20.1 若 $A=LU$ $\det A = \det L \cdot \det U = (\prod L_{ii})(\prod U_{ii})$

A非奇异 $\Rightarrow \det A \neq 0 \Rightarrow L_{ii}\neq 0,\ U_{ii}\neq 0,\ \forall i$

$$A = \begin{pmatrix} A_{1:k,1:k} & \sim \\ \sim & \sim \end{pmatrix} = \begin{pmatrix} L_{1:k,1:k} & 0 \\ \sim & \sim \end{pmatrix}\begin{pmatrix} U_{1:k,1:k} & \sim \\ 0 & \sim \end{pmatrix}$$

$\Rightarrow A_{1:k,1:k} = L_{1:k,1:k}\, U_{1:k,1:k}$

$\det(A_{1:k,1:k}) = \prod_{i=1}^{k} L_{ii} \prod_{i=1}^{k} U_{ii} \neq 0$

否则，要证A可LU分解，只需证明第k步消去时，$x_{kk}\neq 0$.

$k=1$, $x_{11} = A_{11} \neq 0$ (A_{kk}非奇异)

设$k=n$时成立，当$k=n+1$时，

$$\det(A_{1:k+1,1:k+1}) = \det(\tilde{A}_{1:k+1,1:k+1}) = \det\begin{pmatrix} \tilde{A}_{1:k,1:k} & \sim \\ 0 & x_{k+1,k+1} \end{pmatrix} = \prod_{i=1}^{k+1} x_{ii} \neq 0, \Rightarrow x_{k+1,k+1}\neq 0$$

变换后的 $A_{1:k+1,1:k+1}$ 上三角阵.

从而LU分解存在

设 $A=L_1U_1=L_2U_2$.

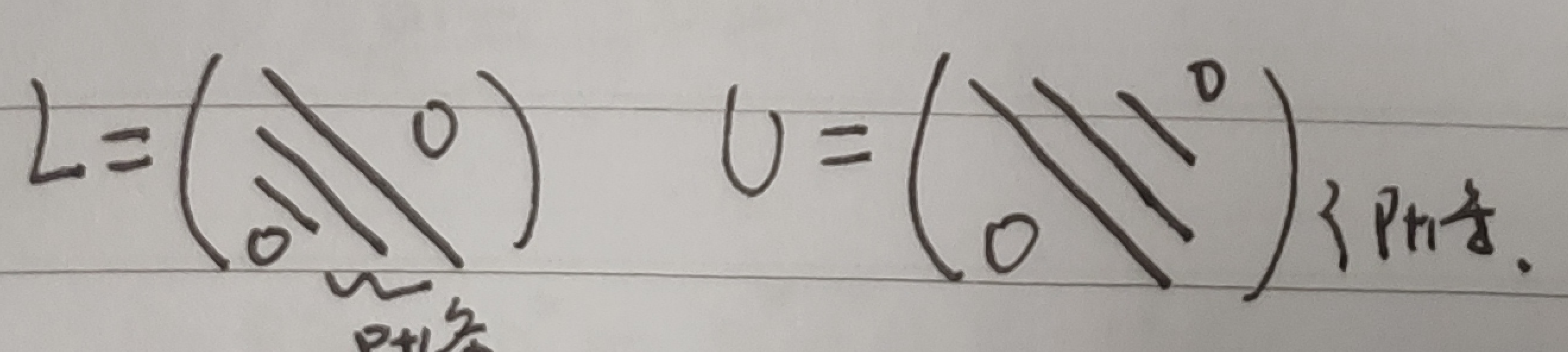
则 $L_2^{-1}L_1 = U_2U_1^{-1}$

L_1, L_2对角元都是1，由L_2^{-1}的形式（对角元仍然全是1）的下三角，有 $L_2^{-1}L_1$是对角元全是1的下三角.

$U_2U_1^{-1}$是上三角，从而 $L_2^{-1}L_1 = U_2U_1^{-1} = I \Rightarrow L_1=L_2,\ U_1=U_2$

20.2 L主对角线下至多有p条次对角线.

U主对角线上至多有p条次对角线



由L的元素的构造方式知L结构

由U的变换方式（k行乘以常数加到$j>k$行）知U结构

20.3 (a)
$$\begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \sim \end{pmatrix}$$

即 $B_2A_{11} + B_3A_{21} = 0$ ($B_3 = I$?)

$\Rightarrow B_2 = -A_{21}A_{11}^{-1}$

$\tilde{A}_{22} = B_2A_{12} + B_3A_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}$

21.1

(a) $\det A = \det L \cdot \det U = 8$

(b) $\det P \det A = \det L \det U = -8$. $\det A = 8$

(c) 由(21.3)式知 $\det P \det A = \det L \det U$

$\det P = 1$ 若偶数次交换，$\det P = -1$ 若奇数次交换

$\det A = \det P \det L \det U$.

21.2 L下三角. U主对角线为1的上三角

21.3 (a) 第k步时，$\hat{A}_{k,i} = 0$，$\forall 1 \le i < k$

但 $\tilde{A}_{kj}$ 至少有一个不为0，$j \ge k$

否则 矩阵第k行全为0，且 $|\det(\tilde{A})| = |\det(A)| \neq 0$.

找出 $\max_{j \ge k} |\hat{A}_{kj}|$ 交换即可

(b) A的第一行全为0，$AQ = LU$ 即不存在.

21.4 (a) $A a_i = e_i$

$A^{-1} = (a_1, a_2 \cdots a_m)$

21.5 (a) ~~$L_2 P_2 L_1 P_1 A P_1 P_2 \cdots P_m = U$~~

$\cdots L_2 P_2 L_1 P_1 A P_1 L_1^T P_2 L_2^T \cdots = D$ （P_i是对称阵）

$L^{-1} P A (L^{-1} P)^T = D$ D是对角阵.

$PAP^T = LDL^T$

~~21.6~~

23.1 $A = QR$. $A^* A = R^* Q^* Q R = R^* R$.

由 Cholesky 分解的唯一性知 $R = U$