

# CX1104: Linear Algebra for Computing

Chap. No : **6.1.3**

Lecture : **Orthogonality**

Topic : **Dot Product**

Concept : **The Dot Product**

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{A \quad m \times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}_{x \quad n \times 1} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b \quad m \times 1}$$

Instructor: **A/P Chng Eng Siong**

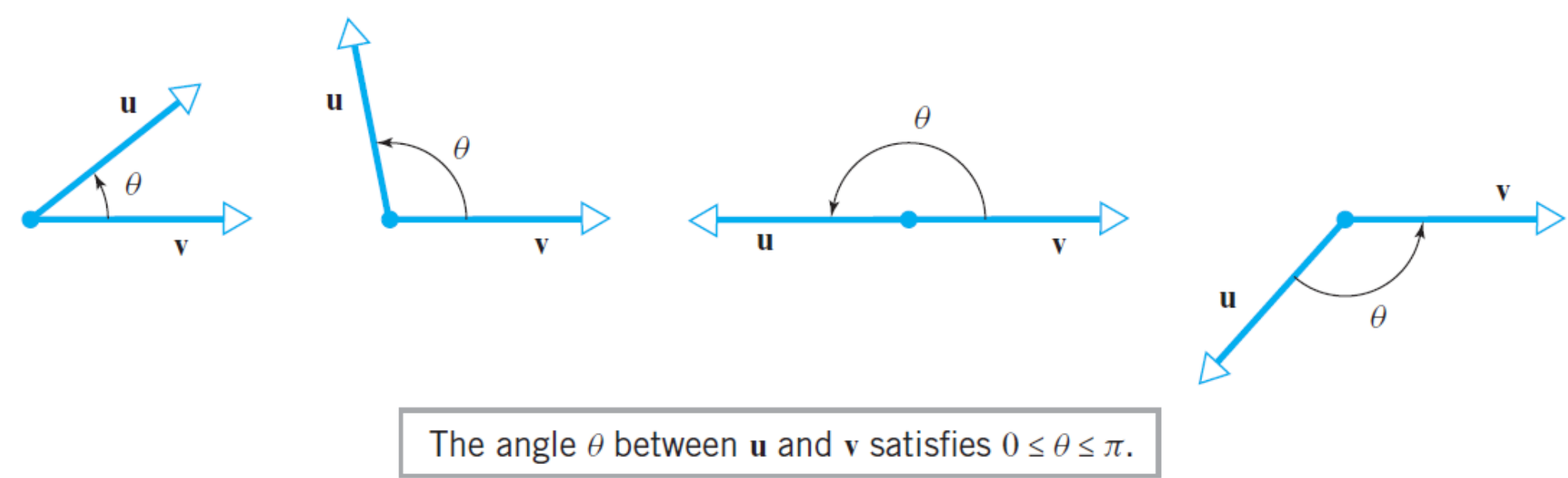
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# Dot Product

**DEFINITION 3** If  $\mathbf{u}$  and  $\mathbf{v}$  are nonzero vectors in  $R^2$  or  $R^3$ , and if  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ , then the *dot product* (also called the *Euclidean inner product*) of  $\mathbf{u}$  and  $\mathbf{v}$  is denoted by  $\mathbf{u} \cdot \mathbf{v}$  and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \tag{12}$$

If  $\mathbf{u} = \mathbf{0}$  or  $\mathbf{v} = \mathbf{0}$ , then we define  $\mathbf{u} \cdot \mathbf{v}$  to be 0.



How to define “angle” between two vectors in  $R^2$  or  $R^3$ ? For this purpose, let  $\mathbf{u}$  and  $\mathbf{v}$  be nonzero vectors in  $R^2$  or  $R^3$  that have been positioned so that their initial points coincide. We define the *angle between  $\mathbf{u}$  and  $\mathbf{v}$*  to be the angle  $\theta$  determined by  $\mathbf{u}$  and  $\mathbf{v}$  that satisfies the inequalities  $0 \leq \theta \leq \pi$  (Figure 3.2.4).

The sign of the dot product reveals information about the angle  $\theta$  that we can obtain by rewriting Formula (12) as

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \tag{13}$$

Since  $0 \leq \theta \leq \pi$ , it follows from Formula (13) and properties of the cosine function studied in trigonometry that

- $\theta$  is acute if  $\mathbf{u} \cdot \mathbf{v} > 0$ .
- $\theta$  is obtuse if  $\mathbf{u} \cdot \mathbf{v} < 0$ .
- $\theta = \pi/2$  if  $\mathbf{u} \cdot \mathbf{v} = 0$ .

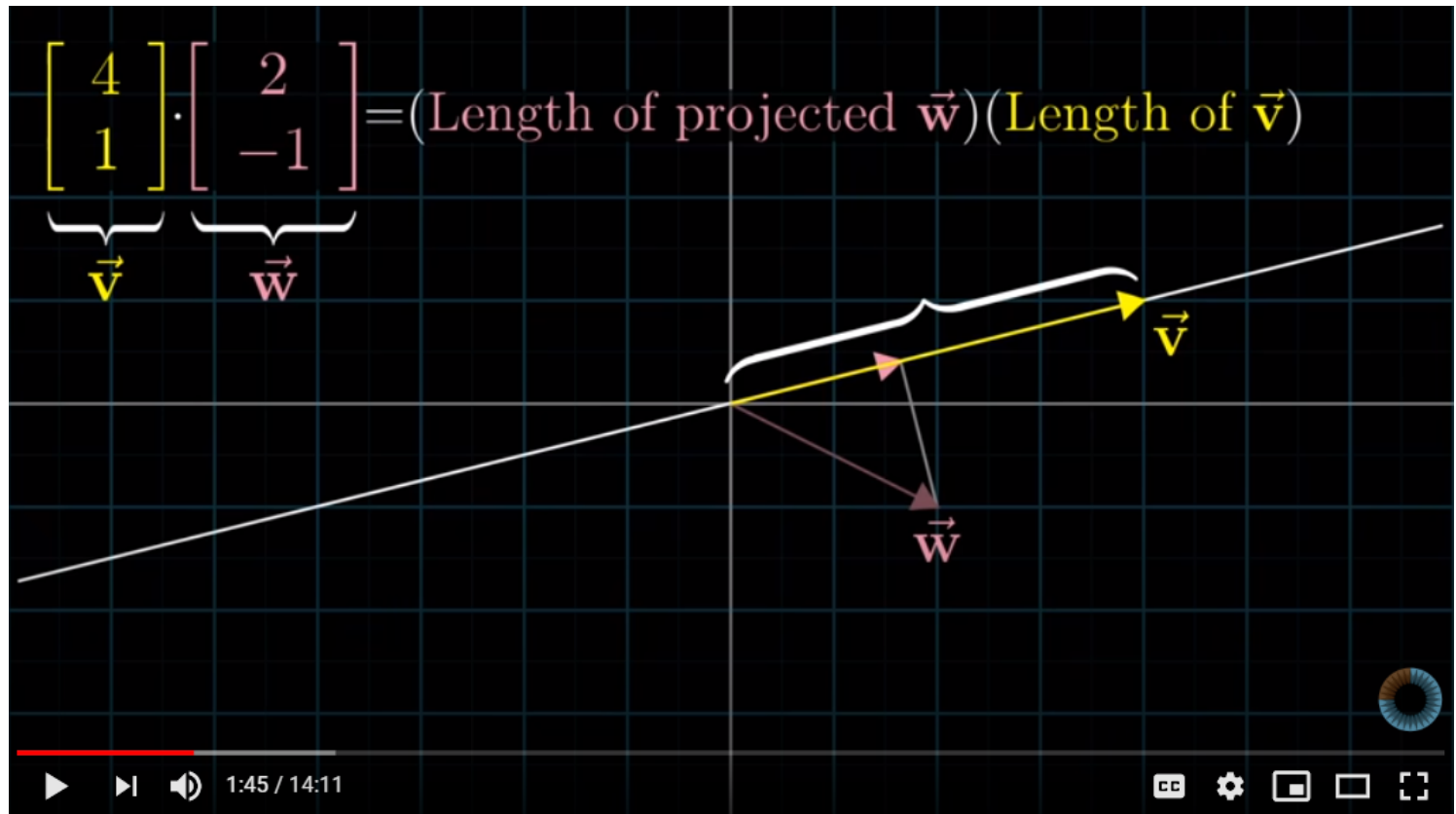
Dot product relates the length of two vectors and the angle ( $\theta$ ) between them.

If vectors  $u$  and  $v$  are unit vectors, i.e,  $\|v\| = \|u\| = 1$ , the dot product is  $u \cdot v = \cos \theta$ .

Ref:

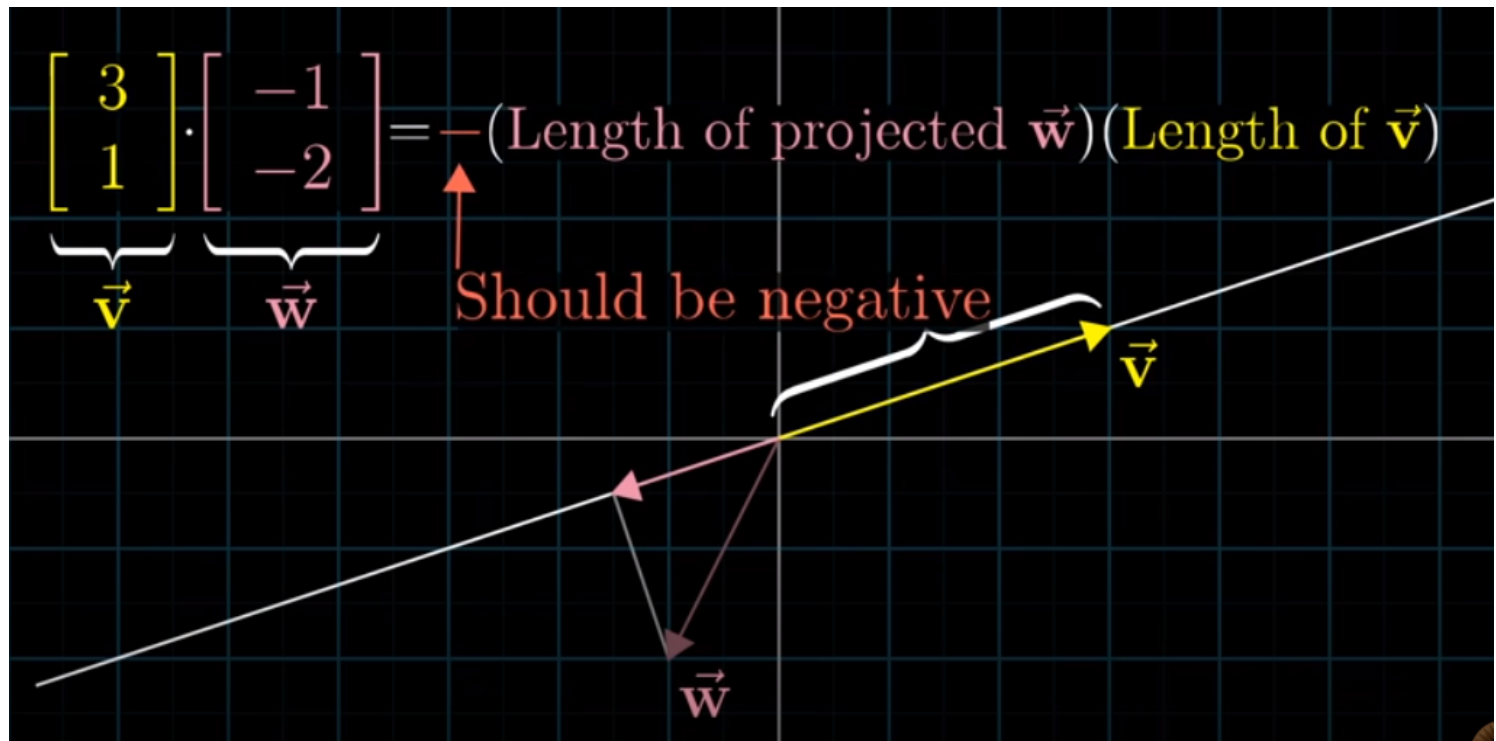
1. [Stack Exchange](#)
2. Khan Academy: <https://www.youtube.com/watch?v=KDHuWxy53uM>
3. 3Blue1Brown, Dot Product and Duality: <https://www.youtube.com/watch?v=LyGKycYT2v0>
4. MathsTheBeautiful: [https://www.youtube.com/watch?v=QPkKWGq\\_V0U](https://www.youtube.com/watch?v=QPkKWGq_V0U)

# Dot Product Interpretation



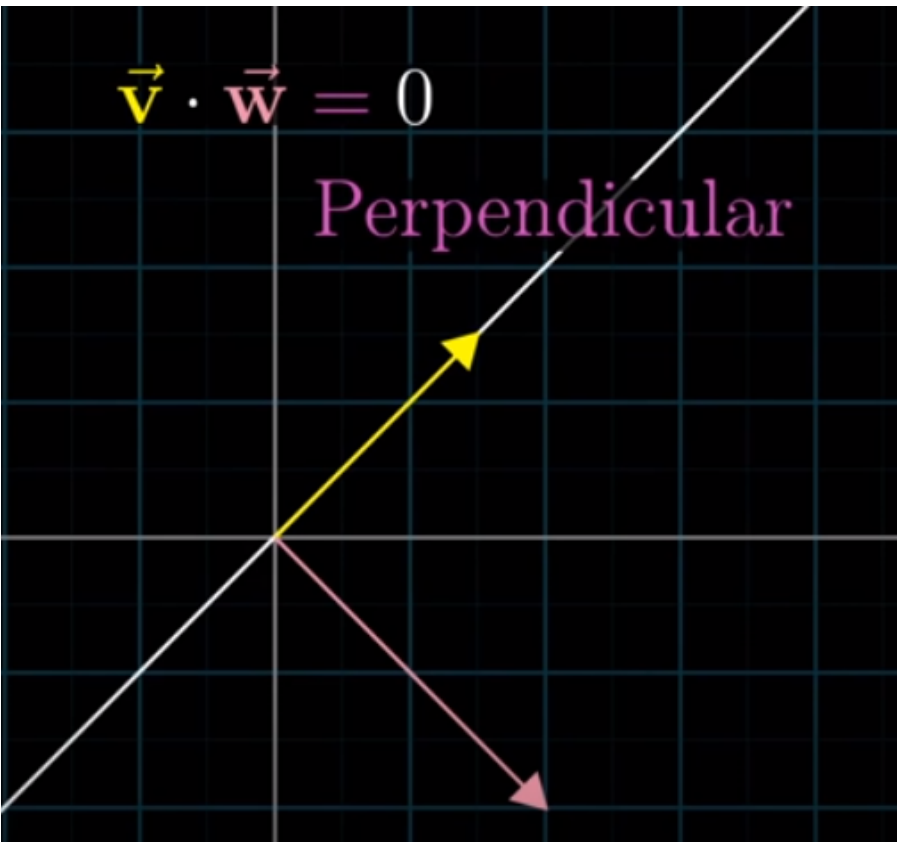
**Case 1:** When two vectors are pointing nearly towards the same direction, their dot product is +ve.

Angle between vectors: acute.



**Case 2:** When two vectors are pointing away from one another, their dot product is -ve.

Angle between vectors: obtuse.



**Case 3:** When two vectors are perpendicular, their dot product is zero.

$$v \cdot w = \sum_{i=1}^n v_i w_i = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$$

$$= ||v|| \times ||w|| \times \cos\theta$$

Derived in Slide 5

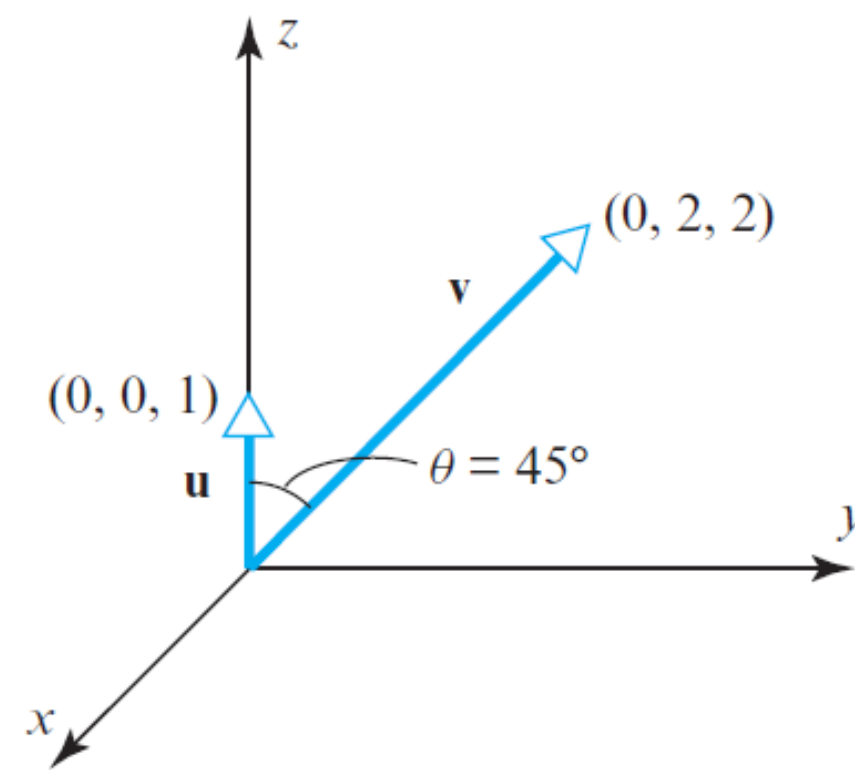
$$v \cdot w = ||v|| \times ||w|| \times \cos\theta$$

$$v \cdot w = \underbrace{(||w|| \times \cos\theta)}_{\text{Length of projection of } w \text{ onto } v} \times \underbrace{||v||}_{\text{Length of vector } v}$$

$$v \cdot w = \underbrace{(||v|| \times \cos\theta)}_{\text{Length of projection of } v \text{ onto } w} \times \underbrace{||w||}_{\text{Length of vector } w}$$

# Example

## 146 Chapter 3 Euclidean Vector Spaces



▲ Figure 3.2.5

### ► EXAMPLE 5 Dot Product

Find the dot product of the vectors shown in Figure 3.2.5.

**Solution** The lengths of the vectors are

$$\|\mathbf{u}\| = 1 \quad \text{and} \quad \|\mathbf{v}\| = \sqrt{8} = 2\sqrt{2}$$

and the cosine of the angle  $\theta$  between them is

$$\cos(45^\circ) = 1/\sqrt{2}$$

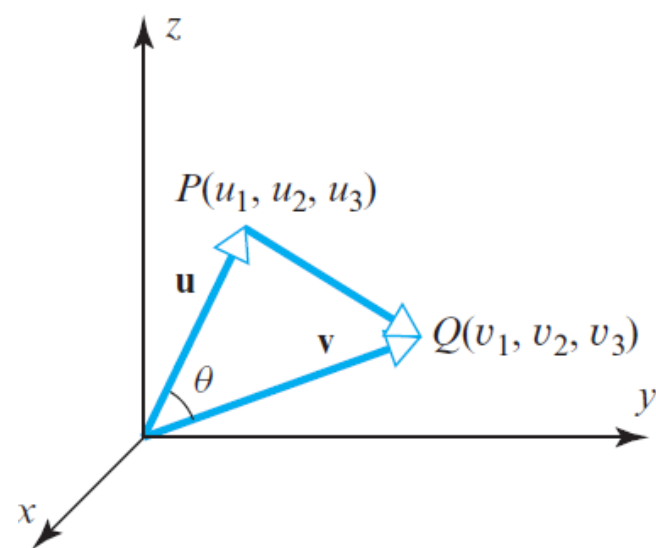
Thus, it follows from Formula (12) that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta = (1)(2\sqrt{2})(1/\sqrt{2}) = 2$$



# Component Form of the Dot Product

## Component Form of the Dot Product



▲ Figure 3.2.6

For computational purposes it is desirable to have a formula that expresses the dot product of two vectors in terms of components. We will derive such a formula for vectors in 3-space; the derivation for vectors in 2-space is similar.

Let  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$  be two nonzero vectors. If, as shown in Figure 3.2.6,  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ , then the law of cosines yields

$$\|\overrightarrow{PQ}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \quad (14)$$

Since  $\overrightarrow{PQ} = \mathbf{v} - \mathbf{u}$ , we can rewrite (14) as

$$\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = \frac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v} - \mathbf{u}\|^2)$$

or

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v} - \mathbf{u}\|^2)$$

Substituting

$$\|\mathbf{u}\|^2 = u_1^2 + u_2^2 + u_3^2, \quad \|\mathbf{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

and

$$\|\mathbf{v} - \mathbf{u}\|^2 = (v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2$$

we obtain, after simplifying,

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 \quad (15)$$

The companion formula for vectors in 2-space is

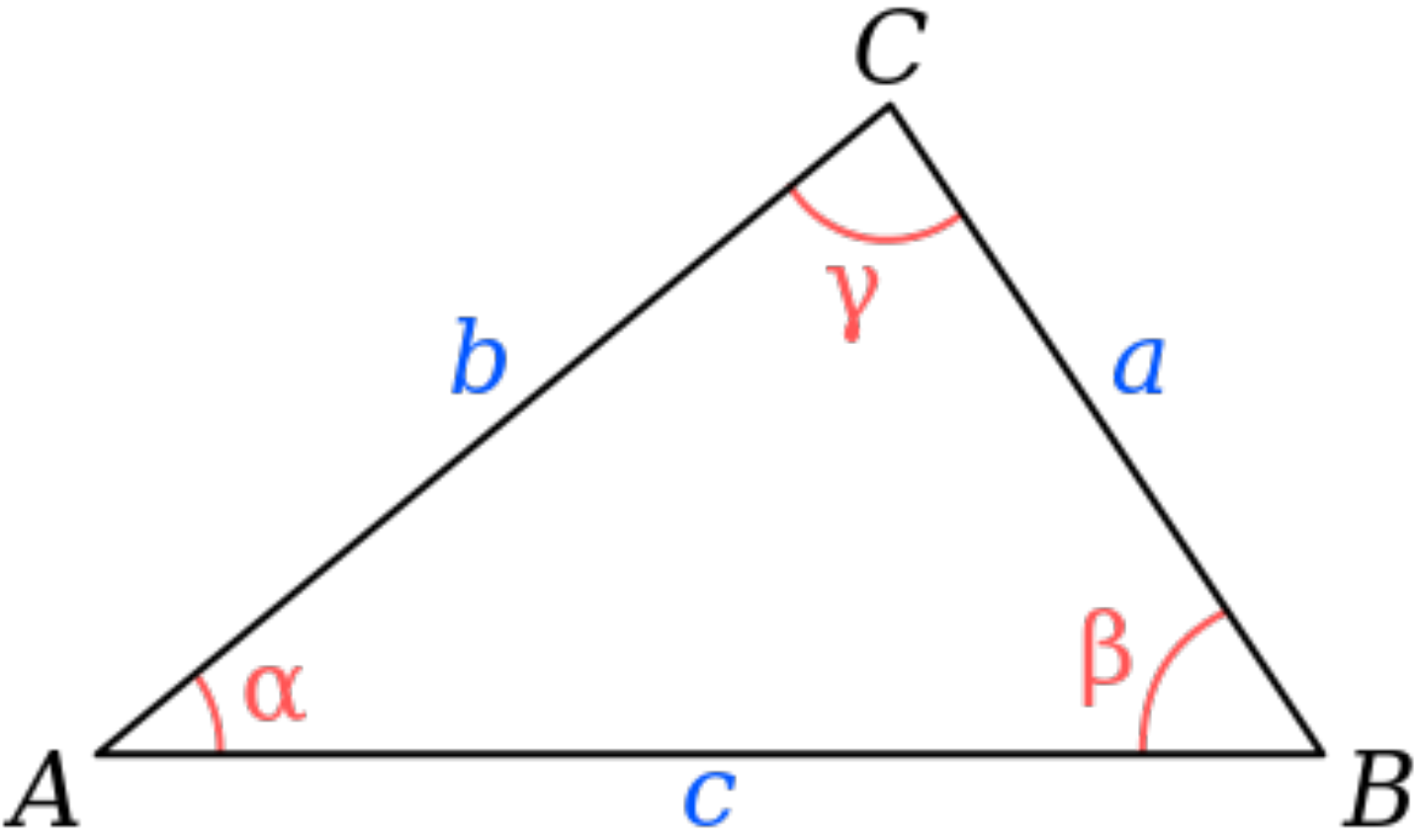
$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 \quad (16)$$

Motivated by the pattern in Formulas (15) and (16), we make the following definition.

**DEFINITION 4** If  $\mathbf{u} = (u_1, u_2, \dots, u_n)$  and  $\mathbf{v} = (v_1, v_2, \dots, v_n)$  are vectors in  $R^n$ , then the *dot product* (also called the *Euclidean inner product*) of  $\mathbf{u}$  and  $\mathbf{v}$  is denoted by  $\mathbf{u} \cdot \mathbf{v}$  and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n \quad (17)$$

## Reviewing Law of Cosines



$$c^2 = a^2 + b^2 - 2ab \cos \gamma,$$

# Example

► **EXAMPLE 6 Calculating Dot Products Using Components**

- (a) Use Formula (15) to compute the dot product of the vectors  $\mathbf{u}$  and  $\mathbf{v}$  in Example 5.
- (b) Calculate  $\mathbf{u} \cdot \mathbf{v}$  for the following vectors in  $R^4$ :

$$\mathbf{u} = (-1, 3, 5, 7), \quad \mathbf{v} = (-3, -4, 1, 0)$$

**Solution (a)** The component forms of the vectors are  $\mathbf{u} = (0, 0, 1)$  and  $\mathbf{v} = (0, 2, 2)$ . Thus,

$$\mathbf{u} \cdot \mathbf{v} = (0)(0) + (0)(2) + (1)(2) = 2$$

which agrees with the result obtained geometrically in Example 5.

**Solution (b)**

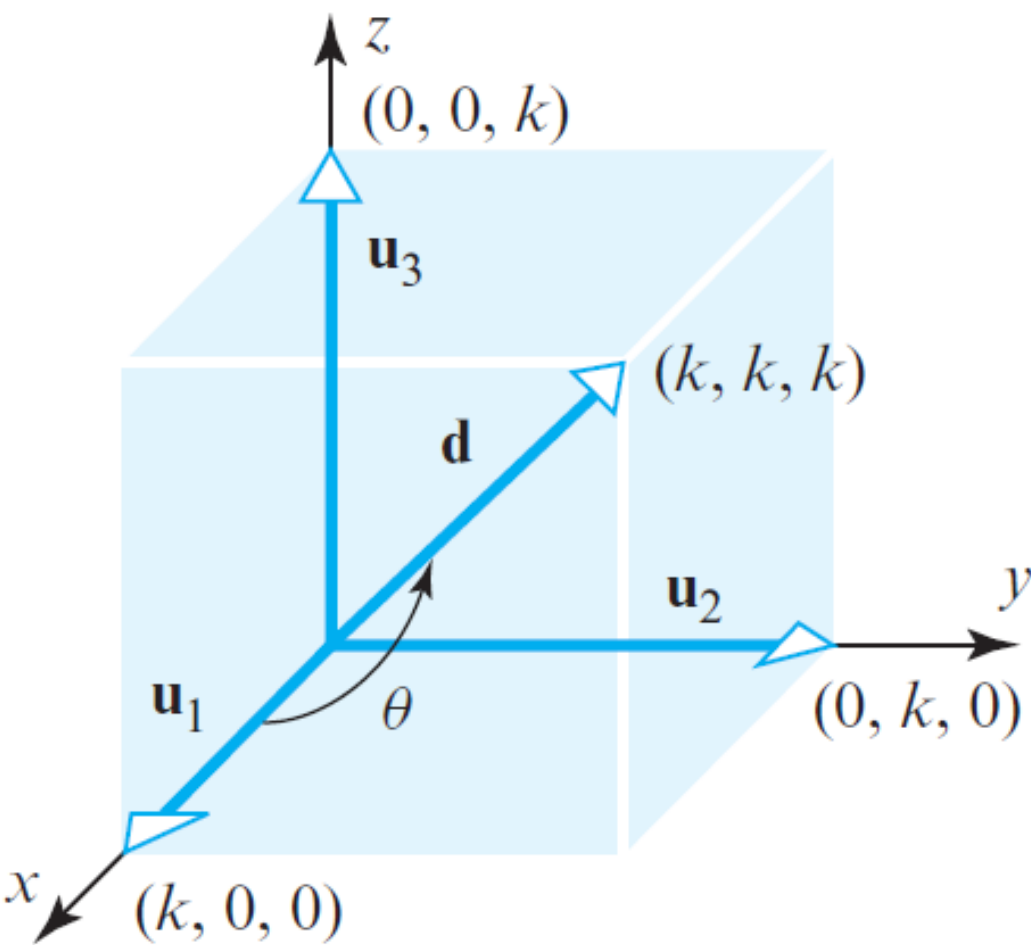
$$\mathbf{u} \cdot \mathbf{v} = (-1)(-3) + (3)(-4) + (5)(1) + (7)(0) = -4$$

In the special case where  $\mathbf{u} = \mathbf{v}$  in Definition 4, we obtain the relationship

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\mathbf{v}\|^2 \tag{18}$$

This yields the following formula for expressing the length of a vector in terms of a dot product:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \tag{19}$$



▲ **Figure 3.2.7**

Note that the angle  $\theta$  obtained in Example 7 does not involve  $k$ . Why was this to be expected?

► **EXAMPLE 7 A Geometry Problem Solved Using Dot Product**

Find the angle between a diagonal of a cube and one of its edges.

**Solution** Let  $k$  be the length of an edge and introduce a coordinate system as shown in Figure 3.2.7. If we let  $\mathbf{u}_1 = (k, 0, 0)$ ,  $\mathbf{u}_2 = (0, k, 0)$ , and  $\mathbf{u}_3 = (0, 0, k)$ , then the vector

$$\mathbf{d} = (k, k, k) = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$

is a diagonal of the cube. It follows from Formula (13) that the angle  $\theta$  between  $\mathbf{d}$  and the edge  $\mathbf{u}_1$  satisfies

$$\cos \theta = \frac{\mathbf{u}_1 \cdot \mathbf{d}}{\|\mathbf{u}_1\| \|\mathbf{d}\|} = \frac{k^2}{(k)(\sqrt{3k^2})} = \frac{1}{\sqrt{3}}$$

With the help of a calculator we obtain

$$\theta = \cos^{-1} \left( \frac{1}{\sqrt{3}} \right) \approx 54.74^\circ \quad \blacktriangleleft$$



# Properties of Dot Product

Dot products have many of the same algebraic properties as products of real numbers.

**THEOREM 3.2.2** If  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are vectors in  $R^n$ , and if  $k$  is a scalar, then:

- (a)  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$  [Symmetry property]
- (b)  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$  [Distributive property]
- (c)  $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$  [Homogeneity property]
- (d)  $\mathbf{v} \cdot \mathbf{v} \geq 0$  and  $\mathbf{v} \cdot \mathbf{v} = 0$  if and only if  $\mathbf{v} = \mathbf{0}$  [Positivity property]

**Proof (c)** Let  $\mathbf{u} = (u_1, u_2, \dots, u_n)$  and  $\mathbf{v} = (v_1, v_2, \dots, v_n)$ . Then

$$\begin{aligned} k(\mathbf{u} \cdot \mathbf{v}) &= k(u_1v_1 + u_2v_2 + \dots + u_nv_n) \\ &= (ku_1)v_1 + (ku_2)v_2 + \dots + (ku_n)v_n = (k\mathbf{u}) \cdot \mathbf{v} \end{aligned}$$

**Proof (d)** The result follows from parts (a) and (b) of Theorem 3.2.1 and the fact that

$$\mathbf{v} \cdot \mathbf{v} = v_1v_1 + v_2v_2 + \dots + v_nv_n = v_1^2 + v_2^2 + \dots + v_n^2 = \|\mathbf{v}\|^2 \quad \blacktriangleleft$$

**THEOREM 3.2.3** If  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are vectors in  $R^n$ , and if  $k$  is a scalar, then:

- (a)  $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$
- (b)  $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- (c)  $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$
- (d)  $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$
- (e)  $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

**Proof (b)**

$$\begin{aligned} (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} &= \mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) && \text{[By symmetry]} \\ &= \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v} && \text{[By distributivity]} \\ &= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} && \text{[By symmetry]} \quad \blacktriangleleft \end{aligned}$$

## ► EXAMPLE 8 Calculating with Dot Products

$$\begin{aligned} (\mathbf{u} - 2\mathbf{v}) \cdot (3\mathbf{u} + 4\mathbf{v}) &= \mathbf{u} \cdot (3\mathbf{u} + 4\mathbf{v}) - 2\mathbf{v} \cdot (3\mathbf{u} + 4\mathbf{v}) \\ &= 3(\mathbf{u} \cdot \mathbf{u}) + 4(\mathbf{u} \cdot \mathbf{v}) - 6(\mathbf{v} \cdot \mathbf{u}) - 8(\mathbf{v} \cdot \mathbf{v}) \\ &= 3\|\mathbf{u}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) - 8\|\mathbf{v}\|^2 \quad \blacktriangleleft \end{aligned}$$

# Property and Example

Table 1

| Form  | Dot Product   | Example   |  |
|---|---|---|--|
| $\mathbf{u}$ a column matrix and $\mathbf{v}$ a column matrix | $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \mathbf{v}^T \mathbf{u}$ | $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$<br>$\mathbf{v} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$ | $\mathbf{u}^T \mathbf{v} = [1 \quad -3 \quad 5] \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$<br>$\mathbf{v}^T \mathbf{u} = [5 \quad 4 \quad 0] \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$ |

If  $A$  is an  $n \times n$  matrix and  $\mathbf{u}$  and  $\mathbf{v}$  are  $n \times 1$  matrices, then it follows from the first row in Table 1 and properties of the transpose that

$$\begin{aligned} A\mathbf{u} \cdot \mathbf{v} &= \mathbf{v}^T(A\mathbf{u}) = (\mathbf{v}^T A)\mathbf{u} = (A^T \mathbf{v})^T \mathbf{u} = \mathbf{u} \cdot A^T \mathbf{v} \\ \mathbf{u} \cdot A\mathbf{v} &= (A\mathbf{v})^T \mathbf{u} = (\mathbf{v}^T A^T)\mathbf{u} = \mathbf{v}^T(A^T \mathbf{u}) = A^T \mathbf{u} \cdot \mathbf{v} \end{aligned}$$

The resulting formulas

$A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T \mathbf{v}$

(26)

$\mathbf{u} \cdot A\mathbf{v} = A^T \mathbf{u} \cdot \mathbf{v}$

(27)

provide an important link between multiplication by an  $n \times n$  matrix  $A$  and multiplication by  $A^T$ .

► **EXAMPLE 9** Verifying that  $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T \mathbf{v}$

Suppose that

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

Then

$$\begin{aligned} A\mathbf{u} &= \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix} \\ A^T \mathbf{v} &= \begin{bmatrix} 1 & 2 & -1 \\ -2 & 4 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ -1 \end{bmatrix} \end{aligned}$$

from which we obtain

$$\begin{aligned} A\mathbf{u} \cdot \mathbf{v} &= 7(-2) + 10(0) + 5(5) = 11 \\ \mathbf{u} \cdot A^T \mathbf{v} &= (-1)(-7) + 2(4) + 4(-1) = 11 \end{aligned}$$

Thus,  $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T \mathbf{v}$  as guaranteed by Formula (26). We leave it for you to verify that Formula (27) also holds. ◀