$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.1.3**

Lecture : **Eigen and Singular Values**

Topic : **Eigenvalue Decomposition**

Concept : **Algebraic and Geometric Multiplicity**

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Rev: 9th July 2020

Algebraic Multiplicity of Eigenvalues

An **eigenvector** of an $N \times N$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

To solve for the eigenvalues λ of a matrix A, we solve the **characteristic equation** in λ: $p(\lambda) = det(A - \lambda I) = 0$.

Note:

- $p(\lambda)$ is the characteristic polynomial in λ.
- $p(\lambda) = 0$ is the characteristic equation.
- $p(\lambda) = 0$ is an N^{th} order polynomial equation in the unknown λ.
- $p(\lambda) = 0$ will have N_λ distinct solutions, where $1 \le N_\lambda \le N$.

The **set of solutions** $(\lambda_1, \lambda_2, \dots, \lambda_{N_\lambda})$, that is, the **eigenvalues**, is called the **spectrum** of A.

$p(\lambda)$ can be factored as follows:

$$p(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_{N_\lambda})^{n_{N_\lambda}} = 0.$$

The integer n_i is termed the **algebraic multiplicity** of eigenvalue λ_i. It is the number of times an eigenvalue appears as a root of the characteristic polynomial.

The algebraic multiplicities sum to N:

$$\sum_{i=1}^{N_\lambda} n_i = N.$$

Ref:
https://en.wikipedia.org/wiki/Eigendecomposition_of_a_matrix

Example

Consider the 2×2 matrix

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix}$$

Their characteristic polynomial:

$$\begin{aligned} f(\lambda) &= \det\left(\begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 4-\lambda & 2 \\ 1 & 2-\lambda \end{bmatrix}\right) \\ &= (4-\lambda)\cdot(2-\lambda) - 2\cdot 1 \\ &= 8 - 4\lambda - 2\lambda + \lambda^2 - 2 \\ &= \lambda^2 - 6\lambda + 6 \end{aligned}$$

Roots of the polynomial, that is, the solutions of $f(\lambda) = 0$ are:

$$\lambda_1 = 3 + \sqrt{3}$$

$$\lambda_2 = 3 - \sqrt{3}$$

Thus, A has two distinct eigenvalues. Their algebraic multiplicities are:

$$\mu(\lambda_1) = 1$$

$$\mu(\lambda_2) = 1$$

because they are not repeated!

Matlab:

```
A =

     4     2
     1     2

>> [U,E] = eig(A)

U =

    0.9391   -0.5907
    0.3437    0.8069


E =

    4.7321         0
         0    1.2679
```

Ref:
https://www.statlect.com/matrix-algebra/algebraic-and-geometric-multiplicity-of-eigenvalues

Example

Consider the 2×2 matrix

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Their characteristic polynomial:

$$\begin{aligned} f(\lambda) &= \det\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{bmatrix}\right) \\ &= (1-\lambda)\cdot(1-\lambda) - 0\cdot 2 \\ &= (1-\lambda)\cdot(1-\lambda) \end{aligned}$$

Roots of the polynomial, that is, the solutions of $f(\lambda) = 0$ are:

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$

Thus, A has one repeated eigenvalue whose algebraic multiplicity is:

$$\mu(\lambda_1) = \mu(\lambda_2) = 2$$

```
A =

     1     0
     2     1

>> [U,E] = eig(A)

U =

         0    0.0000
    1.0000   -1.0000


E =

     1     0
     0     1
```

U = eigenvectors have dependent columns.
Hence inv(U) does not exist.

Hence,
U*E*inv(U) does not exist
And this matrix
IS NOT Diagonalizable.

Ref:
https://www.statlect.com/matrix-algebra/algebraic-and-geometric-multiplicity-of-eigenvalues

Null Space and EigenSpace

Defining **Nullspace (kernel) and EigenSpace**:

Given a nxn matrix A with an eigenvalue λ, its corresponding eigenvectors x are the vectors (non-trivial vectors) that satisfy $(A \ -\lambda I)x$ = **0.** In other words,

i) since the RHS of above equation is the zero vector, the set of all x that satisfy above are the **nullspace** of matrix$(A \ -\lambda I), \mathrm{i.e},$

$\mathcal{N}(A-\lambda I)$ =NULL($A-\lambda I$) = ker($A-\lambda I$)

- Note: The null space of $(A-\lambda_i I)$ is same as the eigenspace of the matrix A corresponding to λ_i.

ii) Because the **nullspace** consists of all vectors that are linear combinations of the eigenvectors, we call this space the e**igenspace of A for eigenvalue =** λ, i.e, $\mathcal{E}_A(\lambda)$.

iii) Note that for a given eigenvalue λ_i, we may have more than one eigen vector (when the value is repeated). Any vectors $x \ \in \mathcal{E}_A(\lambda)$ are eigenvectors of A and $Ax = \lambda x$

Ref:
https://en.wikipedia.org/wiki/Kernel_(linear_algebra)

Ref: NULL Space of a Matrix

Representation as matrix multiplication [edit]

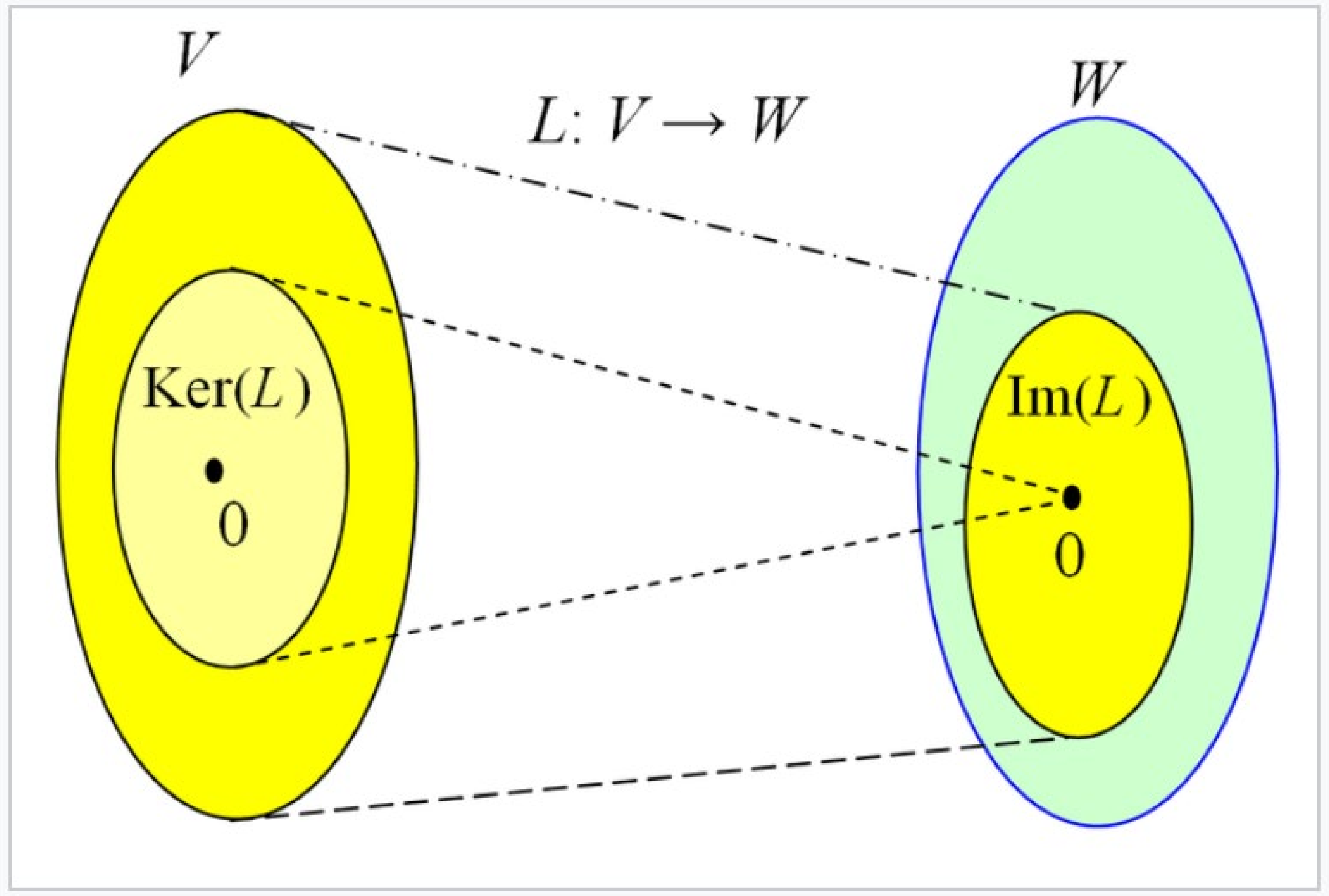
Consider a linear map represented as a $m \times n$ matrix A with coefficients in a field K (typically $\mathbb{R}$ or $\mathbb{C}$), that is operating on column vectors x with n components over K. The kernel of this linear map is the set of solutions to the equation $A\mathbf{x} = \mathbf{0}$, where $\mathbf{0}$ is understood as the zero vector. The dimension of the kernel of A is called the **nullity** of A. In set-builder notation,

$$\mathrm{N}(A) = \mathrm{Null}(A) = \ker(A) = \{\mathbf{x} \in K^n | A\mathbf{x} = \mathbf{0}\} .$$

The matrix equation is equivalent to a homogeneous system of linear equations:

$$A\mathbf{x} = \mathbf{0} \quad \Leftrightarrow \quad \begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0. \end{array}$$

Thus the kernel of A is the same as the solution set to the above homogeneous equations.



Kernel and image of a map L.

Kernel (linear algebra)

From Wikipedia, the free encyclopedia

For other uses, see Kernel (disambiguation).

In mathematics, more specifically in linear algebra and functional analysis, the **kernel** of a linear mapping, also known as the **null space** or **nullspace**, is the set of vectors in the domain of the mapping which are mapped to the zero vector.[1][2] That is, given a linear map $L : V \to W$ between two vector spaces V and W, the kernel of L is the set of all elements $\mathbf{v}$ of V for which $L(\mathbf{v}) = \mathbf{0}$, where $\mathbf{0}$ denotes the zero vector in W,[3] or more symbolically:

$$\ker(L) = \{\mathbf{v} \in V \mid L(\mathbf{v}) = \mathbf{0}\} .$$

Ref: https://en.wikipedia.org/wiki/Kernel_(linear_algebra)

Geometric Multiplicity

Defining **Geometric Multiplicity**:

The geometric multiplicity m_i (of an eigenvalue λ_i) of matrix A is the **dimension of the subspace of vectors** x for which $Ax = \lambda_i x$,or $(A - \lambda_i I)x = 0$.
This dimension also has a name **nullity** $(\boldsymbol{A} - \boldsymbol{\lambda_i I})$

The geometric multiplicity of an eigenvalue can be equal or less than its corresponding algebraic multiplicity $(m_i \leq n_i)$.

Why is Geometric Multiplicity important?
For a matrix to be diagonalizable, the number of independent Eigenvectors that spans the eigenspace must be equal to N, for a NxN square matrix A. (see pg 9)

The *nullity* of a matrix A is the dimension of its null space:

$$\text{nullity}(A) = \dim(N(A)).$$

- Note: The null space of $(A - \lambda_i I)$ is same as the eigenspace of the matrix A corresponding to λ_i.

Ref:
http://www.math.uwaterloo.ca/~jmckinno/Math106/Week12/Lecture5d.pdf
https://people.math.carleton.ca/~kcheung/math/notes/MATH1107/wk10/10_algebraic_and_geometric_multiplicities.html

Example: Algebraic and Geometric Multiplicity

Example: Algebraic multiplicity = 1
Geometric multiplicity = 1

Consider the 2×2 matrix

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Their characteristic polynomial:

$$\begin{aligned} f(\lambda) &= \det\left(\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix}\right) \\ &= (2-\lambda)\cdot(1-\lambda) - 0\cdot 1 \\ &= (2-\lambda)\cdot(1-\lambda) \end{aligned}$$

Roots of the polynomial, that is, the solutions of $f(\lambda) = 0$ are:

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

Thus, A has two distinct eigenvalues. Their algebraic multiplicities are:

$$\mu(\lambda_1) = 1$$
$$\mu(\lambda_2) = 1$$

because they are not repeated!

Now, let us find the geometric multiplicity m_1 for the eigenvalue $\lambda_1 = 2$:

m_1 is the dimension of null space of the matrix $A' = (A - \lambda_1 I)$

$$A' = A - \lambda_1 I = \begin{bmatrix} 2-\lambda_1 & 0 \\ 1 & 1-\lambda_1 \end{bmatrix} = \begin{bmatrix} 2-2 & 0 \\ 1 & 1-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Null space of matrix A' is the subspace of all solution vectors x such that $A'x = 0$.

$$A'x = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Solving this, gives $x_1 = x_2$. Thus the null space of matrix A' contains all vectors x of the form: $x = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where α can be any scalar. As the null space is spanned by a single vector: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the null space has a dimension of 1 and hence, geometric multiplicity $m_1 = 1$.

Example

Consider the 2×2 matrix

Example: Algebraic multiplicity = 2
Geometric multiplicity = 2

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Their characteristic polynomial:

$$\begin{aligned} f(\lambda) &= \det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) \\ &= (2-\lambda)\cdot(2-\lambda) - 0\cdot 0 \\ &= (2-\lambda)\cdot(2-\lambda) \end{aligned}$$

Roots of the polynomial, that is, the solutions of $f(\lambda) = 0$ are:

$$\lambda_1 = 2$$
$$\lambda_2 = 2$$

Ref:
https://www.statlect.com/matrix-algebra/algebraic-and-geometric-multiplicity-of-eigenvalues

Thus, A has one repeated eigenvalue whose algebraic multiplicity is:

$$\mu(\lambda_1) = \mu(\lambda_2) = 2$$

Now, let us find the geometric multiplicity m for eigenvalue $\lambda_1 = \lambda_2 = \lambda = 2$:

m is the dimension of null space of the matrix $A' = (A - \lambda I)$

$$A' = A - \lambda I = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 2-2 & 0 \\ 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Null space of matrix A' is the subspace of all solution vectors x such that $A'x = 0$.

$$A'x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

The above equation is satisfied by any value of x_1, x_2. Thus the null space of matrix A' contains all vectors x of the form: $x = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where α, β can be any scalars. As the null space is spanned by two vectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the null space has a dimension of 2 and hence, geometric multiplicity $m = 2$.

Why is algebraic and geometric multiplicity useful?
Ans: can check if a matrix A is diagonalizable

When is a matrix diagonalizable ?

- Note that the algebraic multiplicity for all eigenvalues, $\sum_{i=1}^{N_\lambda} n_i = N$ where matrix A is of size NxN.
- Fact: The geometric multiplicity of an eigenvalue can be equal or less than its corresponding algebraic multiplicity $(m_i \leq n_i)$.
- If, for each of the eigenvalues λ_i, the algebraic multiplicity equals the geometric multiplicity $(m_i = n_i \quad \forall i)$,
  then the matrix is diagonalizable,
  otherwise it is a defective matrix.

Let U contains the N column of eigenvectors of A

$$AU = U\Lambda$$

// Provide $U^{-1} exist$, then

$$AUU^{-1} = U\Lambda U^{-1}$$
$$A = U\Lambda U^{-1}$$

// When geometric multiplicity is NOT
// equals to algebraic multiplicity, then
// this will result in dependent eigenvectors
// i.e columns of U are dependent implying
// U is singular and has no inverse.
// Therefore we would not be able to perform

$$A = U\Lambda U^{-1}$$

// we call such matrix defective (not diagonalizable).

Ref:https://en.wikipedia.org/wiki/Defective_matrix

Matlab example: Constructing A from given EigenValue and Vectors

$$\begin{aligned} AU &= U\Lambda \\ AUU^{-1} &= U\Lambda U^{-1} \\ A &= U\Lambda U^{-1} \end{aligned}$$

// Provide $U^{-1} exist$

```
% test_create_MyOwnMatrix_fromEigenVectorsValues
% Author: Chng Eng Siong
% Date: 3 Aug 2020

% In this code, we create our own matrix from
% Eigenvectors and eigen values, and then use eig(A) to check if
% we can get what we created.

% Step 1: check how to use eig(A) to get eigenvector and value
% Using eig to find eigenvector and value of a 3x3 matrix
% then reconstruct from U,E to A
disp('\n==================\nStep 1\n');
A = [1 2 3; 1 2 4; 1 3 5]
[U,E] = eig(A)
A_est = U*E*inv(U)
checkOK = norm(A_est-A)    % sanity check, should be error = 0
```

```
\n==================\nStep 1\n

A =

     1     2     3
     1     2     4
     1     3     5


U =

   -0.4578   -0.9463    0.3410
   -0.5457   -0.1372   -0.8456
   -0.7019    0.2928    0.4107

E =

    7.9843         0         0
         0    0.3618         0
         0         0   -0.3461


A_est =

    1.0000    2.0000    3.0000
    1.0000    2.0000    4.0000
    1.0000    3.0000    5.0000


checkOK =

   4.6709e-15
```

Matlab: checking found Eigenvalue and Vector of eig(A)

```
% Step 2: define our own eigenvector U1(make sure it is NOT singular)
% and eigenvalue E1
% Then construct A from U1 and E1
disp('\n==================\nStep 2\n');
U1 = [1 0 0; 0 1 2; 0 1 1]
E1 = [1 0 0; 0 2 0; 0 0 2]
A1 = U1*E1*inv(U1)
[U1_est, E_est] = eig(A1)
% We then see that any vectors spanning
% [0 1 0]' and [ 0 0 1]' are eigen VECTORs.
% NOTE that U1 != U1_est, BUT the eigenvectors associated with \lambda == 2
% spans the same space !!!
A1_est = U1_est*E_est*inv(U1_est)
checkOK = norm(A1_est-A1)    % sanity check, should be error = 0

A1*[0 1 2]' %= 2*[0 1 2]'  ; % where lambda == 2
% sanity check, [0 1 2] is in the eigenSpace(\lambda=2), hence IT should
% be scaled by \lambda ==2
```

```
\n==================\nStep 2\n

U1 =

     1     0     0
     0     1     2
     0     1     1


E1 =

     1     0     0
     0     2     0
     0     0     2


A1 =

     1     0     0
     0     2     0
     0     0     2
```

```
U1_est =

     1     0     0
     0     1     0
     0     0     1


E_est =

     1     0     0
     0     2     0
     0     0     2


A1_est =

     1     0     0
     0     2     0
     0     0     2


checkOK =

     0
```

Remark: we can create our own matrix given our own defined Eigenvectors and values (provided inv(U) exist)
When we use eig(A) to get back the eigenspace, it will agree on the span of the eigenvectors!
BUT the exact values of the eigen vectors may not agree.

Matlab: example of defective matrix

Example [edit]

A simple example of a defective matrix is:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

which has a double eigenvalue of 3 but only one distinct eigenvector

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(and constant multiples thereof).

Above A is a defective
matrix because
algebraic multiplicity of A
For eigen value = 3 is 2.

But geometric multiplicity
Of A for eigen value = 3 is 1.
(Since we only have 1 independent
eigen vector associated with
eigen value = 3)

```
>> A= [3 1; 0 3]

A =

     3     1
     0     3

>> [U,E] = eig(A)

U =

    1.0000   -1.0000
         0    0.0000


E =

     3     0
     0     3
```

```
>> U*E*inv(U)

ans =

    3.0000    0.2500
         0    3.0000
```

```
>> inv(U)

ans =

   1.0e+15 *

    0.0000    1.5012
         0    1.5012
```

```
>> rank(U)

ans =

     2

>> rank(U,1e-6)

ans =

     1
```

```
rank   Matrix rank.
   rank(A) provides an estimate of the number of linearly
   independent rows or columns of a matrix A.

   rank(A,TOL) is the number of singular values of A
   that are larger than TOL. By default, TOL = max(size(A)) * eps(norm(A)).

   Class support for input A:
      float: double, single
```