$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

# CX1104: **Linear Algebra for Computing**

Chap. No : **8.2.1**
Lecture : **Eigen and Singular Values**
Topic : **Similarity Transform and Diagonalisation**
Concept : **Diagonalization Symmetric Matrix**


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What is symmetric matrix
When does it arise – covariance matrix , normal equation,
What nice property does it have (in diagonalization)

# Diagonalisation of Symmetric Matrices

## 1 Symmetric matrices

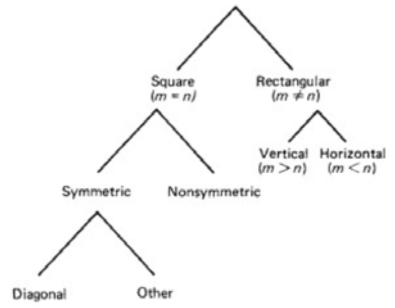
**Definition 1.1** *A symmetric matrix is a matrix $A$ such that $A = A^T$.*

In other words a symmetric matrix is a square matrix $A$ such that $a_{ij} = a_{ji}$.

**Example 1.2**

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 0 & 5 \\ 3 & 0 & 2 & -7 \\ 4 & 5 & -7 & -2 \end{pmatrix}$$

Ref: http://homepage.divms.uiowa.edu/~roseman/m33/m33_chap_7_sec_1.pdf




### 3.13 Eigenvalues and Eigenvectors of Symmetric Matrices

Two remarkable properties come about when we look at the eigenvalues and eigenvectors of a symmetric matrix $A \in \mathbb{S}^n$. First, it can be shown that all the eigenvalues of $A$ are real. Secondly, the eigenvectors of $A$ are orthonormal, i.e., the matrix $X$ defined above is an orthogonal matrix (for this reason, we denote the matrix of eigenvectors as $U$ in this case).

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We can therefore represent $A$ as $A = U\Lambda U^T$, remembering from above that the inverse of an orthogonal matrix is just its transpose.

Using this, we can show that the definiteness of a matrix depends entirely on the sign of its eigenvalues. Suppose $A \in \mathbb{S}^n = U\Lambda U^T$. Then

$$x^T A x = x^T U\Lambda U^T x = y^T \Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2$$

where $y = U^T x$ (and since $U$ is full rank, any vector $y \in \mathbb{R}^n$ can be represented in this form). Because $y_i^2$ is always positive, the sign of this expression depends entirely on the $\lambda_i$'s. If all $\lambda_i > 0$, then the matrix is positive definite; if all $\lambda_i \geq 0$, it is positive semidefinite. Likewise, if all $\lambda_i < 0$ or $\lambda_i \leq 0$, then $A$ is negative definite or negative semidefinite respectively. Finally, if $A$ has both positive and negative eigenvalues, it is indefinite.

http://cs229.stanford.edu/section/cs229-linalg.pdf

https://www.sciencedirect.com/topics/computer-science/symmetric-matrix

**Proofs skipped.**

# Diagonalisation of Symmetric Matrices

**The Spectral Theorem:**

**Proposition 2.7** *(The Spectral Theorem) An $n \times n$ symmetric matrix has the following properties:*

1. *$A$ has $n$ real eigenvalues if we count multiplicity*
2. *For each eigenvalue the dimension of the corresponding eigenspace is equal to the algebraic multiplicity of that eigenvalue.*
3. *The eigenspaces are mutually orthogonal*
4. *$A$ is orthogonally diagonalizable*

Ref: http://homepage.divms.uiowa.edu/~roseman/m33/m33_chap_7_sec_1.pdf

**Proofs skipped.**

# Diagonalisation of Symmetric Matrices

**Takeaways:**

1. Symmetric matrices are always diagonalisable. More specifically, **orthogonally diagonalisable**.

2. The eigenvectors [also eigenspaces] are mutually orthogonal to one another.

3. When a symmetric matrix $A$ has been diagonalised as $A = PDP^{-1}$, the columns of P, which are the eigenvectors of $A$, are mutually orthogonal.

4. $P$ can be an orthogonal matrix if its columns (eigenvectors of $A$) are unit normalised.

Ref: http://homepage.divms.uiowa.edu/~roseman/m33/m33_chap_7_sec_1.pdf

**Example:**

Consider a symmetric matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - 4$$

$$= \lambda^2 - 2\lambda + 1 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

Eigenvector corresponding to $\lambda_1 = 3$:

$q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Upon normalising, $\hat{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

Eigenvector corresponding to $\lambda_2 = -1$:

$q_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ Upon normalising, $\hat{q}_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

Hence, $A = \underbrace{\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{P^{-1}=P^T}$

P is an orthogonal matrix!

# Singular Values

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

Let $A$ be an $m \times n$ matrix. Then $A^TA$ is symmetric and can be orthogonally diagonalized. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A^TA$, and let $\lambda_1, \ldots, \lambda_n$ be the associated eigenvalues of $A^TA$. Then, for $1 \le i \le n$,

$$\begin{aligned} \|A\mathbf{v}_i\|^2 &= (A\mathbf{v}_i)^T A\mathbf{v}_i = \mathbf{v}_i^T A^TA\mathbf{v}_i \\ &= \mathbf{v}_i^T(\lambda_i \mathbf{v}_i) \qquad \text{Since } \mathbf{v}_i \text{ is an eigenvector of } A^TA \\ &= \lambda_i \qquad \text{Since } \mathbf{v}_i \text{ is a unit vector} \end{aligned} \tag{2}$$

So the eigenvalues of $A^TA$ are all nonnegative. By renumbering, if necessary, we may assume that the eigenvalues are arranged so that

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$$

The **singular values** of $A$ are the square roots of the eigenvalues of $A^TA$, denoted by $\sigma_1, \ldots, \sigma_n$, and they are arranged in decreasing order. That is, $\sigma_i = \sqrt{\lambda_i}$ for $1 \le i \le n$. By equation (2), *the singular values of $A$ are the lengths of the vectors $A\mathbf{v}_1, \ldots, A\mathbf{v}_n$.*

Lay, Linear Algebra and its Applications (4th Edition)

7.4 The Singular Value Decomposition **417**

## Example: Find $A$'s singular values.

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

The eigenvalues of $A^TA$ are $\lambda_1 = 360, \lambda_2 = 90$, and $\lambda_3 = 0$.

**(Check for yourself!)**

## Hence, the singular values of $A$ are:

$$\sigma_1 = \sqrt{360} = 6\sqrt{10}, \quad \sigma_2 = \sqrt{90} = 3\sqrt{10}, \quad \sigma_3 = 0$$