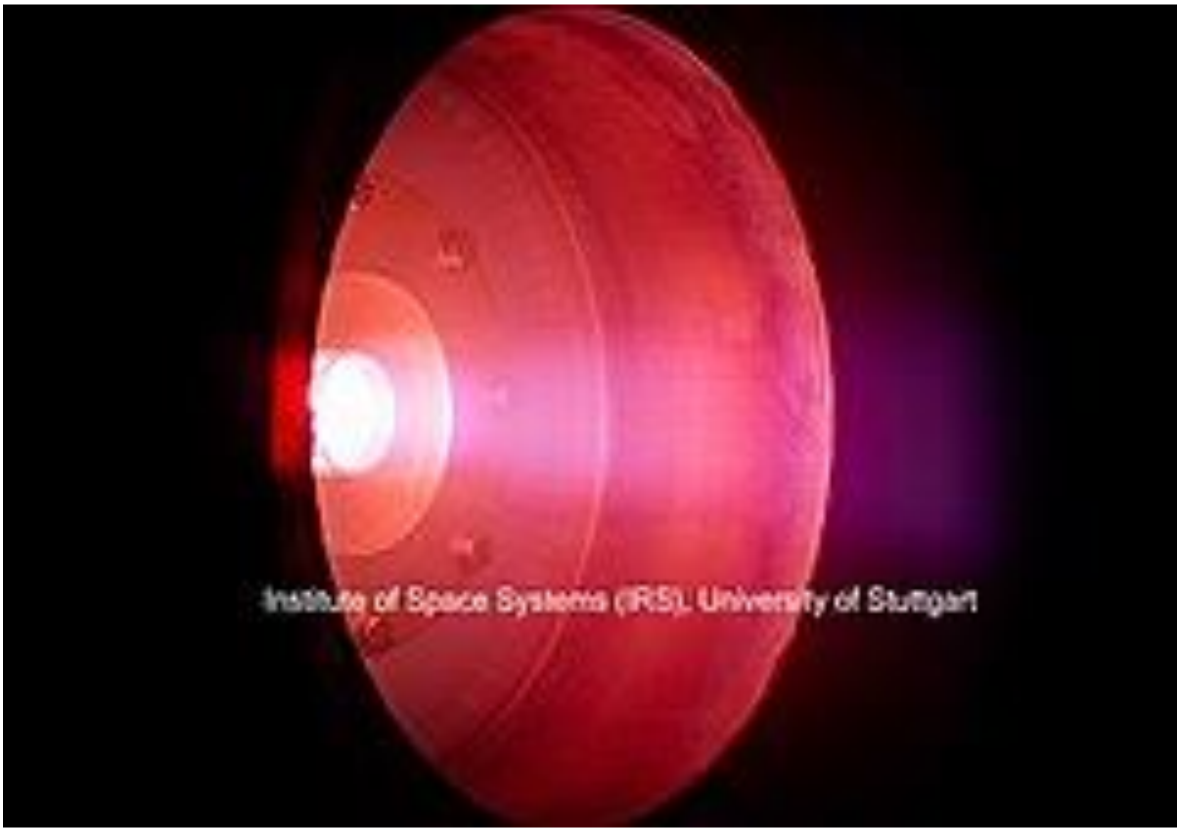
PH – 102 LAB PROJECT

Fourier transformation and Diffraction



Submitted by :

Vinita – 2021CEB1033

Vishal – 2021CEB1034

- **What Does Fourier Transform Mean?**

The Fourier transform is a mathematical function that takes a time-based pattern as input and determines the overall cycle offset, rotation speed and strength for every possible cycle in the given pattern. The Fourier transform is applied to waveforms which are basically a function of time, space or some other variable. The Fourier transform decomposes a waveform into a sinusoid and thus provides another way to represent a waveform.

The Fourier transform is also called a generalization of the Fourier series. This term can also be applied to both the frequency domain representation and the mathematical function used. The Fourier transform helps in extending the Fourier series to non-periodic functions, which allows viewing any function as a sum of simple sinusoids.

The Fourier transform of a function f(x) is given by:

$$f(x) = \int_{-\infty}^{\infty} F(k)\, e^{2\pi i k x}\, dk$$

$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i k x}\, dx$$

Where F(k) can be obtained using inverse Fourier transform.

This equation is derived from Fourier series which state that any wave could be written as the sum of sines and cosines with a constant added to it.

$$F(x) = \frac{1}{2}a_o + a_1 \cos x + a_2 \cos 2x + \ldots\ldots\ldots + a_n \cos nx$$

$$+ b_1 \sin x + b_2 \sin 2x + \ldots\ldots\ldots\ldots + b_n \sin nx .$$

where ,

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, d(x)$ **OR** $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, d(x)$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, d(x)$ **OR** $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, d(x)$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, d(x)$ **OR** $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, d(x)$

And as the equation contains exponential factor with imaginary factor, it follows the Euler's form which is

$$e^{2\pi i\theta} = \cos(2\pi\theta) + i \sin(2\pi\theta)$$

And so it makes the equation rotate about a point in a graph (maybe origin, or some other point).

Due to the properties of sine and cosine, it is possible to recover the amplitude of each wave in a Fourier series using an integral, and similarly getting the signal from each unique wave by applying Inverse Fourier transform which is quite similar to **Fourier Transform.**

- ***How could we apply Fourier transform on light?***

Light has a dual nature

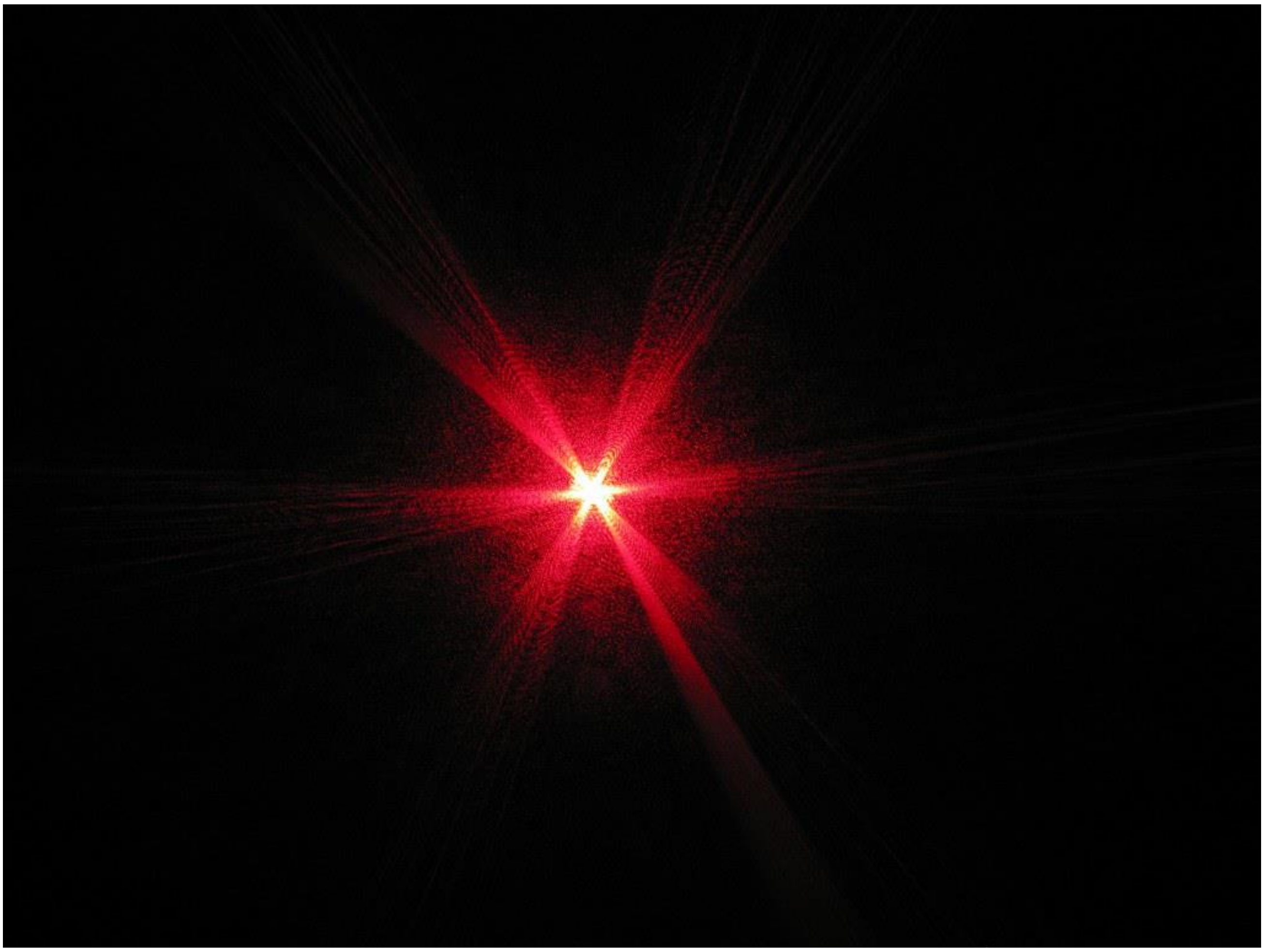
1.Sometimes it behaves like a particle (called a photon), which explains how light travels in straight lines.

2. Sometimes it behaves like a wave, which explains how light bends (or diffracts) around an object.

3. Scientists accept the evidence that supports this dual nature of light (even though it intuitively doesn't make sense to us!)

So as we see that it shows wave property whenever it is near any object, or for our purpose we can say when it passes through a hole. Light is an electromagnetic wave. This is the same type of wave that the sun uses to heat the earth and that radio, televisions, and cell phones utilize to transmit information. Light waves have an electric and magnetic field associated with them since they are electromagnetic waves. In addition, light waves of all types travel at speed of light c=300,000km/s. The frequency f and wavelength (lambda) of light satisfy the following equation

$$c = f\lambda$$



Let there be a time dependent function ‘x(t)’:

A function of time, x(t), is transformed into a function of frequency, X(ω), using the Fourier Transform mathematical approach. This and the Fourier Series are closely connected. The Fourier transform equation is given below:

$$X(f) = \int_{-\infty}^{\infty} x(t) \times e^{-i2\pi ft}\, d(t)$$

<u>OPTICAL METHOD OF FOURIER TRANSFORM</u>

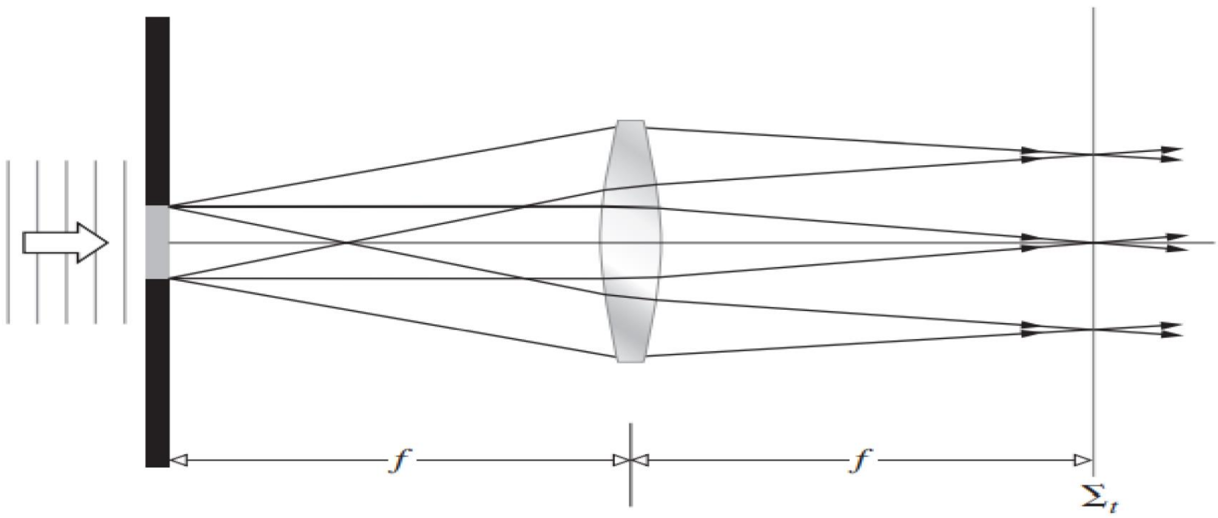
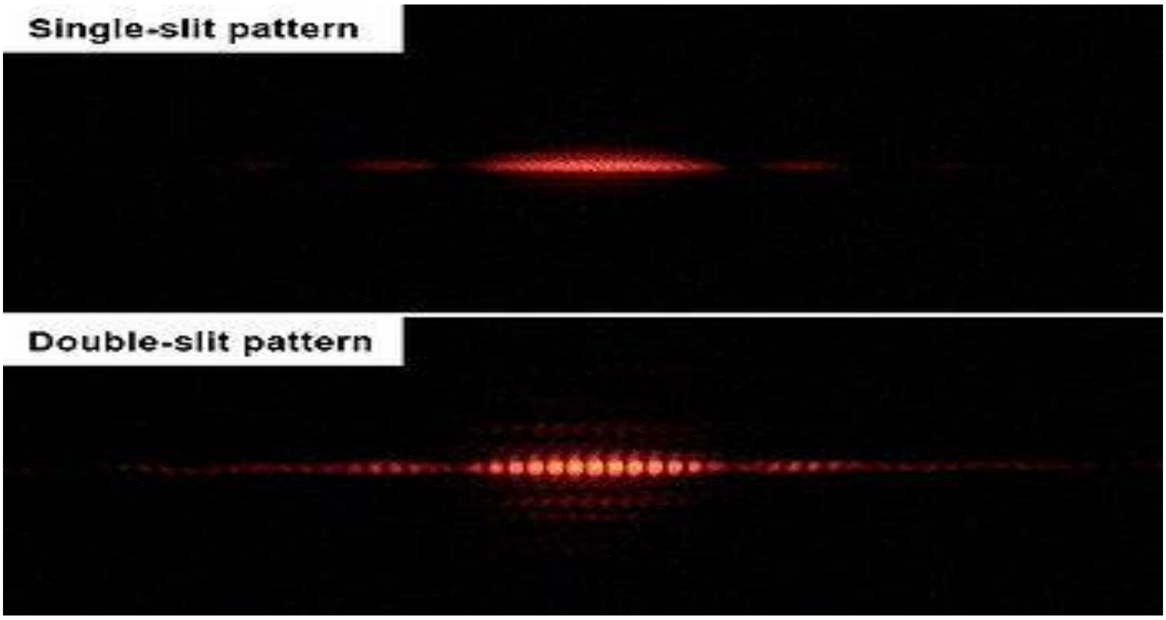


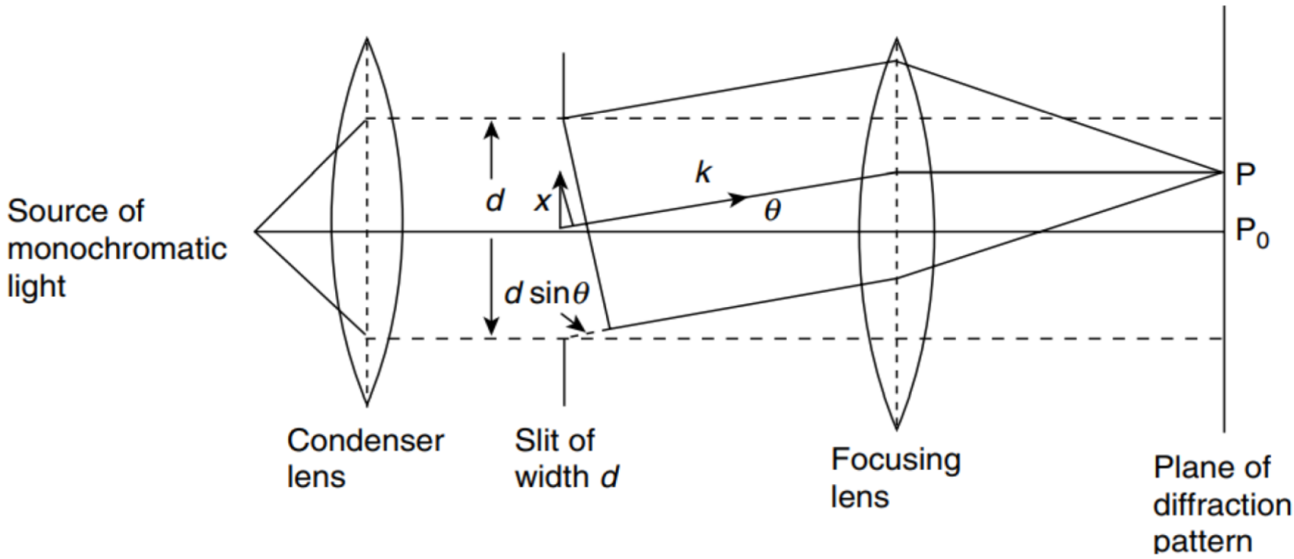
Figure 11.6 The light diffracted by a transparency at the front (or object) focal point of a lens converges to form the far-field diffraction pattern at the back (or image) focal point of the lens.

As we have understood what Fourier transform and Fourier series is, lets look into how we use optical methods to transform certain functions. In the above figure a transparency, located in the front focal plane of a converging lens, being illuminated by parallel light (plane wavefronts). This object, in turn, scatters plane waves, which are collected by the lens, and parallel bundles of rays are brought to convergence at its back focal plane. If a screen were placed there, we would see the far-field diffraction pattern of the object spread across it. Surprisingly, if the electric-field distribution across the object mask, which is known as the aperture function, is $A\{x\}$the image formed on the screen is the Fourier Transform of $A\{x\}$. In other words, aperture function is transformed by the lens into the far-field diffraction pattern which corresponds to the exact Fourier transform of the aperture function.

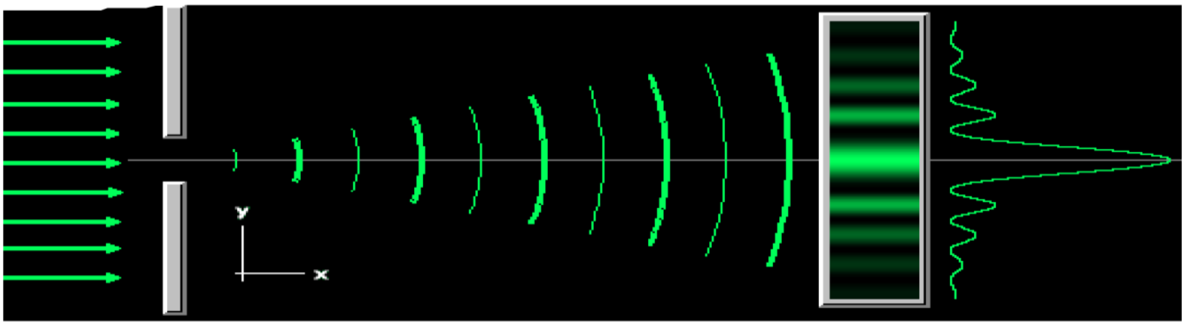


The Fourier Transform of a Single Slit

In order to obtain the Fourier, transform of the single slit aperture function by optical method, we have to create a setup as shown below:



After setting it up we will see following diffraction pattern :



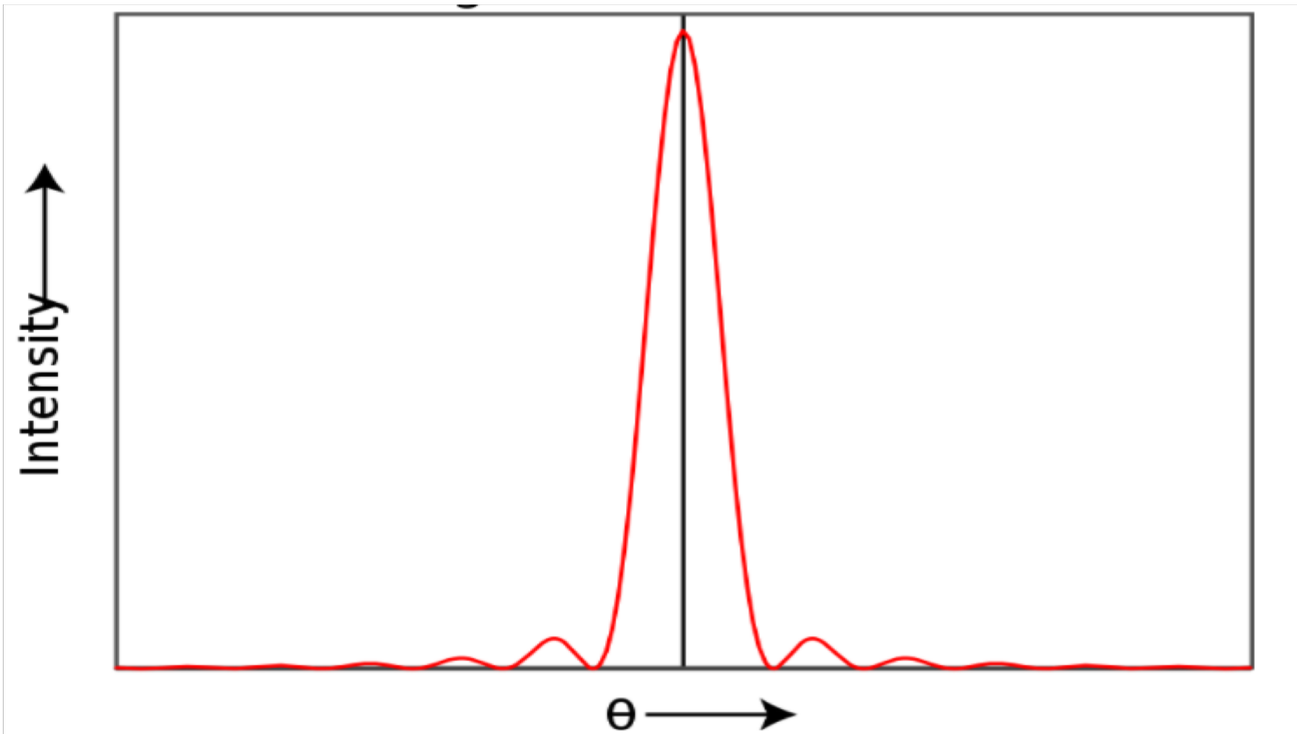


Fig: Graph of Intensity vs Θ

It can be mathematically written as :

Amplitude of light w.r.t. θ on the screen

$$A(\theta) = \frac{dh}{2\pi} \times \frac{\sin(\frac{\pi}{\lambda} dsin\theta)}{\frac{\pi}{\lambda} dsin\theta} \quad = \quad \frac{dh}{2\pi} \frac{\sin(kl\frac{d}{2})}{kl\frac{d}{2}}$$

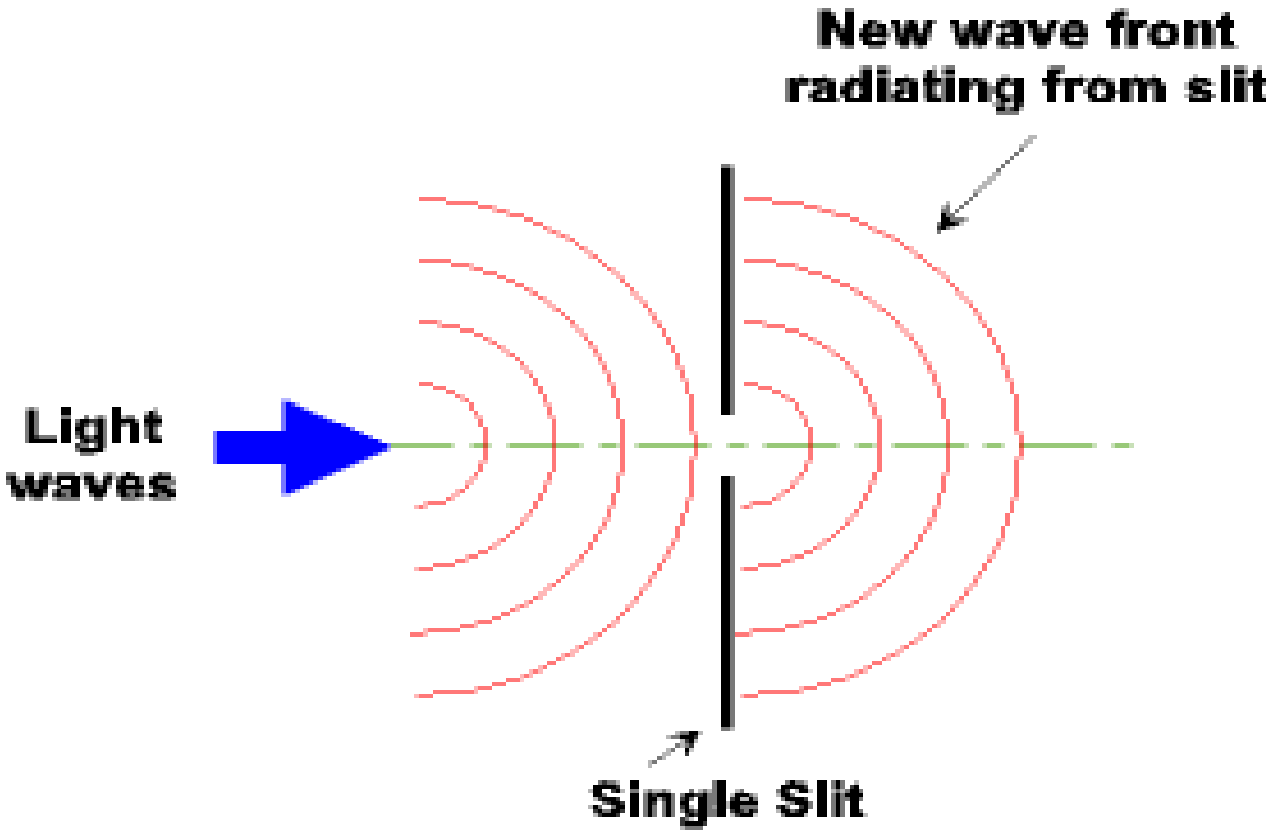
Where k = special frequency = $\frac{2\pi}{\lambda}$; d = slit width

L = sin θ ; h = amplitude of light wave

Therefore , we can say that the fourier transform of the single slit aperture is;

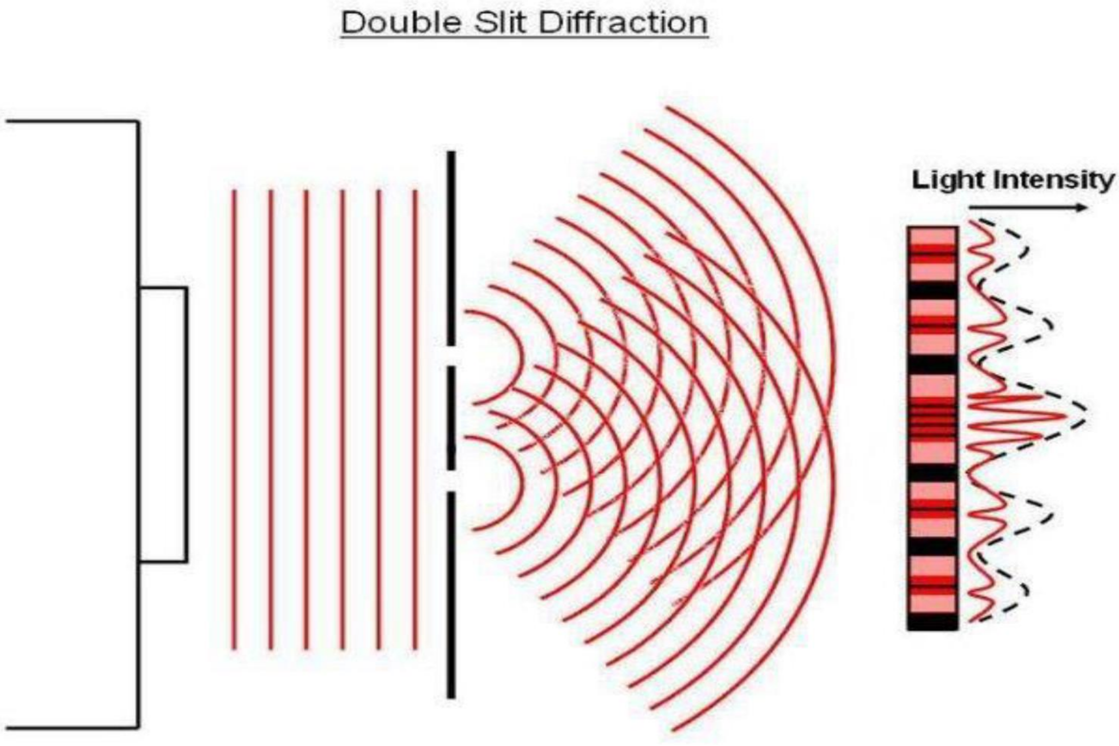
$$f(k) = \frac{dh}{2\pi}\,\frac{\sin\alpha}{\alpha}$$

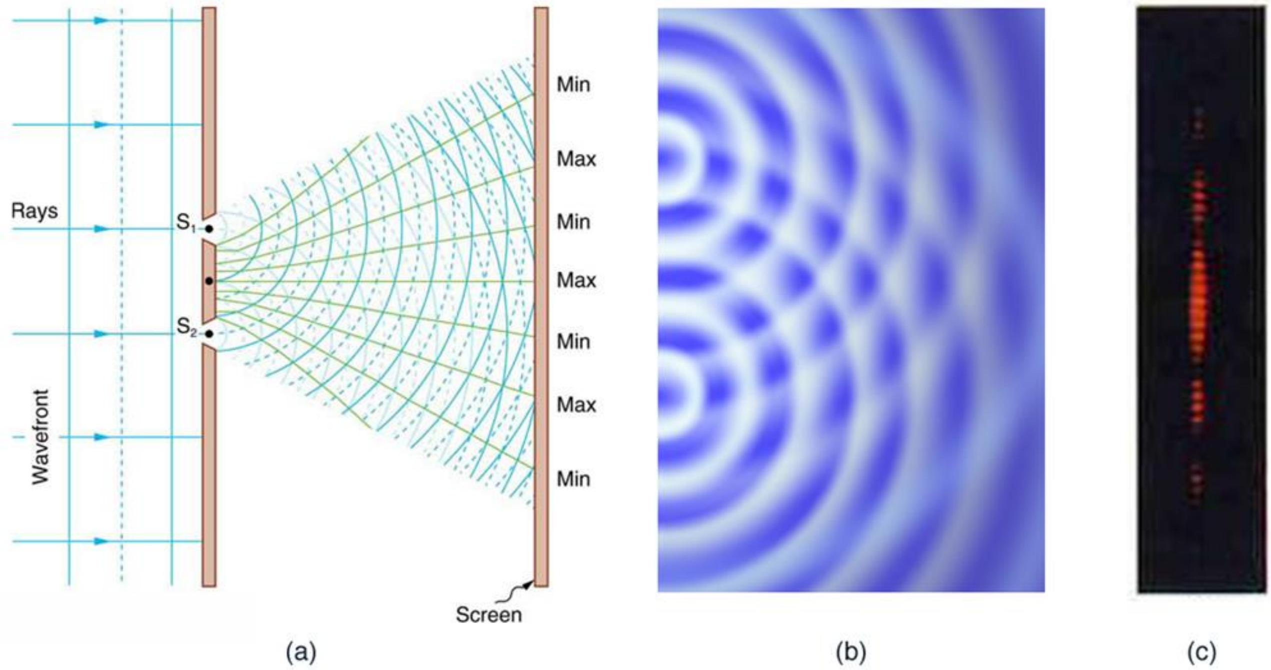
where $\alpha = \frac{kld}{2} = \frac{\pi}{\lambda} d sin\theta$



Two-Slit Diffraction Pattern

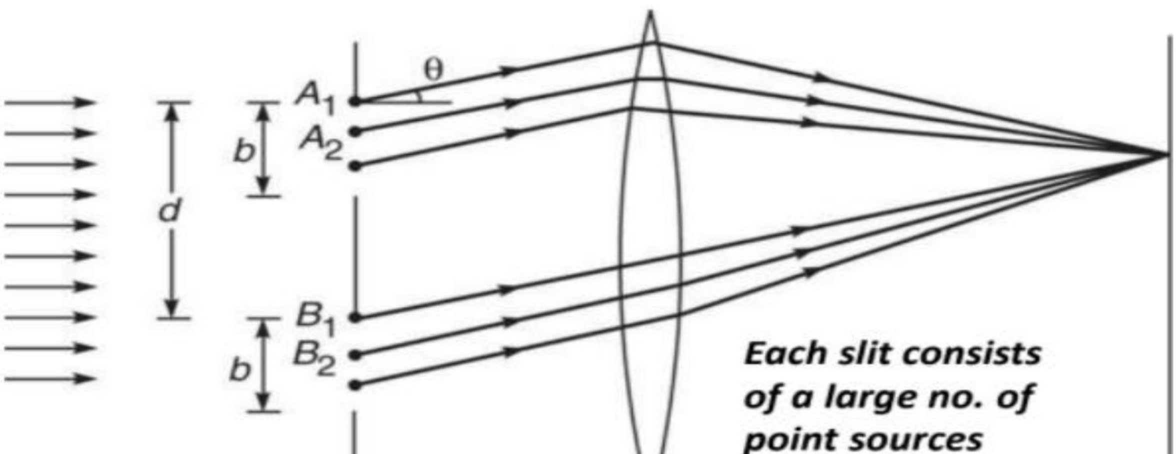
- The interference pattern of two point sources separated by d multiplied by the diffraction pattern of a slit of width a creates the diffraction pattern of two slits of width a.





(a) (b) (c)

The Fourier Transform of a Double Slit



In order to obtain the Fourier, transform of the double slit aperture function by optical method, we have to create a setup as shown above.

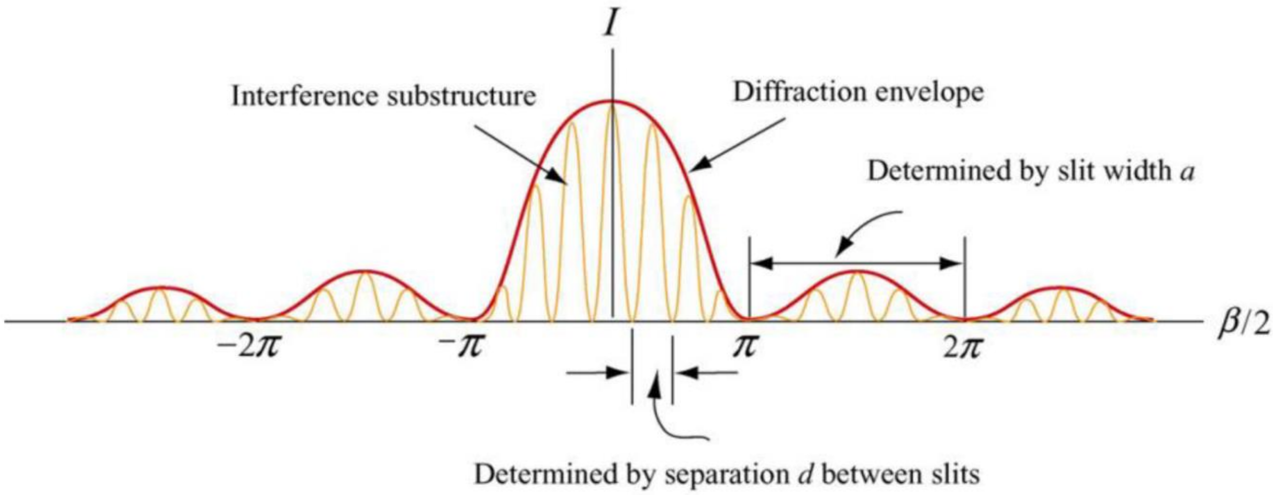


Fig: Graph of Intensity vs $\frac{\beta}{2}$

now it can be mathematically written as;

amplitude of light w.r.t. θ on the screen

$$A(\theta) = 2A\,\frac{\sin\beta}{\beta}\cos\delta$$

Where $\beta = \frac{\pi\, b\, \sin\theta}{\lambda}$; b= slit width ; λ= wavelength

$\delta = \frac{\partial}{2} = \frac{\pi d\, \sin\theta}{\lambda}$; d = distance between two slits

Hence fourier transform for double slit aperture is given by ;

$$\boxed{f(k) = 2A\,\frac{\sin\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\frac{\pi b \sin\theta}{\lambda}} \times \cos\left(\frac{\pi d\, \sin\theta}{\lambda}\right)}$$

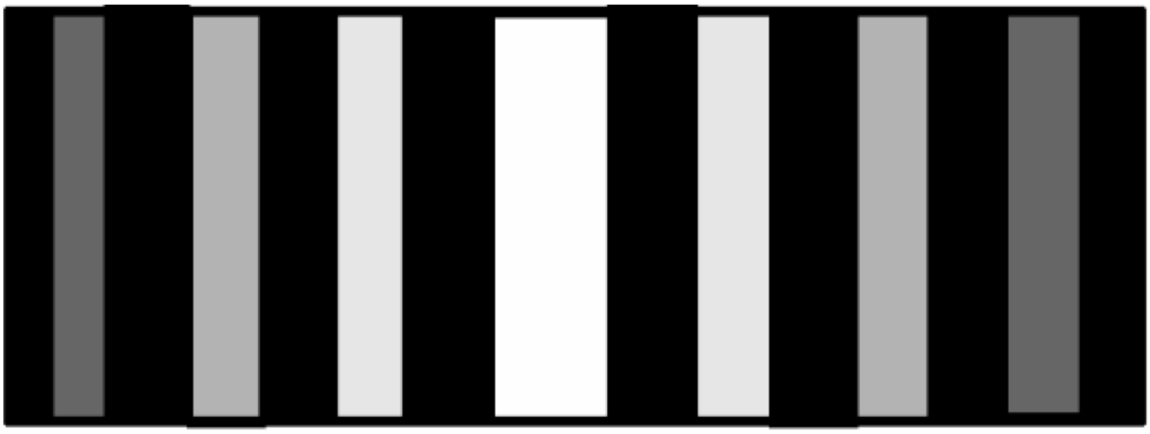
The intensity profile of light diffracted by a double slit is given by :

$$I = 4I_{max} [\frac{\sin p}{p}]^2 \cos^2 q$$

with $p = \frac{\pi}{\lambda} b \sin\alpha$, $q = \frac{\pi}{\lambda} d \sin\alpha$

For rectangular aperture:

Pattern formed on screen:



The Aperture Function A(x) corresponding to figure is given by equation

$$A(x) = rect_w \left(x - \frac{D}{2}\right) + rect_w \left(x + \frac{D}{2}\right)$$

Now to find the Fourier transform we apply the Fourier transform of each rectangular function individually.

Also using the cosine relation:

$$\mathrm{Cos}(t) = \frac{e^{it} + e^{-it}}{2}$$

We get the final result of Fourier transform of the aperture function as

$$A(X) = 2w\,\mathrm{sinc}\left(\frac{w\sin\theta}{\lambda}\right)\cos\left(\frac{\pi D\sin\theta}{\lambda}\right)$$

Intensity;

$$I = 4w^2\sin^2 c\left(\frac{w\sin\theta}{\lambda}\right)\cos^2\left(\frac{\pi D\sin\theta}{\lambda}\right)$$

Conclusion:

In the above discussion we have tried to understand the basic mathematical aspects of Fourier series and Fourier transform. After that we have looked into the topic of Fourier optics where we have specifically focused on using optics to obtain the transform. Later we have that discussed the transform of single slit, double slit and rectangular slit.

Reference:

- Optics by Eugene Hecht .
- The physics of vibration and waves by HJ Pain .
- Fundamentals of Optics by Francis A. Jenkins and Harvey E.White.
- Fourier Series - Definition, Formula, Applications and Examples (byjus.com)
- https://www.thefouriertransform.com/applications/diffraction2.php.