

CX1104: Linear Algebra for Computing

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{A \quad m \times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}_{x \quad n \times 1} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b \quad m \times 1}$$

Chap. No : **6.1.4**

Lecture : **Orthogonality**

Topic : **Dot Product**

Concept : **Important Inequalities**

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Cauchy-Schwarz Inequality

DEFINITION 3 If \mathbf{u} and \mathbf{v} are nonzero vectors in R^2 or R^3 , and if θ is the angle between \mathbf{u} and \mathbf{v} , then the *dot product* (also called the *Euclidean inner product*) of \mathbf{u} and \mathbf{v} is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \quad (12)$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \quad \Rightarrow \quad \theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right)$$

Formula (20) is not defined unless its argument satisfies the inequalities

$$-1 \leq \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \leq 1 \quad (21)$$

Fortunately, these inequalities *do* hold for all nonzero vectors in R^n as a result of the following fundamental result known as the *Cauchy-Schwarz inequality*.

THEOREM 3.2.4 Cauchy-Schwarz Inequality

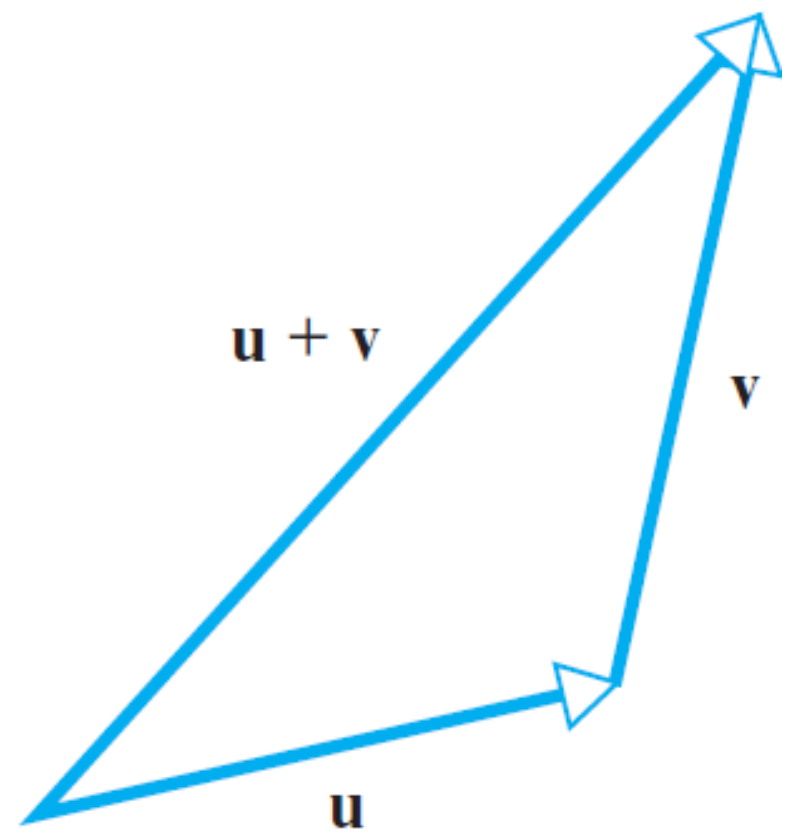
If $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are vectors in R^n , then

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\| \quad (22)$$

or in terms of components

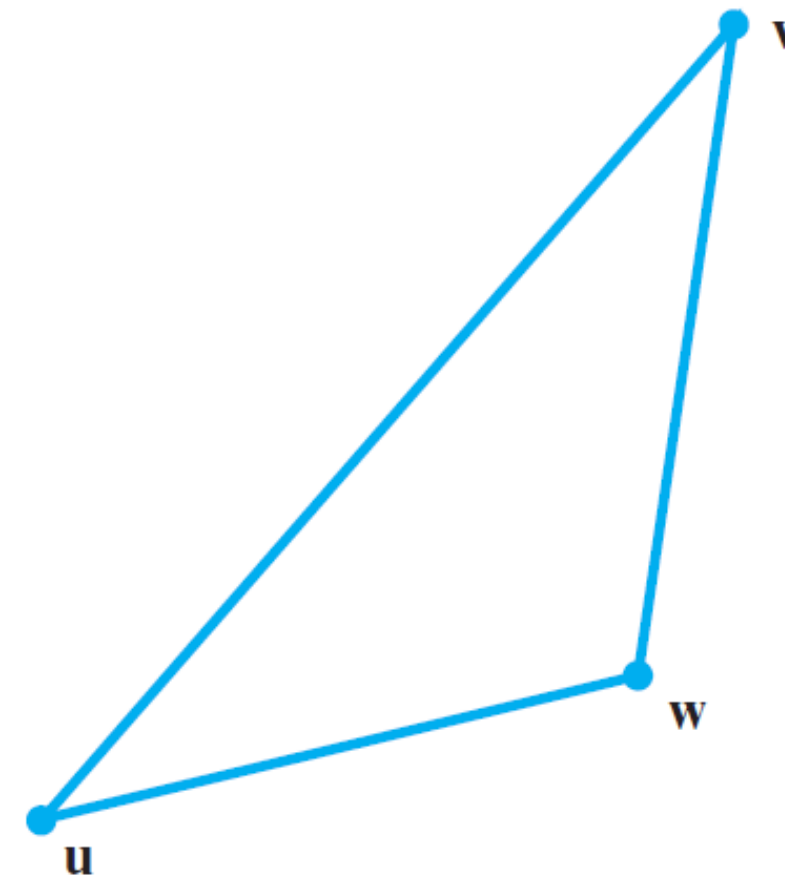
$$|u_1 v_1 + u_2 v_2 + \dots + u_n v_n| \leq (u_1^2 + u_2^2 + \dots + u_n^2)^{1/2} (v_1^2 + v_2^2 + \dots + v_n^2)^{1/2} \quad (23)$$

Triangle Inequality



$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

▲ Figure 3.2.8



$$d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$$

▲ Figure 3.2.9

THEOREM 3.2.5 If \mathbf{u} , \mathbf{v} , and \mathbf{w} are vectors in R^n , then:

- (a) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ [Triangle inequality for vectors]
- (b) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ [Triangle inequality for distances]

Proof:

Proof (a)

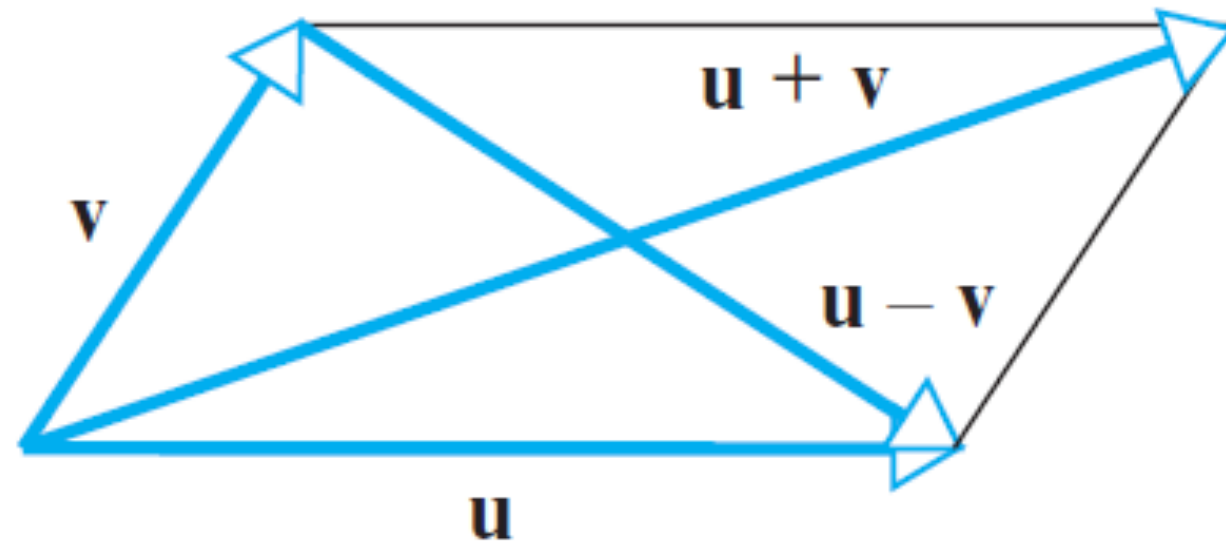
$$\begin{aligned}
 \|\mathbf{u} + \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = (\mathbf{u} \cdot \mathbf{u}) + 2(\mathbf{u} \cdot \mathbf{v}) + (\mathbf{v} \cdot \mathbf{v}) \\
 &= \|\mathbf{u}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2 \\
 &\leq \|\mathbf{u}\|^2 + 2|\mathbf{u} \cdot \mathbf{v}| + \|\mathbf{v}\|^2 && \leftarrow \text{Property of absolute value} \\
 &\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 && \leftarrow \text{Cauchy-Schwarz inequality} \\
 &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2
 \end{aligned}$$

This completes the proof since both sides of the inequality in part (a) are nonnegative.

Proof (b) It follows from part (a) and Formula (11) that

$$\begin{aligned}
 d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| = \|(\mathbf{u} - \mathbf{w}) + (\mathbf{w} - \mathbf{v})\| \\
 &\leq \|\mathbf{u} - \mathbf{w}\| + \|\mathbf{w} - \mathbf{v}\| = d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v}) \quad \blacktriangleleft
 \end{aligned}$$

Parallelogram Equation for Vectors



▲ Figure 3.2.10

THEOREM 3.2.6 Parallelogram Equation for Vectors

If \mathbf{u} and \mathbf{v} are vectors in R^n , then

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2) \quad (24)$$

Proof:

THEOREM 3.2.6 Parallelogram Equation for Vectors

If \mathbf{u} and \mathbf{v} are vectors in R^n , then

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2) \quad (24)$$

Proof

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) + (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= 2(\mathbf{u} \cdot \mathbf{u}) + 2(\mathbf{v} \cdot \mathbf{v}) \\ &= 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2) \quad \blacktriangleleft \end{aligned}$$

THEOREM 3.2.7 If \mathbf{u} and \mathbf{v} are vectors in R^n with the Euclidean inner product, then

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2 \quad (25)$$

Proof

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2 \\ \|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2 \end{aligned}$$

from which (25) follows by simple algebra. \blacktriangleleft

Reference

Materials in these slides have been taken from:

Anton and Rorres, “Linear Algebra”, 11th edition , Wiley.

Chapter: 3.1, 3.2

Euclidean Vector Spaces

3.1 Vectors in 2-Space, 3-Space, and n -Space 131

3.2 Norm, Dot Product, and Distance in R^n 142

