$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.1.1**

Lecture : **Eigen and Singular Values**

Topic : **Eigenvalue Decomposition**

Concept : **Introducing Eigenvectors and Eigenvalues**

Instructor: **A/P Chng Eng Siong**

TAs: **Zhang Su, Vishal Choudhari**

Rev: 3rd July 2020

Eigenvectors and Eigenvalues

DEFINITION

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

Matrix vector multiplication operation can be interpreted as a linear transformation. See: https://mathinsight.org/matrices_linear_transformations

The example on the right shows multiplication of vectors u, v with the matrix A.

Vector u **rotates** upon transformation and hence, is **not an eigenvector** for matrix A.

Vector v **does not rotate** and **retains its direction** upon transformation. Hence, it is **an eigenvector** for matrix A, with an eigenvalue of 2.

Lay, Linear Algebra and its Applications (4th Edition)

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EXAMPLE 1 Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The images of $\mathbf{u}$ and $\mathbf{v}$ under multiplication by A are shown in Fig. 1. In fact, $A\mathbf{v}$ is just $2\mathbf{v}$. So A only "stretches," or dilates, $\mathbf{v}$. ■

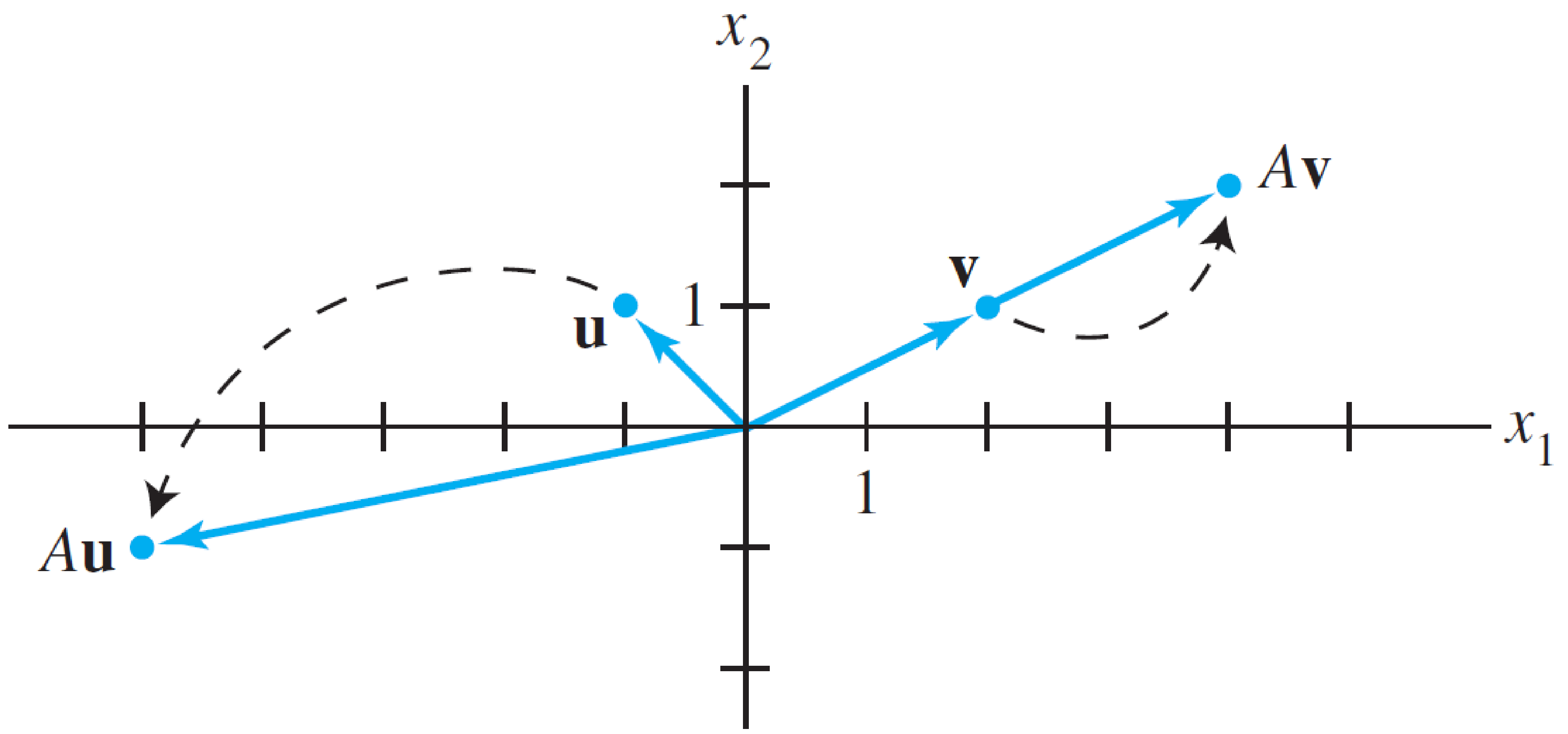


FIGURE 1 Effects of multiplication by A.

Important note:
Eigenvalues and **eigenvectors** are only for **square** matrices. For **rectangular** $m \times n$ matrices, **singular** values are defined. See Lecture 8.4.

Working: Angle between Au and u?

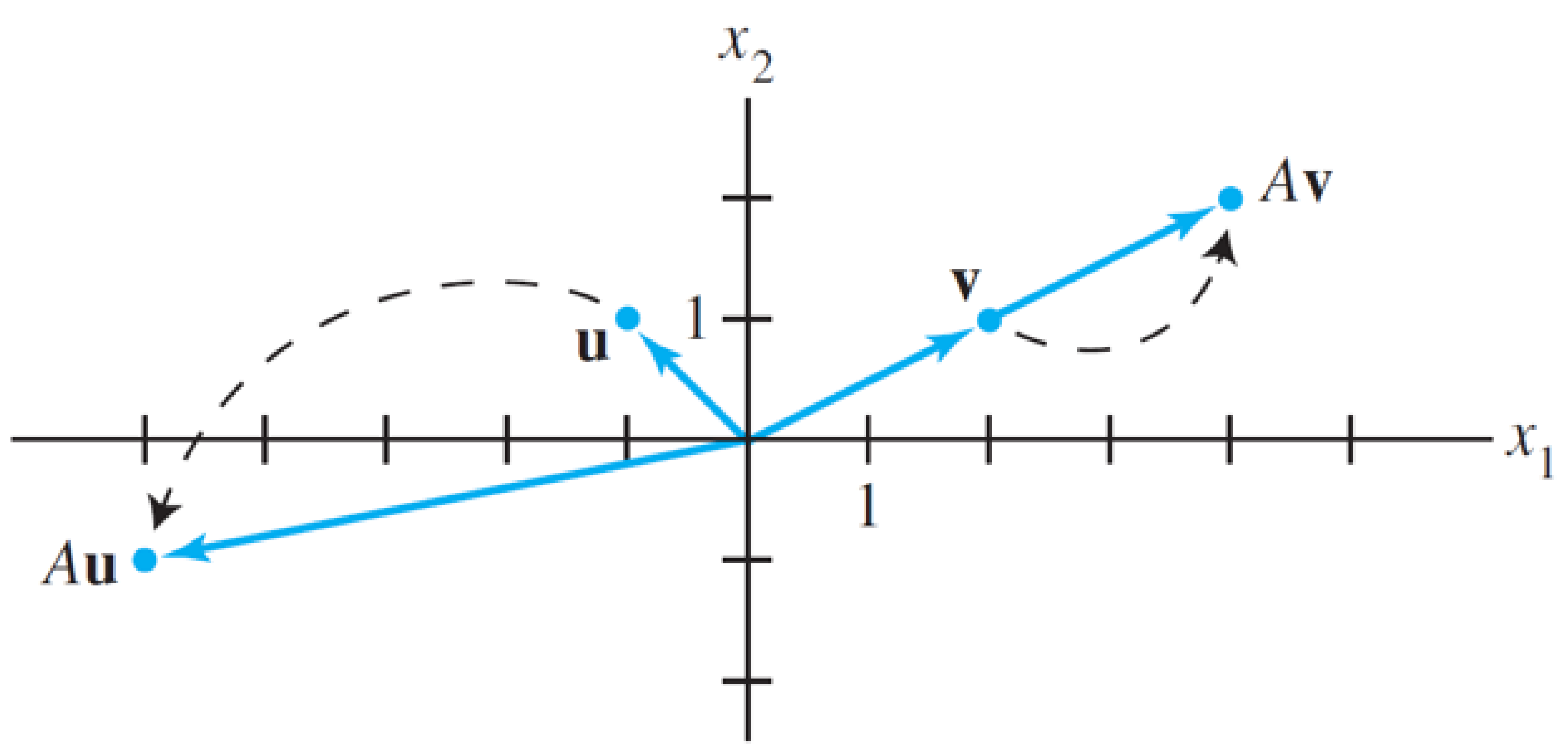


FIGURE 1 Effects of multiplication by A.

L1

$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$; $u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$Au = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$.

Q1) what is $\measuredangle\,\theta$ between Au and u ?

Using dot product :

$(Au)\cdot(u) = \|Au\|\,\|u\|\cos\theta$.

$\therefore \cos\theta = \dfrac{(Au)\cdot(u)}{\|Au\|\,\|u\|}$

$= \dfrac{\begin{bmatrix} -5 \\ -1 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \end{bmatrix}}{\left(\sqrt{-5^2+1^2}\right)\left(\sqrt{1^2+1^2}\right)} = \dfrac{4}{(\sqrt{26})(\sqrt{2})}$

$= 0.5547$.

$\therefore \theta = acos(0.5547) = 56.3^\circ$

Q2) what is $\measuredangle\,\theta$ between Av and v ?

$Av = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$\therefore \cos\theta = \dfrac{(Av)\cdot(v)}{\|Av\|\,\|v\|}$

$= \dfrac{\begin{bmatrix} 4 \\ 2 \end{bmatrix}\cdot\begin{bmatrix} 2 \\ 1 \end{bmatrix}}{\left(\sqrt{4^2+2^2}\right)\left(\sqrt{2^2+1^2}\right)}$

$= \dfrac{10}{\sqrt{20}\,\sqrt{5}} = 1$

$\therefore \theta = acos(1) = 0$ rad

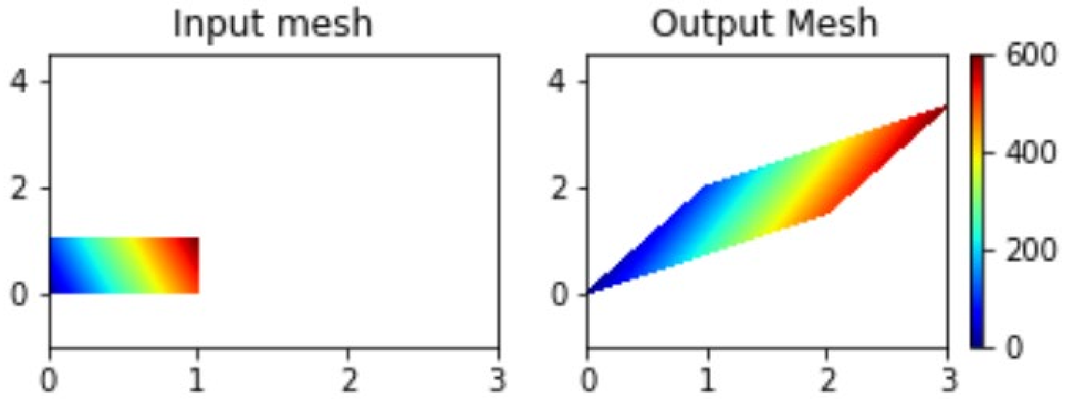
$= 0$ degree

Thinking of Matrix A as a Linear Transformation

```python
# make these smaller to increase the resolution
dx1, dx2 = 0.01, 0.01

# generate 2 2d grids for the x & y bounds
x1, x2 = np.mgrid[slice(0, 1 + dx1, dx1),
                  slice(0, 1 + dx2, dx2)]
y1, y2 = np.mgrid[slice(0, 1 + dx1, dx1),
                  slice(0, 1 + dx2, dx2)]

[nr,nc] =x1.shape
xin     =np.zeros((nr,nc))
yout    =np.zeros((nr,nc))
ipVec   = [0,0]
for index1 in range(nr):
    for index2 in range(nc):
        colorOpVal = index1*5+index2*1
        xin[index1,index2] = colorOpVal  # lets create the input mesh with different height
        ipVec[0]  = x1[index1,index2] # the values at [x1,x2]^T are the real input vector values
        ipVec[1]  = x2[index1,index2]
        y_op = A@ipVec                # Performing y = Ax  , to get the
                                      # transformed vector y using x as input
        y1[index1,index2]  = y_op[0]  # saving where the transformed points land
        y2[index1,index2]  = y_op[1]
        yout[index1,index2]=  colorOpVal    # using the same value as xin[index1,index2] so to generate color


zmin, zmax = xin.min(), xin.max()
print('len(xin) =',len(xin))
print('vmin=',zmin,' vmax=',zmax)
```

$$\begin{array}{ccccc} A & & v & & b \\ \begin{bmatrix} 2 & 1 \\ 1.5 & 2 \end{bmatrix} & * & \begin{matrix} 0.75 \\ 0.25 \end{matrix} & = & \begin{matrix} 1.75 \\ 1.625 \end{matrix} \end{array}$$

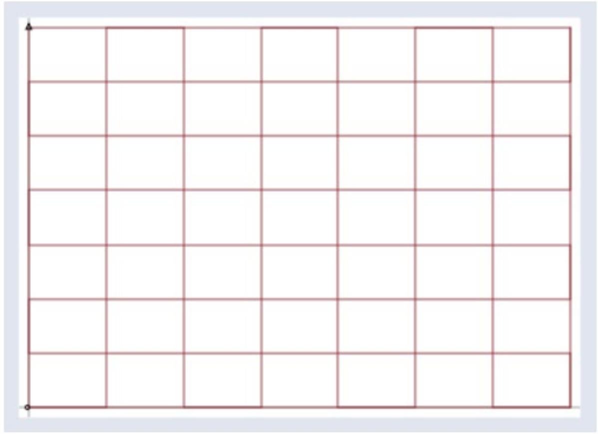
```python
plt.subplot(2, 2, 1)
#plt.pcolor(x1, x2, xin, cmap='RdBu', vmin=zmin, vmax=zmax)
plt.pcolor(x1, x2, xin, cmap='jet', vmin=zmin, vmax=zmax)
plt.title('Input mesh')
# set the limits of the plot to the limits of the data
plt.axis([y1.min(), y1.max(), y2.min()-1, y2.max()+1])

plt.subplot(2, 2, 2)
plt.pcolormesh(y1, y2, yout, cmap='jet', vmin=zmin, vmax=zmax)
plt.title('Output Mesh')
# set the limits of the plot to the limits of the data
plt.axis([y1.min(), y1.max(), y2.min()-1, y2.max()+1])
plt.colorbar()
```



Python code to generate the example on the right:
Ax_asLinearTransform.ipynb

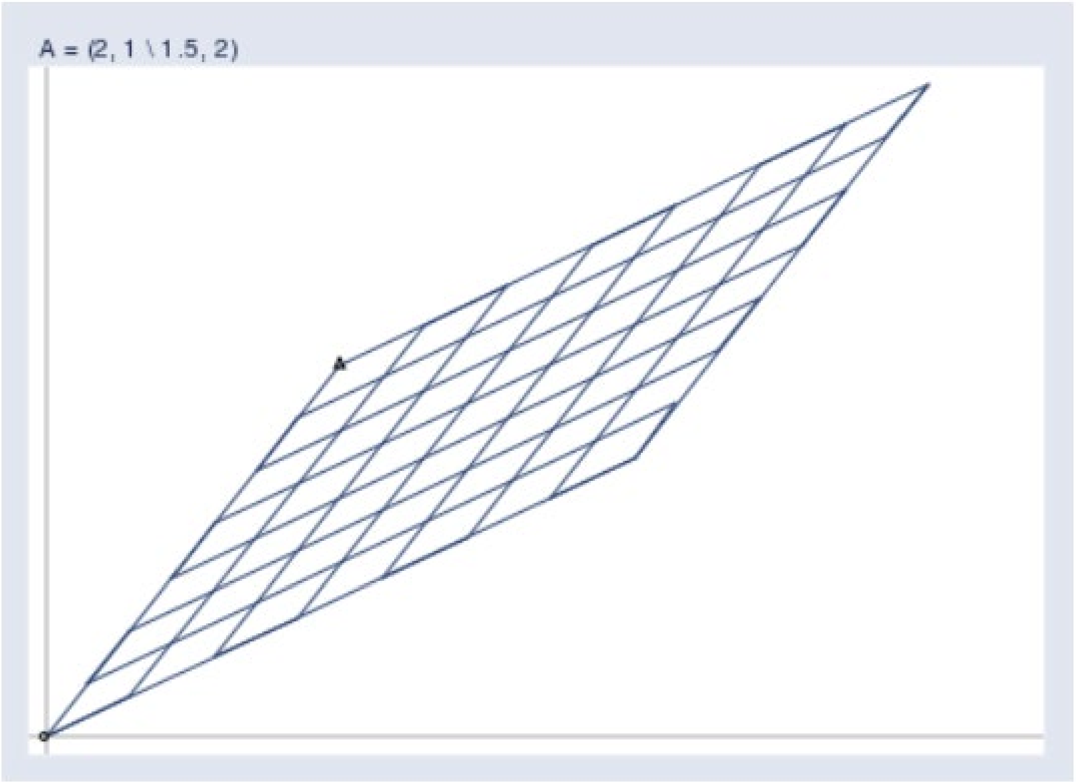
Input Grid

Imagine that all the input vectors *v* live in a normal grid, like this:

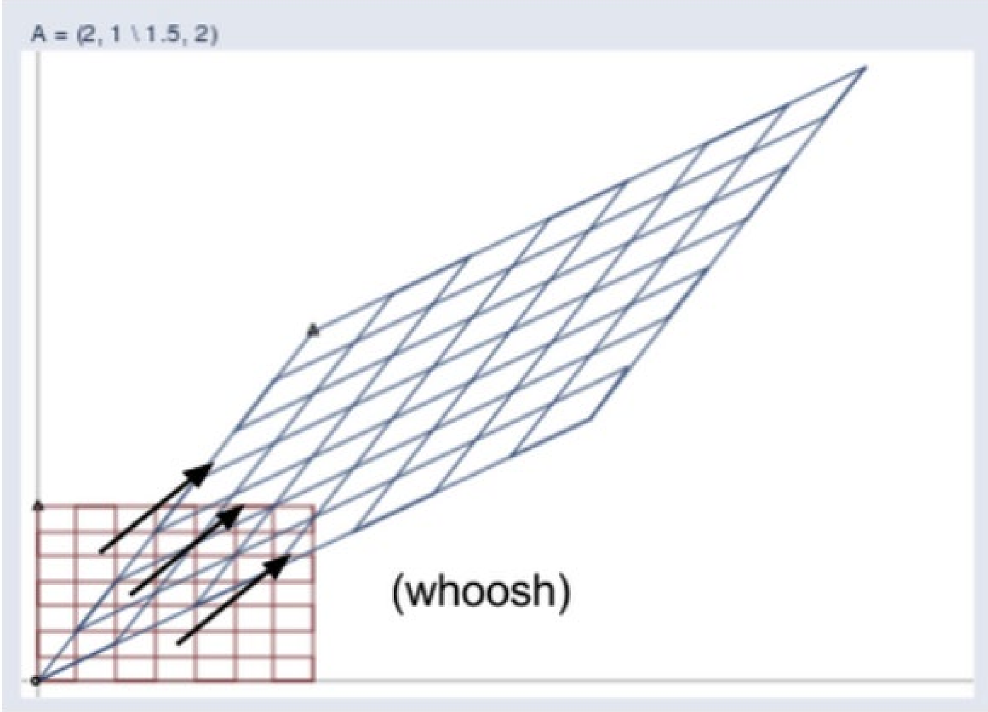


$Av = b$

After Transformation



Here you can see the two spaces juxtaposed:



(*Credit: William Gould, Stata Blog*)

Ref:
https://pathmind.com/wiki/eigenvector
https://www.youtube.com/watch?v=kYB8IZa5AuE
(3Blue1Brown, Linear Transformation, 2:10)

Thinking of Matrix A as a Linear Transformation

Matrices and linear transformations

Let A be a 2×3 matrix, say

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \end{bmatrix}.$$

What do you get if you multiply A by the vector $\mathbf{x} = (x, y, z)$? Remembering matrix multiplication, we see that

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - z \\ 3x + y + 2z \end{bmatrix} = (x - z, 3x + y + 2z).$$

If we define a function $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$, we have created a function of three variables (x, y, z) whose output is a two-dimensional vector $(x - z, 3x + y + 2z)$. Using function notation, we can write $\mathbf{f} : \mathbf{R}^3 \to \mathbf{R}^2$. We have created a vector-valued function of three variables. So, for example, $\mathbf{f}(1, 2, 3) = (1 - 3, 3 \cdot 1 + 2 + 2 \cdot 3) = (-2, 11)$.

In this way, we can associate with every matrix a function. What about going the other way around? Given some function, say $\mathbf{g} : \mathbf{R}^n \to \mathbf{R}^m$, can we associate with $\mathbf{g}(\mathbf{x})$ some matrix? We can only if $\mathbf{g}(\mathbf{x})$ is a special kind of function called a linear transformation. The function $\mathbf{g}(\mathbf{x})$ is a linear transformation if each term of each component of $\mathbf{g}(\mathbf{x})$ is a number times one of the variables. So, for example, the functions $\mathbf{f}(x, y) = (2x + y, y/2)$ and $\mathbf{g}(x, y, z) = (z, 0, 1.2x)$ are linear transformation, but none of the following functions are: $\mathbf{f}(x, y) = (x^2, y, x)$, $\mathbf{g}(x, y, z) = (y, xyz)$, or $\mathbf{h}(x, y, z) = (x + 1, y, z)$. Note that both functions we obtained from matrices above were linear transformations.

Let's take the function $\mathbf{f}(x, y) = (2x + y, y, x - 3y)$, which is a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^3$. The matrix A associated with $\mathbf{f}$ will be a 3×2 matrix, which we'll write as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

We need A to satisfy $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} = (x, y)$.

The easiest way to find A is the following. If we let $\mathbf{x} = (1, 0)$, then $f(\mathbf{x}) = A\mathbf{x}$ is the first column of A. (Can you see that?) So we know the first column of A is simply

$$f(1, 0) = (2, 0, 1) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Similarly, if $\mathbf{x} = (0, 1)$, then $f(\mathbf{x}) = A\mathbf{x}$ is the second column of A, which is

$$f(0, 1) = (1, 1, -3) = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}.$$

Putting these together, we see that the linear transformation $\mathbf{f}(\mathbf{x})$ is associated with the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & -3 \end{bmatrix}.$$

The important conclusion is that every linear transformation is associated with a matrix and vice versa.

Ref:

1. https://mathinsight.org/matrices_linear_transformations
2. https://mathinsight.org/linear_transformation_definition_euclidean
3. https://math.stackexchange.com/questions/1717117/what-does-it-mean-to-write-a-linear-operator-in-a-particular-basis