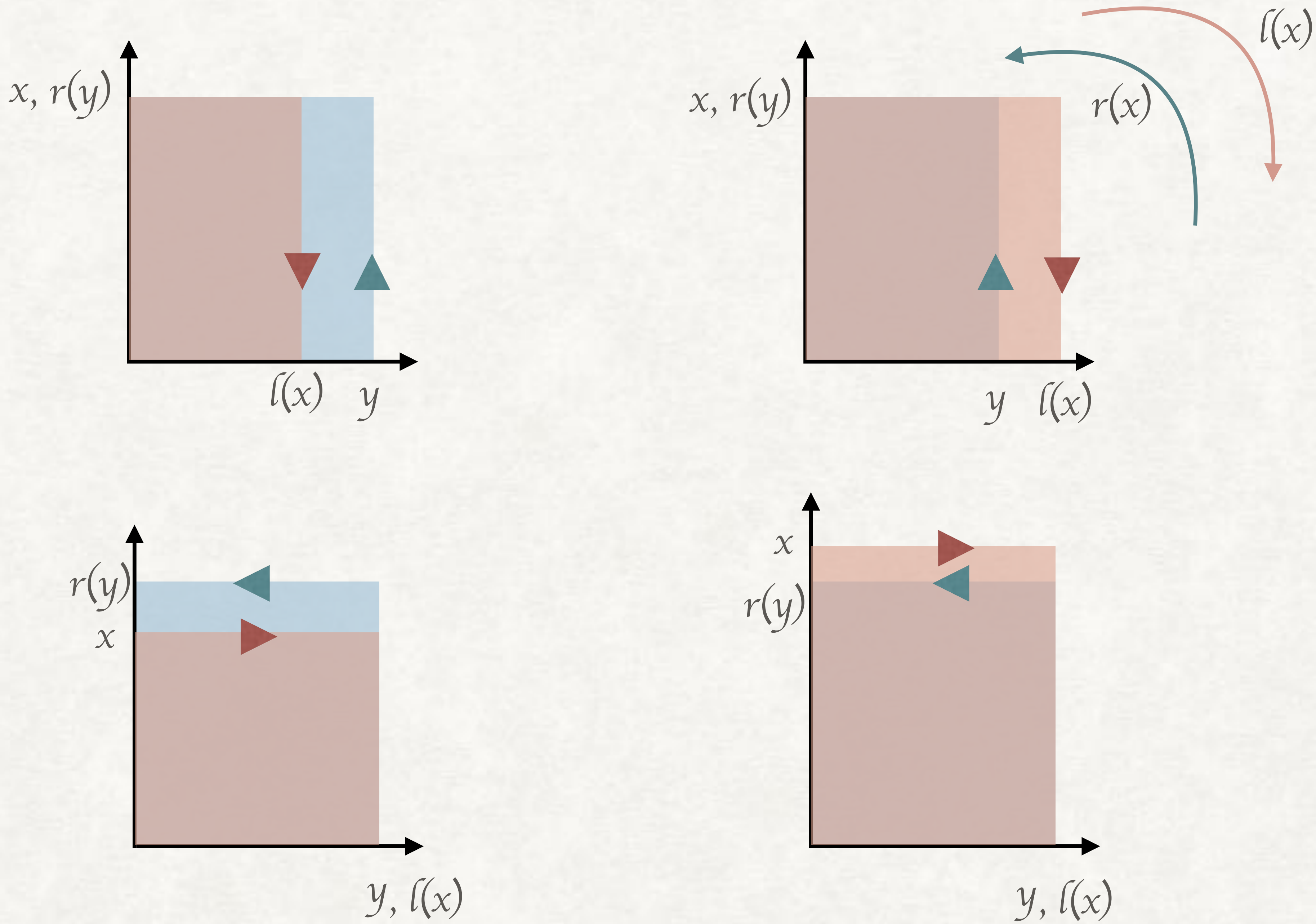


WHICH ONES ARE GALOIS CONNECTIONS?

$$x \le r(y) \Leftrightarrow l(x) \le y$$



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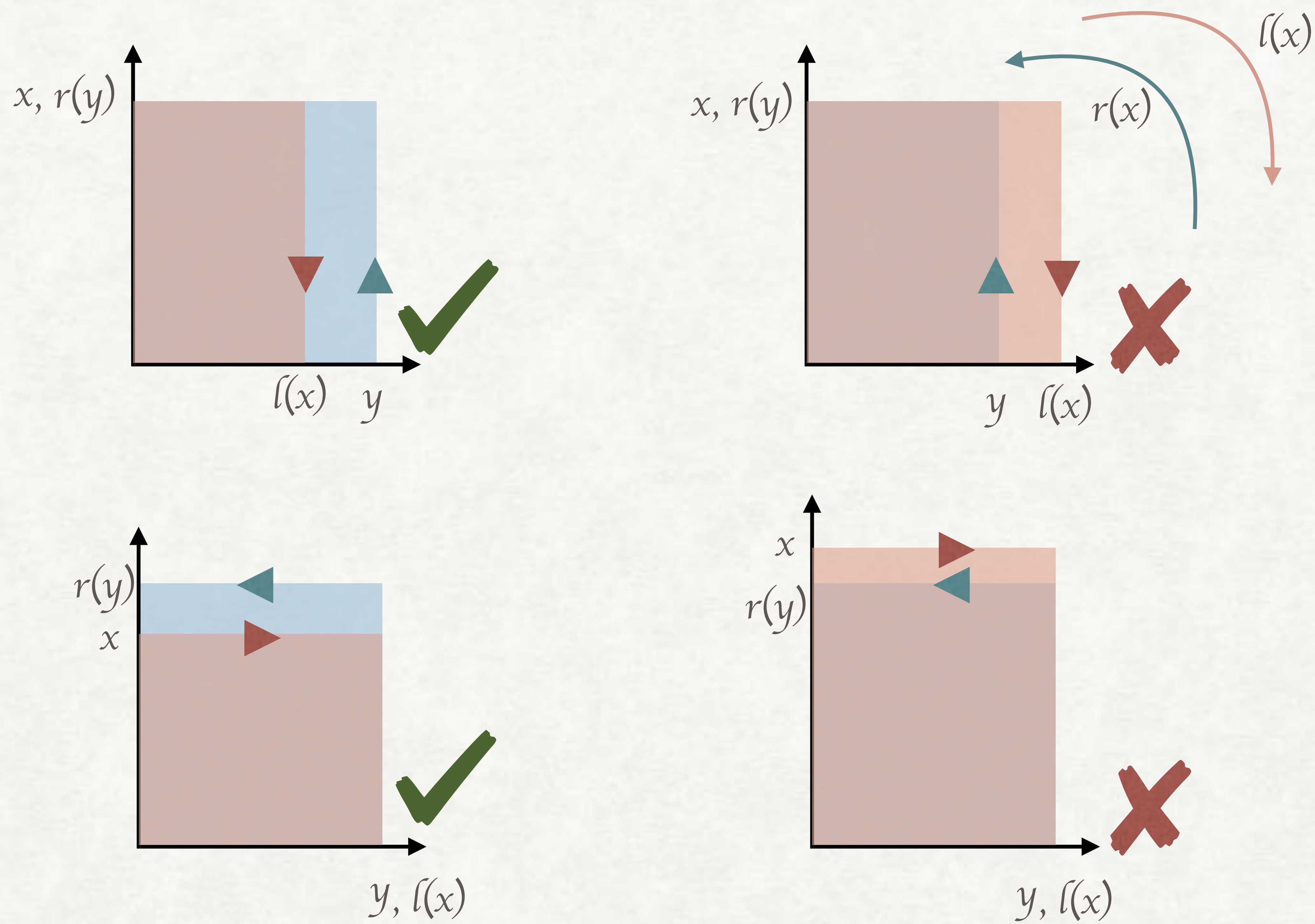
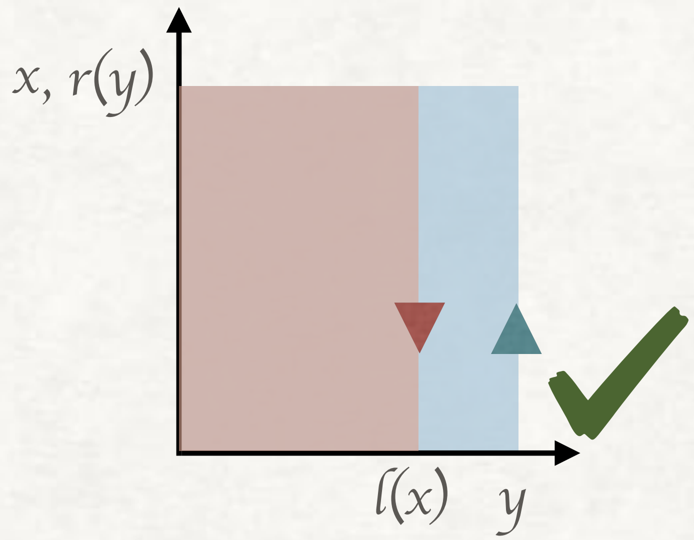
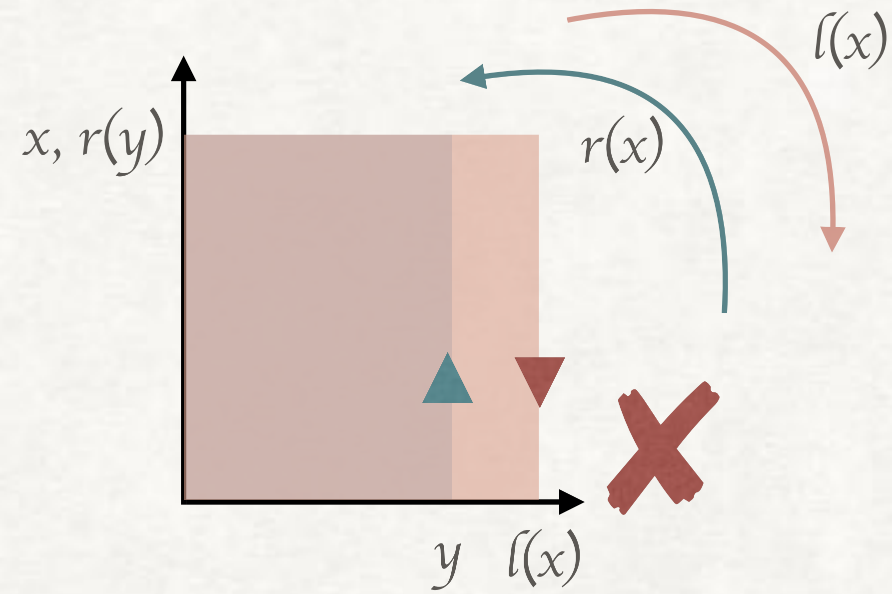
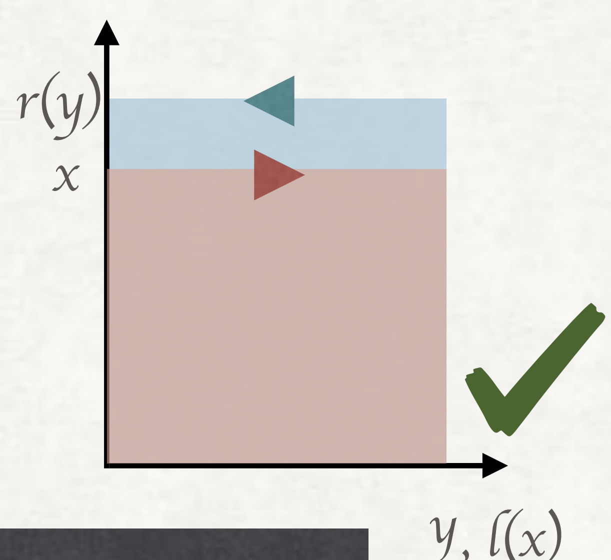


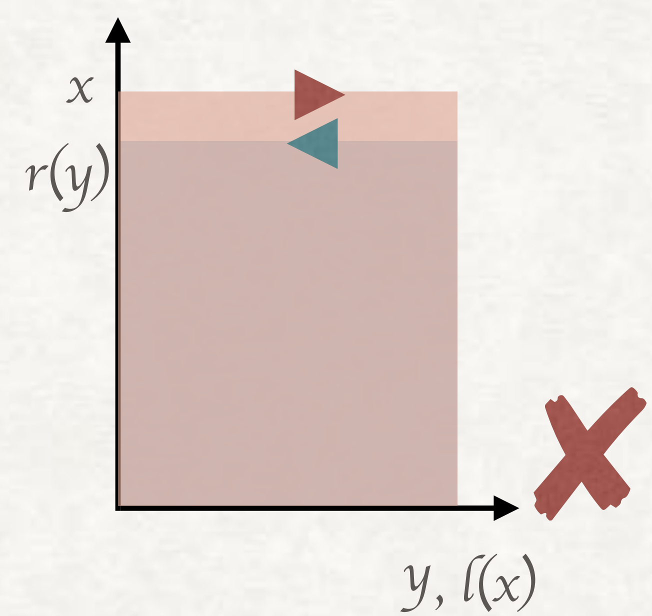
WHICH ONES ARE GALOIS CONNECTIONS?

$$x \le r(y) \Leftrightarrow l(x) \le y$$



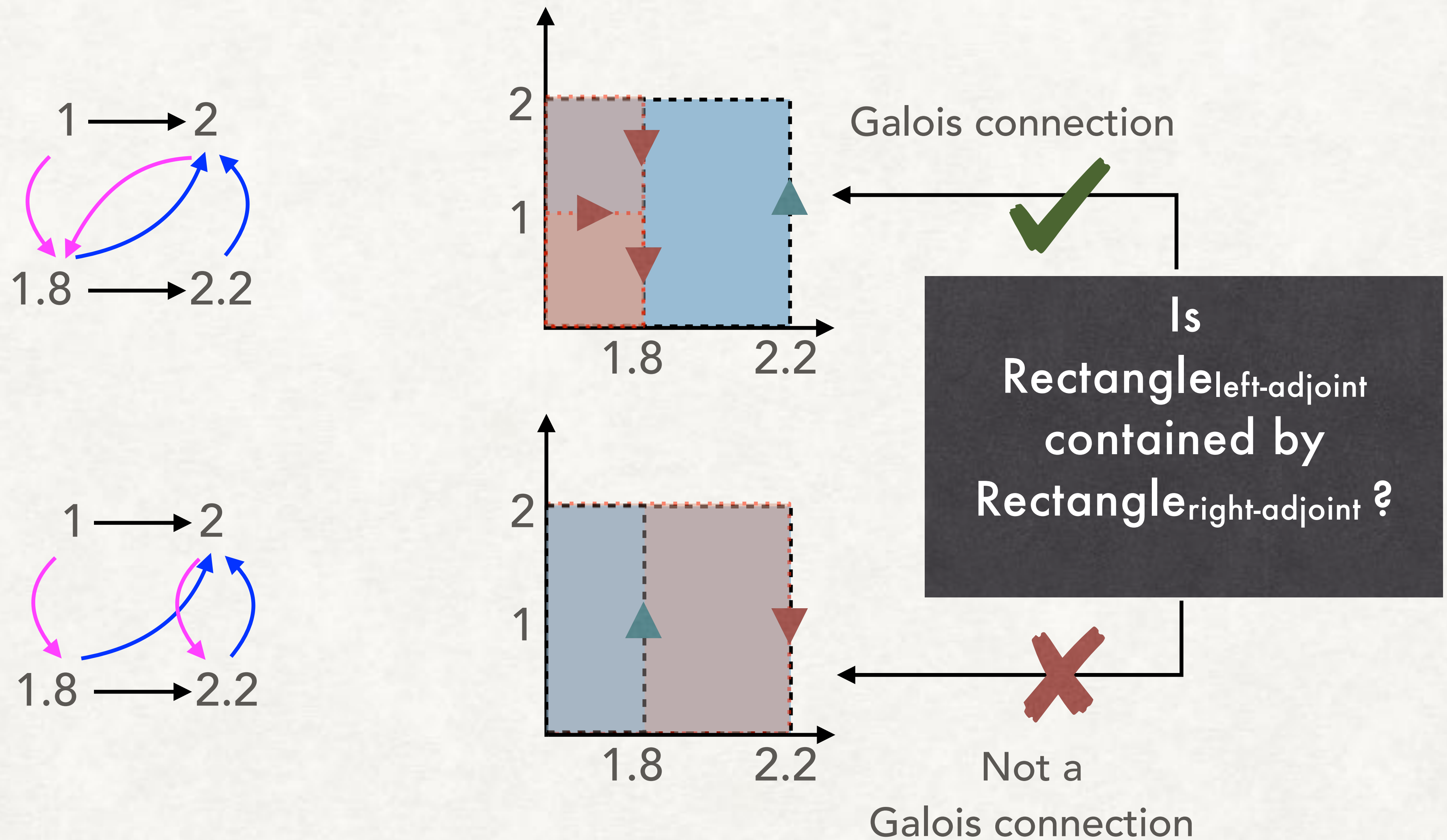




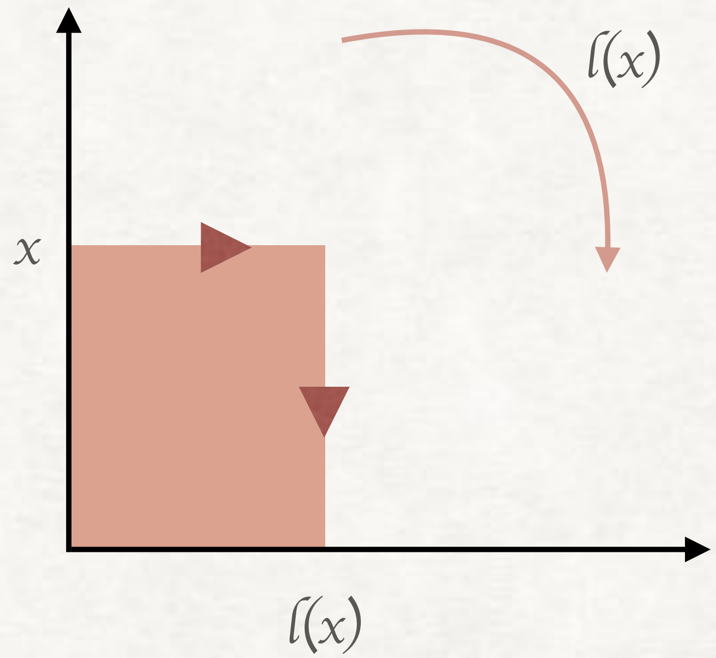


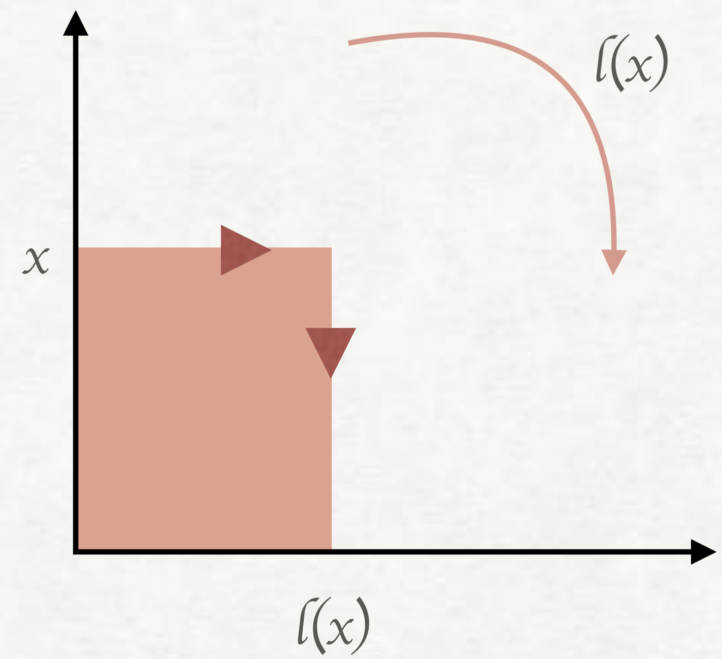
Rectangle$_{\text{left-adjoint}}$
should share an edge with
Rectangle$_{\text{right-adjoint}}$
from the inside

LOOKING BACK AT EXAMPLE #1

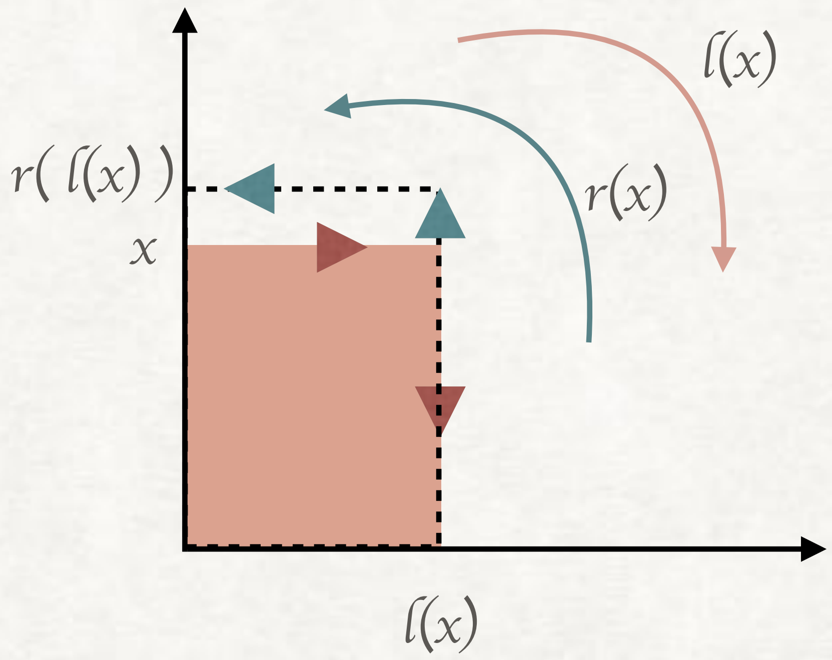


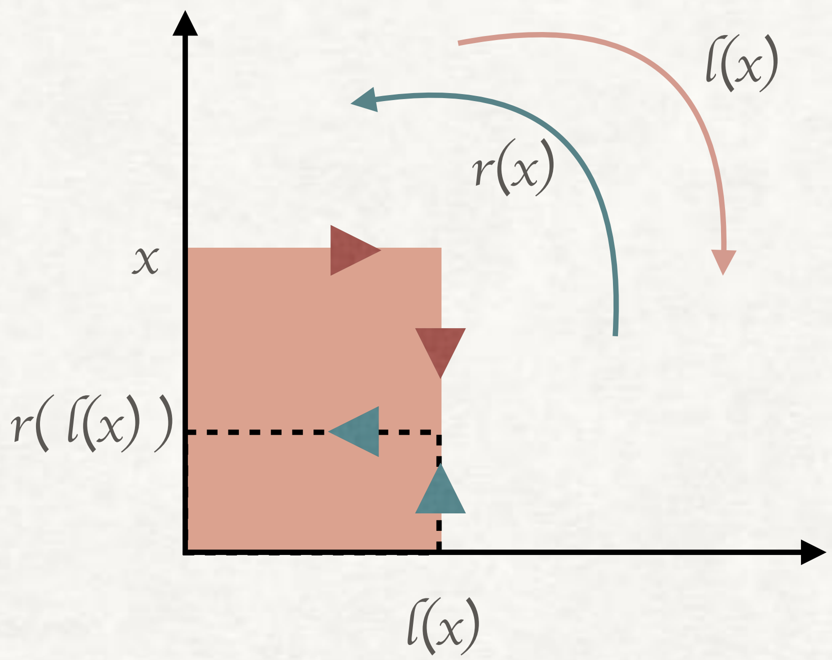
LEFT ADJOINT THEN RIGHT ADJOINT OF AN ELEMENT





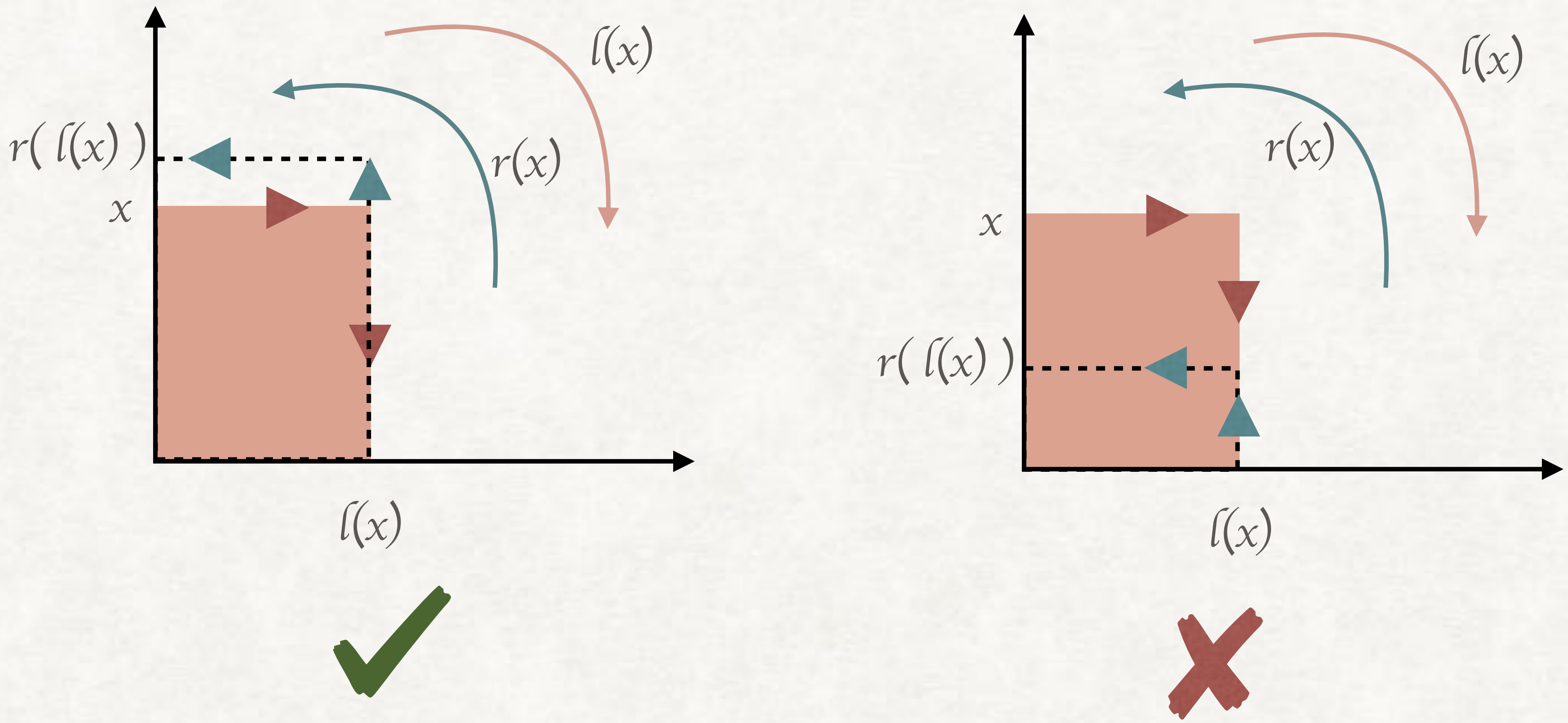
LEFT ADJOINT THEN RIGHT ADJOINT OF AN ELEMENT





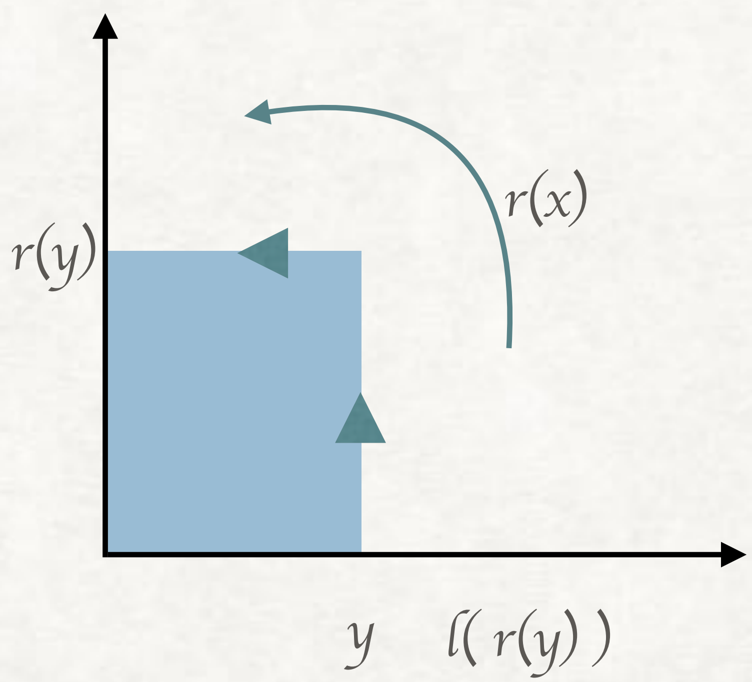
Which one is the right adjoint of a Galois connection?

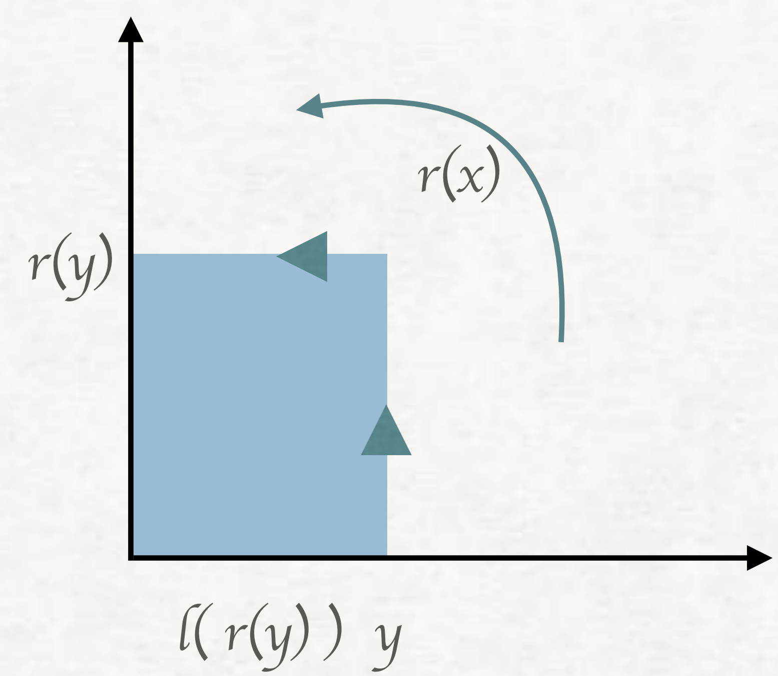
LEFT ADJOINT THEN RIGHT ADJOINT OF AN ELEMENT



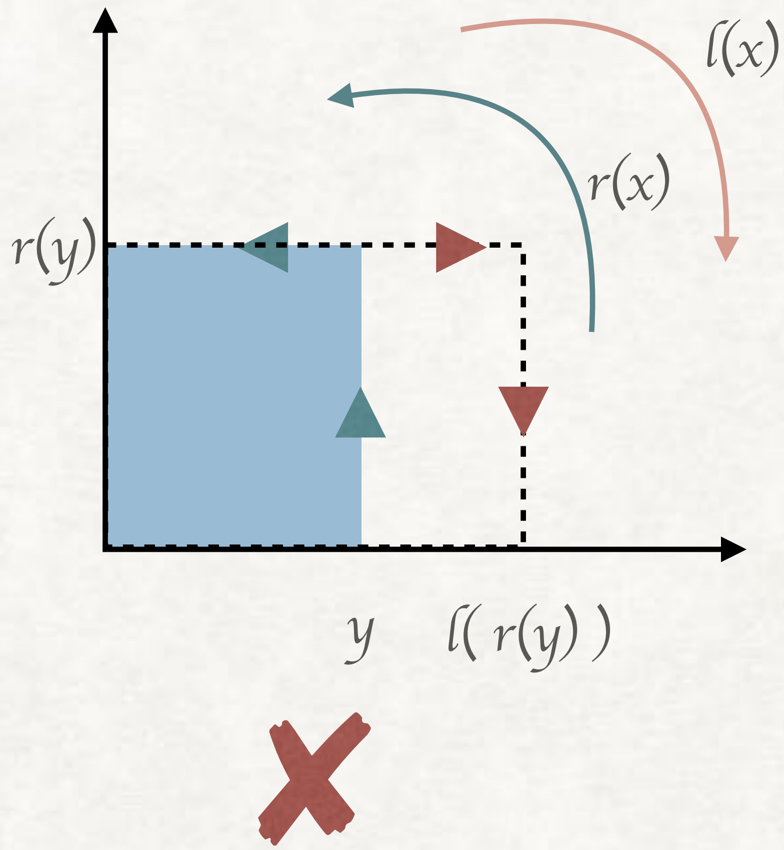
$$x \leq r(\ l(x)\ )$$

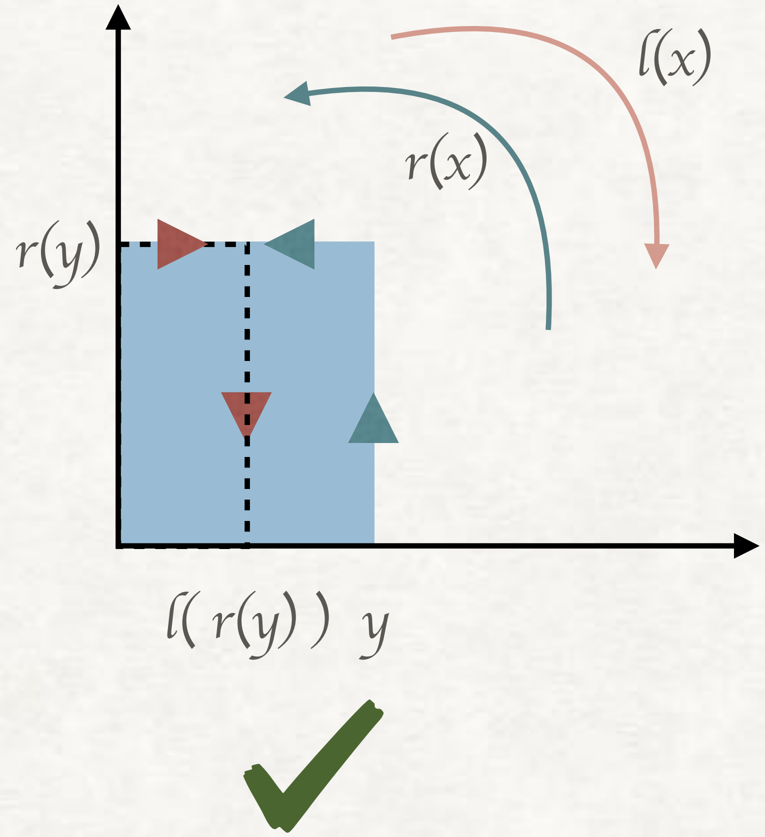
RIGHT ADJOINT THEN LEFT ADJOINT OF AN ELEMENT





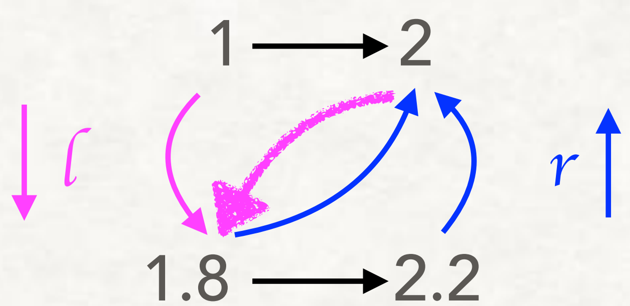
RIGHT ADJOINT THEN LEFT ADJOINT OF AN ELEMENT

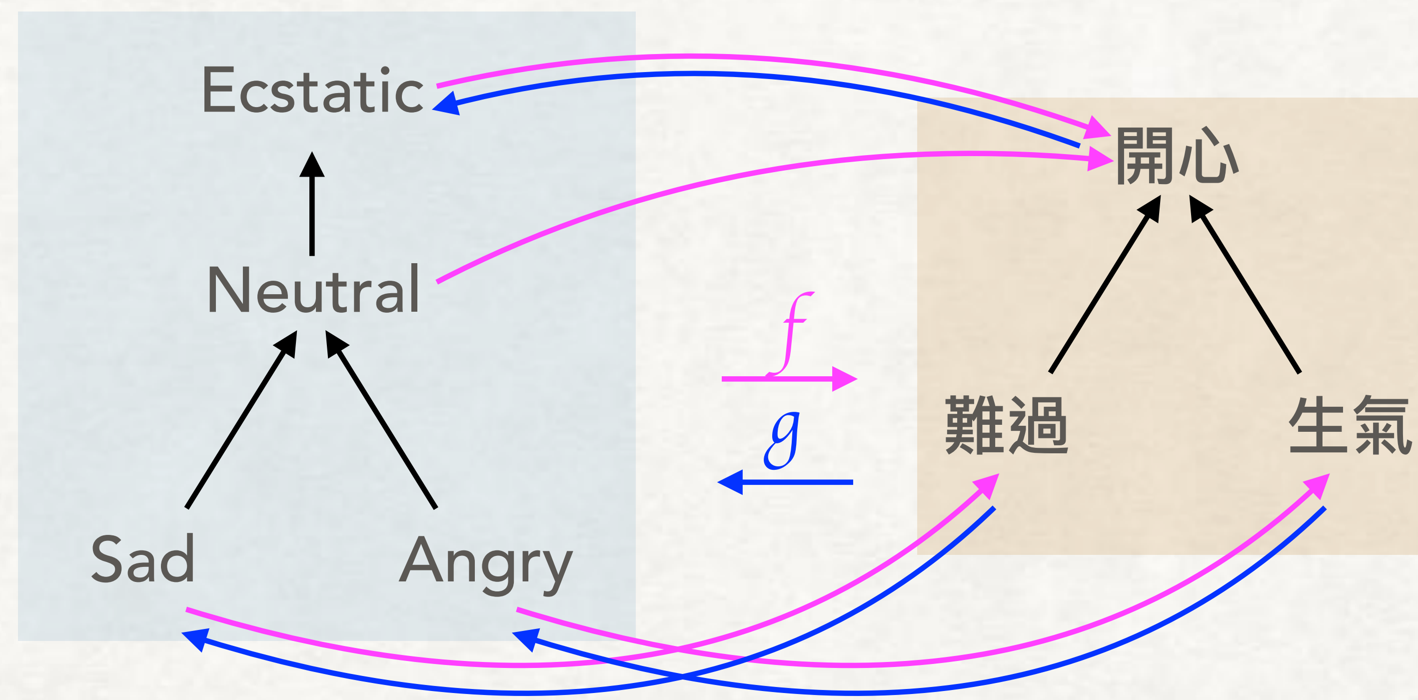




$$l(\,r(y)\,) \leq y$$

LOOKING BACK AT EXAMPLES #1 AND #2





$1 \leq r(\, l(1)\,)$

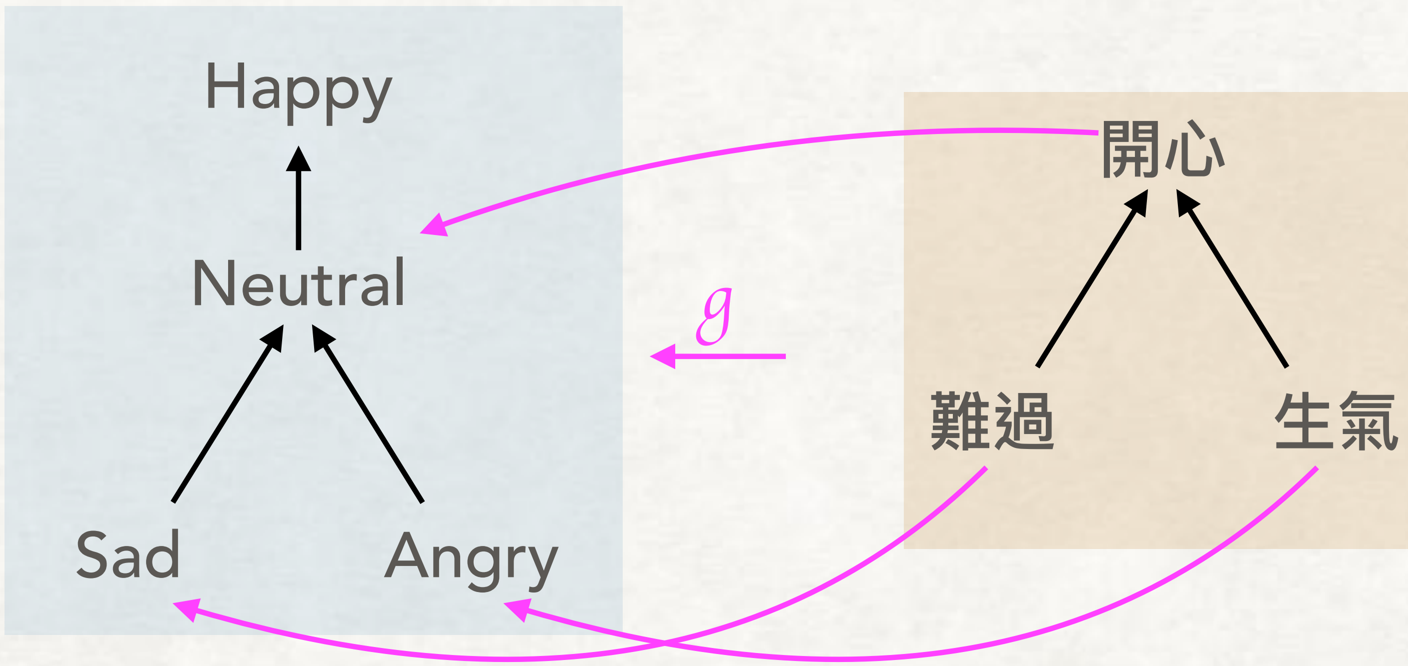
$l(\, r(2.2)\,) \leq 2.2$

$neutral \leq g(\, f(neutral)\,)$

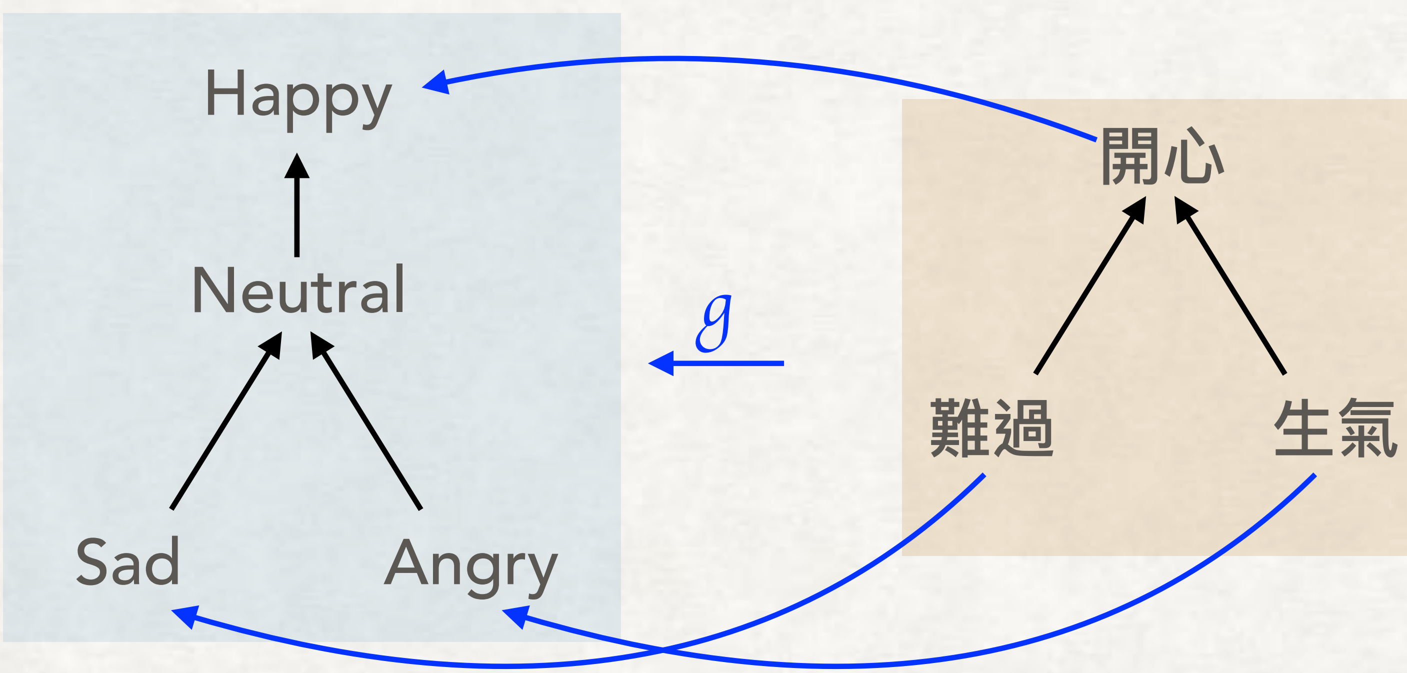
$f(\, g(\text{難過})\,) \leq \text{難過}$

$x \leq r(\, l(x)\,) \qquad l(\, r(y)\,) \leq y$

LEFT ADJOINT PRESERVES JOINS

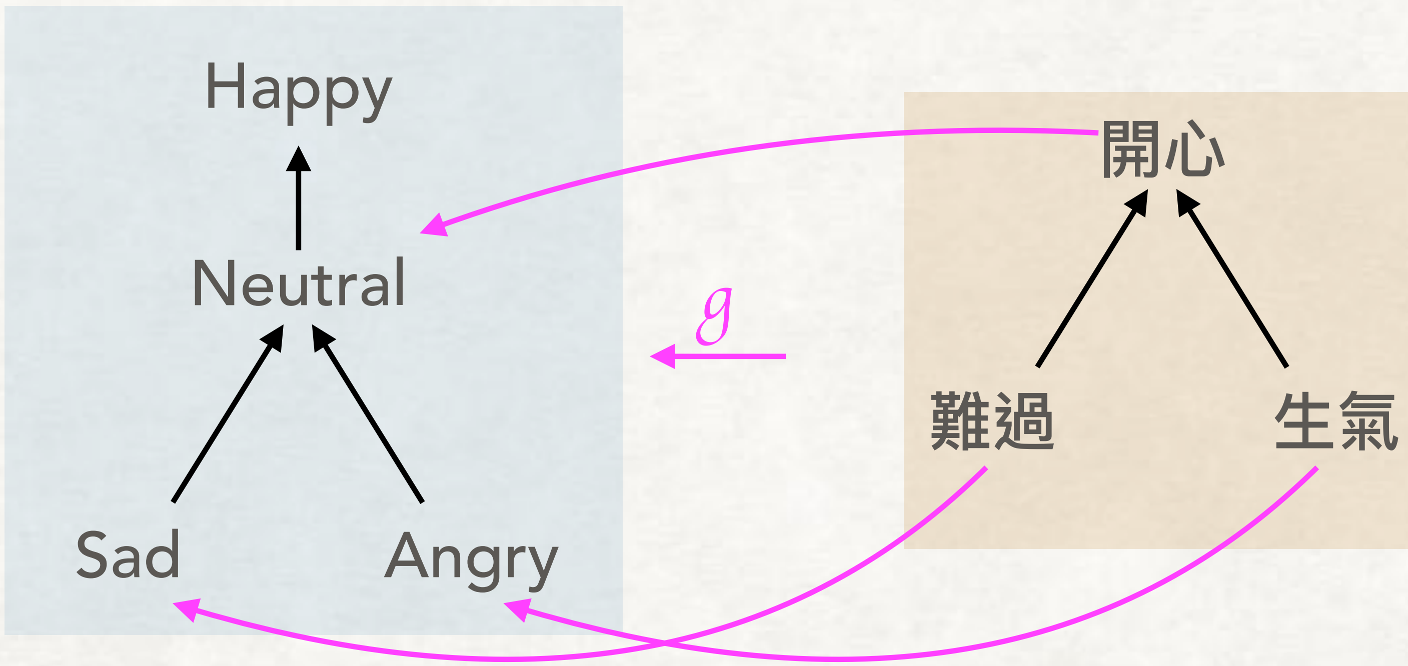


$$g\,(\mathit{join}(\text{難過}, \text{生氣}))$$
$$=$$
$$\mathit{join}(\,g(\text{難過}), g(\text{生氣})\,)$$

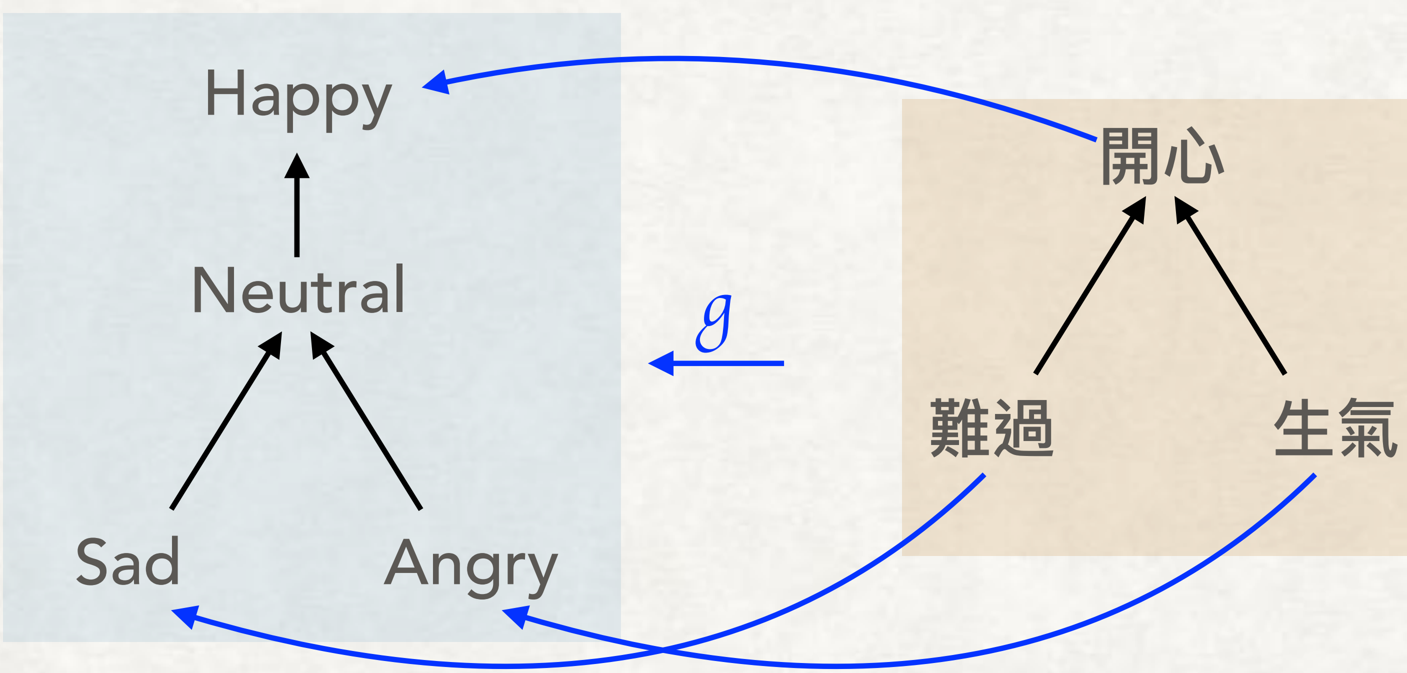


$$g\,(\mathit{join}(\text{難過}, \text{生氣}))$$
$$\neq$$
$$\mathit{join}(\,g(\text{難過}), g(\text{生氣})\,)$$

LEFT ADJOINT PRESERVES JOINS



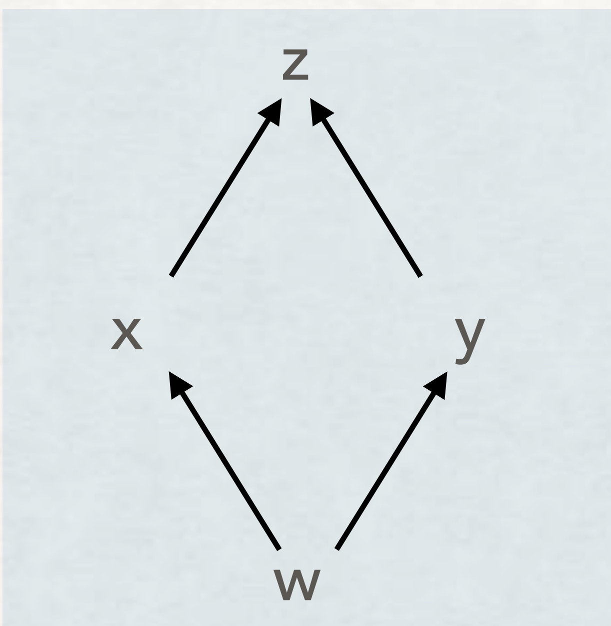
$$g(\mathrm{join}(\text{難過}, \text{生氣})) = \mathrm{join}(g(\text{難過}), g(\text{生氣}))$$



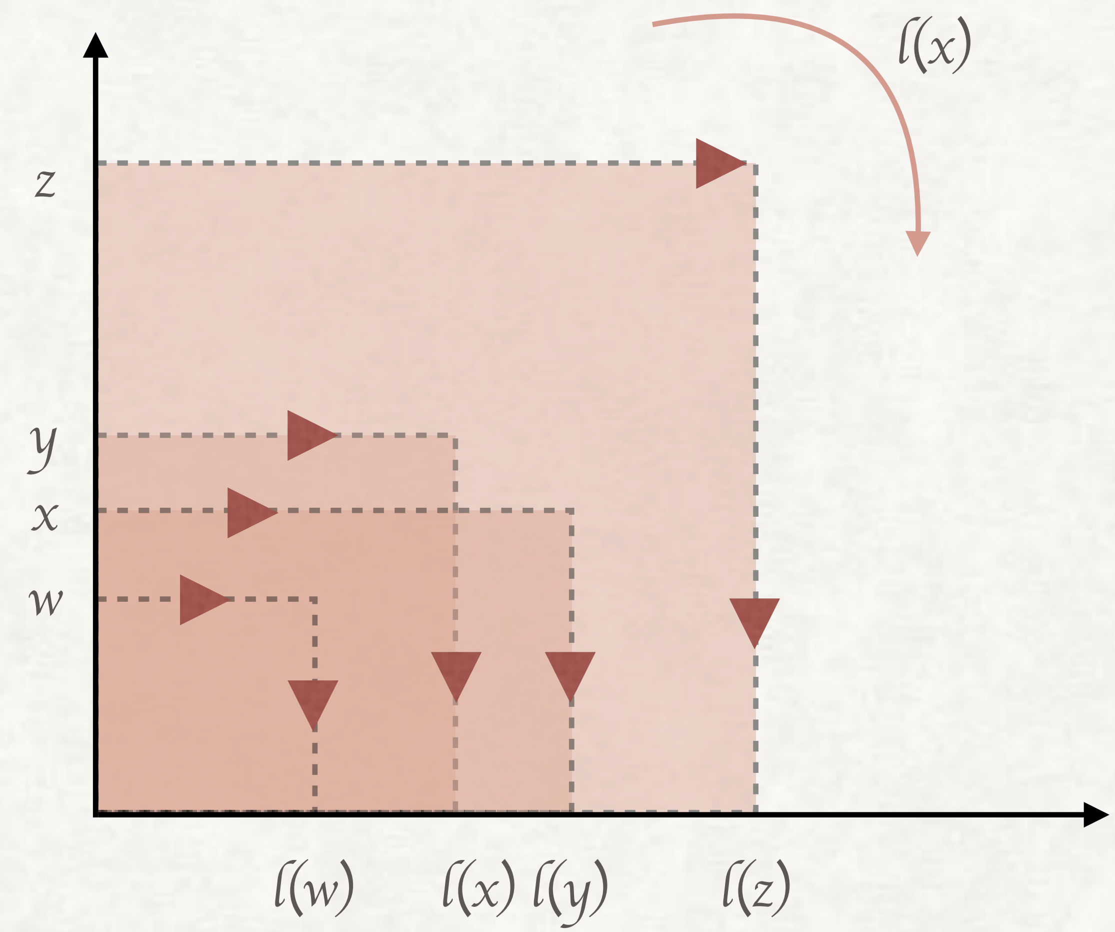
$$g(\mathrm{join}(\text{難過}, \text{生氣})) \neq \mathrm{join}(g(\text{難過}), g(\text{生氣}))$$

SIMILARLY RIGHT ADJOINT PRESERVES MEETS

LEFT ADJOINT PRESERVES JOINS



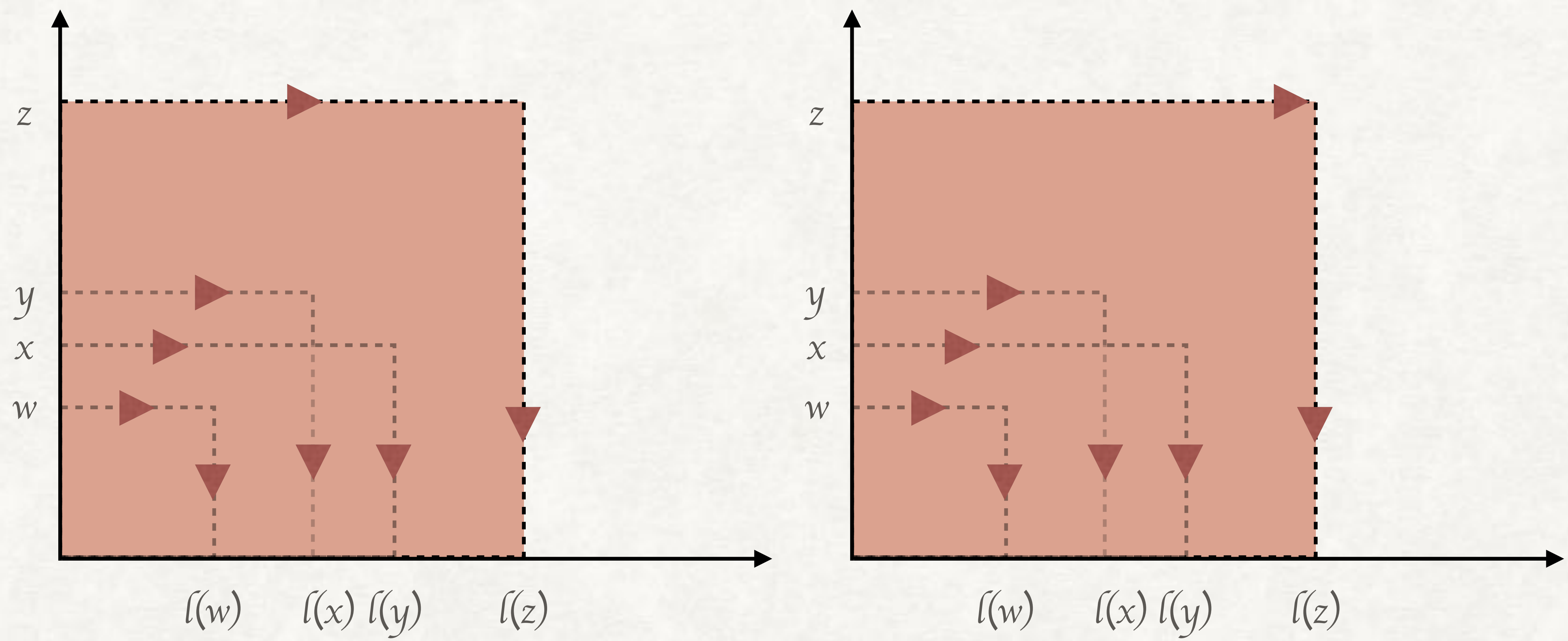
Meet(x, y) = w
Join(x, y) = z



Is this a monotone map?

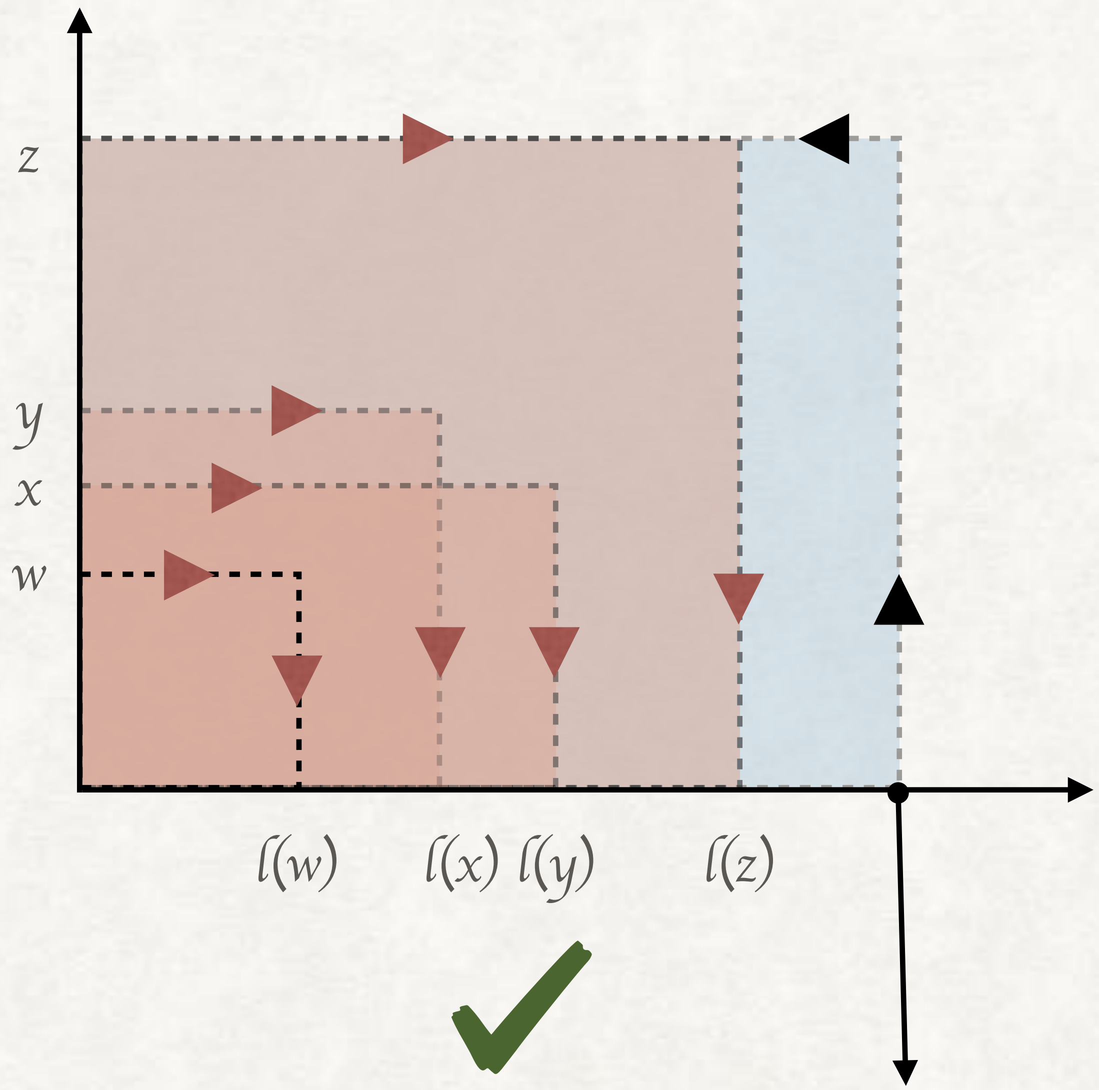
LEFT ADJOINT PRESERVES JOINS

Simply means that:
If join of x and y is z then
join of $l(x)$ and $l(y)$ is $l(z)$

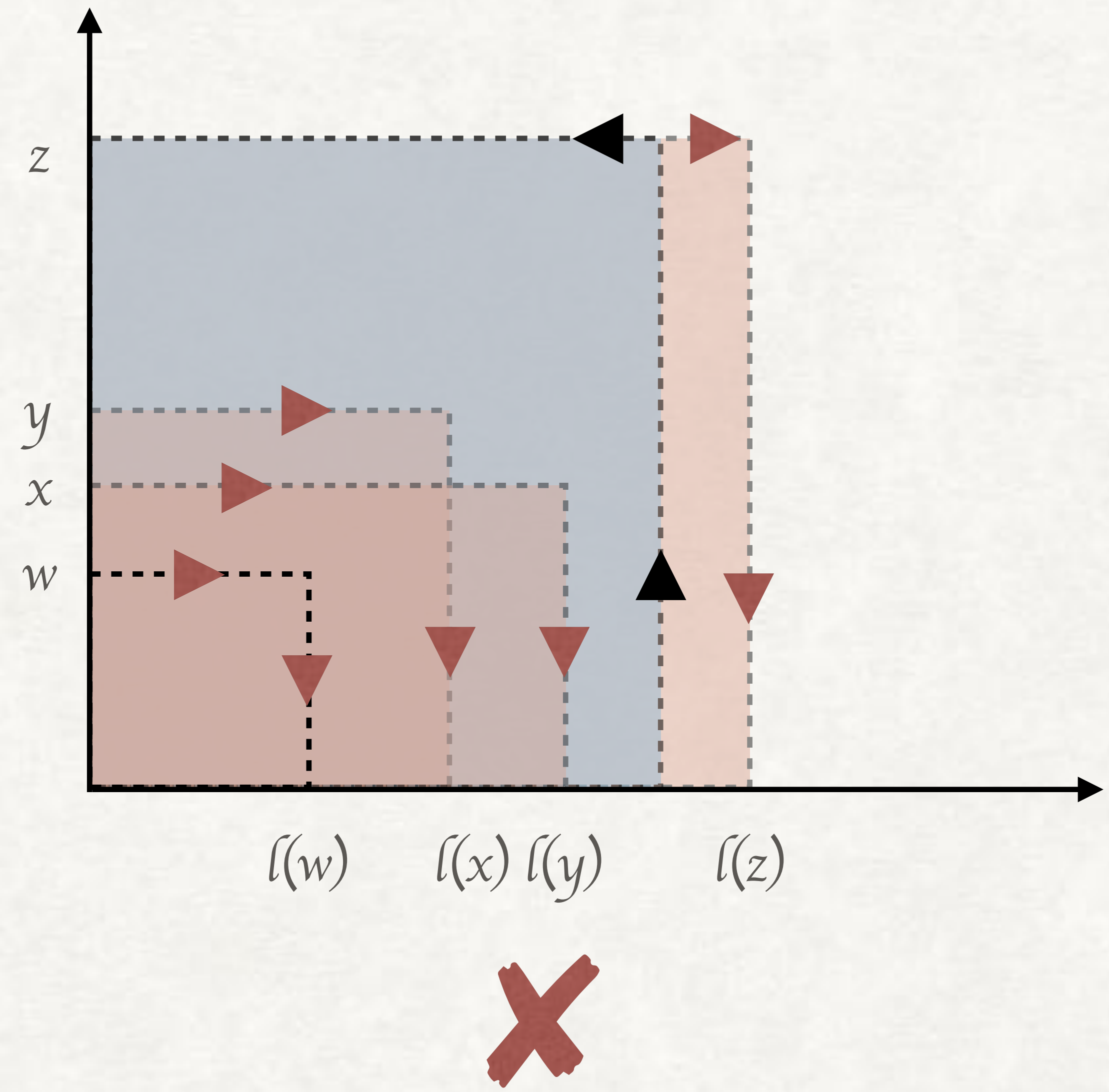


LEFT ADJOINT PRESERVES JOINS

Simply means that:
If join of x and y is z then
join of ℓ(x) and ℓ(y) is ℓ(z)



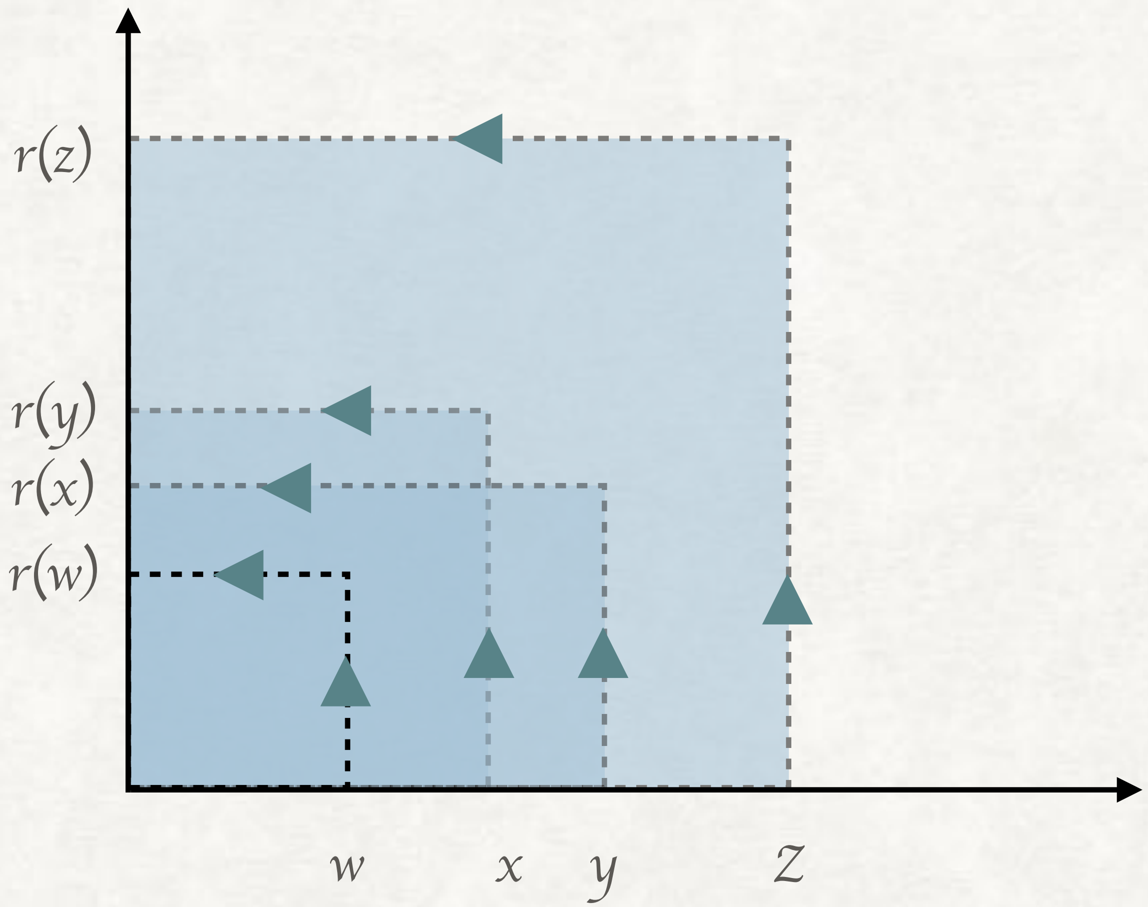
This point can only get as
small as l(z) while maintaining
Galois connection

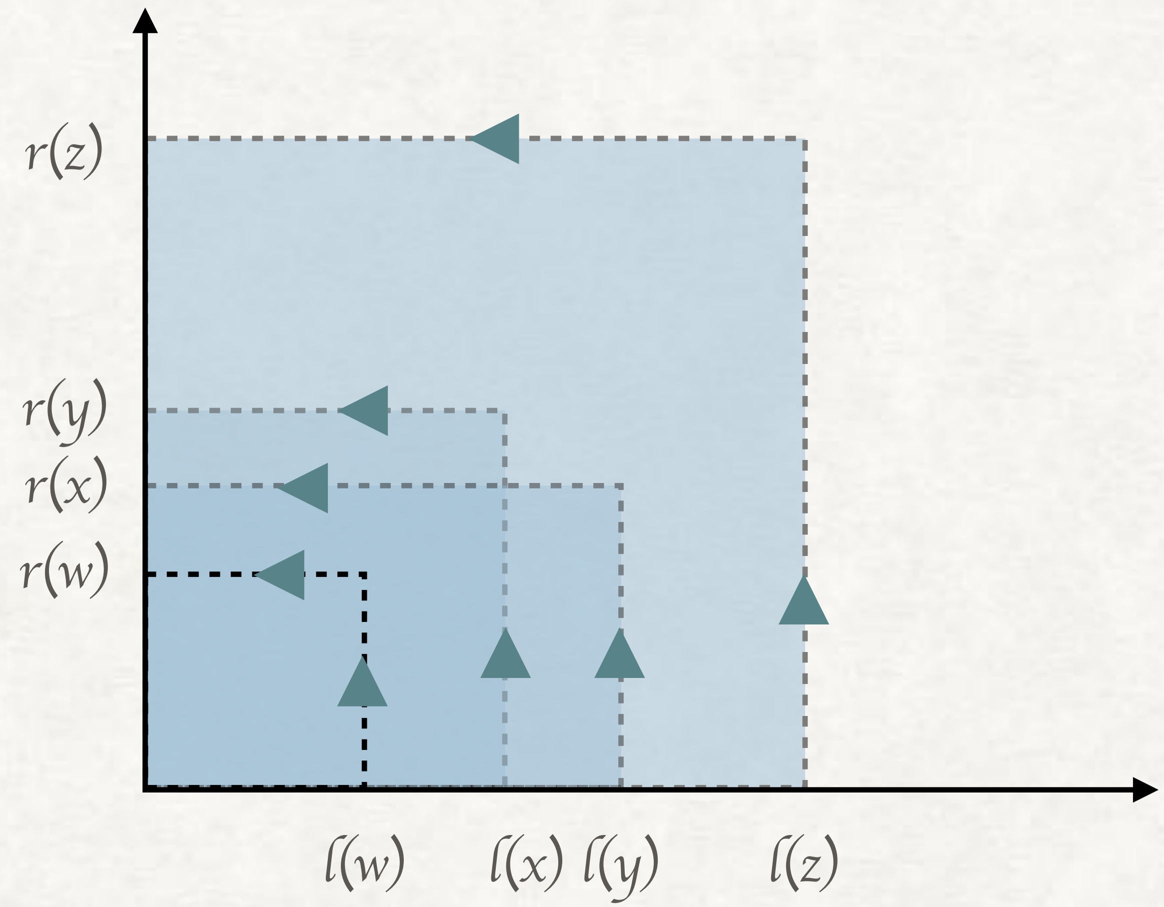


Therefore join of ℓ(x) and ℓ(y)
is ℓ(z)

RIGHT ADJOINT PRESERVES MEETS

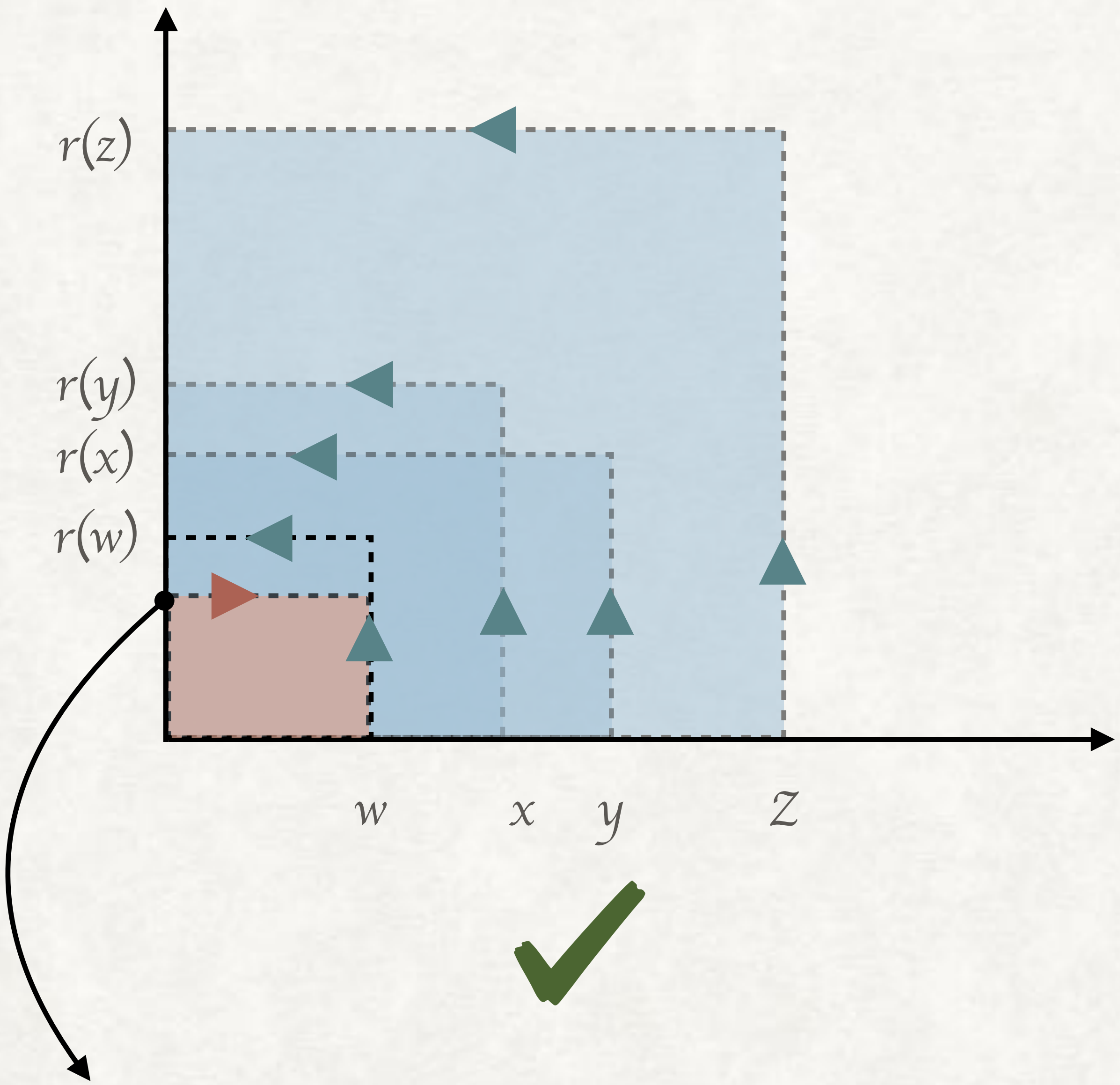
Simply means that:
If meet of x and y is z then
meet of r(x) and r(y) is r(z)

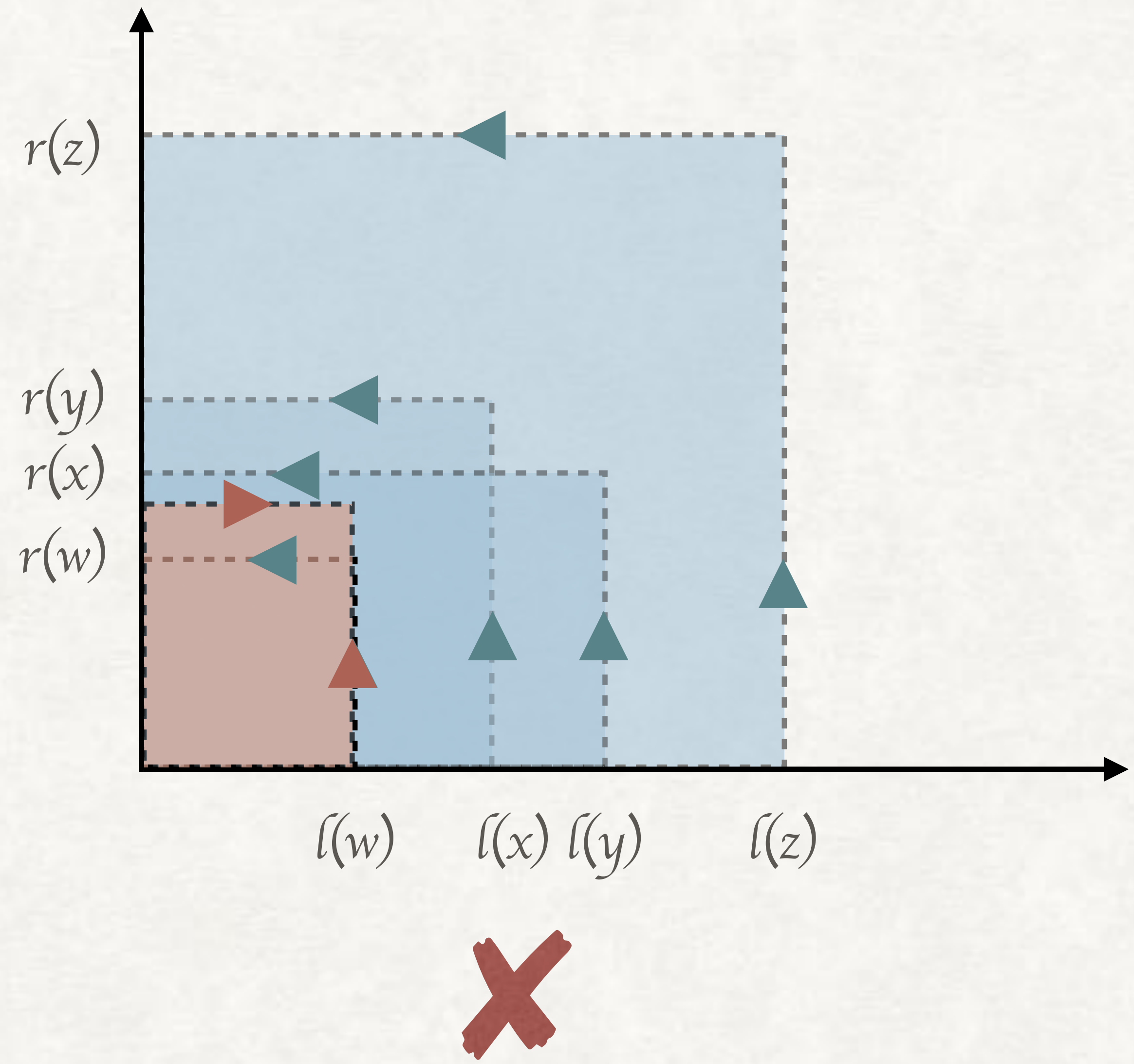




RIGHT ADJOINT PRESERVES MEETS

Simply means that:
If meet of x and y is z then
meet of r(x) and r(y) is r(z)

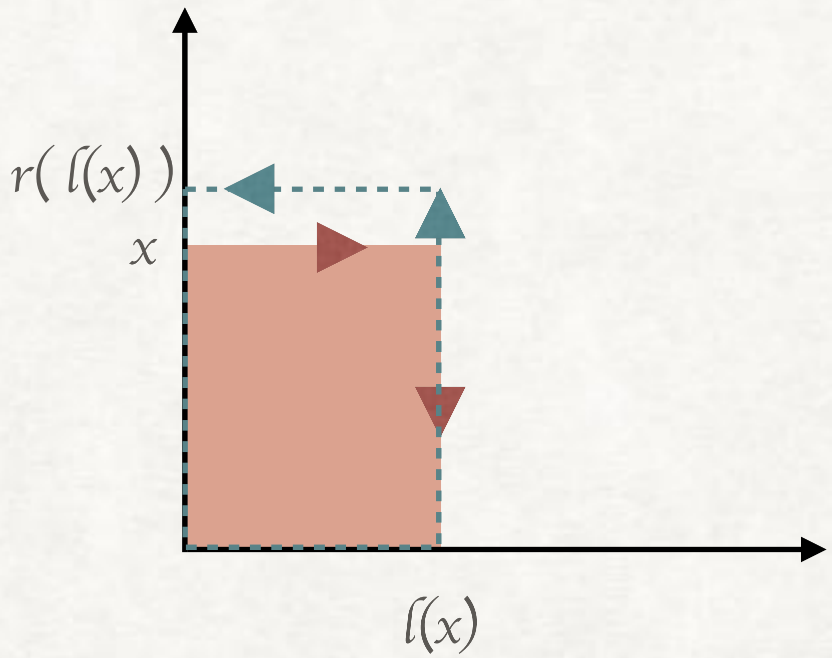


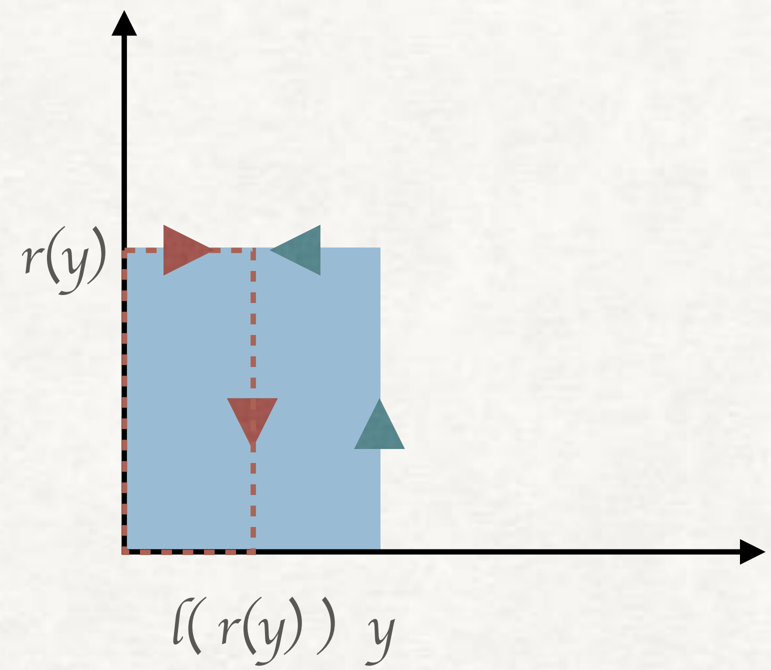


This point can only get as large as r(w) while maintaining Galois connection

Therefore join of r(x) and r(y) is r(z)

CLOSURE





$x \leq r(\,l(x)\,)$

$x \leq l \text{ then } r\ (x)$

$x \leq l\,;r\ (x)$

$l(\,r(y)\,) \leq y$

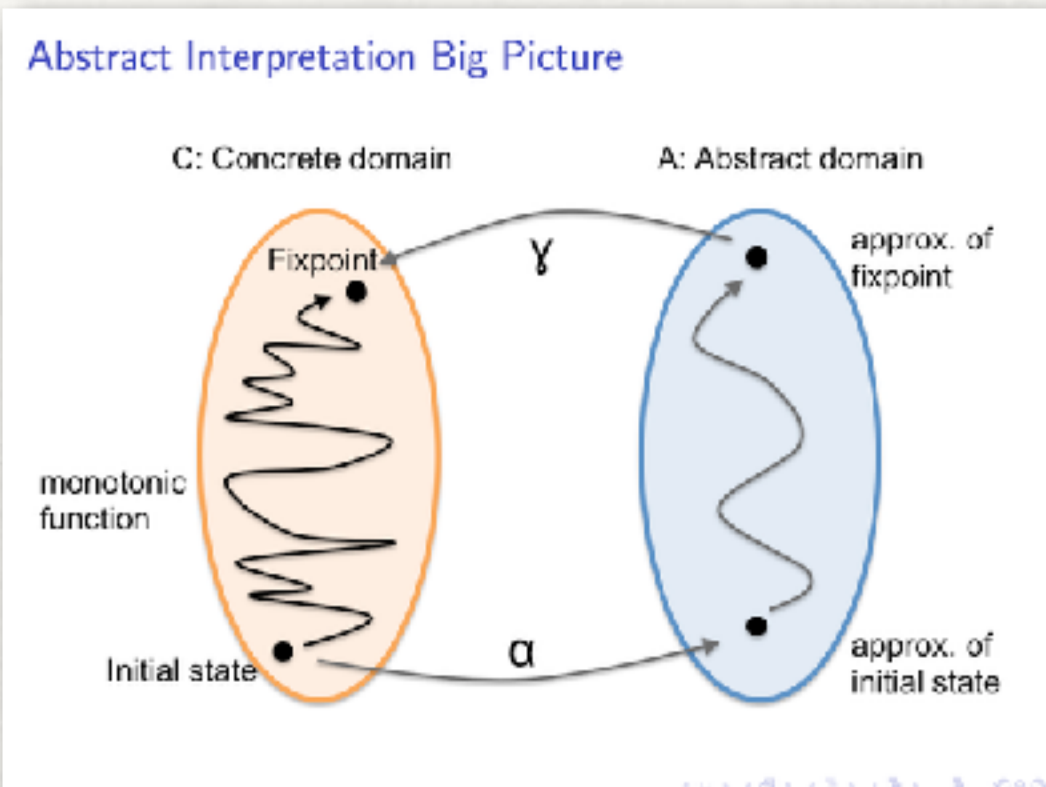
$r \text{ then } l\ (y) \leq y$

$r\,;l\ (y) \leq y$

$$l\,;r\,;l\,;r\ (x) = l\,;r\ (x)$$

APPLICATIONS

Abstract interpretation



Source: https://lara.epfl.ch/w/_media/sav17:lecturecise10.pdf

Syntax and semantics

Grammatical aspect of language

Semantic meaning of language

A statement can be syntactically correct but semantically meaningless. Eg.

Cow eats supremely

Semantics is left adjoint

Syntax is right adjoint

APPLICATIONS

Probability distribution

Cumulative distribution function

Left adjoint

$F_X(x)$: $\mathbb{P}(X \leq x)$

Quantile function

Right adjoint

$Q_X(p)$: $\inf\{x \in \mathbb{R} : p < F(x)\}$

Programming

Programming from Galois Connections

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[a]*Institute of Information Science, Academia Sinica, Taiwan*
[b]*High Assurance Software Lab / INESC TEC and Univ. Minho, Portugal*

lead to specifications made of two parts: one defining a broad class of solutions (the *easy* part) and the other requesting one particular such solution, optimal in some sense (the *hard* part).

… analogous to …

Pragmatic means the speaker wants to be ***correct*** and ***as specific*** as possible

Quantity

Quality

THANK YOU