$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.1.2**
Lecture : **Eigen and Singular Values**
Topic : **Eigenvalue Decomposition**
Concept : **Finding Eigenvalues and Eigenvectors**

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TAs: **Zhang Su, Vishal Choudhari**

Rev: 3rd July 2020

Checking for Eigenvectors

Given a square matrix A and column vectors u, check if u is an eigenvector by showing it follows:

$$Au = \lambda u$$

where, λ is a real (or complex) number.

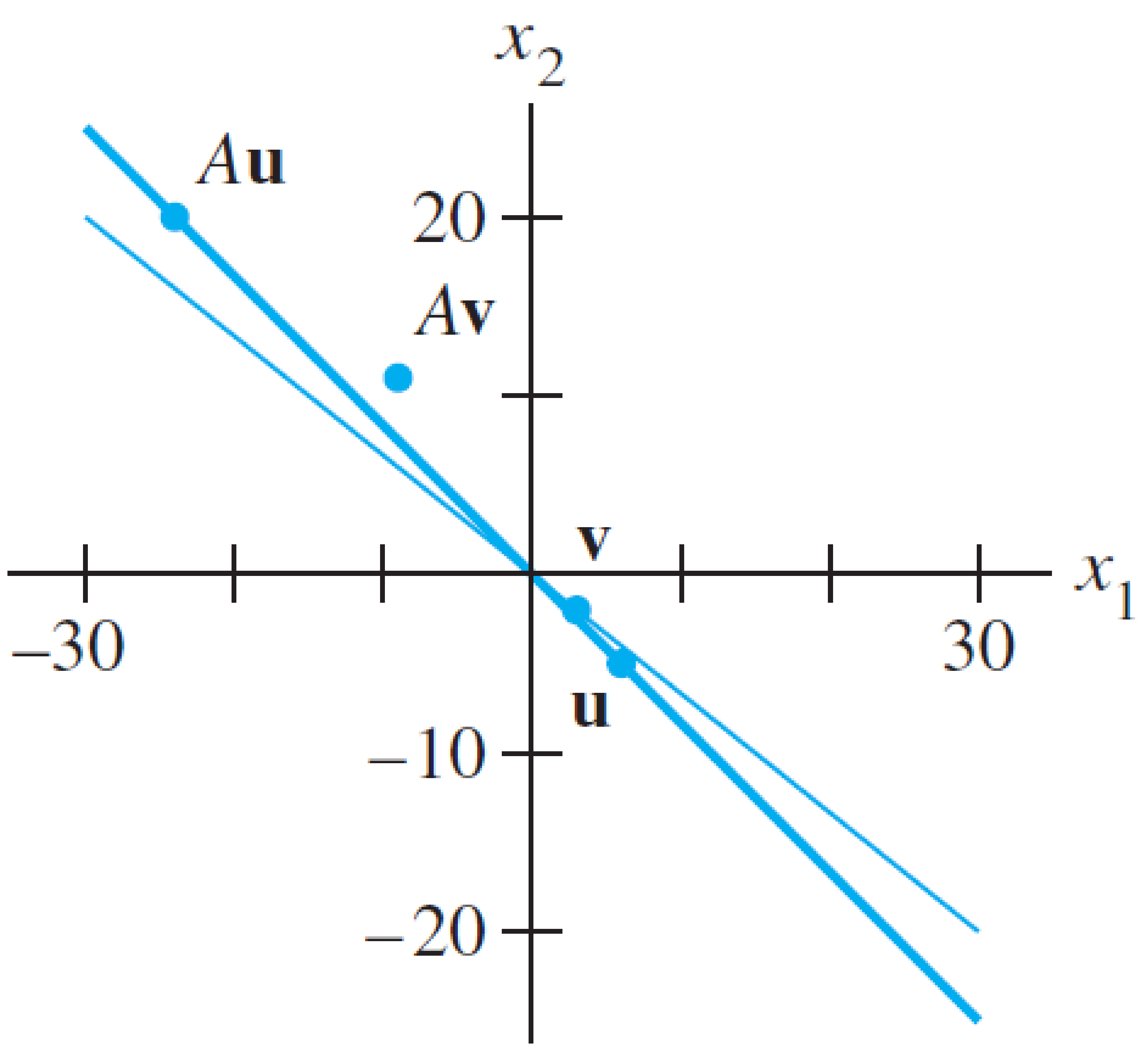
EXAMPLE 2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

SOLUTION

$$A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u}$$

$$A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Thus $\mathbf{u}$ is an eigenvector corresponding to an eigenvalue (-4), but $\mathbf{v}$ is not an eigenvector of A, because $A\mathbf{v}$ is not a multiple of $\mathbf{v}$. ■



$A\mathbf{u} = -4\mathbf{u}$, but $A\mathbf{v} \neq \lambda\mathbf{v}$.

Finding Eigenvectors When Given Eigenvalues

Given a known eigenvalue, it is easy to find its corresponding eigenvector.

But, it is NOT easy to find the eigenvalue!

EXAMPLE 2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

EXAMPLE 3 Show that 7 is an eigenvalue of matrix A in Example 2, and find the corresponding eigenvectors.

SOLUTION The scalar 7 is an eigenvalue of A if and only if the equation

$$A\mathbf{x} = 7\mathbf{x} \qquad (1)$$

has a nontrivial solution. But (1) is equivalent to $A\mathbf{x} - 7\mathbf{x} = \mathbf{0}$, or

$$(A - 7I)\mathbf{x} = \mathbf{0} \qquad (2)$$

To solve this homogeneous equation, form the matrix

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

The columns of $A - 7I$ are obviously linearly dependent, so (2) has nontrivial solutions. Thus 7 *is* an eigenvalue of A. To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution has the form $x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda = 7$. ■

Lay, Linear Algebra and its Applications (4th Edition)

Finding Eigenvalues

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

Qn. **How to find eigenvalues for a square matrix A?**

Eq. 1: $Ax = \lambda x$

Eq. 2: $Ax - \lambda x = 0$

Eq. 3: $(A - \lambda I)x = 0$

Note:

- 0 here is a null column vector of appropriate dimension ($n \times 1$).
- I is the $(n \times n)$ identity matrix.

Eq. 3 has a non-trivial solution x **if and only if** the determinant of the matrix $(A - \lambda I)$ is zero. Therefore, the eigenvalues of A are the values of λ that satisfy the equation:

$$|A - \lambda I| = 0$$

EXAMPLE 2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

Eq. 3 for this example can be written as:

Eq. 4: $\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)x = 0$

Eq. 5: $\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)x = 0$

Eq. 6: $\left(\begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix}\right)x = 0$

Note

When a matrix is **singular**,

-> It also means that the **determinant is zero**.
-> It has **dependent rows** and **columns**.

For more properties, see:
https://math.stackexchange.com/questions/355644/what-does-it-mean-to-have-a-determinant-equal-to-zero
https://en.wikipedia.org/wiki/Invertible_matrix

19

If the determinant of a square matrix $n \times n$ A is zero, then A is *not invertible*. This is a crucial test that helps determine whether a square matrix is invertible, i.e., if the matrix has an inverse. When it does have an inverse, it allows us to find a unique solution, e.g., to the equation $Ax = b$ given some vector b.

When the determinant of a matrix is zero, the system of equations associated with it is linearly dependent; that is, if the determinant of a matrix *is zero*, at least one row of such a matrix is a scalar multiple of another.

[When the determinant of a matrix is nonzero, the linear system it represents is linearly independent.]

When the determinant of a matrix is zero, its rows are linearly dependent vectors, and its columns are linearly dependent vectors.

What does it mean to have a determinant equal to zero?

Asked 7 years, 3 months ago Active 8 months ago Viewed 356k times

91

After looking in my book for a couple of hours, I'm still confused about what it means for a $(n \times n)$-matrix A to have a determinant equal to zero, $\det(A) = 0$.

I hope someone can explain this to me in plain English.

Upcoming Events

2020 Communi
ends in 3 days

116

For an $n \times n$ matrix, each of the following is equivalent to the condition of the matrix having determinant 0:

- The columns of the matrix are dependent vectors in $\mathbb{R}^n$
- The rows of the matrix are dependent vectors in $\mathbb{R}^n$
- The matrix is not invertible.
- The volume of the parallelepiped determined by the column vectors of the matrix is 0.
- The volume of the parallelepiped determined by the row vectors of the matrix is 0.
- The system of homogenous linear equations represented by the matrix has a non-trivial solution.
- The determinant of the linear transformation determined by the matrix is 0.
- The free coefficient in the characteristic polynomial of the matrix is 0.

Depending on the definition of the determinant you saw, proving each equivalence can be more or less hard.

Lay, Linear Algebra and its Applications (4^{th} Edition)

266 CHAPTER 5 Eigenvalues and Eigenvectors

Finding Eigenvalues

EXAMPLE 2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

Eq. 6: $$\underbrace{\left(\begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix}\right)}_{A-\lambda I} x = 0$$

To get RHS of Eq. 6 to be 0 for non-trivial x,
$|A - \lambda I| = 0$ (similarly mean)
$(A - \lambda I)$ must be singular (non-invertible).

Applying $det(A - \lambda I) = 0$ **to Eq. 6:**

$\Longrightarrow (1-\lambda)(2-\lambda) - 30 = 0$

$\Longrightarrow \lambda^2 - 3\lambda - 28 = 0$

$\Longrightarrow (\lambda - 7)(\lambda + 4) = 0$

Hence, solving for λ, we obtain the eigenvalues (λ) as 7 and -4.

Why?
If $(A - \lambda I)$ is singular, then some of its columns are dependent.
Hence, we can find a non-zero x such that LHS = RHS = 0 (null vector).

Example:
If A has dependent columns, then it is singular (non-invertible).

Let A = $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, then to make Ax = $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

we can use:

$$\mathrm{x} = \alpha \begin{bmatrix} 1 \\ -0.5 \end{bmatrix},$$

where $\alpha \in R$, which is a nontrivial solution for the equation.

The characteristic equation sets the characteristic polynomial equal to 00. In symbols, if the characteristic polynomial is p(x)p(x), then the characteristic equation is the equation p(x)=0p(x)=0.

Finding Eigenvalues

Eq. 3: $(A - \lambda I)x = 0$

Eq. 3 has a non-trivial solution x **if and only if** the determinant of the matrix $(A - \lambda I)$ is zero. Therefore, the eigenvalues of A are the values of λ that satisfy the equation:

$$|A - \lambda I| = 0$$

The above equation $det(A - \lambda I) = 0$ is called the ***characteristic equation*** of A. Note you will get the same result if you do $det(\lambda I - A) = 0$.

For a 2×2 Matrix

For a 2×2 matrix (2 rows and 2 columns):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is:

$$|A| = ad - bc$$

"The determinant of A equals a times d minus b times c"

Upon expanding $det(A - \lambda I) = 0$, a polynomial in λ, called the ***characteristic polynomial*** of A is obtained.

Characteristic polynomial and characteristic equation

Asked 5 years, 11 months ago Active 1 year, 3 months ago Viewed 998 times

What is the major difference between the characteristic polynomial and the characteristic equation?

Upcomi

The characteristic equation sets the characteristic polynomial equal to 0. In symbols, if the characteristic polynomial is $p(x)$, then the characteristic equation is the equation $p(x) = 0$.

share cite improve this answer follow

answered Aug 17 '14 at 14:35

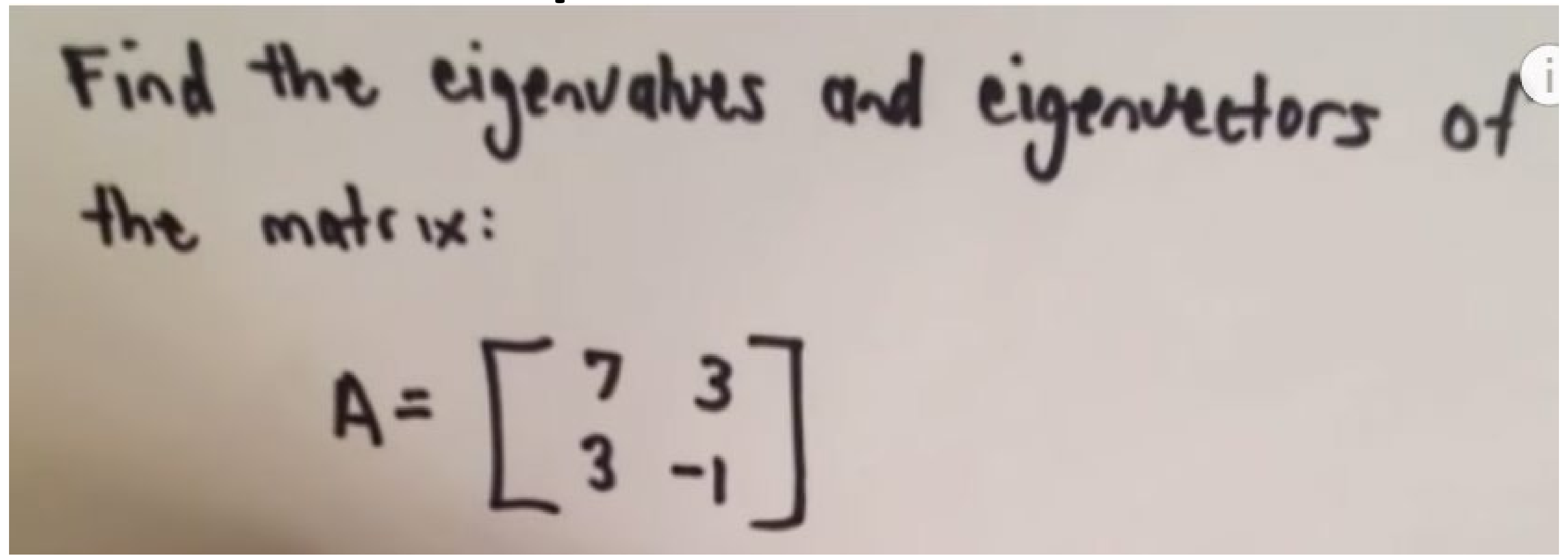
André Nicolas

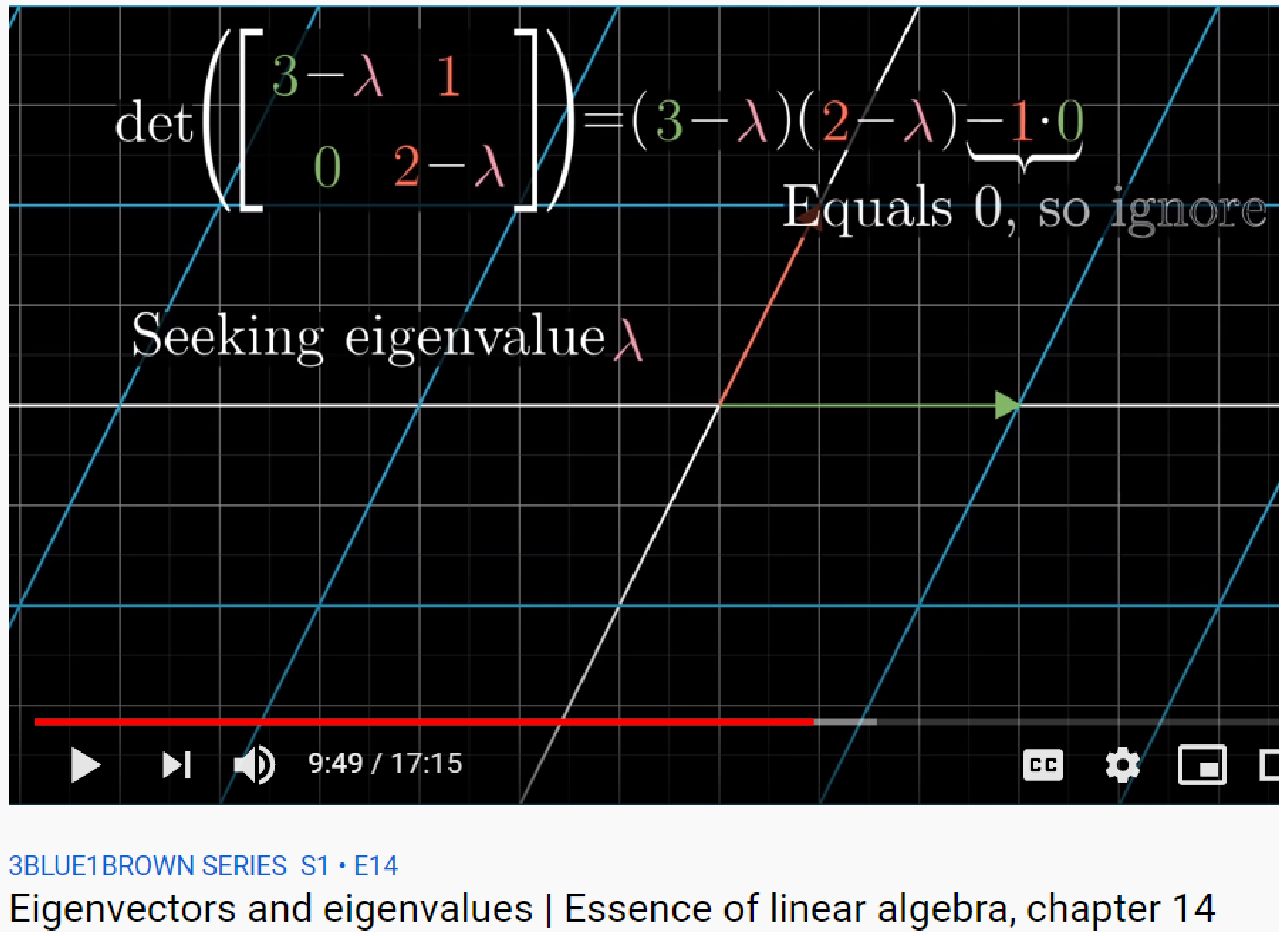
471k 39 470 879

Examples online

1) PatrickJMT: https://www.youtube.com/watch?v=IdsV0RaC9jM

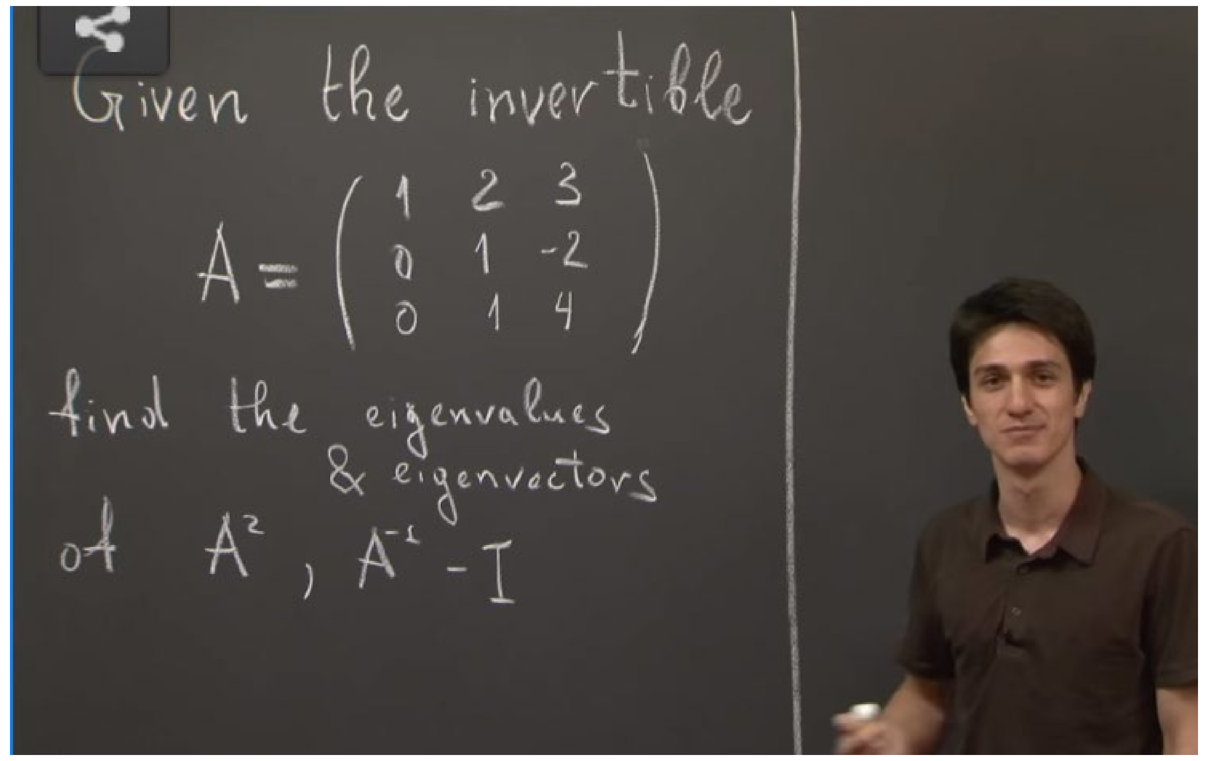
Work examples:





2) 3Blue1Brown: Ch14, https://www.youtube.com/watch?v=PFDu9oVAE-g

Understanding and perspective



3) Some more examples: MIT

https://ocw.mit.edu/courses/mathematics/18-06sc-linear-algebra-fall-2011/resource-index/problem-solving-eigenvalues-and-eigenvectors/