$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.3A**
Lecture : **Eigen and Singular Values**
Topic : **Complex Eigenvalues, Eigenvectors**
Concept : **Complex Eigenvalues, Eigenvectors**

Instructor: **A/P Chng Eng Siong**
TAs: **Zhang Su**, **Vishal Choudhari**

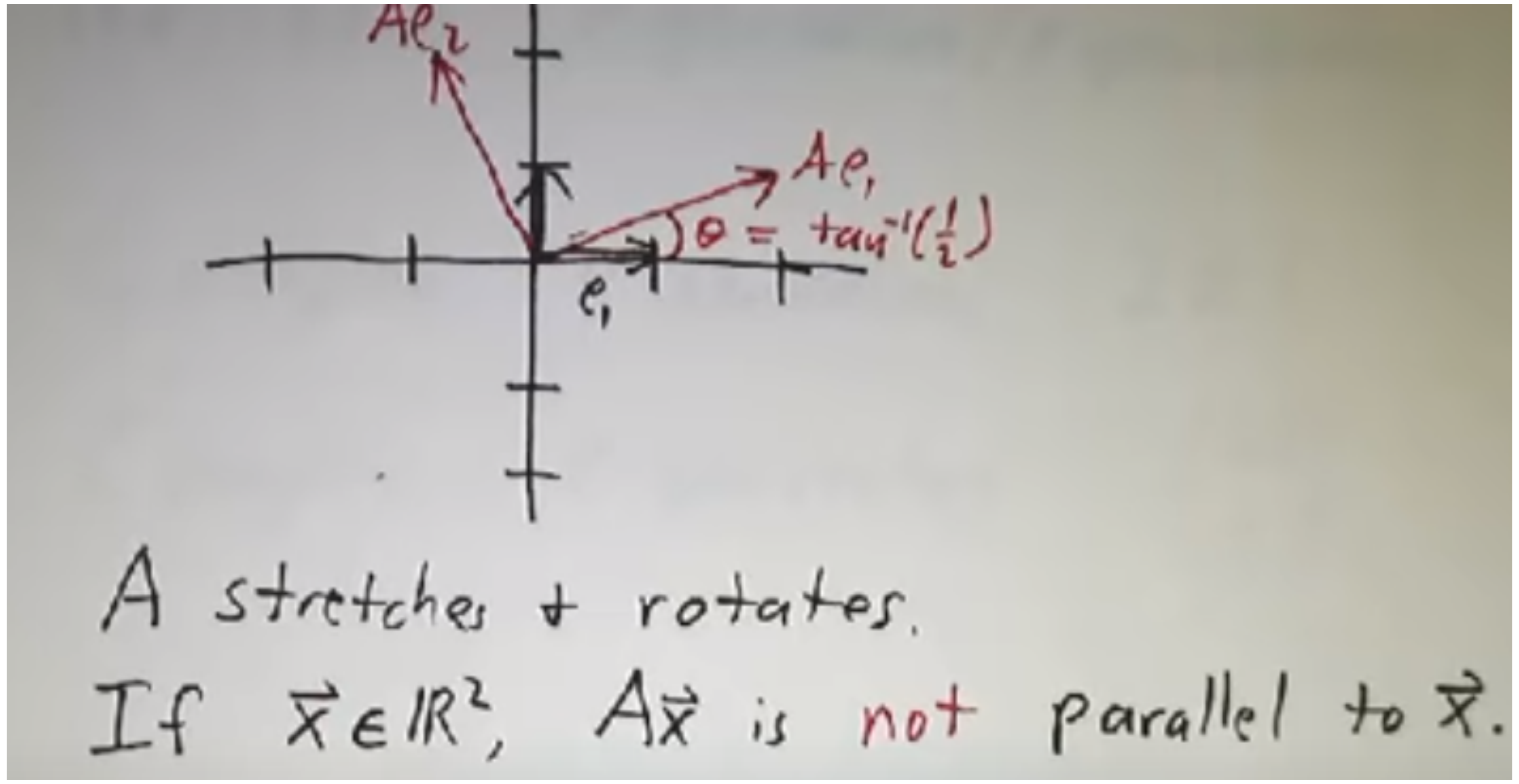
Rev: 18th July 2020

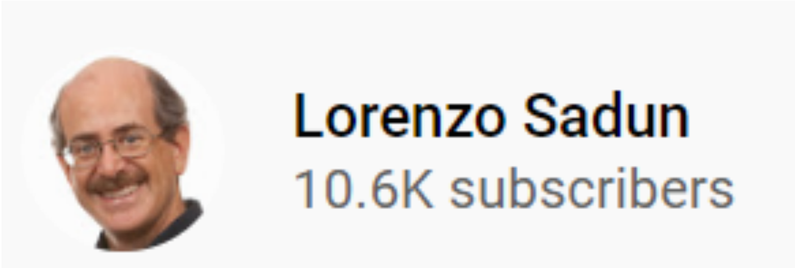
Complex Eigenvalues, Eigenvectors

A real matrix $A \in R^{N \times N}$ may not have real eigenvalues but can have **complex eigenvalues**.

Why do complex eigenvalues and eigenvectors exist?

- Characteristic polynomial $p(\lambda)$ may have complex roots and hence, complex eigenvalues and complex eigenvectors.
- Eigenvectors x are supposed to be in the same direction after the transformation Ax. Such eigenvectors may not necessarily be in R^N, but sometimes in C^N.





Ref:
https://www.youtube.com/watch?v=NmHBGIC1_Z0

Example

EXAMPLE 1 If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ on $\mathbb{R}^2$ rotates the plane counterclockwise through a quarter-turn. The action of A is periodic, since after four quarter-turns, a vector is back where it started. Obviously, no nonzero vector is mapped into a multiple of itself, so A has no eigenvectors in $\mathbb{R}^2$ and hence no real eigenvalues. In fact, the characteristic equation of A is

$$\lambda^2 + 1 = 0$$

The only roots are complex: $\lambda = i$ and $\lambda = -i$. However, if we permit A to act on $\mathbb{C}^2$, then

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i\begin{bmatrix} 1 \\ i \end{bmatrix}$$

Thus i and $-i$ are eigenvalues, with $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$ as corresponding eigenvectors. (A method for *finding* complex eigenvectors is discussed in Example 2.) ■

Complex eigenvalues are obtained when the characteristic polynomial has **complex roots.**

Note: For a matrix $A \in R^{N \times N}$, **complex eigenvalues,** if they occur, occur in **conjugate pairs**!

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5.5 Complex Eigenvalues **295**

Recap:

To solve the eigenvalue problem:

Step 1: Frame the characteristic polynomial: $p(\lambda) = det(A - \lambda I)$.
Step 2: Solve for roots of characteristic equation: $p(\lambda) = det(A - \lambda I) = 0$.

For a 2×2 Matrix

For a 2×2 matrix (2 rows and 2 columns):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is:

$$|A| = ad - bc$$

"The determinant of A equals a times d minus b times c"

Complex Eigenvectors are Also Possible!

```
>> A = [2 -1; 1 2];
>> [U,D] = eig(A)

U =

   0.7071 + 0.0000i   0.7071 + 0.0000i
   0.0000 - 0.7071i   0.0000 + 0.7071i
```

q_1 q_2

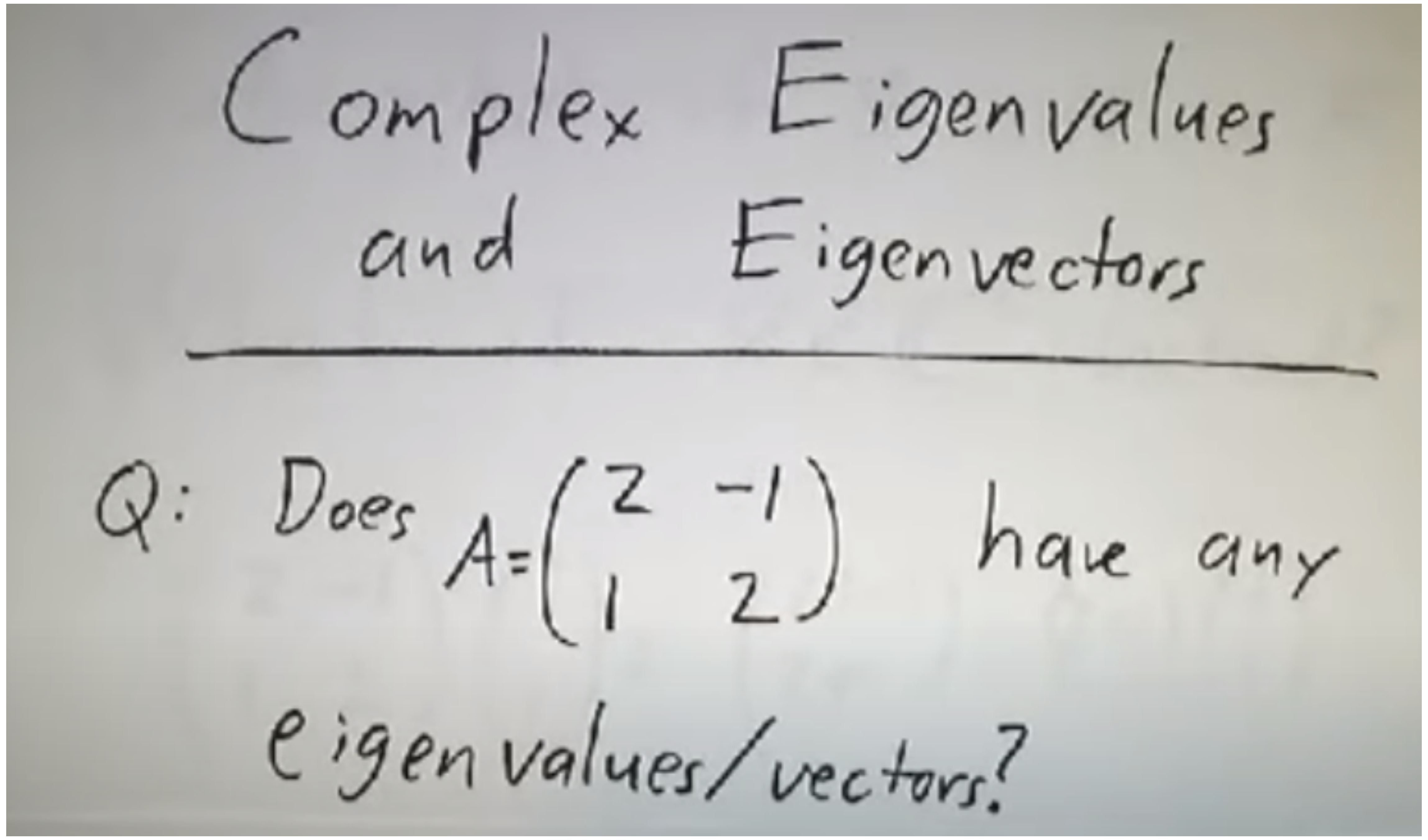
```
D =

   2.0000 + 1.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   2.0000 - 1.0000i

>> U*D*inv(U)

ans =

     2    -1
     1     2
```



Lorenzo Sadun
10.6K subscribers

Ref:
https://www.youtube.com/watch?v=NmHBGIC1_Z0

Example

EXAMPLE 2 Let $A = \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix}$

SOLUTION The characteristic equation of A is

$$0 = \det\begin{bmatrix} .5-\lambda & -.6 \\ .75 & 1.1-\lambda \end{bmatrix} = (.5-\lambda)(1.1-\lambda) - (-.6)(.75)$$
$$= \lambda^2 - 1.6\lambda + 1$$

From the quadratic formula, $\lambda = \frac{1}{2}[1.6 \pm \sqrt{(-1.6)^2 - 4}] = .8 \pm .6i$. For

EXAMPLE 3 One way to see how multiplication by the matrix A in Example 2 affects points is to plot an arbitrary initial point—say, $\mathbf{x}_0 = (2, 0)$—and then to plot successive images of this point under repeated multiplications by A. That is, plot

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.5 \end{bmatrix}$$
$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix}\begin{bmatrix} 1.0 \\ 1.5 \end{bmatrix} = \begin{bmatrix} -.4 \\ 2.4 \end{bmatrix}$$
$$\mathbf{x}_3 = A\mathbf{x}_2, \ldots$$

Figure 1 shows $\mathbf{x}_0, \ldots, \mathbf{x}_8$ as larger dots. The smaller dots are the locations of $\mathbf{x}_9, \ldots, \mathbf{x}_{100}$. The sequence lies along an elliptical orbit. ■

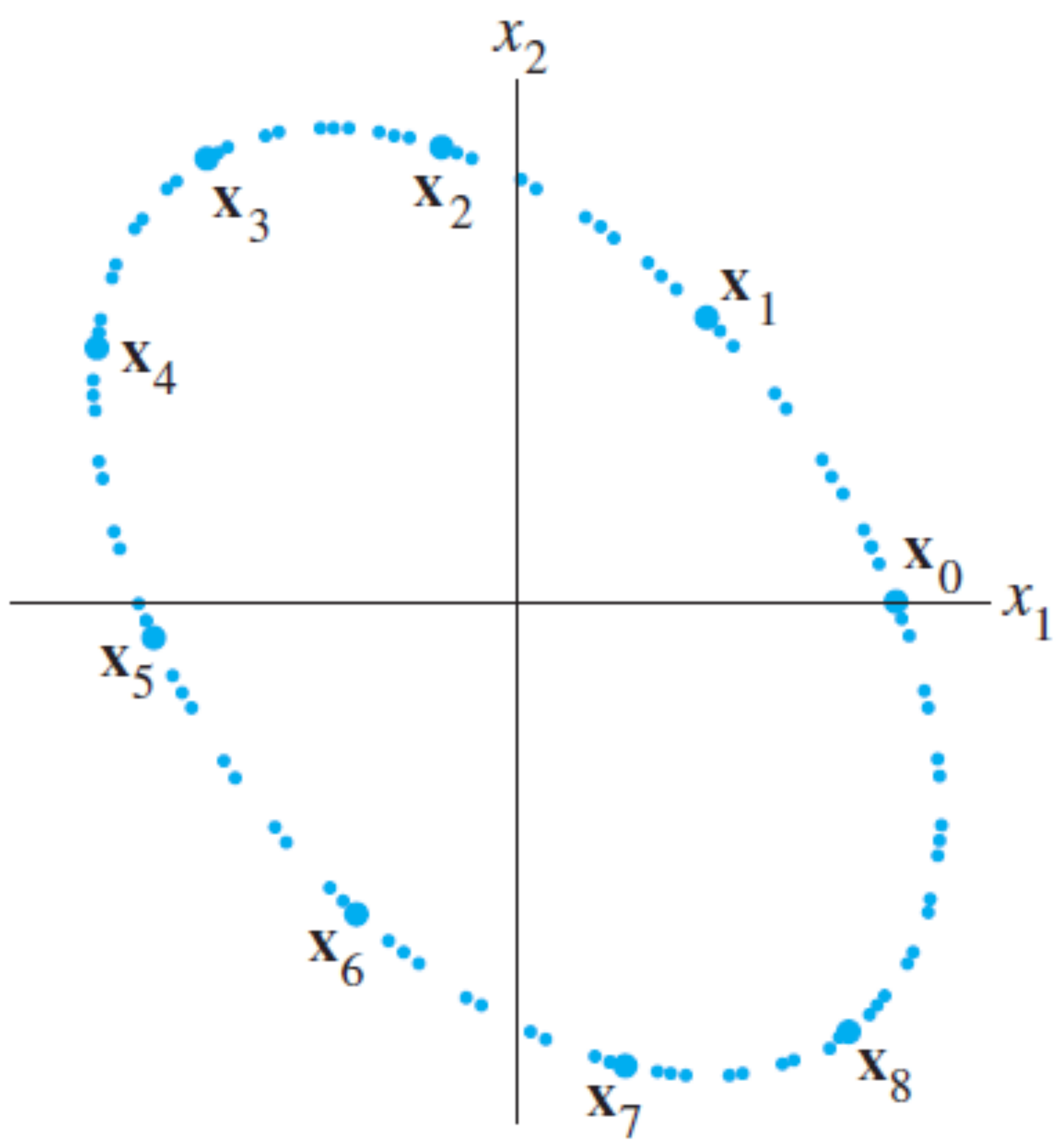


FIGURE 1 Iterates of a point $\mathbf{x}_0$ under the action of a matrix with a complex eigenvalue.

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Example

EXAMPLE 6 If $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where a and b are real and not both zero, then the eigenvalues of C are $\lambda = a \pm bi$. (See the Practice Problem at the end of this section.) Also, if $r = |\lambda| = \sqrt{a^2 + b^2}$, then

$$C = r\begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

where φ is the angle between the positive x-axis and the ray from $(0, 0)$ through (a, b). See Fig. 2 and Appendix B. The angle φ is called the *argument* of $\lambda = a + bi$. Thus the transformation $\mathbf{x} \mapsto C\mathbf{x}$ may be viewed as the composition of a rotation through the angle φ and a scaling by $|\lambda|$ (see Fig. 3). ■

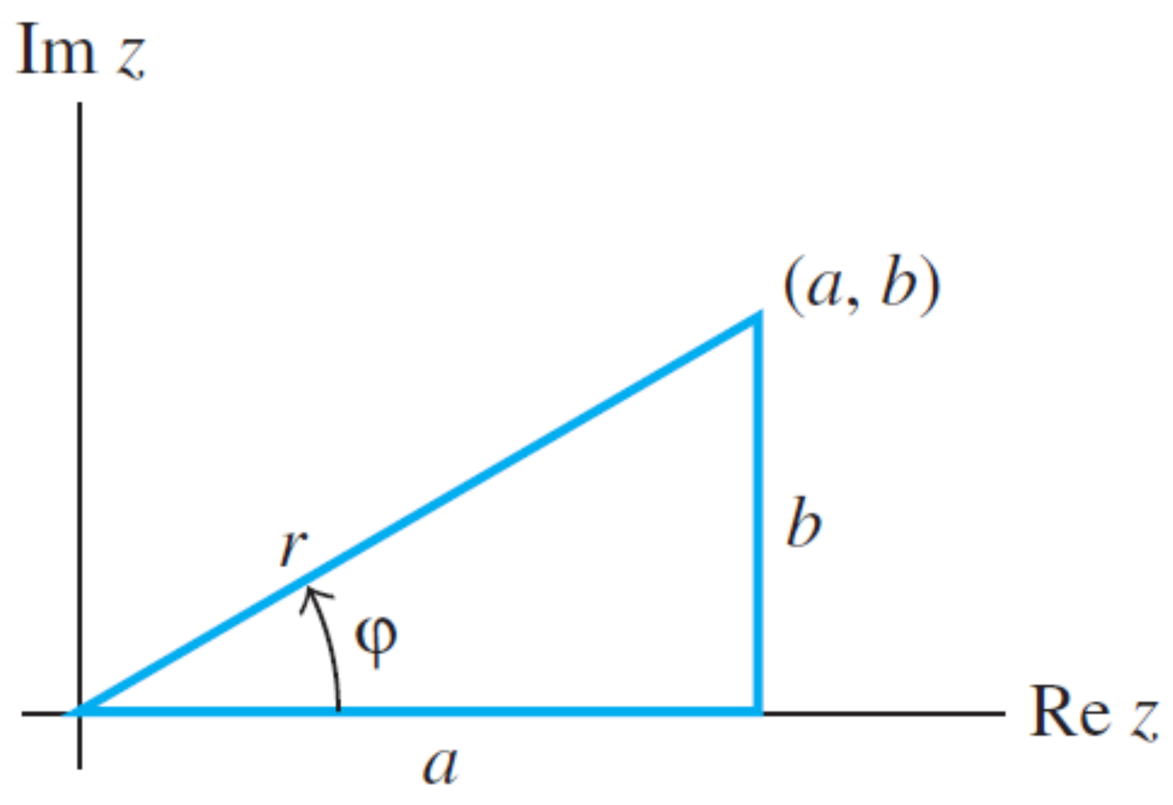

FIGURE 2

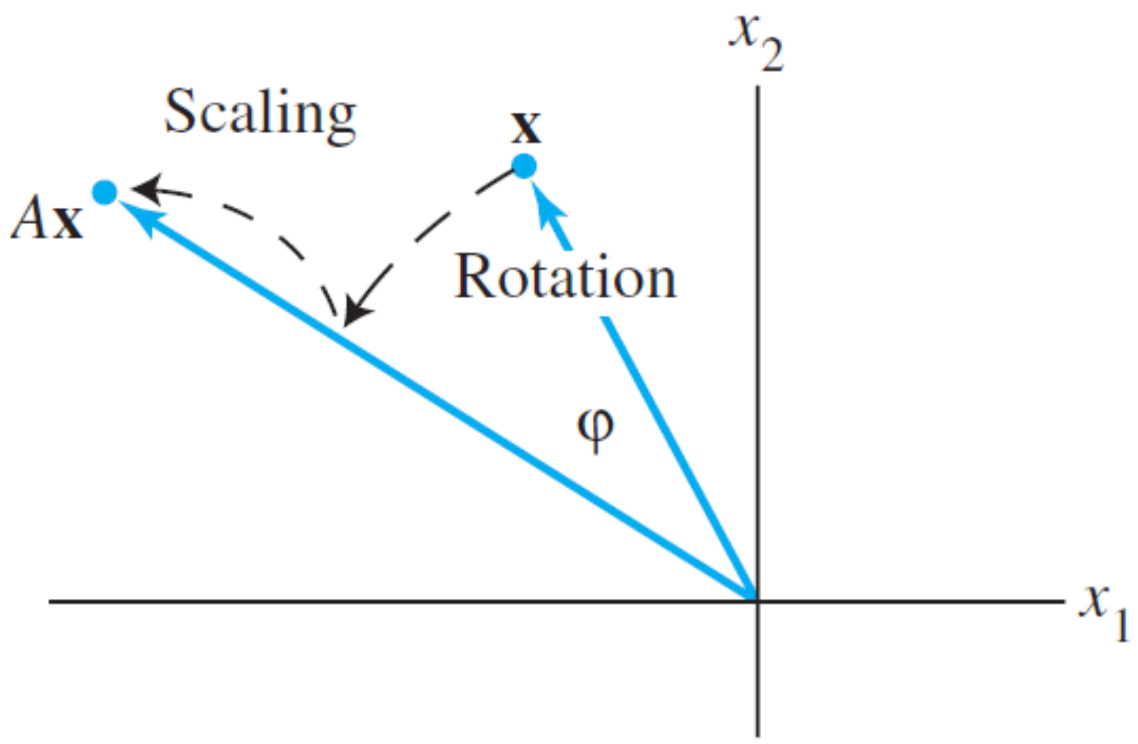

FIGURE 3 A rotation followed by a scaling.

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Example

EXAMPLE 6 If $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where a and b are real and not both zero, then the eigenvalues of C are $\lambda = a \pm bi$. (See the Practice Problem at the end of this section.) Also, if $r = |\lambda| = \sqrt{a^2 + b^2}$, then

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where φ is the angle between the positive x-axis and the ray from $(0,0)$ through (a,b). See Fig. 2 and Appendix B. The angle φ is called the *argument* of $\lambda = a + bi$. Thus the transformation $\mathbf{x} \mapsto C\mathbf{x}$ may be viewed as the composition of a rotation through the angle φ and a scaling by $|\lambda|$ (see Fig. 3). ■

Let's validate the above with the following parameters:

- Starting point:

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- $|\lambda| = 0.995 < 1$. Hence, as $n \uparrow$, trajectory shrinks/spirals towards origin!
- Plot points after every consecutive repeated transformation:

$$x_1 = Cx_0, x_2 = Cx_1 = C^2x_0, \ldots, x_n = C^n x_0$$

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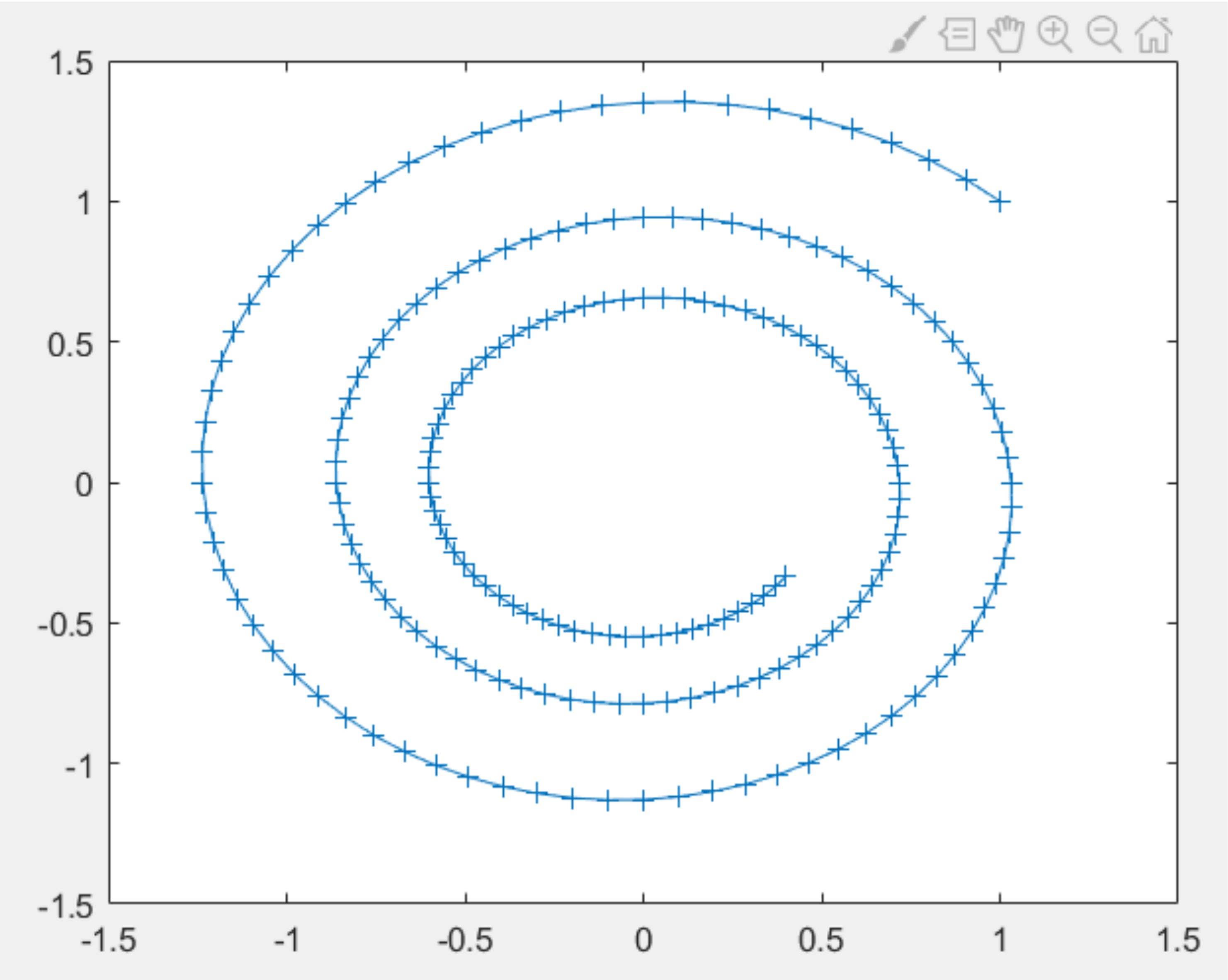
5.5 Complex Eigenvalues 295



1

```
A =

    0.9912   -0.0867
    0.0867    0.9912
```

2

```
>> [U,D] = eig(A)

U =

   0.7071 + 0.0000i   0.7071 + 0.0000i
   0.0000 - 0.7071i   0.0000 + 0.7071i


D =

   0.9912 + 0.0867i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.9912 - 0.0867i
```

3

```
>> abs(D)

ans =

    0.9950         0
         0    0.9950
```

Note: x_0 is not an eigenvector corresponding to λ. Hence the rotation!

MATLAB Code

Code = test_complexVsRealEigenValue.m

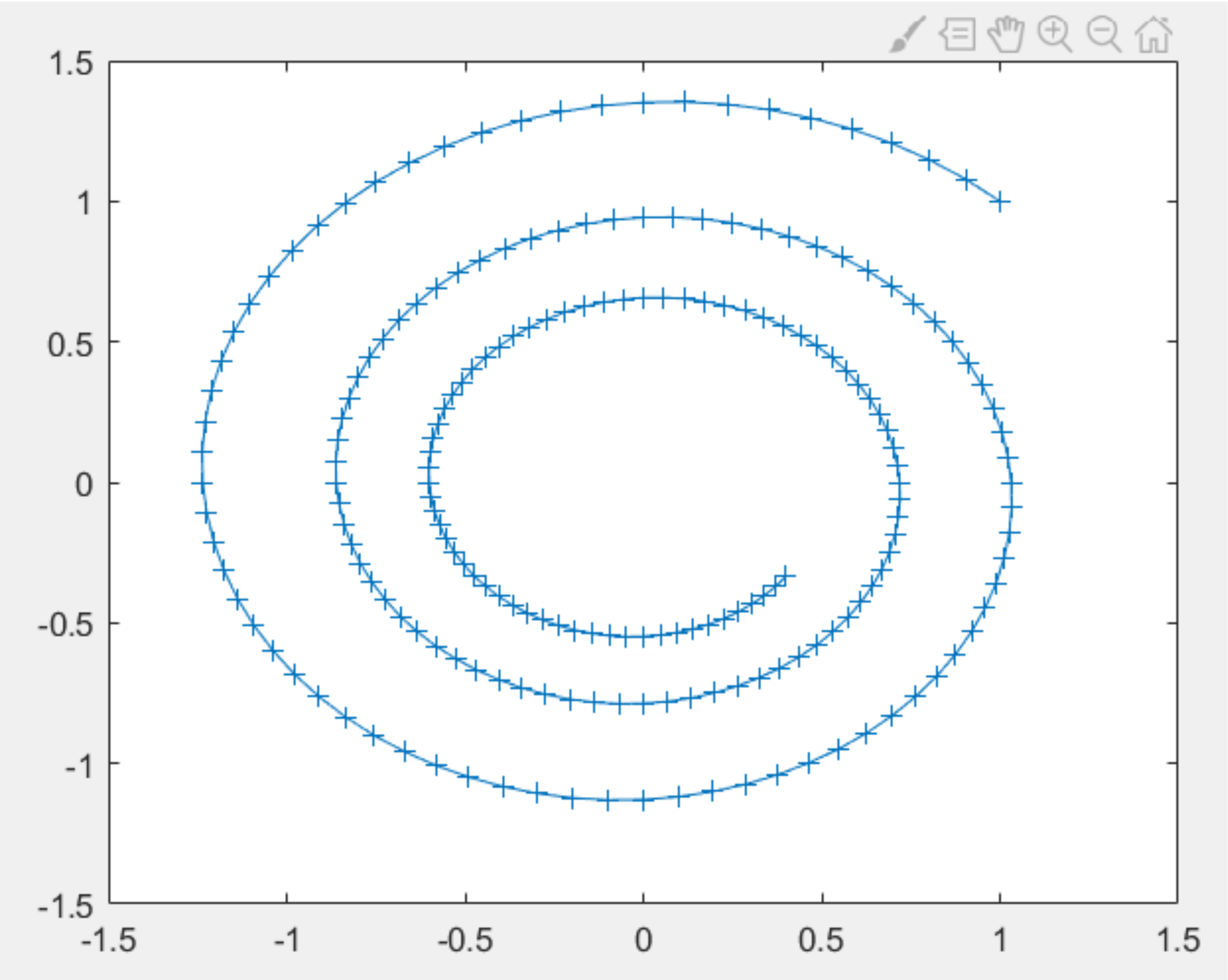
```
a=1;
target_theta = +5;
target_theta_rad  = target_theta*pi/180;
b = a*tan(target_theta_rad);
r = sqrt(a*a+b*b);
A_complex = 0.995*[a/r -b/r; b/r a/r];
%A_complex = [0.78 -0.6; 0.6 0.78];
[U2,D2] = eig(A_complex);

N = 200;
input_x = [1,1]'
[trajectoryComplex, gain_complex] =  genTrajectory(A_complex,input_x,N);
```

```
A_complex =

    0.9912   -0.0867
    0.0867    0.9912
```

```
function [trajectory_seq, gain_seq] = genTrajectory(A,input_x,N)
 trajector_seq = zeros(N,2);
 gain_seq =  zeros(N,2);
 x_old                 = input_x;
 trajectory_seq(1,:) = input_x;
 for (i=2:N)
   x_new = A*x_old;
   trajectory_seq(i,:) = x_new;
   gain_seq(i,:) = x_new./x_old;
   x_old = x_new;
  end
end  % of function
```



1

```
A =

    0.9912   -0.0867
    0.0867    0.9912
```

2

```
>> [U,D] = eig(A)

U =

   0.7071 + 0.0000i   0.7071 + 0.0000i
   0.0000 - 0.7071i   0.0000 + 0.7071i


D =

   0.9912 + 0.0867i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.9912 - 0.0867i
```

3

```
>> abs(D)

ans =

    0.9950         0
         0    0.9950
```

Note: x_0 is not an eigenvector corresponding to λ.

Hence the rotation!