$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **6.1.2**

Lecture : **Orthogonality**

Topic : **Dot Product**

Concept : **Norm of a Vector and Unit Vectors**

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Norm of a Vector

- Indicates the **length** or the **magnitude** of the vector.

In this text we will denote the length of a vector **v** by the symbol $\|\mathbf{v}\|$, which is read as the *norm* of **v**, the *length* of **v**, or the *magnitude* of **v** (the term "norm" being a common mathematical synonym for length). As suggested in Figure 3.2.1*a*, it follows from the Theorem of Pythagoras that the norm of a vector (v_1, v_2) in R^2 is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2} \tag{1}$$

Similarly, for a vector (v_1, v_2, v_3) in R^3, it follows from Figure 3.2.1*b* and two applications of the Theorem of Pythagoras that

$$\|\mathbf{v}\|^2 = (OR)^2 + (RP)^2 = (OQ)^2 + (QR)^2 + (RP)^2 = v_1^2 + v_2^2 + v_3^2$$

and hence that

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \tag{2}$$

Motivated by the pattern of Formulas (1) and (2), we make the following definition.

DEFINITION 1 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the ***norm*** of **v** (also called the ***length*** of **v** or the ***magnitude*** of **v**) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \tag{3}$$

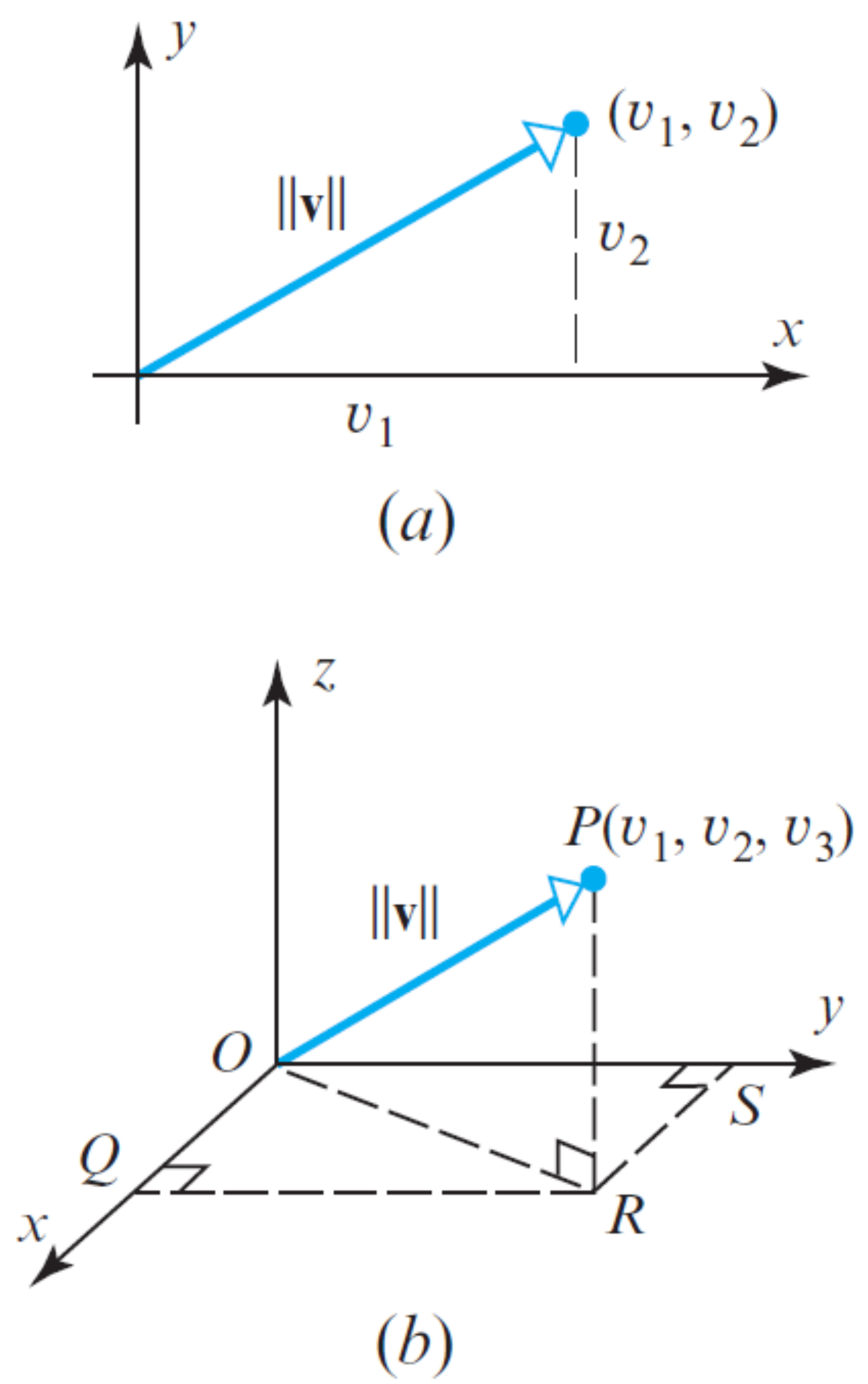


Figure 3.2.1

▶ **EXAMPLE 1 Calculating Norms**

It follows from Formula (2) that the norm of the vector $\mathbf{v} = (-3, 2, 1)$ in R^3 is

$$\|\mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$$

and it follows from Formula (3) that the norm of the vector $\mathbf{v} = (2, -1, 3, -5)$ in R^4 is

$$\|\mathbf{v}\| = \sqrt{2^2 + (-1)^2 + 3^2 + (-5)^2} = \sqrt{39} \ ◀$$

Norm of a Vector

THEOREM 3.2.1 *If $\mathbf{v}$ is a vector in R^n, and if k is any scalar, then:*

(*a*) $\|\mathbf{v}\| \geq 0$

(*b*) $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$

(*c*) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

We will prove part (*c*) and leave (*a*) and (*b*) as exercises.

Proof (c) If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, then $k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n)$, so

$$\begin{aligned} \|k\mathbf{v}\| &= \sqrt{(kv_1)^2 + (kv_2)^2 + \cdots + (kv_n)^2} \\ &= \sqrt{(k^2)(v_1^2 + v_2^2 + \cdots + v_n^2)} \\ &= |k|\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \\ &= |k|\|\mathbf{v}\| \end{aligned}$$

Unit Length Vector

A vector of norm 1 is called a ***unit vector***. Such vectors are useful for specifying a direction when length is not relevant to the problem at hand. You can obtain a unit vector in a desired direction by choosing any *nonzero* vector $\mathbf{v}$ in that direction and multiplying $\mathbf{v}$ by the reciprocal of its length. For example, if $\mathbf{v}$ is a vector of length 2 in R^2 or R^3, then $\frac{1}{2}\mathbf{v}$ is a unit vector in the same direction as $\mathbf{v}$. More generally, if $\mathbf{v}$ is any nonzero vector in R^n, then

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} \tag{4}$$

defines a unit vector that is in the same direction as $\mathbf{v}$. We can confirm that (4) is a unit vector by applying part (*c*) of Theorem 3.2.1 with $k = 1/\|\mathbf{v}\|$ to obtain

$$\|\mathbf{u}\| = \|k\mathbf{v}\| = |k|\|\mathbf{v}\| = k\|\mathbf{v}\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1$$

The process of multiplying a nonzero vector by the reciprocal of its length to obtain a unit vector is called ***normalizing*** $\mathbf{v}$.

EXAMPLE 2 Normalizing a Vector

Find the unit vector $\mathbf{u}$ that has the same direction as $\mathbf{v} = (2, 2, -1)$.

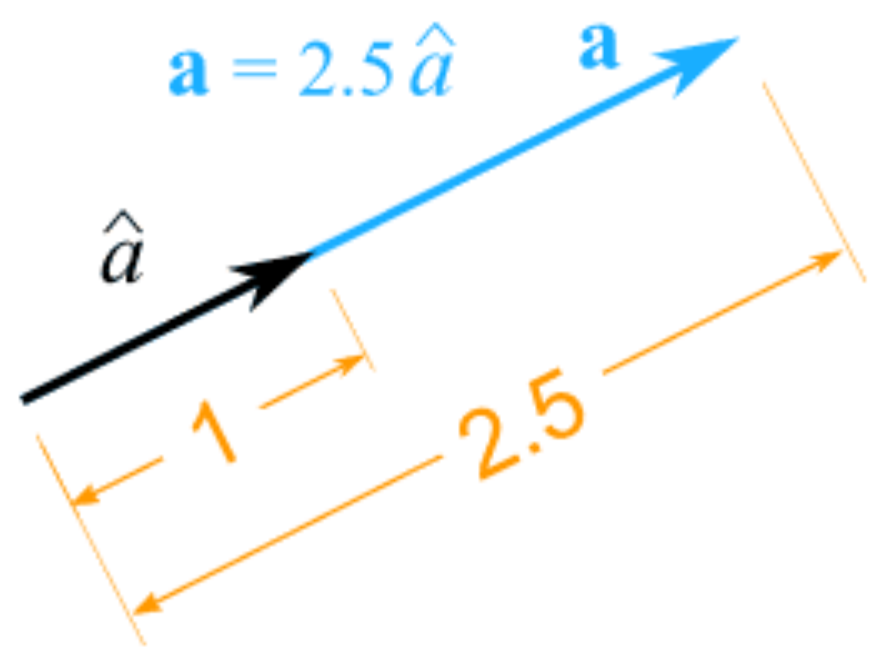
Solution The vector $\mathbf{v}$ has length

$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

Thus, from (4)

$$\mathbf{u} = \tfrac{1}{3}(2, 2, -1) = \left(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right)$$

As a check, you may want to confirm that $\|\mathbf{u}\| = 1$.



$\hat{a}$ is a unit vector with a direction that of vector a

Vector a is normalised to obtain the unit vector $\hat{a}$

Ref: https://www.mathsisfun.com/algebra/vector-unit.html

Examples

A vector whose length is 1 is called a **unit vector**. If we *divide* a nonzero vector $\mathbf{v}$ by its length—that is, multiply by $1/\|\mathbf{v}\|$—we obtain a unit vector $\mathbf{u}$ because the length of $\mathbf{u}$ is $(1/\|\mathbf{v}\|)\|\mathbf{v}\|$. The process of creating $\mathbf{u}$ from $\mathbf{v}$ is sometimes called **normalizing** $\mathbf{v}$, and we say that $\mathbf{u}$ is *in the same direction* as $\mathbf{v}$.

Several examples that follow use the space-saving notation for (column) vectors.

EXAMPLE 2 Let $\mathbf{v} = (1, -2, 2, 0)$. Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$.

SOLUTION First, compute the length of $\mathbf{v}$:

$$\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v} = (1)^2 + (-2)^2 + (2)^2 + (0)^2 = 9$$

$$\|\mathbf{v}\| = \sqrt{9} = 3$$

Then, multiply $\mathbf{v}$ by $1/\|\mathbf{v}\|$ to obtain

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{3}\mathbf{v} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

To check that $\|\mathbf{u}\| = 1$, it suffices to show that $\|\mathbf{u}\|^2 = 1$.

$$\|\mathbf{u}\|^2 = \mathbf{u}\cdot\mathbf{u} = \left(\tfrac{1}{3}\right)^2 + \left(-\tfrac{2}{3}\right)^2 + \left(\tfrac{2}{3}\right)^2 + (0)^2$$

$$= \tfrac{1}{9} + \tfrac{4}{9} + \tfrac{4}{9} + 0 = 1$$

■

Note: $||v||^2 = v\,.\,v$

Derived in Slide 6 of Lecture 6.1.3 on Dot Product

Lay's Linear Algebra and Applications:

332 CHAPTER 6 Orthogonality and Least Squares

DEFINITION For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the **distance between u and v**, written as $\text{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, this definition of distance coincides with the usual formulas for the Euclidean distance between two points, as the next two examples show.

EXAMPLE 4 Compute the distance between the vectors $\mathbf{u} = (7, 1)$ and $\mathbf{v} = (3, 2)$.

SOLUTION Calculate

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$$

The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ are shown in Fig. 4. When the vector $\mathbf{u} - \mathbf{v}$ is added to $\mathbf{v}$, the result is $\mathbf{u}$. Notice that the parallelogram in Fig. 4 shows that the distance from $\mathbf{u}$ to $\mathbf{v}$ is the same as the distance from $\mathbf{u} - \mathbf{v}$ to $\mathbf{0}$. ■

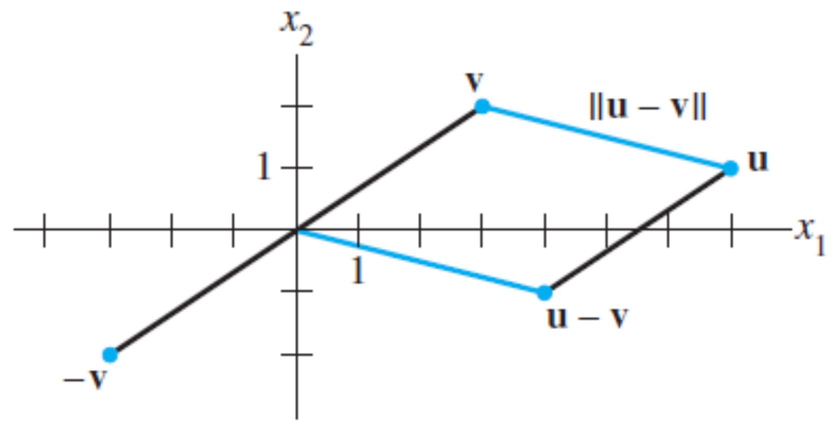


FIGURE 4 The distance between $\mathbf{u}$ and $\mathbf{v}$ is the length of $\mathbf{u} - \mathbf{v}$.

EXAMPLE 5 If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v})\cdot(\mathbf{u} - \mathbf{v})}$$

$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$$