$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

# CX1104: **Linear Algebra for Computing**

Chap. No : **6.2.2**

Lecture : **Orthogonality**

Topic : **Orthogonality**

Concept : **Orthogonal Projections**


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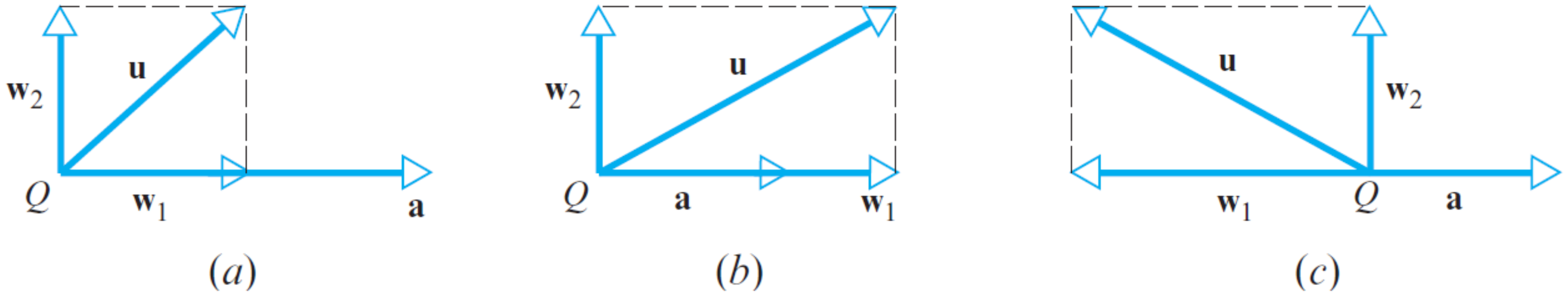
# Orthogonal Projections

## Decomposition in $R^2$ space

In many applications it is necessary to "decompose" a vector $\mathbf{u}$ into a sum of two terms, one term being a scalar multiple of a specified nonzero vector $\mathbf{a}$ and the other term being orthogonal to $\mathbf{a}$. For example, if $\mathbf{u}$ and $\mathbf{a}$ are vectors in $R^2$ that are positioned so their initial points coincide at a point $Q$, then we can create such a decomposition as follows (Figure 3.3.2):




▲ Figure 3.3.2 Three possible cases.

- Drop a perpendicular from the tip of $\mathbf{u}$ to the line through $\mathbf{a}$.
- Construct the vector $\mathbf{w}_1$ from $Q$ to the foot of the perpendicular.
- Construct the vector $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$.

Since

$$\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1 + (\mathbf{u} - \mathbf{w}_1) = \mathbf{u}$$

we have decomposed $\mathbf{u}$ into a sum of two orthogonal vectors, the first term being a scalar multiple of $\mathbf{a}$ and the second being orthogonal to $\mathbf{a}$.



**Decomposing vectors into orthogonal basis has many advantages!**
**The standard basis is an orthogonal basis.**
**Orthogonal basis is explained in 6.2.3.**

### ▶ EXAMPLE 1 The Standard Basis for $R^n$

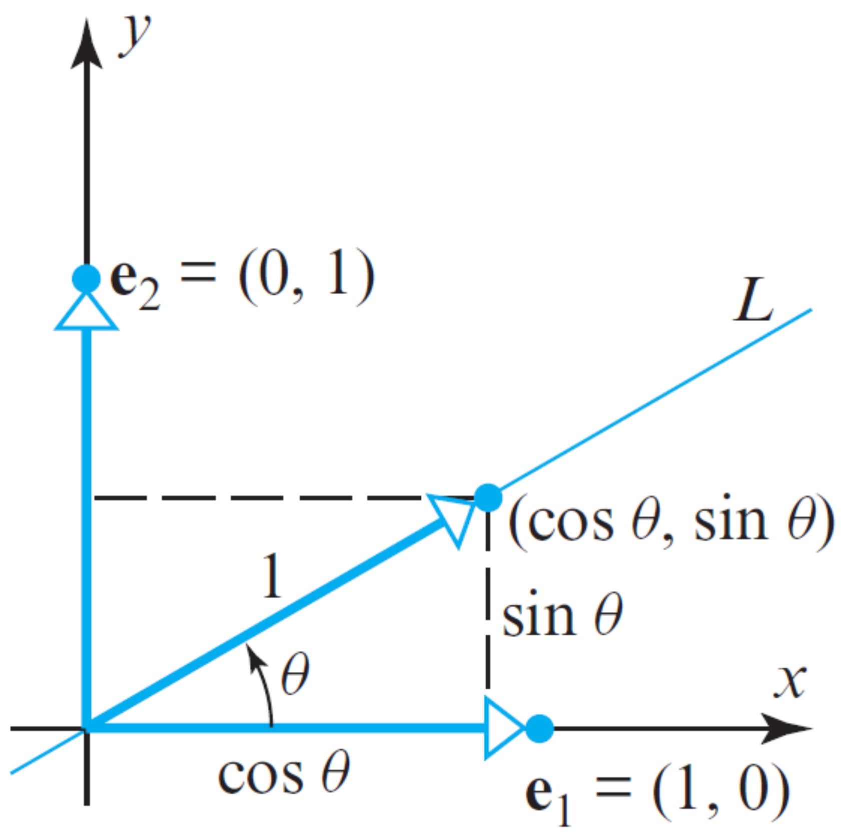
Recall from Example 11 of Section 4.2 that the standard unit vectors

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

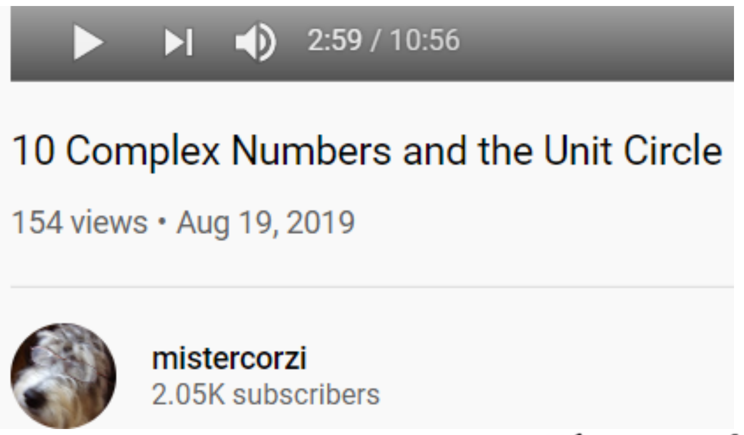
span $R^n$ and from Example 1 of Section 4.3 that they are linearly independent. Thus, they form a basis for $R^n$ that we call the ***standard basis for $R^n$***. In particular,

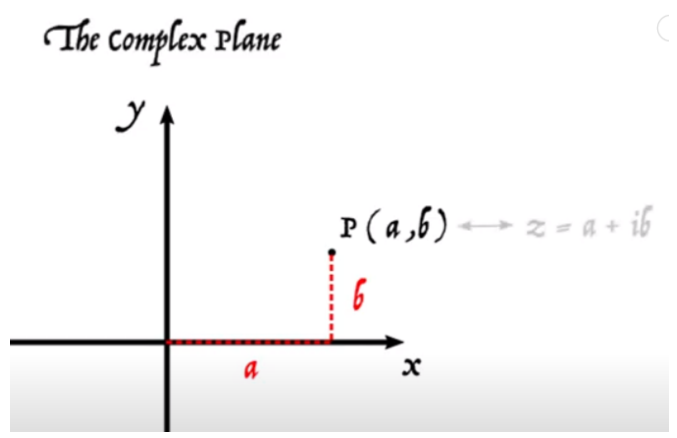
$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

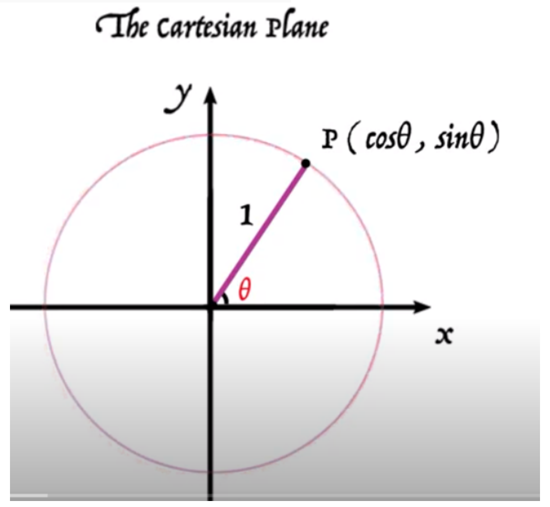
is the standard basis for $R^3$.




▲ Figure 3.3.3




Mister Corzi: https://www.youtube.com/watch?v=UF172DZOiWo

Example above:
Commonly found in complex number of unit circle.
Where $e_1$ is the real line, and $e_2$ is the imaginary line

**Khan Academy:**
https://www.khanacademy.org/math/precalculus/x9e81a4f98389efdf:complex/x9e81a4f98389efdf:complex-polar/v/polar-form-complex-number
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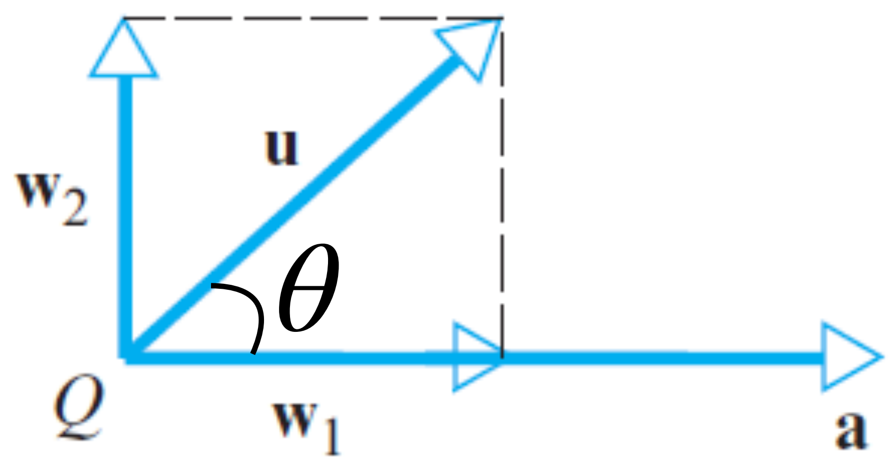
# Orthogonal Projections

**THEOREM 3.3.2 Projection Theorem**

*If* $\mathbf{u}$ *and* $\mathbf{a}$ *are vectors in* $R^n$, *and if* $\mathbf{a} \neq 0$, *then* $\mathbf{u}$ *can be expressed in exactly one way in the form* $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, *where* $\mathbf{w}_1$ *is a scalar multiple of* $\mathbf{a}$ *and* $\mathbf{w}_2$ *is orthogonal to* $\mathbf{a}$.




$(a)$

The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ in the Projection Theorem have associated names—the vector $\mathbf{w}_1$ is called the ***orthogonal projection of*** $\mathbf{u}$ ***on*** $\mathbf{a}$ or sometimes ***the vector component of*** $\mathbf{u}$ ***along*** $\mathbf{a}$, and the vector $\mathbf{w}_2$ is called the ***vector component of*** $\mathbf{u}$ ***orthogonal to*** $\mathbf{a}$. The vector $\mathbf{w}_1$ is commonly denoted by the symbol $\text{proj}_{\mathbf{a}}\mathbf{u}$, in which case it follows from (8) that $\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$. In summary,

$$w_1 = \text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \quad (\textit{vector component of } \mathbf{u} \textit{ along } \mathbf{a}) \quad = ||u||cos\theta \tag{10}$$

$$w_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \quad (\textit{vector component of } \mathbf{u} \textit{ orthogonal to } \mathbf{a}) \tag{11}$$

**EXAMPLE 5 Vector Component of u Along a**

Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{a} = (4, -1, 2)$. Find the vector component of $\mathbf{u}$ along $\mathbf{a}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$.

***Solution***

$$\mathbf{u} \cdot \mathbf{a} = (2)(4) + (-1)(-1) + (3)(2) = 15$$
$$\|\mathbf{a}\|^2 = 4^2 + (-1)^2 + 2^2 = 21$$

Thus the vector component of $\mathbf{u}$ along $\mathbf{a}$ is

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \tfrac{15}{21}(4, -1, 2) = \left(\tfrac{20}{7}, -\tfrac{5}{7}, \tfrac{10}{7}\right)$$

and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (2, -1, 3) - \left(\tfrac{20}{7}, -\tfrac{5}{7}, \tfrac{10}{7}\right) = \left(-\tfrac{6}{7}, -\tfrac{2}{7}, \tfrac{11}{7}\right)$$

As a check, you may wish to verify that the vectors $\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$ and $\mathbf{a}$ are perpendicular by showing that their dot product is zero. ◀

Linear Algebra: Projection onto a Subspace

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# Orthogonal Projections

Fig. 2.5 shows two vectors $\mathbf{u}, \mathbf{v}$ in $R^2$ space. The orthogonal projection of $\mathbf{u}$ on $\mathbf{v}$, i.e. $Proj_{\mathbf{v}}\mathbf{u}$, can be thought of as the shadow formed by vector $\mathbf{u}$ onto $\mathbf{v}$ when an imaginary light is directed to $\mathbf{u}$ along the normal of $\mathbf{v}$; The vector $Proj_{\mathbf{v}}\mathbf{u}$ is the best approximation of $\mathbf{u}$ using $\mathbf{v}$. The following are the definitions of orthogonal projection and the equation to find the orthogonal residual.

$$Proj_{\mathbf{v}}\mathbf{u} = \mathbf{v}\left(\frac{\mathbf{u}^T\mathbf{v}}{||\mathbf{v}||^2}\right), \tag{2.34}$$

$$\mathbf{w}_{\mathbf{v}} = \mathbf{u} - Proj_{\mathbf{v}}\mathbf{u}, \tag{2.35}$$

where $||\mathbf{v}|| = \sqrt{\mathbf{v}^T\mathbf{v}}$ is the Euclidean norm of $\mathbf{v}$ and the vector $\mathbf{w}_{\mathbf{v}}$ is the residual when $\mathbf{v}$ is used to approximate $\mathbf{u}$.

Why is the denominator $||v||^2$? (eqn 2.34).

Ans: it is to normalize vector $v$ to have unit length.

Notice that $v$ occurs twice, to get a vector to have unit length,

We need to divide v by $||v||$

$$\left(\frac{v}{||v||}\right) * u^T\left(\frac{v}{||v||}\right) = v * \frac{u^T v}{(||v||^2)}$$

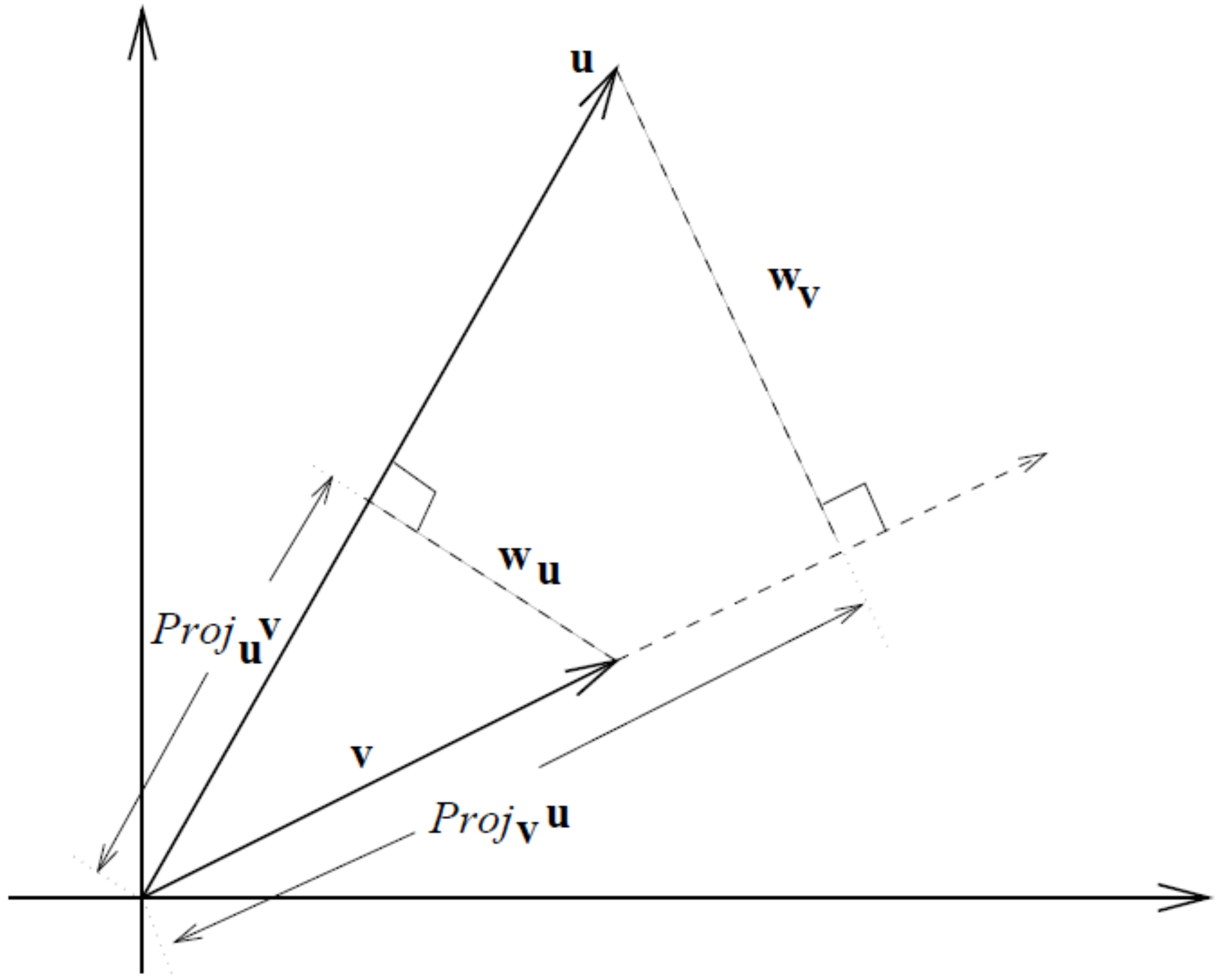
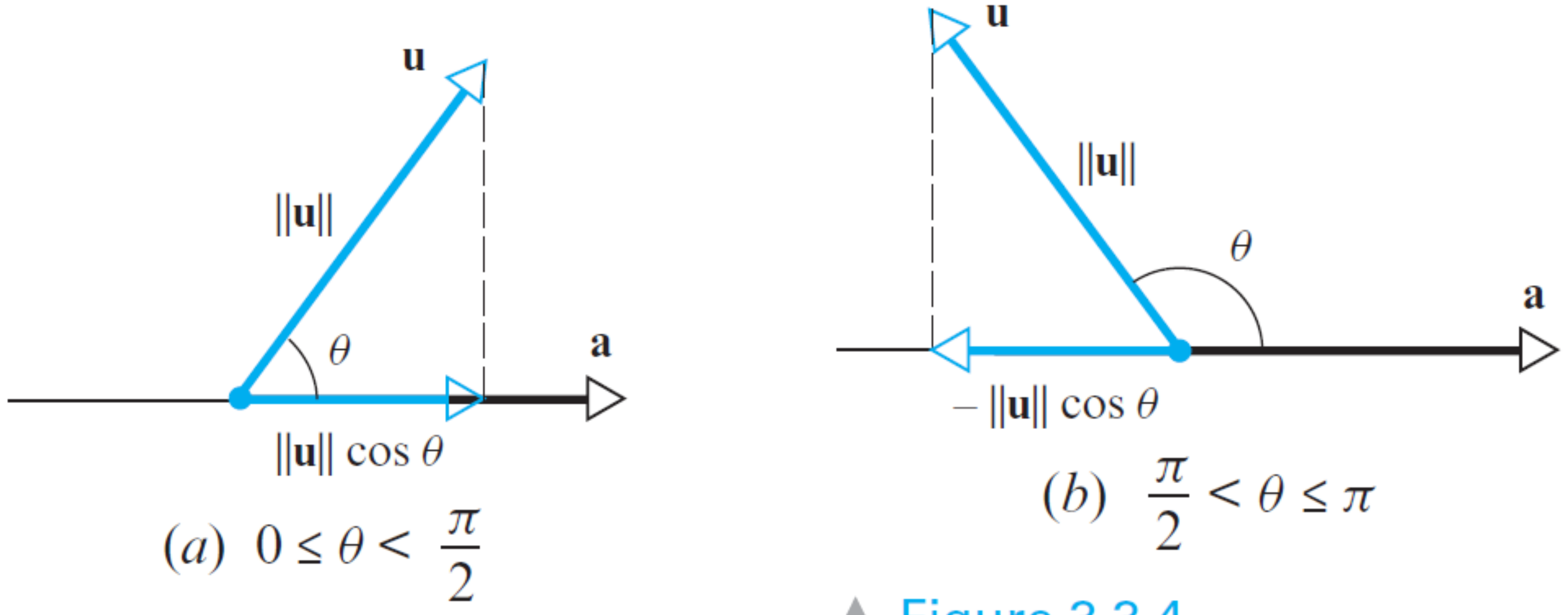



Figure 2.5: Orthogonal projections

# Orthogonal Projections

**Finding the norm of the projected component.**




▲ Figure 3.3.4



Sometimes we will be more interested in the *norm* of the vector component of **u** along **a** than in the vector component itself. A formula for this norm can be derived as follows:

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \left\| \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \right\| = \left| \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2} \right| \|\mathbf{a}\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|^2}\|\mathbf{a}\|$$

where the second equality follows from part (*c*) of Theorem 3.2.1 and the third from the fact that $\|\mathbf{a}\|^2 > 0$. Thus,

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|} \tag{12}$$

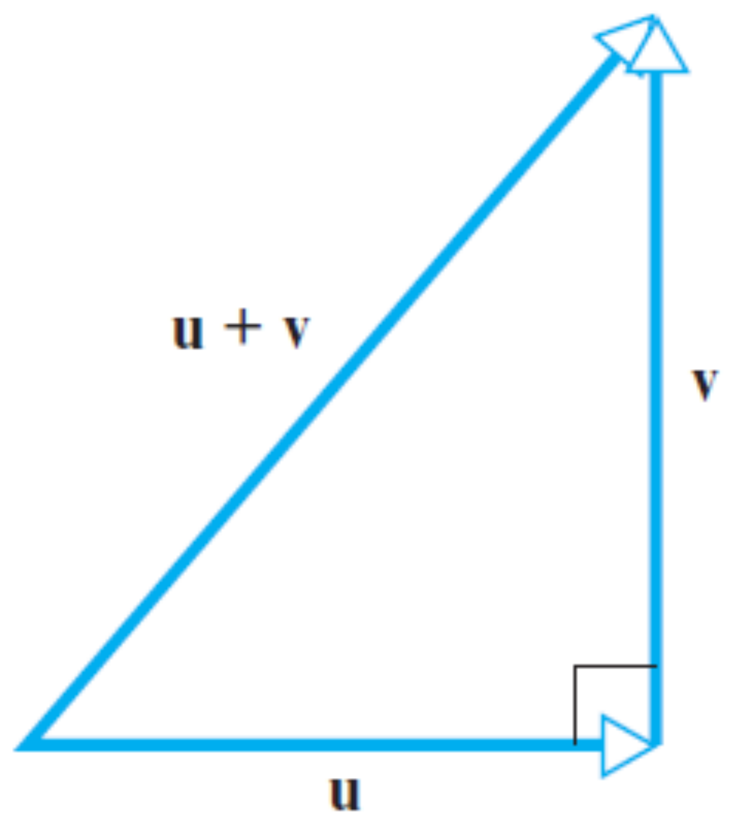
If $\theta$ denotes the angle between **u** and **a**, then $\mathbf{u}\cdot\mathbf{a} = \|\mathbf{u}\|\|\mathbf{a}\|\cos\theta$, so (12) can also be written as

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \|\mathbf{u}\||\cos\theta| \tag{13}$$

(Verify.) A geometric interpretation of this result is given in Figure 3.3.4.

# Pythagoras Theorem

## The Theorem of Pythagoras




▲ Figure 3.3.5

**THEOREM 3.3.3 Theorem of Pythagoras in $R^n$**

*If* **u** *and* **v** *are orthogonal vectors in $R^n$ with the Euclidean inner product, then*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \tag{14}$$

***Proof*** Since **u** and **v** are orthogonal, we have $\mathbf{u}\cdot\mathbf{v} = 0$, from which it follows that

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \quad ◀$$



▶ **EXAMPLE 6 Theorem of Pythagoras in $R^4$**

We showed in Example 1 that the vectors

$$\mathbf{u} = (-2, 3, 1, 4) \quad \text{and} \quad \mathbf{v} = (1, 2, 0, -1)$$

are orthogonal. Verify the Theorem of Pythagoras for these vectors.

***Solution*** We leave it for you to confirm that

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= (-1, 5, 1, 3) \\ \|\mathbf{u}+\mathbf{v}\|^2 &= 36 \\ \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 &= 30 + 6 \end{aligned}$$

Thus, $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ ◀