UNIT II

PROBABILITY AND RANDOM VARIABLES

1. Define Probability and Write Axioms of Probability.

$$P(A) = \frac{n(A)}{n(S)}$$

Axioms :

(i) $0 \le P(A) \le 1$

(ii) $P(S) = 1$.

2. State Baye's Theorem

$$P(A_i/B) = \frac{P(A_i)\,P(B/A_i)}{\sum_i P(A_i)\,P(B/A_i)}$$

where $B \subset \bigcup_i A_i$, A_i's are mutually exclusive events.

3. A Continuous random variable X follows the probability law $f(x) = Ax^2$, $0 \le x \le 1$. Determine A and find the prob. that X lies between 0.2 & 0.5.

Solution

$$\int_0^1 f(x)\,dx = 1$$

$$\int_0^1 Ax^2\,dx = 1$$

$$A\left[\frac{x^3}{3}\right]_0^1 = 1$$

$$A\left[\frac{1}{3}\right] = 1$$

$$\boxed{A = 3}$$

$$P(0.2 < X < 0.5) = \int_{0.2}^{0.5} A x^2\, dx$$

$$= \int_{0.2}^{0.5} 3x^2\, dx = \left[\frac{\cancel{3}x^3}{\cancel{3}}\right]_{0.2}^{0.5}$$

$$= (0.5)^3 - (0.2)^3$$

$$= 0.125 - 0.008$$

$$= 0.117$$

4 If X has a Poisson distribution such that $P(X=1) = P(X=2)$, find $P(X=4)$.

<u>Ans:</u>

$$\frac{e^{-\lambda}\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^2}{2!} \qquad \because \text{For Poisson dist. } P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\Rightarrow 1 = \frac{\lambda}{2} \Rightarrow \boxed{\lambda = 2}$$

$$\therefore P(X=4) = \frac{e^{-\lambda}\lambda^4}{4!} = \frac{e^{-2} \times 2^4}{4!}$$

The prob. that a student passes in Physics test is $2/3$ and the prob that he passes both physics and English test is $\frac{14}{45}$. The prob that he passes atleast one test is $4/5$. What is the prob. that he passes in English test?

Ans:

Given. $P(A) = 2/3$, $P(A \cap B) = \frac{14}{45}$, $P(A \cup B) = 4/5$

w.k.t $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\frac{4}{5} = 2/3 + P(B) - \frac{14}{45}$$

$$P(B) = \frac{4}{5} - 2/3 + \frac{14}{45}$$

$$= \frac{36 - 30 + 14}{45} = \frac{20}{45} = \frac{4}{9}$$

$$P(B) = \frac{4}{9}.$$

6. Find the MGF of Poisson distribution.

Ans:

$$\text{M.G.F } M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} P(x)$$

Poisson pdf is $P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$

$$\therefore M_X(t) = \sum_{x=0}^{\infty} \frac{e^{tx} e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^{tx} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \left\{ \frac{(\lambda e^t)^0}{0!} + \frac{(\lambda e^t)^1}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots \right\}$$

$$= e^{-\lambda} \left\{ 1 + \frac{(\lambda e^t)^1}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots \right\}$$

$$= e^{-\lambda} \left\{ e^{\lambda e^t} \right\} \qquad \left(\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \right)$$

$$= e^{-\lambda + \lambda e^t} = e^{\lambda(e^t - 1)}$$

$$\boxed{\therefore M_x(t) = e^{\lambda(e^t - 1)}}$$

7. Given $P(A) = 1/2$, $P(B) = 1/3$, $P(A \cap B) = 1/4$. State whether the events A & B are (i) mutually exclusive (ii) indepedent.

<u>Ans:</u> Since $P(A \cap B) \neq 0$, the events are not mutually exclusive.

Since $P(A \cap B) \neq P(A) \cdot P(B)$, the events are not independent.

The mean S.D of a binomial dist. are respectively 4 & $\sqrt{8/3}$. Find the values of q & p.

Ans: Given Mean $np = 4$ & S.D $\sigma = \sqrt{8/3}$

$Var = npq = 8/3$.

$$\frac{npq}{np} = \frac{8/3}{4}$$

$$q = 8/3 \times 1/4 = 2/3$$

$$\boxed{q = 2/3}$$

$$\therefore p = 1-q = 1 - 2/3 = 1/3.$$

9. A random variable x has p.d.f $f(x) = \frac{1}{2^x}$, $x = 1, 2, 3, \ldots$ Find its moment generating function.

Ans:

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=1}^{\infty} e^{tx} \times \frac{1}{2^x}$$

$$= \sum_{x=1}^{\infty} \frac{e^{tx}}{2^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(e^t/3\right)^3 + \cdots$$

$$= (1 - e^t/2)^{-1} - 1$$

$\because (1-x)^{-1} = 1 + x + x^2 + \cdots$

$\Rightarrow x + x^2 + x^3 + \cdots = (1-x)^{-1} - 1$

10. A continuous r.v X has p.d.f $f(x)$ given by $f(x) = \begin{cases} 2x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

Find $P(X < 1/2)$.

Ans:

$$P(X < 1/2) = \int_0^{1/2} f(x)\,dx$$

$$= \int_0^{1/2} 2x\,dx = \left[\frac{2x^2}{2}\right]_0^1$$

$$= 1^2 - 0^2 = 1$$

11. The mean of a binomial dist. is 20 and S.D is 4. Find the parameters.

Ans:

Given mean $np = 20$

$S.D = \sqrt{npq} = 4$

$Var = npq = 16$

$$\frac{npq}{np} = \frac{16}{20}$$

$$q = \frac{16}{20} = \frac{4}{5}$$

$$\boxed{q = 4/5}$$

~~$p = 1 - 1/2$~~ $p = 1 - q = 1 - 4/5 = 1/5$.

$\therefore$ Parameters are $p = 1/5, q = 4/5$.