$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.4.1**

Lecture : **Eigen and Singular Values**

Topic : **SVD & Pseudoinverse**

Concept : **Introduction to SVD**

Instructor: **A/P Chng Eng Siong**

TAs: **Zhang Su, Vishal Choudhari**

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Eigenvalues v/s Singular Values

Matrix A

Square

$n \times n$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

Eigenvalues!

Generalise

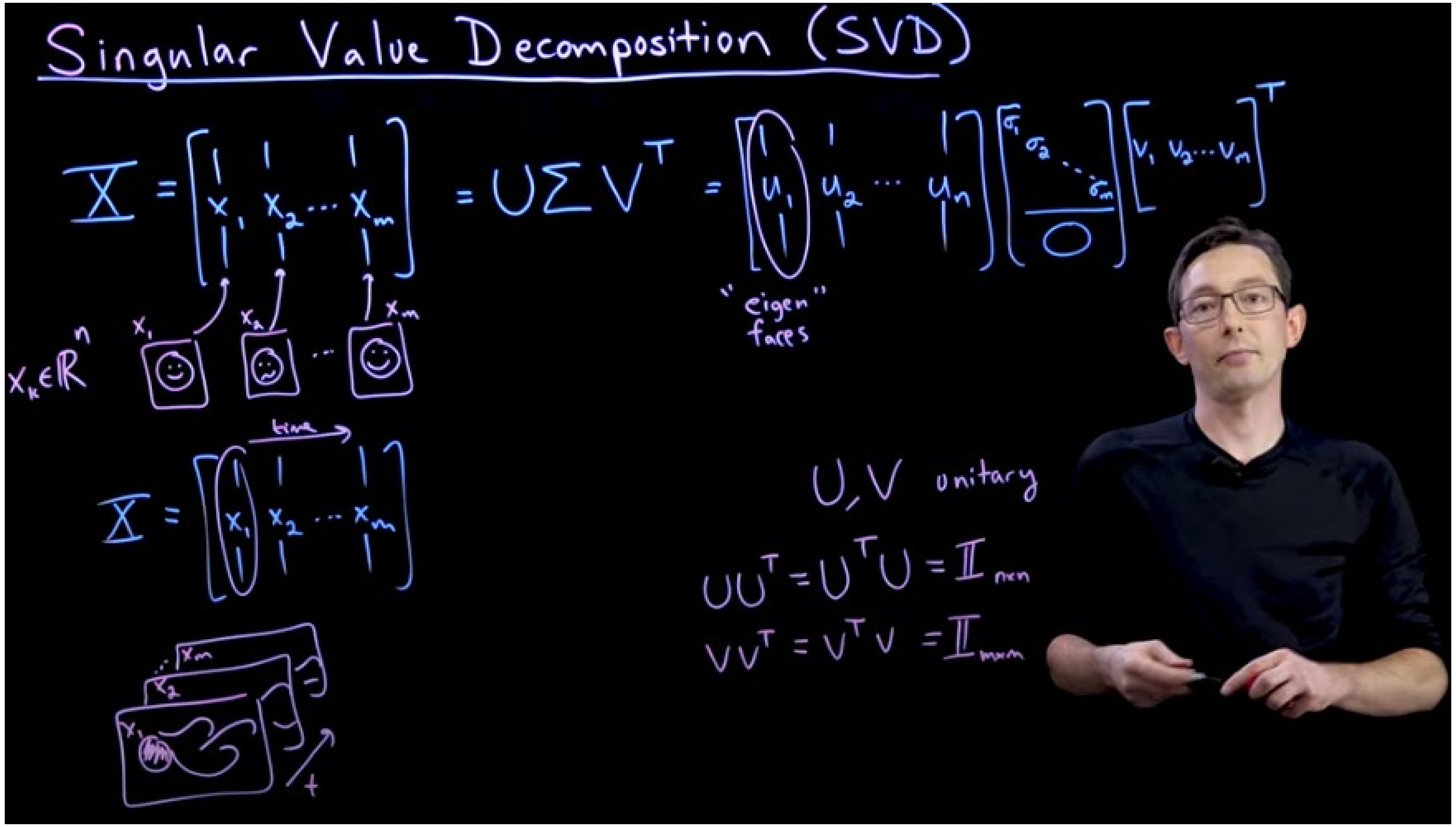
Rectangular

$m \times n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Singular Values!

Steve Brunton: SVD - Math Overview



1) What is SVD
2) How to interpret U, Σ, V

Ref: https://www.youtube.com/watch?v=nbBvuuNVfco

Singular Value Decomposition (SVD)

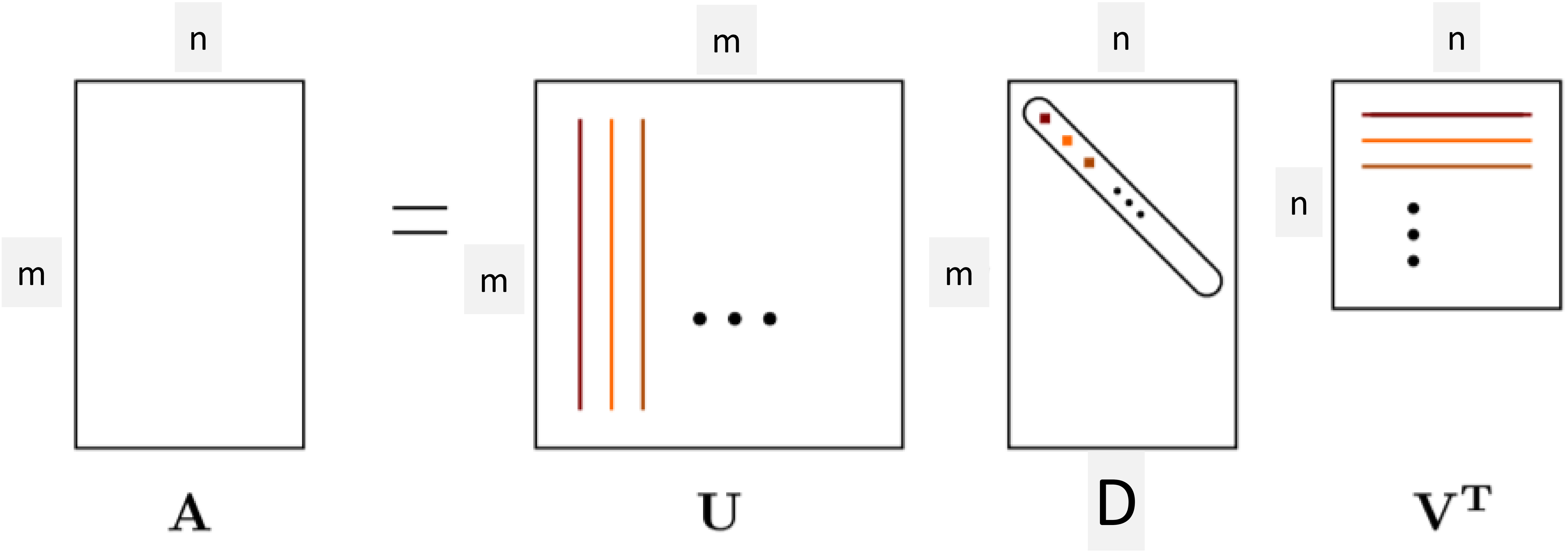


Figure 2: The singular value decomposition (SVD). Each singular value in D has an associated left singular vector in $\mathbf{U}$, and right singular vector in $\mathbf{V}$.

$A = U * D * V'$ where

- U is orthogonal, and $m \times m$;
- D is diagonal, and $m \times n$, with nonnegative diagonal entries σ_i;
- V is orthogonal, and $n \times n$;

The $min(m, n)$ diagonal elements of D, written σ_i, are nonnegative, and in decreasing order. The value σ_1 is the l2 norm of A.

In MATLAB, get the factors by writing:

```
[ U, D, V ] = svd ( A );
```

Ref: theory.stanford.edu/~tim/s15/l/l9.pdf

Singular Value Decomposition (SVD)

The Singular Value Decomposition

Let A be an $m \times n$ matrix with rank r. Then there exists an $\underline{m \times n \text{ matrix } \Sigma}$ as in (3) for which the diagonal entries in D are the first r singular values of A, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, and there exist an $\underline{m \times m \text{ orthogonal matrix } U}$ and an $\underline{n \times n \text{ orthogonal matrix } V}$ such that

$$A = U\Sigma V^T$$

Any factorization $A = U\Sigma V^T$, with U and V orthogonal, Σ as in (3), and positive diagonal entries in D, is called a **singular value decomposition** (or **SVD**) of A. The matrices U and V are not uniquely determined by A, but the diagonal entries of Σ are necessarily the singular values of A. The columns of U in such a decomposition are called **left singular vectors** of A, and the columns of V are called **right singular vectors** of A.

Lay, Linear Algebra and its Applications (4th Edition)

7.4 The Singular Value Decomposition **417**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

The decomposition of A involves an $m \times n$ "diagonal" matrix Σ of the form

$$\Sigma = \begin{array}{cl} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} & \leftarrow m-r \text{ rows} \\ \quad\;\;\uparrow n-r \text{ columns} & \end{array} \tag{3}$$

where D is an $r \times r$ diagonal matrix for some r not exceeding the smaller of m and n.

Review:

If a matrix A has a rank r, it means there are r linearly independent rows or columns in matrix A.

For any matrix A, row rank = column rank = r.

SVD(A) vs SVD(A,’econ’): Tall and Skinny A

```
>> A=[1 2; 3 4; 5 6; 7 8]

A =

     1     2
     3     4
     5     6
     7     8

>> [U,D,V] = svd(A)

U =

   -0.1525   -0.8226   -0.3945   -0.3800
   -0.3499   -0.4214    0.2428    0.8007
   -0.5474   -0.0201    0.6979   -0.4614
   -0.7448    0.3812   -0.5462    0.0407


D =

   14.2691         0
         0    0.6268
         0         0
         0         0


V =

   -0.6414    0.7672
   -0.7672   -0.6414
```

```
>> [U,D,V] = svd(A,'econ')

U =

   -0.1525   -0.8226
   -0.3499   -0.4214
   -0.5474   -0.0201
   -0.7448    0.3812


D =

   14.2691         0
         0    0.6268


V =

   -0.6414    0.7672
   -0.7672   -0.6414
```

```
>> U*D*(V')

ans =

    1.0000    2.0000
    3.0000    4.0000
    5.0000    6.0000
    7.0000    8.0000
```

SVD Examples : Fat/Short A

```
>> FatA = A'

FatA =

     1     3     5     7
     2     4     6     8

>> [U,D,V] = svd(FatA)

U =

   -0.6414    0.7672
   -0.7672   -0.6414


D =

   14.2691         0         0         0
         0    0.6268         0         0


V =

   -0.1525   -0.8226   -0.3945   -0.3800
   -0.3499   -0.4214    0.2428    0.8007
   -0.5474   -0.0201    0.6979   -0.4614
   -0.7448    0.3812   -0.5462    0.0407
```

```
>> U*D*(V')

ans =

    1.0000    3.0000    5.0000    7.0000
    2.0000    4.0000    6.0000    8.0000
```

```
>> [U,D,V] = svd(FatA,'econ')

U =

   -0.6414    0.7672
   -0.7672   -0.6414


D =

   14.2691         0
         0    0.6268


V =

   -0.1525   -0.8226
   -0.3499   -0.4214
   -0.5474   -0.0201
   -0.7448    0.3812

>> U*D*(V')

ans =

    1.0000    3.0000    5.0000    7.0000
    2.0000    4.0000    6.0000    8.0000
```