CP213 APPLIED MATHEMATICS AND PROBLEM SOLVING

MODEL SOLUTIONS

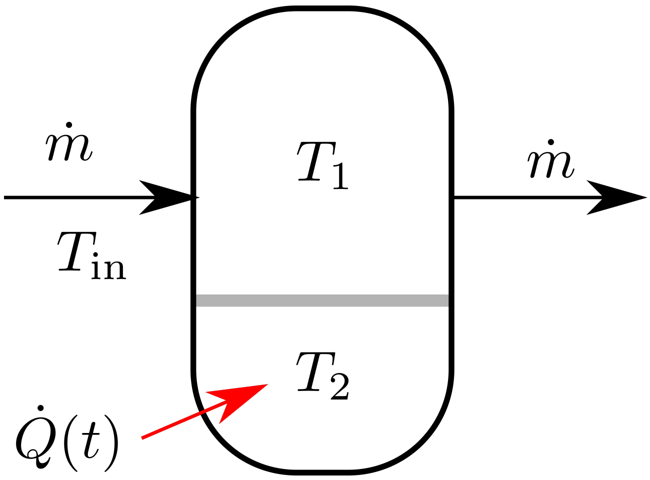
Q1. A tank is divided into two separate sections, as shown in the figure below. Section 1 of the tank contains $m_1 = 10\,\mathrm{kg}$ of water that has a heat capacity $C_1 = 4.2\,\mathrm{kJ\,kg^{-1}\,K^{-1}}$. The water in section 1 of the tank is initially at $10°\mathrm{C}$ and is fully mixed. Water at at $T_{\mathrm{in}} = 20°\mathrm{C}$ flows into section 1 of the tank at a rate of $\dot{m} = 1\,\mathrm{kg\,s^{-1}}$, and water exits the section at the same mass flow rate.

Section 2 of the tank contains $m_2 = 5\,\mathrm{kg}$ of an oil that has a heat capacity $C_2 = 1.8\,\mathrm{kJ\,kg^{-1}\,K^{-1}}$, which is fully mixed. The temperature of the oil is initially $10°\mathrm{C}$. This section is connected to an electric heater which supplies power according to

$$\dot{Q}(t) = \dot{Q}_0(1 - e^{-t/\tau})$$

where $\dot{Q}_0 = 0.1\,\mathrm{kW}$, and $\tau = 5\,\mathrm{s}$.

The two sections of the tank are separated by a metal plate. No mass is transferred beteen the two sections, but there is a heat transfer coefficient between the two sections is $UA = 0.1\,\mathrm{kW\,K^{-1}}$.



(a) Perform an energy balance around section 1 of the tank to develop a differential equation that describes the rate of change of the temperature of its contents.

[6 marks]

(b) Perform an energy balance around section 2 of the tank to develop a differential equation that describes the rate of change of the temperature of its contents.

[6 marks]

(c) Write these differential equations in matrix/vector form:

$$\frac{d}{dt}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = -\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} + \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

by providing values for the elements in the matrix $\mathbf{A}$ and vector $\mathbf{q}$. **[3 marks]**

(d) Determine the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. What is the matrix that diagonalizes $\mathbf{A}$ (i.e. the matrix $\mathbf{P}$ such that $\mathbf{P}^{-1} \cdot \mathbf{A} \cdot \mathbf{P}$ is diagonal)? **[5 marks]**

(e) Solve the differential equation to determine the variation with time of the temperature T_1 of the liquid in section 1 of the tank and the temperature T_2 of the liquid in section 2 of the tank. Plot the variation of the two tanks with time.

[10 marks]

(a) The energy balance around the first section of the tank is

$$
\begin{aligned}
m_1 C_1 \frac{dT_1}{dt} &= \dot{m} H_{\text{in}} - \dot{m} H_1 - UA(T_1 - T_2) \\
&= \dot{m} C_1 (T_{\text{in}} - T_1) - UA(T_1 - T_2) \\
\frac{dT_1}{dt} &= \frac{\dot{m}}{m_1}(T_{\text{in}} - T_1) - \frac{UA}{m_1 C_1}(T_1 - T_2) \\
&= -\left(\frac{\dot{m}}{m_1} + \frac{UA}{m_1 C_1}\right) T_1 + \frac{UA}{m_1 C_1} T_2 + \frac{\dot{m}}{m_1} T_{\text{in}}
\end{aligned}
$$

(b) The energy balance on section 2 of the tank

$$
\begin{aligned}
m_2 C_2 \frac{dT_2}{dt} &= -UA(T_2 - T_1) + \dot{Q} \\
\frac{dT_2}{dt} &= -\frac{UA}{m_2 C_2}(T_2 - T_1) + \frac{\dot{Q}(t)}{m_2 C_2} \\
&= \frac{UA}{m_2 C_2} T_1 - \frac{UA}{m_2 C_2} T_2 + \frac{\dot{Q}(t)}{m_2 C_2}
\end{aligned}
$$

(c) Comparing the two differential energy balances with the matrix/vector form of the equations leads to

$$
\begin{aligned}
A_{11} &= \left(\frac{\dot{m}}{m_1} + \frac{UA}{m_1 C_1}\right) \\
A_{12} &= -\frac{UA}{m_1 C_1} \\
A_{21} &= -\frac{UA}{m_2 C_2} \\
A_{22} &= \frac{UA}{m_2 C_2} \\
q_1 &= \frac{\dot{m}}{m_1} T_{\text{in}} \\
q_2 &= \frac{\dot{Q}(t)}{m_2 C_2} = \frac{\dot{Q}_0}{m_2 C_2}(1 - e^{-t/\tau})
\end{aligned}
$$

(d) We can determine the eigenvalues and eigenvectors of the problem by using the `Numpy` library. The required matrix $\mathbf{P}$ is just a matrix with columns equal to the eigenvectors.

```
from numpy.linalg import eig, inv
import pylab as plt

Tin = 20.0
Tinit = np.array([10.0, 10.0])


dotm = 1  # mass flow rate of water into and out of section 1
m1 = 10.0  # mass of water in section 1
m2 =  5.0  # mass of oil in section 2
UA = 0.1  # heat transfer coefficient / kW K^{-1}
C1 = 4.2
```

```
C2 = 1.8

dotQ0 = 0.1  # kW
tau = 5  # time constant for heater / s



A11 = (dotm/m1 + UA/(m1*C1))
A12 = - UA/(m1*C1)
A21 = - UA/(m2*C2)
A22 = UA/(m2*C2)
q1 = Tin * dotm/m1
q2 = dotQ0/(m2*C2)


A = np.array([[A11, A12], [A21, A22]])
q = np.array([q1, q2]).reshape(2,1)

print(A)
print(q)

lam, P = eig(A)
print()
print('eigenvalues: lambda = ')
print(lam)
print('eigenvectors: P = ')
print(P)

Pinv = inv(P)
print('P^{-1} =')
print(Pinv)
```

The eigenvalues of the matrix $\mathbf{A}$ are $\lambda_1 = 0.10266989$ and $\lambda_2 = 0.01082217$, and the matrix of eigenvectors is:

$$\mathbf{P} = \begin{pmatrix} -0.99271683 & -0.02599584 \\ 0.1204711 & -0.99966205 \end{pmatrix}$$

The inverse of the matrix $\mathbf{P}$ is

$$\mathbf{P}^{-1} = \begin{pmatrix} -1.00416766 & 0.02611301 \\ -0.12101408 & -0.99719114 \end{pmatrix}$$

(e) To solve this matrix equation, we define the "rotated" vector $\mathbf{y}$ in terms of the original temperature vector:

$$\mathrm{y} = \mathbf{P}^{-1} \cdot \mathbf{T}.$$

In terms of this new variable, the equations become decoupled:

$$\begin{aligned} \frac{d}{dt}\mathbf{T} &= -\mathbf{A} \cdot \mathbf{T} + \mathbf{q} \\ \mathbf{P}^{-1} \cdot \frac{d}{dt}\mathbf{T} &= -\mathbf{P}^{-1} \cdot \mathbf{A} \cdot \mathbf{P} \cdot \mathbf{P}^{-1} \cdot \mathbf{T} + \mathbf{P}^{-1} \cdot \mathbf{q} \\ \frac{d}{dt}\mathbf{y} &= -\mathbf{D} \cdot \mathbf{y} + \mathbf{q}' \end{aligned}$$

This is a set of decoupled first order differential equations that we can solve individually. This problem is a bit more difficult to solve than previous the coupled

ODEs we have encountered because the inhomogeneous term depends on time. However, we have seen these kinds of equations before (first order linear) and solved them using an integrating factor:

$$\begin{aligned}
\frac{dy_k}{dt} &= -\lambda_k y_k(t) + q'_k(t) \\
\left[\frac{dy_k}{dt} + \lambda_k y_k(t)\right] e^{\lambda_k t} &= q'_k(t) e^{\lambda_k t} \\
\frac{d}{dt}\left[y_k(t) e^{\lambda_k t}\right] &= q'_k(t) e^{\lambda_k t} \\
y_k(t) e^{\lambda_k t} - y_k(0) &= \int_0^t dt' q'_k(t') e^{\lambda_k t'} \\
y_k(t) &= y_k(0) e^{-\lambda_k t} + e^{-\lambda_k t} \int_0^t dt' q'_k(t') e^{\lambda_k t'}
\end{aligned}$$

To perform the final integral, we note that

$$\begin{aligned}
q'_1(t) &= P^{-1}_{11} q_1 + P^{-1}_{12} q_2(t) \\
&= P^{-1}_{11} q_1 + P^{-1}_{12} \frac{\dot{Q}_0}{m_2 C_2}(1 - e^{-t/\tau}) \\
q'_2(t) &= P^{-1}_{21} q_1(t) + P^{-1}_{22} q_2(t) \\
&= P^{-1}_{21} q_1 + P^{-1}_{22} \frac{\dot{Q}_0}{m_2 C_2}(1 - e^{-t/\tau})
\end{aligned}$$

These terms have the general form

$$q'_k(t) = B_k + C_k e^{-t/\tau}$$

where

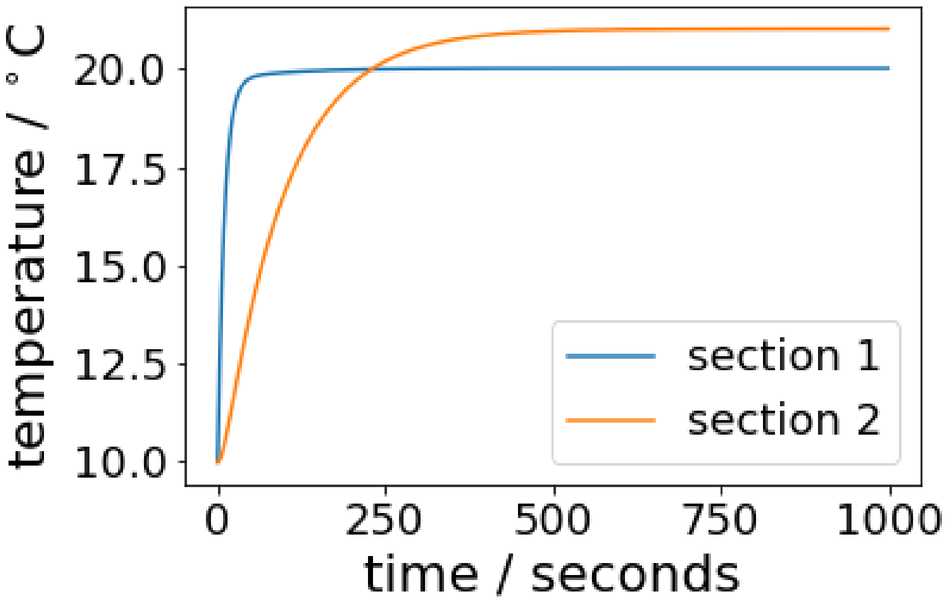
$$\begin{aligned}
B_1 &= P^{-1}_{11} q_1 + P^{-1}_{12} \frac{\dot{Q}_0}{m_2 C_2} \\
C_1 &= -P^{-1}_{12} \frac{\dot{Q}_0}{m_2 C_2} \\
B_2 &= P^{-1}_{21} q_1 + P^{-1}_{12} \frac{\dot{Q}_0}{m_2 C_2} \\
C_2 &= -P^{-1}_{22} \frac{\dot{Q}_0}{m_2 C_2}.
\end{aligned}$$

This can be integrated to give

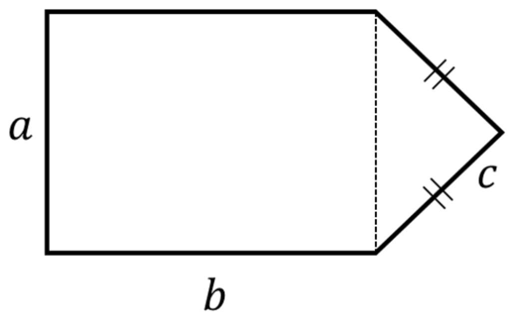
$$\begin{aligned}
y_k(t) &= y_k(0) e^{-\lambda_k t} + e^{-\lambda_k t} \int_0^t dt' (B_k + C_k e^{-t'/\tau}) e^{\lambda_k t'} \\
&= y_k(0) e^{-\lambda_k t} + e^{-\lambda_k t} \int_0^t dt' (B_k e^{\lambda_k t'} + C_k e^{-(1/\tau - \lambda_k) t'}) \\
&= y_k(0) e^{-\lambda_k t} + e^{-\lambda_k t} \left[-\frac{B_k}{-\lambda_k} e^{\lambda_k t'} - \frac{C_k}{1/\tau - \lambda_k} e^{-(1/\tau - \lambda_k) t'})\right]_0^t \\
&= y_k(0) e^{-\lambda_k t} + e^{-\lambda_k t} \left[-\frac{B_k}{-\lambda_k} e^{\lambda_k t'} - \frac{C_k}{1/\tau - \lambda_k} e^{-(1/\tau - \lambda_k) t'})\right]_0^t \\
&= y_k(0) e^{-\lambda_k t} + e^{-\lambda_k t} \left[\frac{B_k}{-\lambda_k}(1 - e^{\lambda_k t}) + \frac{C_k}{1/\tau - \lambda_k}(1 - e^{-(1/\tau - \lambda_k) t})\right] \\
&= y_k(0) e^{-\lambda_k t} - \frac{B_k}{\lambda_k}(e^{-\lambda_k t} - 1) + \frac{C_k}{1/\tau - \lambda_k}(e^{-\lambda_k t} - e^{-t/\tau})
\end{aligned}$$

The final solution can then be obtained by "rotating back" the vector $\mathbf{y}$ to the vector $\mathbf{T}(t) = \mathbf{P} \cdot \mathbf{y}(t)$.

```
# compute initial value of y vector
yinit = np.dot(Pinv, Tinit)

# compute B and C constants for solution of y(t)
B1 = Pinv[0,0] * q1 + Pinv[0,1]*dotQ0/(m2*C2)
C1 = -Pinv[0,1]*dotQ0/(m2*C2)

B2 = Pinv[1,0]*q1 + Pinv[1,1]*dotQ0/(m2*C2)
C2 = -Pinv[1,1]*dotQ0/(m2*C2)

# create a list of times
tmax = 1000.0
dt = 1.0e0
t_data = np.arange(0, tmax, dt)

y1_data = []
y2_data = []
T1_data = []
T2_data = []
for t in t_data:

    y1 = yinit[0]*np.exp(-lam[0]*t)
    y1 += -B1*(np.exp(-lam[0]*t)-1.0)/lam[0]
    y1 +=  C1*(np.exp(-lam[0]*t)-np.exp(-t/tau)) / (1/tau-lam[0])
    y1_data.append(y1)

    y2 = yinit[1]*np.exp(-lam[1]*t)
    y2 += -B2*(np.exp(-lam[1]*t)-1.0) / lam[1]
    y2 +=  C2*(np.exp(-lam[1]*t)-np.exp(-t/tau)) / (1/tau-lam[1])
    y2_data.append(y2)

    yvec = np.array([y1, y2]).reshape(2,1)
    Tvec = np.dot(P, yvec)
    T1_data.append( Tvec[0,0] )
    T2_data.append( Tvec[1,0] )


# plot results
plt.rcParams['font.size'] = 18
plt.rcParams['axes.labelsize'] = 'large'
plt.rcParams.update({'figure.autolayout': True})

plt.plot(t_data, T1_data, label='section 1')
plt.plot(t_data, T2_data, label='section 2')
plt.xlabel(r'time / seconds')
plt.ylabel(r'temperature / $^\circ$C')
plt.legend()
plt.savefig('heatedtank-fig.png')
plt.show()
```


temperature / °C
20.0
17.5
15.0
12.5
10.0
0
250
500
750
1000
time / seconds
section 1
section 2

Q2. An area on land, as shown below, is to be enclosed with $100 \mathrm{~m}$ of fencing.



The enclosed area, comprised of a rectangular and triangular element, must be maximised subject to these constraints.

(a) Determine a function and constraint to be maximised in terms of lengths a, b, and c. **[5 marks]**

(b) By using the Lagrange method for optimisation show how your answer from part (a), can be represented as a system of equations. **[7 marks]**

(c) Considering each equation from part (b) and the constraint, solve the system of equations and determine the lengths a, b, and c which maximises the area. **[8 marks]**

(d) The density, $\rho(x, y)$ in $\mathrm{g~m}^{-2}$, and mass, M in g, of a thin laminate are

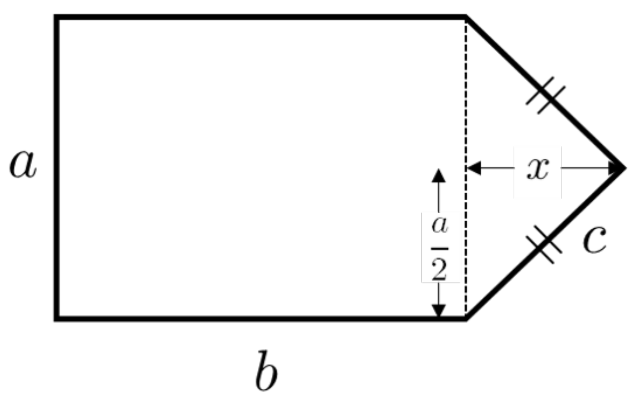
$$\rho(x, y) = \frac{\cos y}{y}$$

and

$$M = \int_0^{\frac{\pi}{4}} \int_{2x}^{\frac{\pi}{2}} \rho(x, y)\, dy\, dx$$

respectively. Sketch the region of integration and determine the centre of mass, $(\bar{x}, \bar{y})$, of the laminate.

[10 marks]



(a) From the diagram, we observe that the function to be optimised is given by

$$\begin{aligned}A &= A_{rectangle} + A_{isos-triangle}\\&= ab + \frac{a}{2}\times x\\&= ab + \frac{a}{2}\times\sqrt{c^2 - \left(\frac{a}{2}\right)^2}\\&= ab + \frac{1}{4}a\sqrt{4c^2 - a^2}\end{aligned}$$

The constraint is simply the perimeter

$$P = a + 2b + 2c = 100 \text{ m}$$

(b) Using the method of Lagrange multipliers the a derivative is given by

$$\begin{aligned}A_a &= \lambda P_a\\b + \frac{1}{4}\sqrt{4c^2 - a^2} + \frac{1}{4}\frac{1}{2}\frac{-2a^2}{\sqrt{4c^2 - a^2}} &= \lambda\\b + \frac{1}{4}\sqrt{4c^2 - a^2} - \frac{1}{4}\frac{a^2}{\sqrt{4c^2 - a^2}} &= \lambda\\b + \frac{1}{4\sqrt{4c^2 - a^2}}\left(4c^2 - a^2 - a^2\right) &= \lambda\\b + \frac{2c^2 - a^2}{2\sqrt{4c^2 - a^2}} &= \lambda. \qquad (1)\end{aligned}$$

The b derivative gives:

$$\begin{aligned}A_b &= \lambda P_b\\a &= 2\lambda \qquad (2)\end{aligned}$$

The c derivative gives:

$$\begin{aligned}A_c &= \lambda P_c\\\frac{1}{4}a\times 8c^2\times\frac{1}{2}\times\frac{1}{\sqrt{4c^2 - a^2}} &= 2\lambda\\\frac{ac}{\sqrt{4c^2 - a^2}} &= 2\lambda \qquad (3)\end{aligned}$$

Each of these represent equations that form the system of equations to be solved. The final equation is

$$a + 2b + 2c = 100 \qquad (4)$$

(c) Considering each equation in turn, we can solve the system of equations. From equation (2) and equation (3)

$$\begin{aligned}\frac{2\lambda c}{\sqrt{4c^2 - 4\lambda^2}} &= 2\lambda\\4\lambda^2 c^2 &= 4\lambda^2\left(4c^2 - 4\lambda^2\right)\\4\lambda^2 c^2 - 16\lambda^2 c^2 + 16\lambda^4 &= 0\\4\lambda^2\left(4\lambda^2 - 3c^2\right) &= 0\end{aligned}$$

So either $\lambda = 0$ or $c = \sqrt{\frac{4}{3}}\lambda$. $\lambda = 0$ is not possible as this would imply that $a = 0$, which is unphysical. Therefore $c = \sqrt{\frac{4}{3}}\lambda$.

Equation (1) gives:

$$b + \frac{2c^2 - a^2}{2\sqrt{4c^2 - a^2}} = \lambda$$
$$b = \frac{3+\sqrt{3}}{3}\lambda$$

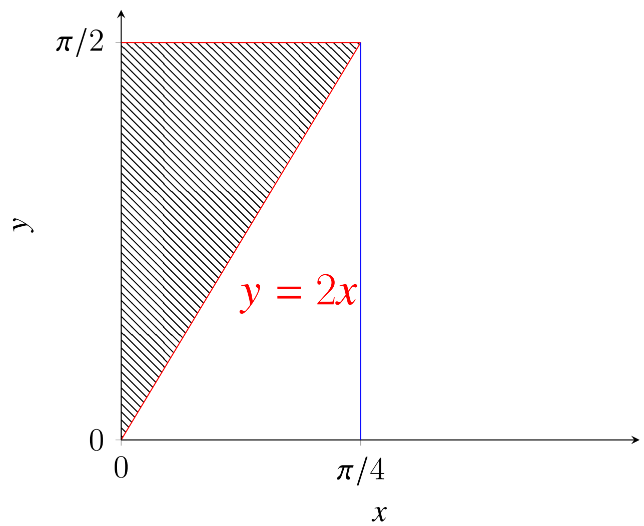
Finally, equation (4) gives:

$$2\lambda + \frac{2\left(3+\sqrt{3}\right)}{3}\lambda + \frac{4}{\sqrt{3}}\lambda = 100$$
$$\lambda = \frac{50}{2+\sqrt{3}}$$

Solving for a, b, and c gives

$$a = \frac{100}{2+\sqrt{3}} \approx 26.8 \text{ m}$$
$$b = \frac{150 - 50\sqrt{3}}{3} \approx 21.1 \text{ m}$$
$$c = \frac{100}{2\sqrt{3}+3} \approx 15.5 \text{ m}$$

(d) Sketch of the region:



To determine the centre of mass we consider the ratio of first moments to the mass of the laminate, $\bar{x} = M_y/M$ and $\bar{y} = M_x/M$ in turn.

The mass of the laminate, M, must be determined by changing the order of the variables:

$$
\begin{aligned}
M &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{y}{2}} \rho(x,y)\,dx\,dy \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\frac{y}{2}} \frac{\cos y}{y}\,dx\,dy \\
&= \int_0^{\frac{\pi}{2}} \left[x\frac{\cos y}{y}\right]_0^{\frac{y}{2}} dy \\
&= \int_0^{\frac{\pi}{2}} \frac{\cos y}{2}\,dy \\
&= \left[\frac{\sin y}{2}\right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{2}
\end{aligned}
$$

The first moment in x is given by:

$$
\begin{aligned}
M_x &= \int_0^{\frac{\pi}{4}} \int_{2x}^{\frac{\pi}{2}} y\rho(x,y)\,dy\,dx \\
&= \int_0^{\frac{\pi}{4}} \int_{2x}^{\frac{\pi}{2}} \cos y\,dy\,dx \\
&= \int_0^{\frac{\pi}{4}} [\sin y]_{2x}^{\frac{\pi}{2}}\,dx \\
&= \int_0^{\frac{\pi}{4}} 1 - \sin 2x\,dx \\
&= \left[x + \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{4} - \frac{1}{2}
\end{aligned}
$$

$\bar{y}$ is given by

$$\bar{y} = \frac{M_x}{M} = \frac{\pi - 2}{2}$$

The first moment in y requires a change in order (similar to the calculation for M).

Also, note that the second integration can be evaluated by integrating by parts.

$$\begin{aligned}
M_y &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{y}{2}} x\rho(x, y)\, dx\, dy \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\frac{y}{2}} x \frac{\cos y}{y}\, dx\, dy \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{x^2}{2y} \cos y \right]_0^{\frac{y}{2}} dy \\
&= \frac{1}{8} \int_0^{\frac{\pi}{2}} y \cos y\, dy \\
&= \frac{1}{8} \left[y \sin y \right]_0^{\frac{\pi}{2}} - \frac{1}{8} \int_0^{\frac{\pi}{2}} \sin y\, dy \\
&= \frac{\pi}{16} + \frac{1}{8} \left[\cos y \right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi - 2}{16}
\end{aligned}$$

$\bar{x}$ is given by

$$\bar{x} = \frac{M_y}{M} = \frac{\pi - 2}{8}$$

The centre of mass occurs at

$$\left(\frac{\pi - 2}{8}, \frac{\pi - 2}{2} \right)$$