$$
\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}
$$

CX1104: **Linear Algebra for Computing**

Chap. No : **7.2.3**

Lecture : **Least Squares**

Topic : **Reviewing Basic Matrix Algebra**

Concept : **Orthogonal Matrix and Examples**

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Orthogonal Matrix

In linear algebra, an **orthogonal matrix** is a square matrix whose columns and rows are orthogonal unit vectors (orthonormal vectors).

One way to express this is

$$Q^{\mathrm{T}}Q = QQ^{\mathrm{T}} = I,$$

where Q^{T} is the transpose of Q and I is the identity matrix.

This leads to the equivalent characterization: a matrix Q is orthogonal if its transpose is equal to its inverse:

$$Q^{\mathrm{T}} = Q^{-1},$$

where Q^{-1} is the inverse of Q.

In decomposing $A = QR$,

- Remember that Q is an orthogonal matrix.
- And R is an upper triangular matrix!

Let A be a square invertible matrix:

$$AA^{-1} = A^{-1}A = I$$

then A^{-1} is its inverse if left or right multiplication produces the identity matrix of appropriate size.

Let U be an orthogonal matrix (orthogonal matrices are usually square), then:

$$UU^T = I \text{ and } U^TU = I$$

Hence orthogonal matrix are very special because its inverse is simply its transpose!, i.e, $U^{-1} = U^T$.

Ref:

1. https://en.wikipedia.org/wiki/Orthogonal_matrix
2. Basic examples: https://www.nagwa.com/en/explainers/476190725258/
3. Advance: https://mathworld.wolfram.com/OrthogonalMatrix.html

Properties of an Orthogonal Matrix

An important property:

We know that if U is an orthogonal matrix (has orthonormal columns and is square) then $\|U^T x\| = \|x\|$ because

$$\begin{aligned} \|U^T x\|^2 &= x^T U U^T x \\ &= x^T x \qquad \text{(since } UU^T = I\text{)}. \\ &= \|x\|^2. \end{aligned}$$

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Other properties:

Consider a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$, $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ matrix.

Theorem The following conditions are equivalent:

(i) $\|L(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$;

(ii) $L(\mathbf{x}) \cdot L(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;

(iii) the transformation L preserves distance between points: $\|L(\mathbf{x}) - L(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;

(iv) L preserves length of vectors and angle between vectors;

(v) the matrix A is orthogonal;

Example 1

Solve for X, given:

$$AX(D+BX)^{-1} = C$$

For equation to agree, assume all matrixes are square and invertible.

from (1),

$$AX = C(D+BX);$$

then

$$AX = CD + CBX,$$

or

$$(A - CB)X = CD;$$

if now we know--or assume--that $A - CB$ is invertible, we have

$$X = (A - CB)^{-1}CD.$$

Ref:

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Sanity Check:

```
m =3; n =3;
A = rand(m,n)-0.5;
B = rand(m,n)-0.5;
D = rand(m,n)-0.5;
X = rand(m,n)-0.5;

C = A*X*(inv(D+B*X))
X_est = (inv(A-C*B))*(C*D)

X-X_est
```

```
X_est =

   -0.1326    0.3852   -0.4013
    0.4880    0.4133   -0.2381
   -0.4623    0.2962   -0.1646


ans =

   1.0e-13 *

   -0.2254    0.0966    0.0816
   -0.3597    0.2026    0.1055
    0.0344   -0.0189   -0.0017
```

Example 2

Solve for X, given:

$$(AX)^T \left((D + BX)^{-1} \right)^T = I$$

For equation to agree, assume all matrixes are square and invertible.

$$(AX)^T = I * \left(\left((D + BX)^{-1} \right)^T \right)^{-1}$$

$$X^T A^T =$$

$$I * \left(\left((D + BX)^T \right)^{-1} \right)^{-1}$$

$$X^T A^T = I * (D + BX)^T$$

$$X^T A^T = I * D^T + X^T B^T$$

$$X^T A^T - X^T B^T = I * D^T$$

$$X^T (A^T - B^T) = I * D^T$$

$$X^T =$$

$$I * D^T * \left(A^T - B^T \right)^{-1}$$

Ref:

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3. Advance: https://mathworld.wolfram.com/OrthogonalMatrix.html

Sanity Check:

```
m =3; n =3;
%%%%%%%%%%%%%%% Example 1

A = rand(m,n)-0.5;
B = rand(m,n)-0.5;
D = rand(m,n)-0.5;
```

```
%%%%%%%%%%%%%%% Example 2
X_t = D'*(inv(A'-B'))
X = X_t'
checkI = (A*X)'*((inv(D+B*X))')
```

```
>> checkI = (A*X)'*((inv(D+B*X))')

checkI =

    1.0000   -0.0000    0.0000
    0.0000    1.0000   -0.0000
    0.0000    0.0000    1.0000
```

Summary

0.1 basic formulae

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC} \tag{1a}$$

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T \tag{1b}$$

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \tag{1c}$$

$$\text{if individual inverses exist} \quad (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{1d}$$

$$(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} \tag{1e}$$

0.2 trace, determinant and rank

$$|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| \tag{2a}$$

$$|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|} \tag{2b}$$

$$|\mathbf{A}| = \prod \text{evals} \tag{2c}$$

$$\mathrm{Tr}\left[\mathbf{A}\right] = \sum \text{evals} \tag{2d}$$

if the cyclic products are well defined,

$$\mathrm{Tr}\left[\mathbf{ABC}\ldots\right] = \mathrm{Tr}\left[\mathbf{BC}\ldots\mathbf{A}\right] = \mathrm{Tr}\left[\mathbf{C}\ldots\mathbf{AB}\right] = \ldots \tag{2e}$$

$$\mathrm{rank}\left[\mathbf{A}\right] = \mathrm{rank}\left[\mathbf{A}^T\mathbf{A}\right] = \mathrm{rank}\left[\mathbf{AA}^T\right] \tag{2f}$$

$$\text{condition number} = \gamma = \sqrt{\frac{\text{biggest eval}}{\text{smallest eval}}} \tag{2g}$$

Ref:

1. https://cs.nyu.edu/~roweis/notes/matrixid.pdf

evals = eigenValues

Supplementary

- The Matrix Cookbook: https://www.math.uwaterloo.ca/~hwolkowi/matrixcookbook.pdf