$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **6.3.3**

Lecture : **Orthogonality**

Topic : **Gram–Schmidt Process**

Concept : **Using GS Process for QR Factorisation**

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Rev: 29th June 2020

Gram–Schmidt Process for QR Factorisation

The application of the Gram–Schmidt process to the column vectors of a **full column rank** matrix yields the QR decomposition.

QR decomposition helps in solving linear least squares problem (next chapter).

If $A \in R^{m \times n}$ has full column rank,
i.e, A has linearly independent columns,
A can be factored as follows:

$$A = QR$$

$$A = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ 0 & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{nn} \end{bmatrix}$$

The Q Factor:

- Q is $m \times n$ with orthonormal columns $(Q^T Q = I)$
- If A is square $(m = n)$, then Q is orthogonal, i.e, $Q^T Q = QQ^T = I$

The R Factor:

- R is $n \times n$ upper triangular, with nonzero diagonal elements
- R is nonsingular (diagonal elements are nonzero)

- Vectors $q_1, q_2, \ldots, q_n$ are orthonormal m-vectors: $||q_i|| = 1 \;\; and \;\; q_i^T q_j = 0 \; if \;\; i \neq j$
- Diagonal elements R_{ii} are nonzero.

NOTE:

Q is obtained by performing GS Process on A

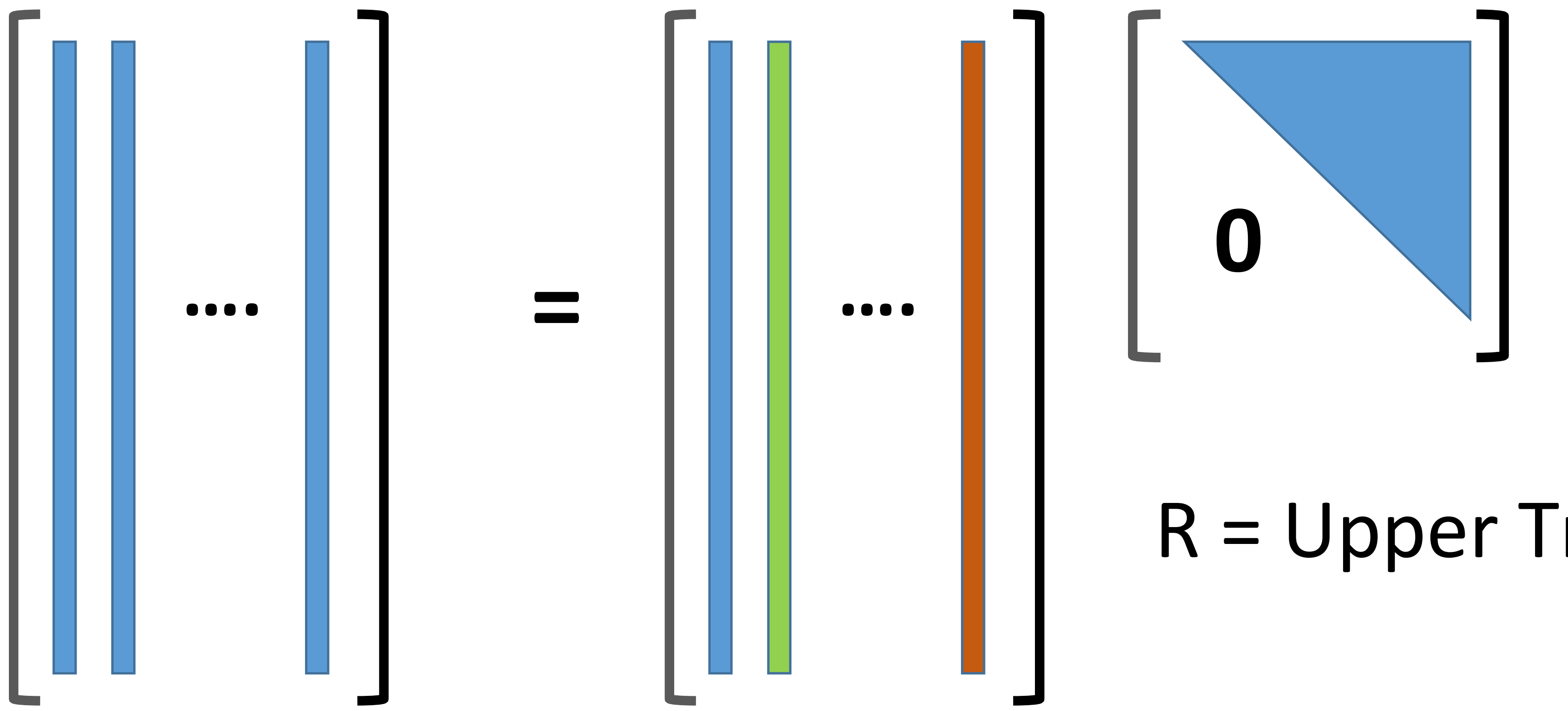
Gram–Schmidt Process for QR Factorisation

THEOREM 12

The QR Factorization

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for Col A and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

$$A = [\,\mathbf{x}_1 \quad \cdots \quad \mathbf{x}_n\,] = [\,Q\mathbf{r}_1 \quad \cdots \quad Q\mathbf{r}_n\,] = QR$$



R = Upper Triangle

$A \in R^{m*n}$ Q = columns are orthogonal

```
% Gram Schmit - Chng Eng Siong, for QR decomposition
% 20 May 2020
function [Q,R]= my_GS(A)

[m,n] = size(A);
Q=zeros(m,n);
R=zeros(n,n);

x1     = A(:,1);
Q(:,1) = x1/norm(x1);
R(1,1) = norm(x1);

    for i=2:n
        x_i = A(:,i);
        v_i = x_i;
        for j=1:i-1
            v_j = Q(:,j);
            RVal = (x_i'*v_j)/(v_j'*v_j);
            v_i  = v_i - RVal*v_j;
            R(j,i) = RVal;
        end  % of for j
        Q(:,i) = v_i/norm(v_i);
        R(i,i) = norm(v_i);
    end  % of for i
end  % of funcyion
```

Gram–Schmidt Process for QR Factorisation

Consider a full column rank matrix A:

$A = [\, x_1, x_2, x_3, \dots, x_n]$

Gram–Schmidt Process

$$\mathrm{Proj}_v x = \frac{< v, x >}{< v, v >}\, v$$

$$v_1 = x_1 \qquad e_1 = \frac{v_1}{||v_1||}$$

$$v_2 = x_2 - \mathrm{Proj}_{v_1} x_2$$

$$v_3 = x_3 - \mathrm{Proj}_{v_1} x_3 - \mathrm{Proj}_{v_2} x_3$$

$$v_i = x_i - \sum_{j=1}^{i-1} \mathrm{Proj}_{v_j} x_i \qquad e_i = \frac{v_i}{||v_i||}$$

We can now express the $\mathbf{a}_i$s over our newly computed orthonormal basis:

$$x_1 = < e_1, x_1 > \mathrm{e}_1$$

$$x_2 = < e_1, x_2 > \mathrm{e}_1 + < e_2, x_2 > e_2$$

$$x_3 = < e_1, x_3 > \mathrm{e}_1 + < e_2, x_3 > e_2 + < e_3, x_3 > e_3$$

$$\vdots$$

$$x_n = \sum_{j=1}^{n} < e_j, x_n > e_j$$

This can be written in matrix form:

$$A = QR$$

where:

$$Q = [\, e_1, e_2, e_3, \dots, e_n]$$

and

$$R = \begin{pmatrix} < e_1, x_1 > & < e_1, x_2 > & < e_1, x_3 > & \dots \\ 0 & < e_2, x_2 > & < e_2, x_3 > & \dots \\ 0 & 0 & < e_3, x_3 > & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Ref: https://en.wikipedia.org/wiki/QR_decomposition#Using_the_Gram–Schmidt_process

Gram–Schmidt Process for QR Factorisation

Example [edit]

Consider the decomposition of

$$A = \begin{pmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{pmatrix}.$$

Recall that an orthonormal matrix Q has the property

$$Q^{\mathsf{T}} Q = I.$$

Then, we can calculate Q by means of Gram–Schmidt as follows:

$$U = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix} = \begin{pmatrix} 12 & -69 & -58/5 \\ 6 & 158 & 6/5 \\ -4 & 30 & -33 \end{pmatrix};$$

$$Q = \begin{pmatrix} \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} & \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} & \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \end{pmatrix} = \begin{pmatrix} 6/7 & -69/175 & -58/175 \\ 3/7 & 158/175 & 6/175 \\ -2/7 & 6/35 & -33/35 \end{pmatrix}$$

Ref: https://en.wikipedia.org/wiki/QR_decomposition#Using_the_Gram–Schmidt_process

Thus, we have

$$Q^{\mathsf{T}} A = Q^{\mathsf{T}} Q\, R = R;$$

$$R = Q^{\mathsf{T}} A = \begin{pmatrix} 14 & 21 & -14 \\ 0 & 175 & -70 \\ 0 & 0 & 35 \end{pmatrix}$$

Hence,

$$A = \begin{pmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{pmatrix} =$$

$$\begin{pmatrix} 6/7 & -69/175 & -58/175 \\ 3/7 & 158/175 & 6/175 \\ -2/7 & 6/35 & -33/35 \end{pmatrix} \times \begin{pmatrix} 14 & 21 & -14 \\ 0 & 175 & -70 \\ 0 & 0 & 35 \end{pmatrix}$$

Q: Orthogonal Matrix $\quad$ R: Upper Triangular Matrix