$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

# CX1104: **Linear Algebra for Computing**

Chap. No : **8.2.1**

Lecture : **Eigen and Singular Values**

Topic : **Similarity Transform and Diagonalisation**

Concept : **Similarity Transform**

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Rev: 17th July 2020

# Similar Matrices

**DEFINITION 1** If $A$ and $B$ are square matrices, then we say that ***$B$ is similar to $A$*** if there is an invertible matrix $P$ such that $B = P^{-1}AP$.

Note that if $B$ is similar to $A$, then it is also true that $A$ is similar to $B$ since we can express $A$ as $A = Q^{-1}BQ$ by taking $Q = P^{-1}$. This being the case, we will usually say that $A$ and $B$ are ***similar matrices*** if either is similar to the other.

## Matrix similarity

From Wikipedia, the free encyclopedia

*For other uses, see Similarity (geometry) and Similarity transformation (disambiguation).*
*Not to be confused with similarity matrix.*

In linear algebra, two $n$-by-$n$ matrices $A$ and $B$ are called **similar** if there exists an invertible $n$-by-$n$ matrix $P$ such that

$$B = P^{-1}AP.$$

Similar matrices represent the same linear map under two (possibly) different bases, with $P$ being the change of basis matrix.[1][2]

$$B = P^{-1}AP$$
$$PB = AP$$
$$PBP^{-1} = A$$

The goal: Find a matrix $B$ that is **diagonal**.

Because diagonal matrices have such a simple form, it is natural to inquire whether a given $n \times n$ matrix $A$ is similar to a matrix of this type. Should this turn out to be the case, and should we be able to actually find a diagonal matrix $B$ that is similar to $A$, then we would be able to ascertain many of the similarity invariant properties of $A$ directly from the diagonal entries of $B$. For example, the diagonal entries of $B$ will be the eigenvalues of $A$ (Theorem 5.1.2), and the product of the diagonal entries of $B$ will be the determinant of $A$ (Theorem 2.1.2). This leads us to introduce the following terminology.

**DEFINITION 2** A square matrix $A$ is said to be ***diagonalizable*** if it is similar to some diagonal matrix; that is, if there exists an invertible matrix $P$ such that $\underbrace{P^{-1}AP}_{B}$ is diagonal. In this case the matrix $P$ is said to ***diagonalize*** $A$.

Note: This is what **eigendecomposition** achieves!

Anton, Elementary Linear Algebra: Applications Version (11th Edition)

# Eigenvectors and Similarity Transform

**Eigenvector & Similarity Transform.**

$$Ax = \lambda x \quad ; \quad A \in R^{N\times N}$$
$$x \in R^N, \quad \lambda \in R$$

$$\therefore \underbrace{\begin{bmatrix} A \end{bmatrix}}_{N\times N} \underbrace{\begin{bmatrix} \uparrow \\ x_1 \\ \downarrow \end{bmatrix}}_{N\times 1} = \underbrace{\lambda}_{(R)} \underbrace{\begin{bmatrix} \uparrow \\ x_1 \\ \downarrow \end{bmatrix}}_{(N\times 1)}$$

Now consider $P = \begin{bmatrix} \uparrow & \uparrow \\ x_1 & x_2 \\ \downarrow & \downarrow \end{bmatrix}$ holding 2 col^n of eigenvector of A

$$\therefore \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ x_1 & x_2 \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow \\ \lambda_1 x_1 & \lambda_2 x_2 \\ \downarrow & \downarrow \end{bmatrix}$$
$$= \begin{bmatrix} \uparrow & \uparrow \\ x_1 & x_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$A \quad P \quad = \quad P \quad D$$

where $D \in R^{N\times N}$ diagonal of eigenvalue.

$$AP = PD.$$
$$\therefore A = PDP^{-1}$$

similar matrix $A, B$.

$$\therefore P^{-1}AP = D$$
$$\therefore P^{-1}A = DP^{-1}$$

1st eigenvector of $A$, normalised to unit vector.

2nd eigenvector of $A$, normalised to unit vector.

```
>> A = [1 2; 3 4];
>> [P,D] = eig(A)

P =

   -0.8246   -0.4160
    0.5658   -0.9094


D =

   -0.3723         0
         0    5.3723
```

$\lambda_1$ (above $-0.3723$), $\lambda_2$ (below $5.3723$)

Checking $P^{-1}AP = D$

```
>> inv(P)*A*P

ans =

   -0.3723    0.0000
    0.0000    5.3723

>> D

D =

   -0.3723         0
         0    5.3723
```

Checking $AP = PD$

```
>> A*P - P*D

ans =

   1.0e-16 *

         0         0
    0.5551         0
```

Checking $A = PDP^{-1}$

```
>> P*D*inv(P)

ans =

    1.0000    2.0000
    3.0000    4.0000
```

Checking $P^{-1}A = DP^{-1}$

```
>> inv(P)*A - D*inv(P)

ans =

   1.0e-15 *

   -0.1665   -0.2498
   -0.4441         0
```

# Why is Similarity Transform Important?

## Diagonalization

Essentially, Eigendecomposition!

In this section we will be concerned with the problem of finding a basis for $R^n$ that consists of eigenvectors of an $n \times n$ matrix $A$. Such bases can be used to study geometric properties of $A$ and to simplify various numerical computations. These bases are also of physical significance in a wide variety of applications, some of which will be considered later in this text.

Products of the form $P^{-1}AP$ in which $A$ and $P$ are $n \times n$ matrices and $P$ is invertible will be our main topic of study in this section. There are various ways to think about such products, one of which is to view them as transformations

$$A \to P^{-1}AP$$

in which the matrix $A$ is mapped into the matrix $P^{-1}AP$. These are called ***similarity transformations***. Such transformations are important because they preserve many properties of the matrix $A$. For example, if we let $B = P^{-1}AP$, then $A$ and $B$ have the same determinant since

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P)$$

$$= \frac{1}{\det(P)}\det(A)\det(P) = \det(A)$$

Anton, Elementary Linear Algebra: Applications Version (11th Edition)



In general, any property that is preserved by a similarity transformation is called a ***similarity invariant*** and is said to be ***invariant under similarity***. Table 1 lists the most important similarity invariants. The proofs of some of these are given as exercises.

Table 1 Similarity Invariants

| Property | Description |
|---|---|
| Determinant | $A$ and $P^{-1}AP$ have the same determinant. |
| Invertibility | $A$ is invertible if and only if $P^{-1}AP$ is invertible. |
| Rank | $A$ and $P^{-1}AP$ have the same rank. |
| Nullity | $A$ and $P^{-1}AP$ have the same nullity. |
| Trace | $A$ and $P^{-1}AP$ have the same trace. |
| Characteristic polynomial | $A$ and $P^{-1}AP$ have the same characteristic polynomial. |
| Eigenvalues | $A$ and $P^{-1}AP$ have the same eigenvalues. |
| Eigenspace dimension | If $\lambda$ is an eigenvalue of $A$ (and hence of $P^{-1}AP$) then the eigenspace of $A$ corresponding to $\lambda$ and the eigenspace of $P^{-1}AP$ corresponding to $\lambda$ have the same dimension. |

Note: **Having the same determinant is a necessary condition but not sufficient condition. For $A$ and $B$ to be called similar matrices, all the properties above must be satisfied.**

# How to Tell if Two Matrices are Similar?

## How do I tell if matrices are similar?

Asked 9 years, 5 months ago Active 17 days ago Viewed 65k times

I have two $2 \times 2$ matrices, $A$ and $B$, with the same determinant. I want to know if they are similar or not.

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There is something called "canonical forms" for a matrix; they are special forms for a matrix that can be obtained intrinsically from the matrix, and which will allow you to easily compare two matrices of the same size to see if they are similar or not. They are indeed based on eigenvalues and eigenvectors.

At this point, without the necessary machinery having been covered, the answer is that it is difficult to know if the two matrices are the same or not. The simplest test you can make is to see whether their characteristic polynomials are the same. This is *necessary*, but **not sufficient** for similarity (it is related to having the same eigenvalues).

Once you have learned about canonical forms, one can use either the Jordan canonical form (if the characteristic polynomial splits) or the rational canonical form (if the characteristic polynomial does not split) to compare the two matrices. They will be similar if and only if their rational forms are equal (up to some easily spotted differences; exactly analogous to the fact that two diagonal matrices are the same if they have the same diagonal entries, though the entries don't have to appear in the same order in both matrices).

The reduced row echelon form and the reduced column echelon form will not help you, because any two invertible matrices have the same forms (the identity), but need not have the same determinant (so they will not be similar).

Ref: https://math.stackexchange.com/questions/14075/how-do-i-tell-if-matrices-are-similar

# Why Use Eigendecomposition to Find Similar Matrices?

Review:

**DEFINITION 1** If $A$ and $B$ are square matrices, then we say that ***B* is similar to *A*** if there is an invertible matrix $P$ such that $B = P^{-1}AP$.

$$B = P^{-1}AP$$
$$PB = AP$$
$$PBP^{-1} = A$$

The goal: Find a matrix $B$ that is **diagonal**.

**Why?**

It allows us to study a complex matrix such as $A$, in terms of it similar matrix $B$, which is a simple (diagonal) matrix.

Ref:
https://math.stackexchange.com/questions/14075/how-do-i-tell-if-matrices-are-similar

As mentioned earlier, eigendecomposition helps achieve diagonalisation.

Matrix $A$ can be **eigendecomposed** as:

$$A = Q\Lambda Q^{-1}$$

Hence, matrix $A$ is similar to the diagonal matrix $\Lambda$.

- Similarity is an equivalence relation on the space of square matrices.
- Why? Because matrices are similar if and only if they represent the same linear operator with respect to (possibly) different bases, similar matrices share all properties of their shared underlying operator.

Ref:
https://en.wikipedia.org/wiki/Matrix_similarity
https://math.stackexchange.com/questions/3010998/why-do-similar-matrices-represent-the-same-linear-transformation

# Why do Similar Matrices Represent the Same Linear Transformation?

StackExchange Search on Mathematics...

MATHEMATICS

This is quite a long post, and it goes into a lot more detail than is perhaps necessary, but I felt it was important to be thorough. For the sake of simplicity, I'm going to restrict to linear transformations from a vector space $V$ to itself, also known as linear operators. This is really the only situation in which talking about similarity of matrices is meaningful, since it assumes that each matrix will use only one basis for both its input and its output.

## Part I Setting Up the Stage

It's important to understand here *in what sense* $A$ and $B$ are representing the same linear transformation, and really in what sense they represent a linear transformation at all. The crucial thing to keep in mind is that each matrix is representing a particular linear transformation *with respect to a specific basis*. Generally, when we go writing down matrices, the basis we're working with is implicit and often it isn't even actually mentioned since there is an implicit understanding of what the basis is that we're working with. This can be quite confusing however when you learn linear algebra for the first time as it can create a lot of potential for glossing over important distinctions, or even completely failing to notice them.

Ref:
https://math.stackexchange.com/questions/3010998/why-do-similar-matrices-represent-the-same-linear-transformation

## Part II Change of Basis

To see why this works requires understanding the concept of *change of basis*. That is, we need to understand *how* the matrix that represents a particular transformation changes as we change the basis with respect to which we want to express that linear transformation. So, given some transformation $T$ which is represented by the matrix $T_1$ with respect to a basis $\beta_1$, and given some other basis $\beta_2$ which we would like to express $T$ with respect to, how can we find the matrix that does so? To do this we have to construct the *change of basis matrix*.

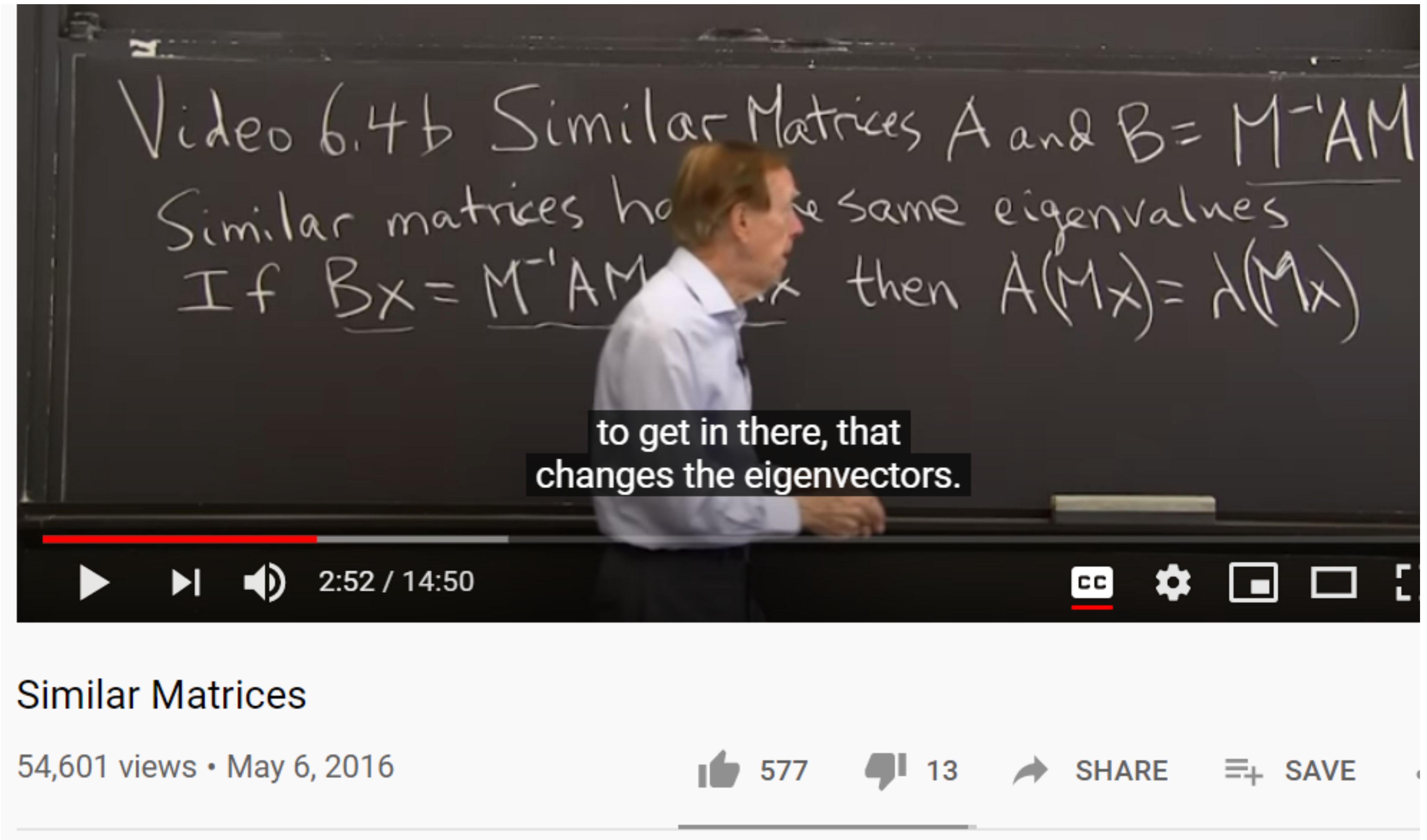
Let's say that the matrix that expresses $T$ relative to $\beta_2$ is $T_2$. Given some vector $v \in V$ *expressed with respect to* $\beta_2$ as a column vector, we expect that when we multiply the matrix $T_2$ by $v$ we get a column vector $T_2 v$ that expresses $T(v)$ with respect to $\beta_2$. We should think of the matrix $T_2$ as taking in a vector expressed with respect to $\beta_2$ and spitting out what the transformation $T$ does to that vector, expressed with respect to $\beta_2$.

What we have so far is a way of taking in a vector expressed with respect to $\beta_1$, and spitting out what the transformation does to that vector expressed with respect to $\beta_1$. This is given by the matrix $T_1$. So what we need is some way of taking a vector expressed with respect to $\beta_2$, and finding out how to express it with respect to $\beta_1$. That way, we can feed in that vector expressed with respect to $\beta_1$ to our matrix $T_1$, which will then dutifully spit out what $T$ does to that vector expressed with respect to $\beta_1$, when we multiply $T_1$ by that vector. Since the output is now expressed with respect to $\beta_1$, we also need some way of taking that output and expressing it with respect to $\beta_2$, which will then complete the process.

## Part III Wrapping It Up

Now that we know how to express a linear transformation with respect to arbitrary bases, it should really seem quite obvious why similarity is necessary for two matrices two represent the same transformation, and you have mentioned that that part seems to make sense to you. It's essentially baked into the process! What about the other direction? Well, if we can find a matrix $M$ such that $B = M^{-1}AM$, then we can think of $M$ as being the change of coordinates matrix. That is, we think of $M$ as taking vectors expressed with respect to the basis we are using for $B$, and expressing them with respect to a new basis with respect to which $A$ represents the same transformation. So any two matrices that are similar can represent the same linear transformation if you pick the right bases.

# Example of a Similarity Invariant Property




Prof. Strang shows similar matrices $A$ and $B$ have the same eigenvalues in the first 3 minutes and works out some examples of similar matrices later!

Ref:
https://www.youtube.com/watch?v=LKMGo8G7-vk

In general, any property that is preserved by a similarity transformation is called a ***similarity invariant*** and is said to be ***invariant under similarity***. Table 1 lists the most important similarity invariants. The proofs of some of these are given as exercises.

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