Date.

Lecture 1

1.1 (a) $\begin{pmatrix} 1 & -1 & & \\ & 1 & & \\ & -1 & 1 & \\ & -1 & & 1 \end{pmatrix}\begin{pmatrix} 1 & & 1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \frac{1}{2} & \\ & & & 1 \end{pmatrix} B \begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 0 \end{pmatrix}\begin{pmatrix} 0 & & \\ 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & -1 & & \\ & 1 & & \\ & -1 & 1 & \\ & -1 & & 1 \end{pmatrix}\begin{pmatrix} 1 & & 1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \frac{1}{2} & \\ & & & 1 \end{pmatrix}$

$C = \begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 0 \end{pmatrix}\begin{pmatrix} 0 & & \\ 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

1.2 (a) $\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} -k_{12} & k_{12} & 0 & 0 \\ k_{12} & -(k_{12}+k_{23}) & k_{23} & 0 \\ 0 & k_{23} & -(k_{23}+k_{34}) & k_{34} \\ 0 & 0 & k_{34} & -k_{34} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} -l_{12}k_{12} \\ l_{12}k_{12} - l_{23}k_{23} \\ l_{23}k_{23} - l_{34}k_{34} \\ l_{34}k_{34} \end{pmatrix}$

1.3. Proof. 由(1.8)知，$e_j = \sum_i z_{ij} r_i$，r_i是R的第i列. $e_j = (0, \dots 0, \underset{\text{第j位}}{1}, 0 \dots 0)^T$

不妨设 $j < m$. 由 $0 = (e_j)_m = \sum_i z_{ij} r_{im} = z_{mj} r_{mm}$

R可逆. $\det R = \prod_i r_{ii} \neq 0$，从而 $z_{mj} = 0$.

类似可得 $z_{ij} = 0$，$\forall i > j$.

从而 $R^{-1} = Z$ 上三角.

1.4 (a). Proof. $F = (f_{ij})_{8\times 8}$

则 $\sum_j c_j f_j(i) = (Fc)_i = d_i$. $Fc = d$

由 $\mathrm{range}(F) = \mathbb{C}^8$ 知，F列满秩，从而可逆. $c = F^{-1}d$.

(b) $A^{-1} = F$. $(A^{-1})_{ij} = F_{ij} = \cancel{\text{自}}\, f_j(i)$.

2.1. Proof: 不妨设 A 上三角. A 是酉矩阵, 从而 $A^* = A^{-1}$. A^* 是下三角; 由 Exercises 1.4 (3) 知, A^{-1} 上三角.

从而 A 是对角阵.

2.2 (a) Proof. $\|x_1+x_2\|^2 = (x_1+x_2)^*(x_1+x_2)$

$= x_1^*x_2 + x_2^*x_1 + x_1^*x_1 + x_2^*x_2$

$= x_1^*x_1 + x_2^*x_2$

$= \|x_1\|^2 + \|x_2\|^2$

(b) 设 $n \le k$ 时成立.

当 $n=k+1$ 时, $\|\sum_{i=1}^{k+1} x_i\|_2^2 = \|\sum_{i=1}^{k} x_i\|^2 + \|x_{k+1}\|^2$ $(n=2)$

$= \sum_{i=1}^{k}\|x_i\|^2 + \|x_{k+1}\|^2$ $(n=k)$

$= \sum_{i=1}^{k+1}\|x_i\|^2$

2.3 (a) Proof. A hermitian $\Rightarrow A = A^*$

$Ax=\lambda x \Rightarrow x^*A^* = \lambda^* x^* \Rightarrow x^*A = \lambda^* x^*$

$\Rightarrow x^*Ax = \lambda x^*x = \lambda^* x^*x \Rightarrow \lambda = \lambda^* \Rightarrow \therefore \lambda \in R^m$

(b) $Ax=\lambda_1 x,\ Ay=\lambda_2 y,\ \lambda_1 \ne \lambda_2$

$\cancel{y^*A^*Ax = y^*A^*\lambda_1 x = \lambda_1 y^*Ax}$

$y^*Ax = y^*\lambda_1 x = \lambda_1 y^*x$

$y^*A^*x = (Ay)^*x = (\lambda_2 y)^*x = \lambda_2 y^*x$

$\Rightarrow (\lambda_1-\lambda_2)y^*x = 0 \Rightarrow y^*x = 0$

2.4 $\|\lambda\| = 1$

Proof. $Ax=\lambda x$, $x^*A^* = \lambda^* x^*$

$\Rightarrow x^*A^*Ax = x^*\lambda^*\lambda x = \|\lambda\|^2\|x\|^2$

而 $A^*A = I$, 从而 $\|x\|^2 = \|\lambda\|^2\|x\|^2$

$\Rightarrow \|\lambda\| = 1$

2.5 (a) Proof. $(iS)^* = -iS^* = iS$

$\Rightarrow$ iS 是酉矩阵, 由 2.3 知, $iSx = \lambda x$, $\lambda \in R^m$

从而 S 的特征值为一纯虚数.

(b) $(I-S)x = x-Sx$

由(a)知，1不是S的特征值. 从而 $\forall x\neq 0$, $x-Sx\neq 0$

即 0不是 $I-S$ 的特征值. 从而 $I-S$ 非奇异

(c) $QQ^* = (I-S)^{-1}(I+S)(I+S)^*(I-S)^{-*} = I$

只需证 $(I+S)(I+S)^* = (I-S)(I-S)^*$

$I+SS^* = I+SS^*$ 成立

2.6 Proof. $(I+uv^*)(I+\alpha uv^*)$

$= I+(\alpha+1)uv^* + \alpha uv^*uv^*$

$= I+[(\alpha+1)+\alpha v^*u]uv^*$

$= I$

$\Rightarrow \alpha = -\frac{1}{1+v^*u}$

当 $v^*u=-1$ 时, A 奇异.

$\mathrm{Null}(A) = \{\alpha u\}$.

2.7 $H_{k+1}^T H_{k+1} = \begin{bmatrix} H_k^T & H_k^T \\ H_k^T & -H_k^T \end{bmatrix} \begin{bmatrix} H_k & H_k \\ H_k & -H_k \end{bmatrix}$

$= \begin{bmatrix} H_k^T H_k & 0 \\ 0 & H_k^T H_k \end{bmatrix}$

$= \alpha_{k+1} I.$