Week 7 Class Problems

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1. Find a function f and a number a such that

$$lim_{h\to 0}\frac{(2+h)^6-64}{h}=f^{'}(a) \tag{1}$$

From the definition of a derivative,

$$lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f^{'}(x) \tag{2}$$

We can see that the function is $f(x)=x^6$ and $a=2$.

2. Suppose that the number of calories of heat required to raise 1 gram of water (or ice) from -40°C to x°C is given by

$$f(x)=\begin{cases}\frac{1}{2}x+20 & \text{if } -40\leq x<0\\ x+100 & \text{if } 0\leq x\end{cases} \tag{3}$$

Is the function continuous $\forall x\in[-40,\infty)$? What happens to water at 0°C that account for the behavior of the function at 0 °C?

The function is not contiuous $\forall x\in[-40,\infty)$. At $x=0$,

$$lim_{x\to 0^-}\frac{1}{2}x+20=20 \tag{4}$$

and

$$lim_{x\to 0^+}x+100=100 \tag{5}$$

Therefore, since the $lim_{x\to 0^-}f(x)\neq lim_{x\to 0^+}f(x)$ the function is not continuous at $x=0$. At 0°C water goes through a phase change from liquid to solid.

3. Find the asymptotes of the graph of $f(x)=\frac{4-x}{3+x}$ and use them to sketch the graph and the graph of $f^{'}$. Find $f^{'}$ and graph it in R.

We can see that the function is not continuous at $x=-3$. So there may be an asymptote there. We need to look at the limits at $x=-3$

$$lim_{x\to -3^-}\frac{4-x}{3+x}=\frac{lim_{x\to -3^-}4-x}{lim_{x\to -3^-}3+x}=\frac{7}{0_-}=-\infty \tag{6}$$

and

$$lim_{x\to -3^+}\frac{4-x}{3+x}=\frac{lim_{x\to -3^+}4-x}{lim_{x\to -3^+}3+x}=\frac{7}{0_+}=+\infty \tag{7}$$

What about the behavior as $x \to -\infty$ and $x \to +\infty$.

$$lim_{x \to -\infty} \frac{4-x}{3+x} \text{ and } lim_{x \to +\infty} \frac{4-x}{3+x} \tag{8}$$

To calculate these, we have to follow L'Hospital's Rule which says that if we have

$$lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\pm\infty}{\pm\infty} \tag{9}$$

with a as any real number or $\pm\infty$. We can have

$$lim_{x \to a} \frac{f(x)}{g(x)} = lim_{x \to a} \frac{f^{'}(x)}{g^{'}(x)} \tag{10}$$

Using this, we have

$$lim_{x \to -\infty} \frac{4-x}{3+x} = lim_{x \to -\infty} \frac{-x}{x} = -1 \text{ and } lim_{x \to +\infty} \frac{-x}{x} = -1 \tag{11}$$

Overall, there are two/three asymptotes.

At $x = -3$ where from the left $f(x) \to -\infty$ and from the right $f(x) \to +\infty$.

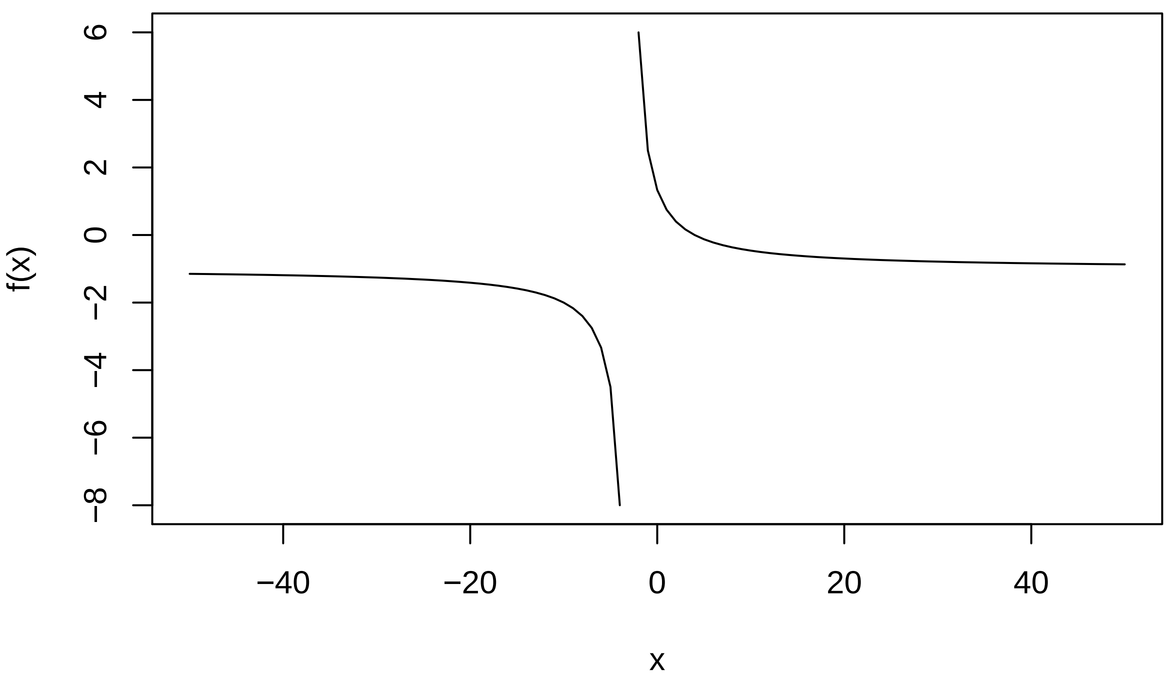
As $x \to -\infty$, $f(x) \to -1$ from the bottom (as at $x \to -3^{-} = -\infty$).

As $x \to +\infty$, $f(x) \to -1$ from the top (as at $x \to -3^{+} = +\infty$).

```r
n.3 = function(x){return((4-x)/(3+x))}

x.seq = seq(-50, 50)

plot(x.seq, n.3(x.seq), type="l", xlab="x", ylab="f(x)")
```



4. A bacteria culture contains 200 cells initially and grows at a rate proportional to its size. After half an hour the population has increased to 360 cells.

- Find the number of bacteria after t hours.
- Find the number of bacteria after 4 hours.
- Find the rate of growth after 4 hours
- When will the population reach 10,000?

Since we are given that the population grows at a rate that is proportional to its current size, we can use the model

$$N(t) = N_0 e^{rt} \tag{12}$$

what we do not know and need to find is r. We can rearrange the equation to

$$r = \frac{ln(\frac{N_t}{N_0})}{t} \tag{13}$$

Using the information from the problem

$$r = \frac{ln(\frac{360}{200})}{0.5} = 1.175573 \tag{14}$$

Which means our model is

$$N(t) = 200e^{1.175573*t} \tag{15}$$

After 4 hours,

$$N(4) = 200e^{1.175573*4} \approx 22040 \tag{16}$$

To find the rate of growth, we need to take a derivative

$$N^{'}(t) = 1.175573 * 200e^{1.175573*t} = r * N(t) \tag{17}$$

So, if we set t=4,

$$N^{'}(4) = r * N(t) = 1.175573 * 22040 \approx 25910 \text{ bacteria/hour} \tag{18}$$

To find the time when the population will reach a certain size we need to rearrange $N(t)$

$$N(t) = N_0 e^{rt} \tag{19}$$

$$t = \frac{ln(\frac{N(t)}{N_0})}{r} \tag{20}$$

We need to determine when the population reaches 10000 individuals.

$$t = \frac{ln(\frac{10000}{200})}{1.175573} \approx 3.33 \text{ hours} \tag{21}$$