$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.1.5**

Lecture : **Eigen and Singular Values**

Topic : **Eigen Decomposition and Diagonalization**

Concept : **Examples and Special Cases**

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Summary for Finding Eigenvalues, Eigenvectors

The procedure we will follow to find the eigenvalues and eigenvectors (eigenspaces) of a matrix:

Let *A* be an $n \times n$ matrix.

1. a) Compute the characteristic polynomial $\det(A - \lambda I)$ of *A*.

   b) Find the eigenvalues of *A* by solving the characteristic equation: $\det(A - \lambda I) = 0$ for λ.

2. a) For each eigenvalue λ, find the null space of the matrix $A - \lambda I$. This is the eigenspace E_λ, the nonzero vectors of which are the eigenvectors of *A* corresponding to λ.

   b) Find a basis for each eigenspace.

Example 1: Finding Eigenvectors given eigenvalue

EXAMPLE 2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

EXAMPLE 3 Show that 7 is an eigenvalue of matrix A in Example 2, and find the corresponding eigenvectors.

SOLUTION The scalar 7 is an eigenvalue of A if and only if the equation

$$A\mathbf{x} = 7\mathbf{x} \tag{1}$$

has a nontrivial solution. But (1) is equivalent to $A\mathbf{x} - 7\mathbf{x} = \mathbf{0}$, or

$$(A - 7I)\mathbf{x} = \mathbf{0} \tag{2}$$

To solve this homogeneous equation, form the matrix

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

The columns of $A - 7I$ are obviously linearly dependent, so (2) has nontrivial solutions. Thus 7 *is* an eigenvalue of A. To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution has the form $x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda = 7$. ■

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The equivalence of equations (1) and (2) obviously holds for any λ in place of $\lambda = 7$. Thus λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \tag{3}$$

has a nontrivial solution. The set of *all* solutions of (3) is just the null space of the matrix $A - \lambda I$. So this set is a *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ. The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.

Example 3 shows that for matrix A in Example 2, the eigenspace corresponding to $\lambda = 7$ consists of *all* multiples of (1, 1), which is the line through (1, 1) and the origin. From Example 2, you can check that the eigenspace corresponding to $\lambda = -4$ is the line through (6, −5). These eigenspaces are shown in Fig. 2, along with eigenvectors (1, 1) and (3/2, −5/4) and the geometric action of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ on each eigenspace.

Note: Upon transformation (Ax) of a vector in the eigenspace, the direction of the resulting vector is same as the original vector.

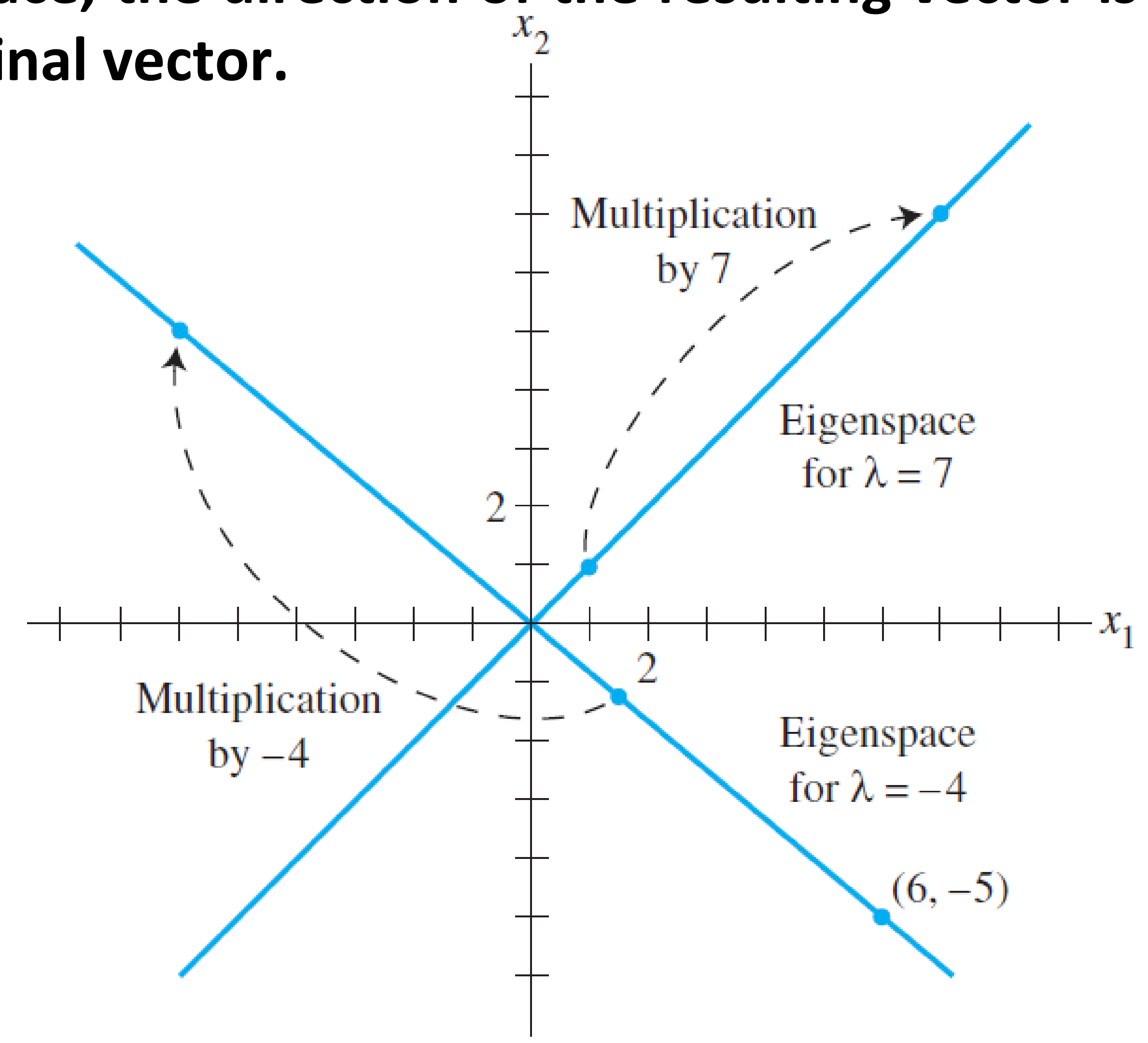

FIGURE 2 Eigenspaces for $\lambda = -4$ and $\lambda = 7$.

Example 2: repeated eigen values and eigenspace span by 2 eigenvectors

EXAMPLE 4 Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

SOLUTION Form

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and row reduce the augmented matrix for $(A - 2I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

At this point, it is clear that 2 is indeed an eigenvalue of A because the equation $(A - 2I)\mathbf{x} = \mathbf{0}$ has free variables. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad x_2 \text{ and } x_3 \text{ free}$$

The eigenspace, shown in Fig. 3, is a two-dimensional subspace of $\mathbb{R}^3$. A basis is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$ ■

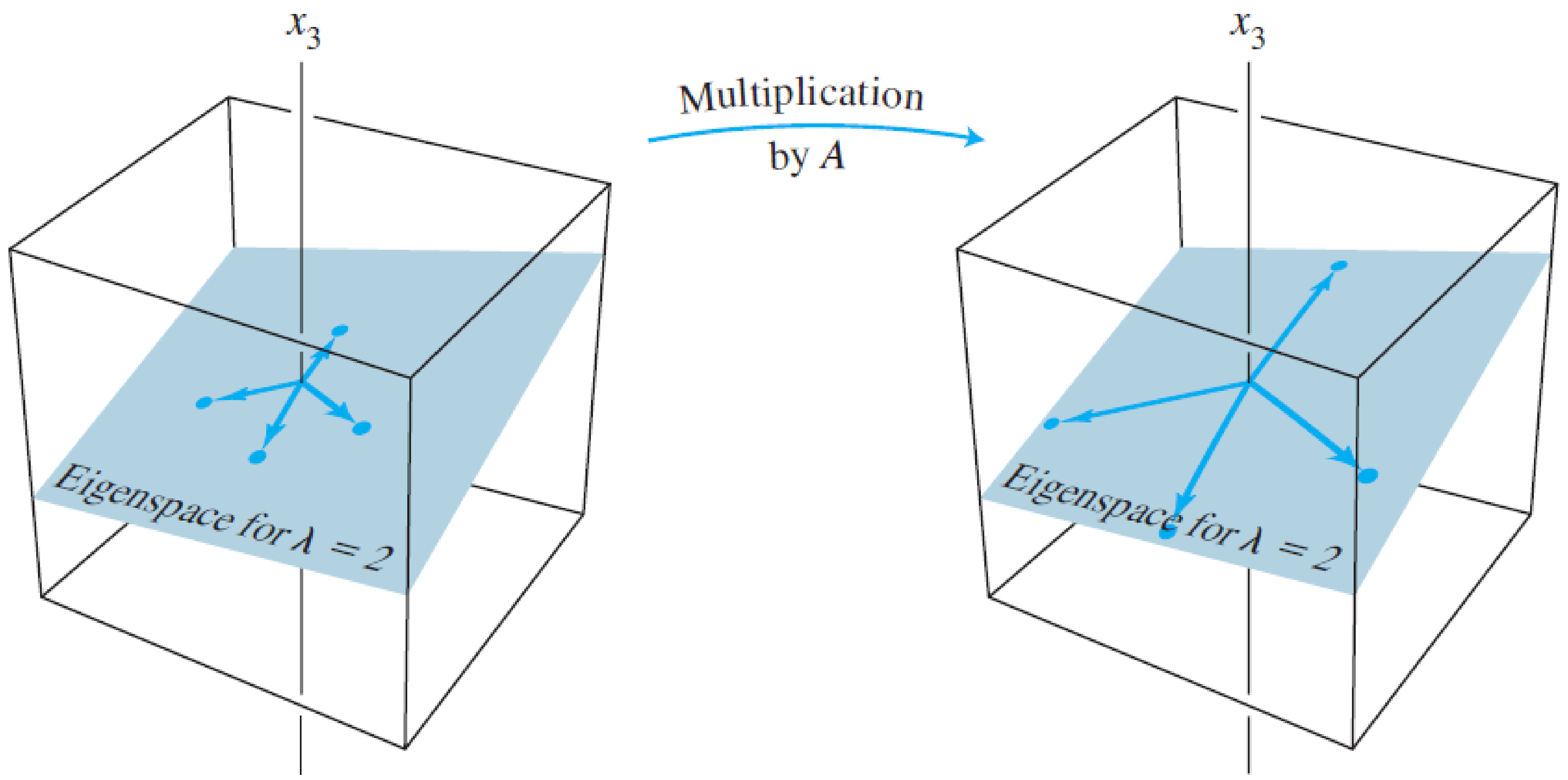

FIGURE 3 A acts as a dilation on the eigenspace.

MATLAB

```
A =

     4    -1     6
     2     1     6
     2    -1     8

>> [U,D] = eig(A)

U =

   -0.5774   -0.6122    0.3205
   -0.5774   -0.7873   -0.9112
   -0.5774    0.0728   -0.2587

D =

    9.0000         0         0
         0    2.0000         0
         0         0    2.0000
```

Eigenvector corresponding to λ_1

Eigenvector corresponding to λ_2

Eigenvector corresponding to λ_3

λ_1, λ_2, λ_3

Note: Here, $\lambda_2 = \lambda_3 = 2$. Hence, there are two eigenvectors corresponding to the eigenvalue of 2. The eigen space can be formed by any set of basis vectors that span it. Hence, the eigen space spanned by the basis on LHS is the same eigen space spanned by the 2nd and 3rd columns of matrix U.

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Example 3: using Matlab eig function to perform eigendecomposition

```
A =

     4    -1     6
     2     1     6
     2    -1     8

>> [U,D] = eig(A)

U =

   -0.5774   -0.6122    0.3205
   -0.5774   -0.7873   -0.9112
   -0.5774    0.0728   -0.2587
```

Eigenvector corresponding to λ_1

Eigenvector corresponding to λ_2

Eigenvector corresponding to λ_3

```
D =

    9.0000         0         0
         0    2.0000         0
         0         0    2.0000
```

λ_1 λ_2 λ_3

```
>> x = 2*U(:,2)+3*U(:,3)

x =

   -0.2627
   -4.3083
   -0.6305

>> A*x

ans =

   -0.5255
   -8.6167
   -1.2610

>> (A*x)./x

ans =

    2.0000
    2.0000
    2.0000
```

NOTE:

Here, x is a vector in the eigen space of A corresponding to $\lambda = 2$. It is formed by the linear combination of the two eigenvectors corresponding to $\lambda = 2$.

The above shows that vector x in the eigen space of $\lambda = 2$,
'grows' by 2, when acted upon by A!

```
>> x = 2*U(:,1)

x =

   -1.1547
   -1.1547
   -1.1547

>> A*x

ans =

  -10.3923
  -10.3923
  -10.3923

>> (A*x)./x

ans =

    9.0000
    9.0000
    9.0000
```

NOTE:

Here, x is a vector in the eigen space of A corresponding to $\lambda = 9$. It is formed by scalar multiplication of the eigenvector corresponding to $\lambda = 9$.

The above shows that vector x in the eigen space of $\lambda = 9$,
'grows' by 9, when acted upon by A!

Diagonalizing a matrix == Finding EigenValues and EigenVectors

How to diagonalize a matrix [edit]

Diagonalizing a matrix is the same process as finding its eigenvalues and eigenvectors, in the case that the eigenvectors form a basis.

The following theorem and the ideas used in its proof will provide us with a roadmap for devising a technique for determining whether a matrix is diagonalizable and, if so, for finding a matrix P that will perform the diagonalization.

THEOREM 5.2.1 *If A is an $n \times n$ matrix, the following statements are equivalent.*

(a) A is diagonalizable.

(b) A has n linearly independent eigenvectors.

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Proof (a) ⇒ (b) Since A is assumed to be diagonalizable, it follows that there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ or, equivalently,

$$AP = PD \tag{1}$$

If we denote the column vectors of P by $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, and if we assume that the diagonal entries of D are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then by Formula (6) of Section 1.3 the left side of (1) can be expressed as

$$AP = A[\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n] = [A\mathbf{p}_1 \quad A\mathbf{p}_2 \quad \cdots \quad A\mathbf{p}_n]$$

and, as noted in the comment following Example 1 of Section 1.7, the right side of (1) can be expressed as

$$PD = [\lambda_1\mathbf{p}_1 \quad \lambda_2\mathbf{p}_2 \quad \cdots \quad \lambda_n\mathbf{p}_n]$$

Thus, it follows from (1) that

$$A\mathbf{p}_1 = \lambda_1\mathbf{p}_1, \quad A\mathbf{p}_2 = \lambda_2\mathbf{p}_2, \ldots, \quad A\mathbf{p}_n = \lambda_n\mathbf{p}_n \tag{2}$$

Since P is invertible, we know from Theorem 5.1.5 that its column vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are linearly independent (and hence nonzero). Thus, it follows from (2) that these n column vectors are eigenvectors of A.

Proof (b) ⇒ (a) Assume that A has n linearly independent eigenvectors, $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, and that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues. If we let

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]$$

and if we let D be the diagonal matrix that has $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal entries, then

$$\begin{aligned} AP &= A[\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n] = [A\mathbf{p}_1 \quad A\mathbf{p}_2 \quad \cdots \quad A\mathbf{p}_n] \\ &= [\lambda_1\mathbf{p}_1 \quad \lambda_2\mathbf{p}_2 \quad \cdots \quad \lambda_n\mathbf{p}_n] = PD \end{aligned}$$

Since the column vectors of P are linearly independent, it follows from Theorem 5.1.5 that P is invertible, so that this last equation can be rewritten as $P^{-1}AP = D$, which shows that A is diagonalizable. ◀

Example 1: Diagonalization is same as EigenDecomposition

Diagonalizing Matrices

EXAMPLE 3 Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

SOLUTION There are four steps to implement the description in Theorem 5.

Step 1. ***Find the eigenvalues of A.*** As mentioned in Section 5.2, the mechanics of this step are appropriate for a computer when the matrix is larger than 2×2. To avoid unnecessary distractions, the text will usually supply information needed for this step. In the present case, the characteristic equation turns out to involve a cubic polynomial that can be factored:

$$\begin{aligned} 0 = \det(A - \lambda I) &= -\lambda^3 - 3\lambda^2 + 4 \\ &= -(\lambda - 1)(\lambda + 2)^2 \end{aligned}$$

The eigenvalues are $\lambda = 1$ and $\lambda = -2$.

Step 2. ***Find three linearly independent eigenvectors of A.*** *Three* vectors are needed because A is a 3×3 matrix. This is the critical step. If it fails, then Theorem 5 says that A cannot be diagonalized. The method in Section 5.1 produces a basis for each eigenspace:

$$\text{Basis for } \lambda = 1: \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Basis for } \lambda = -2: \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

You can check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

Step 3. ***Construct P from the vectors in step 2.*** The order of the vectors is unimportant. Using the order chosen in step 2, form

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 4. ***Construct D from the corresponding eigenvalues.*** In this step, it is essential that the order of the eigenvalues matches the order chosen for the columns of P. Use the eigenvalue $\lambda = -2$ twice, once for each of the eigenvectors corresponding to $\lambda = -2$:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

It is a good idea to check that P and D really work. To avoid computing P^{-1}, simply verify that $AP = PD$. This is equivalent to $A = PDP^{-1}$ when P is invertible. (However, be sure that P is invertible!) Compute

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$ ■

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Example 2: Diagonalization is same as EigenDecomposition

EXAMPLE 1 Finding a Matrix P That Diagonalizes a Matrix A

Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution In Example 7 of the preceding section we found the characteristic equation of A to be

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

and we found the following bases for the eigenspaces:

$$\lambda = 2: \quad \mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \qquad \lambda = 1: \quad \mathbf{p}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

There are three basis vectors in total, so the matrix

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagonalizes A. As a check, you should verify that

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In general, there is no preferred order for the columns of P. Since the ith diagonal entry of $P^{-1}AP$ is an eigenvalue for the ith column vector of P, changing the order of the columns of P just changes the order of the eigenvalues on the diagonal of $P^{-1}AP$. Thus, had we written

$$P = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

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in the preceding example, we would have obtained

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

MATLAB:

```
A =

     0     0    -2
     1     2     1
     1     0     3

>> [P,D] = eig(A)

P =

         0   -0.8165    0.7066
    1.0000    0.4082    0.0395
         0    0.4082   -0.7066


D =

     2     0     0
     0     1     0
     0     0     2
```

```
>> P*D*inv(P)

ans =

   -0.0000         0   -2.0000
    1.0000    2.0000    1.0000
    1.0000         0    3.0000
```

```
>> inv(P)*A*P

ans =

    2.0000   -0.0000   -0.0000
         0    1.0000    0.0000
         0    0.0000    2.0000
```

Example 3: Diagonalization is same as EigenDecomposition

```
A =

     4    -1     6
     2     1     6
     2    -1     8

>> [U,D] = eig(A)

U =

   -0.5774   -0.6122    0.3205
   -0.5774   -0.7873   -0.9112
   -0.5774    0.0728   -0.2587
```

Eigenvector corresponding to λ_1 | Eigenvector corresponding to λ_2 | Eigenvector corresponding to λ_3

```
D =

    9.0000         0         0
         0    2.0000         0
         0         0    2.0000
```

λ_1 λ_2 λ_3

```
>> x = 2*U(:,2)+3*U(:,3)

x =

   -0.2627
   -4.3083
   -0.6305

>> A*x

ans =

   -0.5255
   -8.6167
   -1.2610

>> (A*x)./x

ans =

    2.0000
    2.0000
    2.0000
```

NOTE:

Here, x is a vector in the eigen space of A corresponding to $\lambda = 2$. It is formed by the linear combination of the two eigenvectors corresponding to $\lambda = 2$.

The above shows that vector x in the eigen space of $\lambda =$ 2,
'grows' by 2, when acted upon by A!

```
>> x = 2*U(:,1)

x =

   -1.1547
   -1.1547
   -1.1547

>> A*x

ans =

  -10.3923
  -10.3923
  -10.3923

>> (A*x)./x

ans =

    9.0000
    9.0000
    9.0000
```

NOTE:

Here, x is a vector in the eigen space of A corresponding to $\lambda = 9$. It is formed by scalar multiplication of the eigenvector corresponding to $\lambda = 9$.

The above shows that vector x in the eigen space of $\lambda =$ 9,
'grows' by 9, when acted upon by A!

Some Matrices Cannot Be Diagonalised!

Some matrices are not diagonalizable over any field, most notably nonzero nilpotent matrices. This happens more generally if the algebraic and geometric multiplicities of an eigenvalue do not coincide. For instance, consider

$$C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Matrices that are not diagonalizable [edit]

In general, a rotation matrix is not diagonalizable over the reals, but all rotation matrices are diagonalizable over the complex field. Even if a matrix is not diagonalizable, it is always possible to "do the best one can", and find a matrix with the same properties consisting of eigenvalues on the leading diagonal, and either ones or zeroes on the superdiagonal – known as Jordan normal form.

Some real matrices are not diagonalizable over the reals. Consider for instance the matrix

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The matrix B does not have any real eigenvalues, so there is no **real** matrix Q such that $Q^{-1}BQ$ is a diagonal matrix. However, we can diagonalize B if we allow complex numbers. Indeed, if we take

$$Q = \begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix},$$

then $Q^{-1}BQ$ is diagonal. It is easy to find that B is the rotation matrix which rotates counterclockwise by angle $\theta = \frac{3\pi}{2}$

What is a nilpotent matrix?

In linear algebra, a **nilpotent matrix** is a square matrix *N* such that

$$N^k = 0$$

for some positive integer k. The smallest such k is sometimes called the **index** of N.[1]

More generally, a **nilpotent transformation** is a linear transformation L of a vector space such that $L^k = 0$ for some positive integer k (and thus, $L^j = 0$ for all $j \geq k$).

MATLAB:

```
>> C = [ 0 1; 0 0];
>> [U,D] = eig(C)

U =

    1.0000   -1.0000
         0    0.0000


D =

     0     0
     0     0
```

```
>> B = [0 1; -1 0];
>> [U,D] = eig(B)

U =

   0.7071 + 0.0000i   0.7071 + 0.0000i
   0.0000 + 0.7071i   0.0000 - 0.7071i


D =

   0.0000 + 1.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 - 1.0000i
```

```
>> U*D*inv(U)

ans =

     0     1
    -1     0
```

Ref:
https://en.wikipedia.org/wiki/Diagonalizable_matrix
https://en.wikipedia.org/wiki/Nilpotent_matrix

Some Matrices Cannot Be Diagonalised!

▶ **EXAMPLE 2 A Matrix That Is Not Diagonalizable**

Show that the following matrix is not diagonalizable:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

Solution The characteristic polynomial of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 2 & 0 \\ 3 & -5 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2)^2$$

so the characteristic equation is

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

and the distinct eigenvalues of A are $\lambda = 1$ and $\lambda = 2$. We leave it for you to show that bases for the eigenspaces are

$$\lambda = 1: \quad \mathbf{p}_1 = \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \\ 1 \end{bmatrix}; \qquad \lambda = 2: \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Since A is a 3×3 matrix and there are only two basis vectors in total, A is not diagonalizable.

Another way of looking at it: $n_i \neq m_i \forall i$.

Geometric and algebraic multiplicities are not equal for all eigenvalues!

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```
>> A = [1 0 0; 1 2 0; -3 5 2]

A =

     1     0     0
     1     2     0
    -3     5     2

>> [P,D] = eig(A)

P =

         0         0    0.1231
         0    0.0000   -0.1231
    1.0000   -1.0000    0.9847


D =

     2     0     0
     0     2     0
     0     0     1

>> P*D*inv(P)
Warning: Matrix is close to singular or badly scaled.
RCOND =   3.607798e-17.

ans =

     1     0     0
     1     2     0
    -8     0     2
```

Be wary of MATLAB's output because P is singular!

Note that, in P, the second column is a scaled version of the first column!

Some Matrices Cannot Be Diagonalised!

...continued:

Alternative Solution If you are concerned only in determining whether a matrix is diagonalizable and not with actually finding a diagonalizing matrix P, then it is not necessary to compute bases for the eigenspaces—it suffices to find the dimensions of the eigenspaces. For this example, the eigenspace corresponding to $\lambda = 1$ is the solution space of the system

$$\overbrace{\begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 3 & -5 & -1 \end{bmatrix}}^{\lambda I - A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the coefficient matrix has rank 2 (verify), the nullity of this matrix is 1 by Theorem 4.8.2, and hence the eigenspace corresponding to $\lambda = 1$ is one-dimensional.

The eigenspace corresponding to $\lambda = 2$ is the solution space of the system

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 3 & -5 & 0 \end{bmatrix}}^{\lambda I - A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This coefficient matrix also has rank 2 and nullity 1 (verify), so the eigenspace corresponding to $\lambda = 2$ is also one-dimensional. Since the eigenspaces produce a total of two basis vectors, and since three are needed, the matrix A is not diagonalizable.

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Recap:

THEOREM 4.8.2 Dimension Theorem for Matrices

If A is a matrix with n columns, then

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$$

Rank and Nullity

The dimensions of the row space, column space, and null space of a matrix are such important numbers that there is some notation and terminology associated with them.

DEFINITION 1 The common dimension of the row space and column space of a matrix A is called the ***rank*** of A and is denoted by $\operatorname{rank}(A)$; the dimension of the null space of A is called the ***nullity*** of A and is denoted by $\operatorname{nullity}(A)$.

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Special Case 1: Having an Eigenvalue as 0

If matrix A has an eigenvalue of 0, then A will not be invertible.

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

s. The number 0 is *not* an eigenvalue of A.

t. The determinant of A is *not* zero.

Ref:

1. https://qr.ae/pNK3bf
2. https://textbooks.math.gatech.edu/ila/eigenvectors.html
3. https://math.stackexchange.com/questions/755780/is-a-matrix-a-with-an-eigenvalue-of-0-invertible

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5.2 The Characteristic Equation 275

Note. Eigenvalues and eigenvectors are only for square matrices.

Eigenvectors are *by definition nonzero*. Eigenvalues may be equal to zero.

We do not consider the zero vector to be an eigenvector: since $A0 = 0 = \lambda 0$ for *every* scalar λ, the associated eigenvalue would be undefined.

David Joyce, Professor of Mathematics and Computer Science

Answered October 27, 2017 · Upvoted by Allan Steinhardt, Codeveloped the hyperbolic householder transform with C Rader. Widely cited

0 is one of the two most important eigenvalues, the other being 1.

Having a 0 eigenvalue tells you that a square matrix is singular; not having it as an eigenvalue tells you the matrix is invertible.

Suppose A is square matrix and has an eigenvalue of 0. For the sake of contradiction, lets assume A is invertible.

Consider, $Av = \lambda v$, with $\lambda = 0$ means there exists a non-zero v such that $Av = 0$. This implies $Av = 0v \Rightarrow Av = 0$

For an invertible matrix A, $Av = 0$ implies $v = 0$. So, $Av = 0 = A \cdot 0$. Since v cannot be 0,this means A must not have been one-to-one. Hence, our contradiction, A must not be invertible.

linear-algebra eigenvalues-eigenvectors proof-verification determinant

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edited Jan 16 '16 at 23:53
nbro
5,103 ● 12 ■ 46 ▲ 94

asked Apr 16 '14 at 3:32
Derrick J.
391 ● 1 ■ 3 ▲ 5

When a matrix is invertible, it also means

THEOREM 5.1.5 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(a) A *is invertible.*

(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(c) *The reduced row echelon form of A is I_n.*

(d) A *is expressible as a product of elementary matrices.*

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every $n \times 1$ matrix $\mathbf{b}$.*

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.*

(g) $\det(A) \neq 0$.

(h) *The column vectors of A are linearly independent.*

(i) *The row vectors of A are linearly independent.*

(j) *The column vectors of A span R^n.*

(k) *The row vectors of A span R^n.*

(l) *The column vectors of A form a basis for R^n.*

(m) *The row vectors of A form a basis for R^n.*

(n) A *has rank n.*

(o) A *has nullity 0.*

(p) *The orthogonal complement of the null space of A is R^n.*

(q) *The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.*

(r) *The kernel of T_A is $\{\mathbf{0}\}$.*

(s) *The range of T_A is R^n.*

(t) T_A *is one-to-one.*

(u) $\lambda = 0$ *is not an eigenvalue of A.*

Anton, Elementary Linear Algebra: Applications Version (11^{th} Edition)

5.1 Eigenvalues and Eigenvectors 299

Special Case 2: Eigenvalues for Triangular Matrices

EXAMPLE 4 Eigenvalues of an Upper Triangular Matrix

Find the eigenvalues of the upper triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Solution Recalling that the determinant of a triangular matrix is the product of the entries on the main diagonal (Theorem 2.1.2), we obtain

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - a_{11} & -a_{12} & -a_{13} & -a_{14} \\ 0 & \lambda - a_{22} & -a_{23} & -a_{24} \\ 0 & 0 & \lambda - a_{33} & -a_{34} \\ 0 & 0 & 0 & \lambda - a_{44} \end{bmatrix}$$

$$= (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44})$$

Thus, the characteristic equation is

$$(\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}) = 0$$

and the eigenvalues are

$$\lambda = a_{11}, \quad \lambda = a_{22}, \quad \lambda = a_{33}, \quad \lambda = a_{44}$$

which are precisely the diagonal entries of A. ◀

Anton, Elementary Linear Algebra: Applications Version (11th Edition)

THEOREM 2.1.2 *If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is, $\det(A) = a_{11}a_{22}\cdots a_{nn}$.*

EXAMPLE 6 Determinant of a Lower Triangular Matrix

The following computation shows that the determinant of a 4×4 lower triangular matrix is the product of its diagonal entries. Each part of the computation uses a cofactor expansion along the first row.

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11}a_{22} \begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33}|a_{44}| = a_{11}a_{22}a_{33}a_{44}$$ ◀

References

1. Good reads:

a. https://pathmind.com/wiki/eigenvector: Good overview of the following:

i. Interpreting Ax (eigenvector of A)

ii. Intepreting A= $X^T X$, the covariance matrix A's interpretation of eigenvector/value

b. https://machinelearningmastery.com/introduction-to-eigendecomposition-eigenvalues-and-eigenvectors/

2. How to find eigenvalues/vectors (process):

a. Chasnov (L33): https://www.youtube.com/watch?v=29keVZGvqME&list=PLkZjai-2Jcxlg-Z1roB0pUwFU-P58tvOx&index=33