$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.3B**

Lecture : **Eigen and Singular Values**

Topic : **Dynamical Systems**

Concept : **Dynamical Systems**

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Calculating A^k with Eigenvalue Decomposition

EXAMPLE 2 Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k, given that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

SOLUTION The standard formula for the inverse of a 2×2 matrix yields

$$P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

Then, by associativity of matrix multiplication,

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD\underbrace{(P^{-1}P)}_{I}DP^{-1} = PDDP^{-1}$$

$$= PD^2P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

Again,

$$A^3 = (PDP^{-1})A^2 = (PD\underbrace{P^{-1})P}_{I}D^2P^{-1} = PDD^2P^{-1} = PD^3P^{-1}$$

In general, for $k \geq 1$,

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ 2 \cdot 3^k - 2 \cdot 5^k & 2 \cdot 3^k - 5^k \end{bmatrix}$$

Important! ■

A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D. The next theorem gives a characterization of diagonalizable matrices and tells how to construct a suitable factorization.

MATLAB:

```
>> P = [1 1; -1 -2];   D=[5 0; 0 3];
>> P*D*inv(P)

ans =

     7     2
    -4     1
```

Lay, Linear Algebra and its Applications (4th Edition)

5.3 Diagonalization **281**

Finding Trajectory by Repeated Transformation [Brute Force]

You are given:

1. Transformation Matrix A
2. Initial Starting Point x_0

$$A = \begin{bmatrix} .80 & 0 \\ 0 & .64 \end{bmatrix}$$

$$x_0 = (3,3)$$

Asked to plot the trajectory spanned by repeatedly multiplying the initial starting point x_0 with the transformation matrix A:

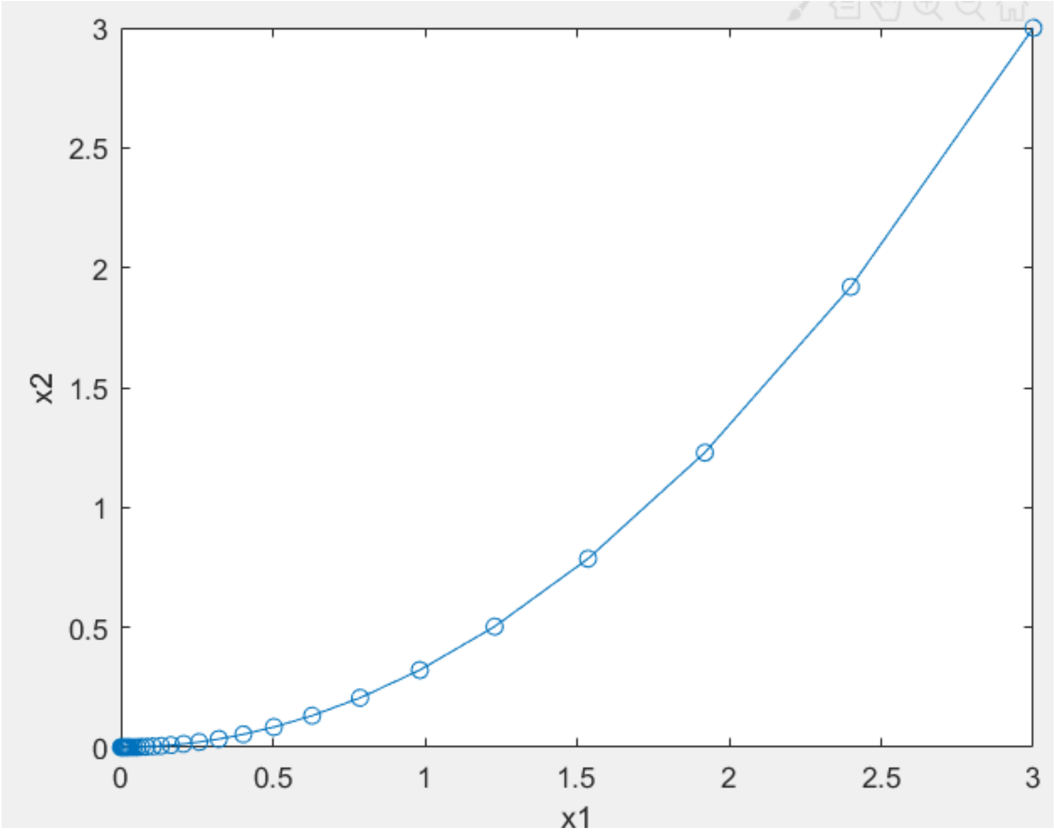
- $x_0 \rightarrow$ Starting Point
- $x_1 = Ax_0$
- $x_2 = Ax_1 = AAx_0 = A^2x_0$
- $\vdots$
- $x_k = Ax_{k-1} = A^kx_0$

```
% FileName: test_trajectory1
% Lay's example 2, complex exponential, pg 296
% Trajectory of x_(k+1) = Ax_k

A = [ 0.8 0; 0 0.64];
[V,D] = eig(A)

% sort the eigenvalues - remember Matlab does not sort eigenvalue
[d,ind] = sort(diag(D),'descend')
Ds = D(ind,ind)
Vs = V(:,ind)

x0 = [3 3]';
```

The brute force way to do this is to write a for loop. Code here shows how it can be done.

Drawback: This is **computationally expensive!**

```
% Brute Force
N = 100;
hist_x = zeros(2,N);
hist_x(:,1) = x0;
for k=2:N
  curr_x = hist_x(:,k-1);
  hist_x(:,k) =  A*curr_x;
end
figure
plot(hist_x(1,:),hist_x(2,:),'-o');
xlabel('x1');
ylabel('x2');hold on;
```



Finding Trajectory by Repeated Transformation [EVD]

You are given:

1. Transformation Matrix A
2. Initial Starting Point x_0

$$A = \begin{bmatrix} .80 & 0 \\ 0 & .64 \end{bmatrix}$$

$$x_0 = (3,3)$$

Asked to plot the trajectory spanned by repeatedly multiplying the initial starting point x_0 with the transformation matrix A:

- $x_0 \rightarrow$ Starting Point
- $x_1 = Ax_0$
- $x_2 = Ax_1 = AAx_0 = A^2x_0$
- $\vdots$
- $x_k = Ax_{k-1} = A^kx_0$

EVD can be used to efficiently compute A^k and thereby bringdown computational complexity!

Eigenvalues and eigenvectors provide the key to understanding the long-term behavior, or *evolution*, of a dynamical system described by a difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

Until Example 6, we assume that A is diagonalizable, with n linearly independent eigenvectors, $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and corresponding eigenvalues, $\lambda_1, \ldots, \lambda_n$. For convenience, assume the eigenvectors are arranged so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$, any initial vector $\mathbf{x}_0$ can be written uniquely as

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \tag{1}$$

This *eigenvector decomposition* of $\mathbf{x}_0$ determines what happens to the sequence $\{\mathbf{x}_k\}$. The next calculation generalizes the simple case examined in Example 5 of Section 5.2. Since the $\mathbf{v}_i$ are eigenvectors,

$$\begin{aligned} \mathbf{x}_1 = A\mathbf{x}_0 &= c_1A\mathbf{v}_1 + \cdots + c_nA\mathbf{v}_n \\ &= c_1\lambda_1\mathbf{v}_1 + \cdots + c_n\lambda_n\mathbf{v}_n \end{aligned}$$

In general,

$$\mathbf{x}_k = c_1(\lambda_1)^k\mathbf{v}_1 + \cdots + c_n(\lambda_n)^k\mathbf{v}_n \qquad (k = 0, 1, 2, \ldots) \tag{2}$$

Eigenvector decomposition: Rewriting a vector x_0 as a linear combination of eigenvectors $\{v_1, v_2, \ldots, v_n\}$ of a matrix A.

Finding Trajectory by Repeated Transformation [EVD]

EVD can be used to efficiently compute A^k and thereby bringdown computational complexity!

$$A = \begin{bmatrix} .80 & 0 \\ 0 & .64 \end{bmatrix} \qquad x_0 = (3,3)$$

Goal: **Compute** $x_k = A^k x_0$.

Solution:

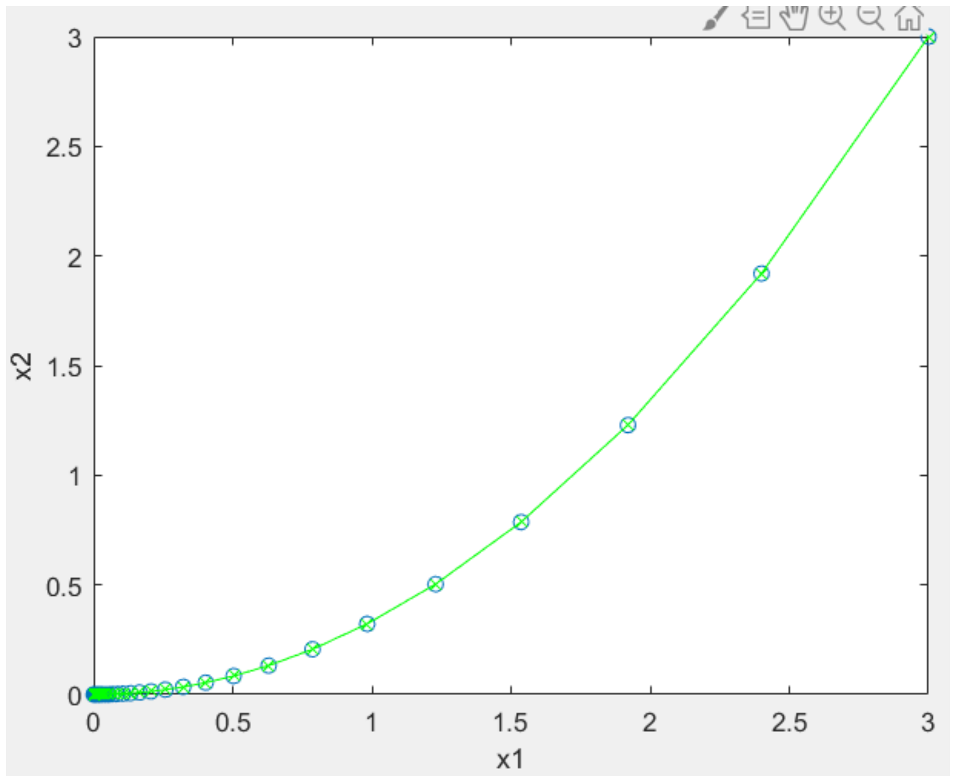
Step 1: Find eigenvectors of matrix A. Let matrix V_s contain the eigenvectors as columns, sorted by their corresponding eigenvalues.

Step 2: Decompose x_0 into scaled components of eigenvalues of A. How? x_0 would be a linear combination of eigenvectors. Hence, $V_s\ c = x_0$, for some column vector c. Solve for c by: $c = V_s^{-1} x_0$.

Step 3: With $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ obtained, use the result of previous slide to generalise an expression for x_k.

$$\mathbf{x}_k = c_1 \underbrace{(.8)}_{\lambda_1}{}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \underbrace{(.64)}_{\lambda_2}{}^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

```
% FileName: test_trajectory1
% Lay's example 2, complex exponential, pg 296
% Trajectory of x_(k+1) = Ax_k

A = [ 0.8 0; 0 0.64];
[V,D] = eig(A)

% sort the eigenvalues - remember Matlab does not sort eigenvalue
[d,ind] = sort(diag(D),'descend')
Ds = D(ind,ind)
Vs = V(:,ind)

x0 = [3 3]';


% Using Example 2 process Lay, pg 303
% x_k = c1*(lambda1)^k*Vs(1) + c2*lambda2^k*vs2

% Decompose x into basis span by Vs
% x = Vs*c, therefore c = inv(Vs)*x
c = inv(Vs)*x0

N = 100;
hist_x2 = zeros(2,N);
hist_x2(:,1) = x0;
for k=2:N
  hist_x2(:,k) =  (c(1)*Ds(1,1)^(k-1)*Vs(:,1))+(c(2)*Ds(2,2)^(k-1)*Vs(:,2));
end
plot(hist_x2(1,:),hist_x2(2,:),'g-x');
```



Computationally less expensive!

Lay, Linear Algebra and its Applications (4th Edition)

5.6 Discrete Dynamical Systems 303

Example

In Example 2, the origin is called an **attractor** of the dynamical system because all trajectories tend toward $\mathbf{0}$. This occurs whenever both eigenvalues are less than 1 in magnitude. The direction of greatest attraction is along the line through $\mathbf{0}$ and the eigenvector $\mathbf{v}_2$ for the eigenvalue of smaller magnitude.

```
% plotting many starting point
figure
xlim([-4 4]); ylim([-4 4]); xlabel('x1'); ylabel('x2')
grid on;
% plotting several trajectories
for x2 = -3:6:3
    for x1 = -3:1.5:3  % horizontal axis
        x0 = [x1 x2]'
        c = inv(Vs)*x0
        N = 20;
        hist_x2 = zeros(2,N);
        hist_x2(:,1) = x0;
        for k=2:N
          hist_x2(:,k) =  (c(1)*Ds(1,1)^(k-1)*Vs(:,1))+(c(2)*Ds(2,2)^(k-1)*Vs(:,2));
        end
        plot(hist_x2(1,:),hist_x2(2,:),'g-x'); hold on;
        pause(0.1); grid on;
    end % of x2
end
```

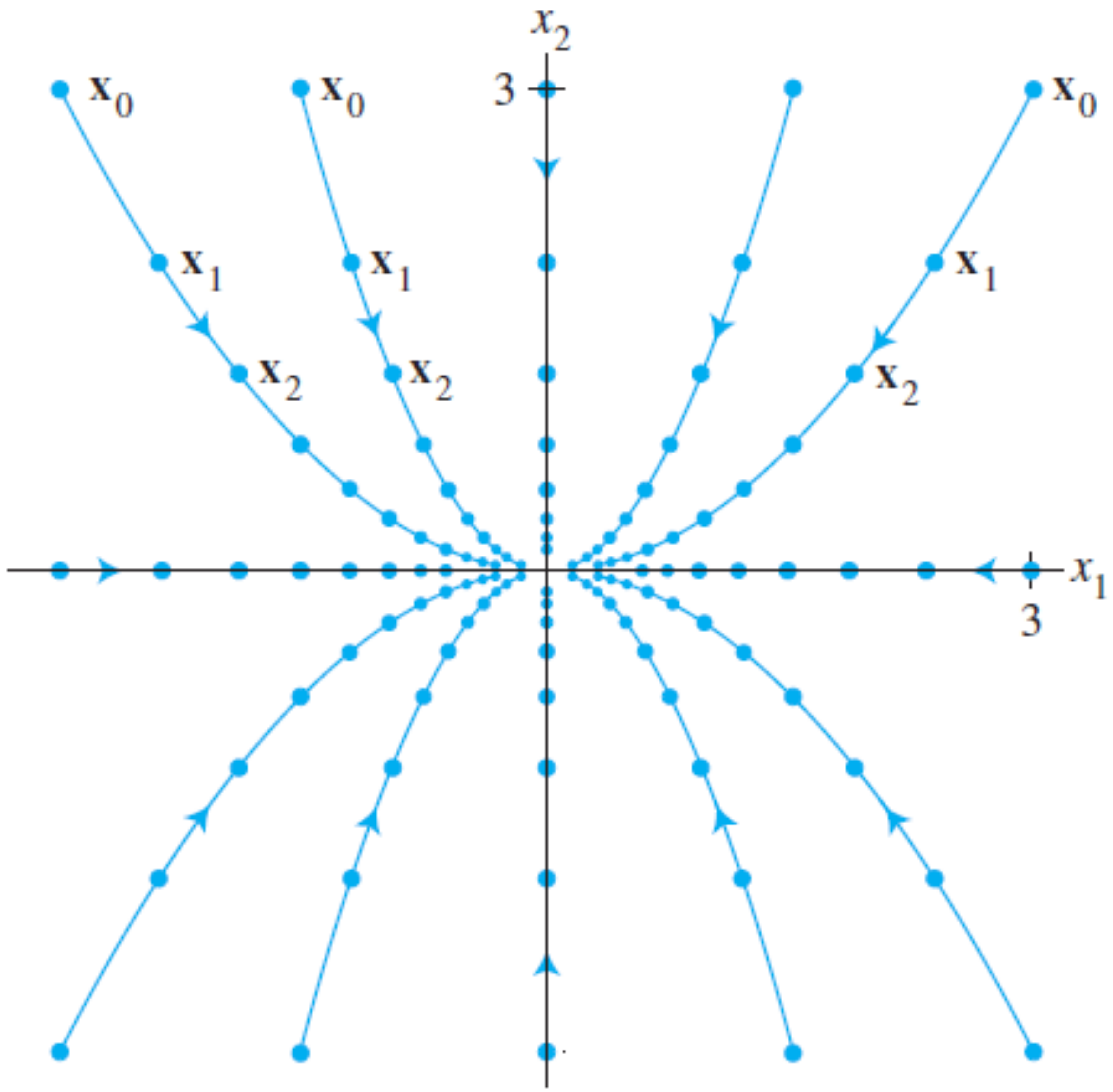


FIGURE 1 The origin as an attractor.

Example

When A^k happens?
We hope A is diagonal.

The following example illustrates that powers of a diagonal matrix are easy to compute.

EXAMPLE 1 If $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, then $D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$
and

$$D^3 = DD^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}$$

In general,

$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \quad \text{for } k \geq 1$$

If $A = PDP^{-1}$ for some invertible P and diagonal D, then A^k is also easy to compute, as the next example shows.

Ref: https://ocw.mit.edu/courses/mathematics/18-06sc-linear-algebra-fall-2011/least-squares-determinants-and-eigenvalues/diagonalization-and-powers-of-a/#?w=535

Problem Solving Video

- Watch the recitation video on Powers of a Matrix (00:09:06)