$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.3C**

Lecture : **Eigen and Singular Values**

Topic : **Eigenvalues**

Concept : **Properties and Theorems**

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Important Properties of Eigenvalues

Let A be an arbitrary $n \times n$ matrix of complex numbers with eigenvalues $\lambda_1, \ldots, \lambda_n$. Each eigenvalue appears $\mu_A(\lambda_i)$ times in this list, where $\mu_A(\lambda_i)$ is the eigenvalue's algebraic multiplicity. The following are properties of this matrix and its eigenvalues:

- The trace of A, defined as the sum of its diagonal elements, is also the sum of all eigenvalues,
- : $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.[27][28][29]
- The determinant of A is the product of all its eigenvalues,
- : $\det(A) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 \cdots \lambda_n$.[27][30][31]
- The eigenvalues of the k^{th} power of A; i.e., the eigenvalues of A^k, for any positive integer k, are $\lambda_1^k, \ldots, \lambda_n^k$.
- The matrix A is invertible if and only if every eigenvalue is nonzero.
- If A is invertible, then the eigenvalues of A^{-1} are $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$ and each eigenvalue's geometric multiplicity coincides. Moreover, since the characteristic polynomial of the inverse is the reciprocal polynomial of the original, the eigenvalues share the same algebraic multiplicity.
- If A is equal to its conjugate transpose A^*, or equivalently if A is Hermitian, then every eigenvalue is real. The same is true of any symmetric real matrix.
- If A is not only Hermitian but also positive-definite, positive-semidefinite, negative-definite, or negative-semidefinite, then every eigenvalue is positive, non-negative, negative, or non-positive, respectively.
- If A is unitary, every eigenvalue has absolute value $|\lambda_i| = 1$.
- if A is a $n \times n$ matrix and $\{\lambda_1, \ldots, \lambda_k\}$ are its eigenvalues, then the eigenvalues of matrix $I + A$ (where I is the identity matrix) are $\{\lambda_1 + 1, \ldots, \lambda_k + 1\}$. Moreover, if $\alpha \in \mathbb{C}$, the eigenvalues of $\alpha I + A$ are $\{\lambda_1 + \alpha, \ldots, \lambda_k + \alpha\}$. More generally, for a polynomial P the eigenvalues of matrix $P(A)$ are $\{P(\lambda_1), \ldots, P(\lambda_k)\}$.

Ref: https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors

Important Eigenvalue Theorems - Theorem 1

The eigenvalues of a triangular matrix are the entries on its main diagonal.

PROOF For simplicity, consider the 3×3 case. If A is upper triangular, then $A - \lambda I$ has the form

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$

The scalar λ is an eigenvalue of A if and only if the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, that is, if and only if the equation has a free variable. Because of the zero entries in $A - \lambda I$, it is easy to see that $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a free variable if and only if at least one of the entries on the diagonal of $A - \lambda I$ is zero. This happens if and only if λ equals one of the entries a_{11}, a_{22}, a_{33} in A. For the case in which A is lower triangular, see Exercise 28. ■

EXAMPLE 5 Let $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$. The eigenvalues of A are $3, 0$, and 2. The eigenvalues of B are 4 and 1. ■

What does it mean for a matrix A to have an eigenvalue of 0, such as in Example 5? This happens if and only if the equation

$$A\mathbf{x} = 0\mathbf{x} \tag{4}$$

has a nontrivial solution. But (4) is equivalent to $A\mathbf{x} = \mathbf{0}$, which has a nontrivial solution if and only if A is not invertible. Thus 0 *is an eigenvalue of A if and only if A is not invertible*. This fact will be added to the Invertible Matrix Theorem in Section 5.2.

The following important theorem will be needed later. Its proof illustrates a typical calculation with eigenvectors. One way to prove the statement "If P then Q" is to show that P and the negation of Q leads to a contradiction. This strategy is used in the proof of the theorem.

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

s. The number 0 is *not* an eigenvalue of A.

t. The determinant of A is *not* zero.

Important Eigenvalue Theorems - Theorem 2

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

PROOF Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly dependent. Since $\mathbf{v}_1$ is nonzero, Theorem 7 in Section 1.7 says that one of the vectors in the set is a linear combination of the preceding vectors. Let p be the least index such that $\mathbf{v}_{p+1}$ is a linear combination of the preceding (linearly independent) vectors. Then there exist scalars $c_1, \ldots, c_p$ such that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{v}_{p+1} \tag{5}$$

Multiplying both sides of (5) by A and using the fact that $A\mathbf{v}_k = \lambda_k\mathbf{v}_k$ for each k, we obtain

$$\begin{aligned} c_1A\mathbf{v}_1 + \cdots + c_pA\mathbf{v}_p &= A\mathbf{v}_{p+1} \\ c_1\lambda_1\mathbf{v}_1 + \cdots + c_p\lambda_p\mathbf{v}_p &= \lambda_{p+1}\mathbf{v}_{p+1} \end{aligned} \tag{6}$$

Multiplying both sides of (5) by λ_{p+1} and subtracting the result from (6), we have

$$c_1(\lambda_1 - \lambda_{p+1})\mathbf{v}_1 + \cdots + c_p(\lambda_p - \lambda_{p+1})\mathbf{v}_p = \mathbf{0} \tag{7}$$

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent, the weights in (7) are all zero. But none of the factors $\lambda_i - \lambda_{p+1}$ are zero, because the eigenvalues are distinct. Hence $c_i = 0$ for $i = 1, \ldots, p$. But then (5) says that $\mathbf{v}_{p+1} = \mathbf{0}$, which is impossible. Hence $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ cannot be linearly dependent and therefore must be linearly independent. ■

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Important Eigenvalue Theorems - Theorem 3

Properties of Determinants

Let A and B be $n \times n$ matrices.

a. A is invertible if and only if $\det A \neq 0$.

b. $\det AB = (\det A)(\det B)$.

c. $\det A^T = \det A$.

d. If A is triangular, then $\det A$ is the product of the entries on the main diagonal of A.

e. A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

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CHAPTER 5 Eigenvalues and Eigenvectors

Important Eigenvalue Theorems - Theorem 4

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

PROOF If $B = P^{-1}AP$, then

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$$

Using the multiplicative property (b) in Theorem 3, we compute

$$\begin{aligned} \det(B - \lambda I) &= \det[P^{-1}(A - \lambda I)P] \\ &= \det(P^{-1}) \cdot \det(A - \lambda I) \cdot \det(P) \qquad (2) \end{aligned}$$

Since $\det(P^{-1}) \cdot \det(P) = \det(P^{-1}P) = \det I = 1$, we see from equation (2) that $\det(B - \lambda I) = \det(A - \lambda I)$. ■

WARNINGS:

1. The matrices
$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
are not similar even though they have the same eigenvalues.
2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E.) Row operations on a matrix usually change its eigenvalues.

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CHAPTER 5 Eigenvalues and Eigenvectors

Important Eigenvalue Theorems - Theorem 5

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

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CHAPTER 5 Eigenvalues and Eigenvectors

PROOF First, observe that if P is any $n \times n$ matrix with columns $\mathbf{v}_1, \dots, \mathbf{v}_n$, and if D is any diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$, then

$$AP = A[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n\,] = [\,A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n\,] \tag{1}$$

while

$$PD = P\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\,\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n\,] \tag{2}$$

Now suppose A is diagonalizable and $A = PDP^{-1}$. Then right-multiplying this relation by P, we have $AP = PD$. In this case, equations (1) and (2) imply that

$$[\,A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n\,] = [\,\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n\,] \tag{3}$$

Equating columns, we find that

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \quad A\mathbf{v}_2 = \lambda_2\mathbf{v}_2, \quad \dots, \quad A\mathbf{v}_n = \lambda_n\mathbf{v}_n \tag{4}$$

Since P is invertible, its columns $\mathbf{v}_1, \dots, \mathbf{v}_n$ must be linearly independent. Also, since these columns are nonzero, the equations in (4) show that $\lambda_1, \dots, \lambda_n$ are eigenvalues and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are corresponding eigenvectors. This argument proves the "only if" parts of the first and second statements, along with the third statement, of the theorem.

Finally, given any n eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, use them to construct the columns of P and use corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ to construct D. By equations (1)–(3), $AP = PD$. This is true without any condition on the eigenvectors. If, in fact, the eigenvectors are linearly independent, then P is invertible (by the Invertible Matrix Theorem), and $AP = PD$ implies that $A = PDP^{-1}$. ■

Important Eigenvalue Theorems - Theorem 6

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

PROOF Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be eigenvectors corresponding to the n distinct eigenvalues of a matrix A. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent, by Theorem 2 in Section 5.1. Hence A is diagonalizable, by Theorem 5. ■

It is not *necessary* for an $n \times n$ matrix to have n distinct eigenvalues in order to be diagonalizable. The 3×3 matrix in Example 3 is diagonalizable even though it has only two distinct eigenvalues.

Theorem 2

If $\mathbf{v}_1, \dots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.

Theorem 5

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

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CHAPTER 5 Eigenvalues and Eigenvectors

Important Eigenvalue Theorems - Theorem 7

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \dots, \lambda_p$.

a. For $1 \le k \le p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \dots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

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CHAPTER 5 Eigenvalues and Eigenvectors

Important Eigenvalue Theorems - Theorem 9

Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then

$$A = PCP^{-1}, \quad \text{where} \quad P = [\operatorname{Re}\mathbf{v} \quad \operatorname{Im}\mathbf{v}] \quad \text{and} \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

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CHAPTER 5 Eigenvalues and Eigenvectors