$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **7.2.2**

Lecture : **Least Squares**

Topic : **Reviewing Basic Matrix Algebra**

Concept : **Unitary Matrix Operations**

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Transpose of a Matrix

What is the transpose of a matrix?

Let $[A]$ be a $m \times n$ matrix. Then $[B]$ is the transpose of the $[A]$ if $b_{ji} = a_{ij}$ for all i and j. That is, the i^{th} row and the j^{th} column element of $[A]$ is the j^{th} row and i^{th} column element of $[B]$. Note, $[B]$ would be a $n \times m$ matrix. The transpose of $[A]$ is denoted by $[A]^{\mathrm{T}}$.

Example 1

Find the transpose of

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

Solution

The transpose of $[A]$ is

$$[A]^{\mathrm{T}} = \begin{bmatrix} 25 & 5 & 6 \\ 20 & 10 & 16 \\ 3 & 15 & 7 \\ 2 & 25 & 27 \end{bmatrix}$$

Note, the transpose of a row vector is a column vector and the transpose of a column vector is a row vector.

Also, note that the transpose of a transpose of a matrix is the matrix itself, that is, $\left([A]^{\mathrm{T}}\right)^{\mathrm{T}} = [A]$. Also, $(A+B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}; (cA)^{\mathrm{T}} = cA^{\mathrm{T}}$.

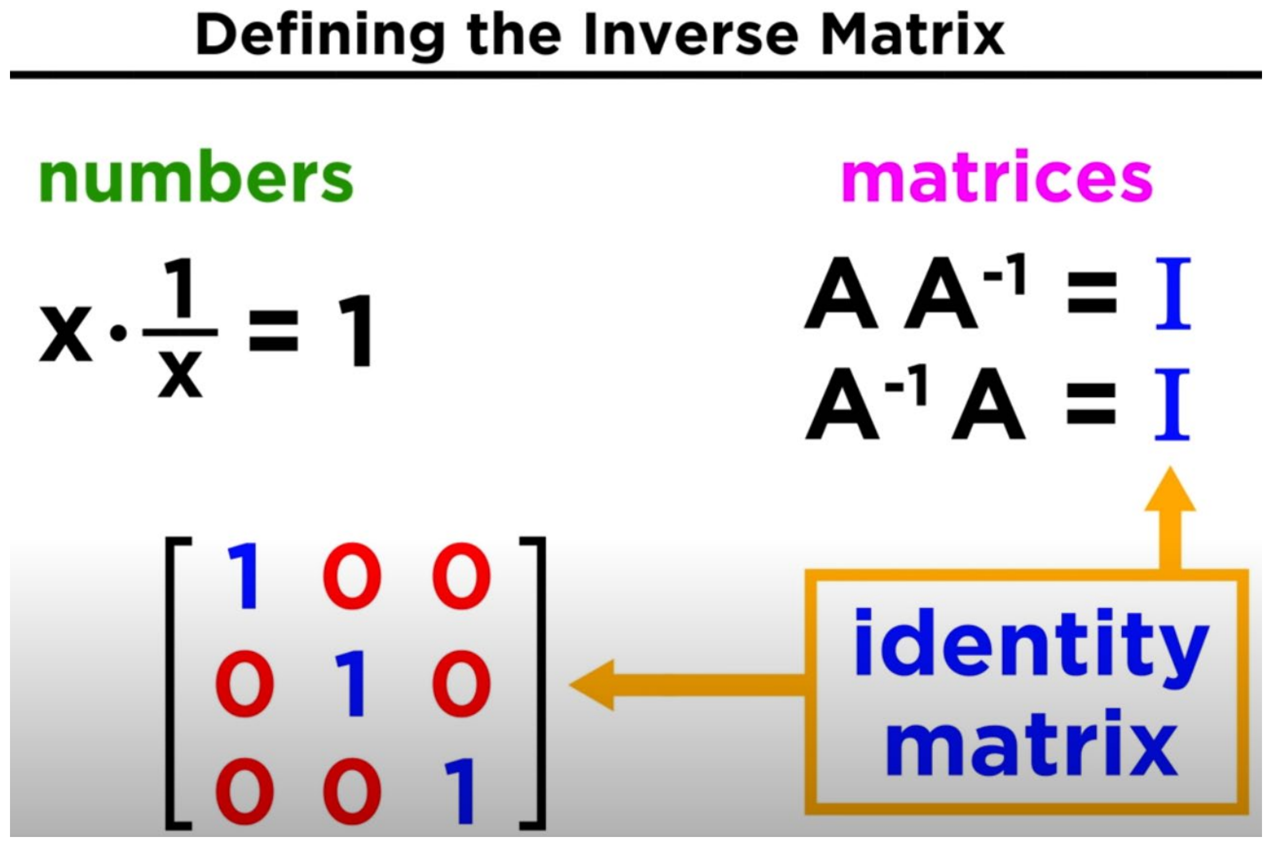
Transpose of a product: $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.

Transpose of an extended product: $(\mathbf{ABC})' = \mathbf{C}'\mathbf{B}'\mathbf{A}'$.

Inverse of a Matrix

- Defined only for **square** matrices.
- A square matrix A has an inverse if there exists a matrix A^{-1} such that:

$$AA^{-1} = I = A^{-1}A$$



When A has an inverse, we say:

- A is invertible
- A is non-singular
- $Det(A) \neq 0$

Three Properties of the Inverse

1. If A is a square matrix and B is the inverse of A, then A is the inverse of B, since $AB = I = BA$. Then we have the identity:

$$(A^{-1})^{-1} = A$$

2. Notice that $B^{-1}A^{-1}AB = B^{-1}IB = I = ABB^{-1}A^{-1}$. Then:

$$(AB)^{-1} = B^{-1}A^{-1}$$

   Then much like the transpose, taking the inverse of a product *reverses* the order of the product.

3. Finally, recall that $(AB)^T = B^TA^T$. Since $I^T = I$, then $(A^{-1}A)^T = A^T(A^{-1})^T = I$. Similarly, $(AA^{-1})^T = (A^{-1})^TA^T = I$. Then:

$$(A^{-1})^T = (A^T)^{-1}$$

Inverse is NOT distributive over addition!
That is:

$$(\mathrm{A} + \mathrm{B})^{-1} \neq A^{-1} + B^{-1}$$

e.g, It is NOT even correct if A and B are just real numbers (a 1x1 matrix)!

Ref:
1. https://people.richland.edu/james/lecture/m116/matrices/inverses.html
2. Prof. Dave: https://www.youtube.com/watch?v=kWorj5BBy9k

Transpose and Inverse

Is $(A^{-1})^T = (A^T)^{-1}$ you ask.

Well

$$\begin{aligned} A^T(A^{-1})^T &= (A^{-1}A)^T = I^T = I \\ (A^{-1})^T A^T &= (AA^{-1})^T = I^T = I \end{aligned}$$

This proves that the inverse of A^T is $(A^{-1})^T$. So the answer to your question is yes.

Here I have used that

$$A^T B^T = (BA)^T.$$

And we have used that the inverse of a matrix A is exactly (by definition) the matrix B such that $AB = BA = I$.

This property has also been used in proving 2nd and 3rd properties of inverse in Slide 2.

Ref: https://math.stackexchange.com/questions/340233/transpose-of-inverse-vs-inverse-of-transpose

Trace of a Matrix

What is the trace of a matrix?

The trace of a $n \times n$ matrix $[A]$ is the sum of the diagonal entries of $[A]$, that is,

$$\mathrm{tr}[A] = \sum_{i=1}^{n} a_{ii}$$

Find the trace of

$$[A] = \begin{bmatrix} 15 & 6 & 7 \\ 2 & -4 & 2 \\ 3 & 2 & 6 \end{bmatrix}$$

Solution

$$\begin{aligned} \mathrm{tr}[A] &= \sum_{i=1}^{3} a_{ii} \\ &= (15) + (-4) + (6) \\ &= 17 \end{aligned}$$

Advance References:

1. https://www.quora.com/What-is-a-trace-as-in-trace-of-a-matrix-and-why-is-it-used
2. https://math.stackexchange.com/questions/2931926/trace-of-a-matrix-when-to-use-what-is-trace-trick
3. https://jsteinhardt.wordpress.com/2012/12/17/algebra-trick-of-the-day/
4. Lorenzo Sadun: (Tricks of the Trade I: Traces and Determinants) https://www.youtube.com/watch?v=xyBPEoXqba4

Properties of Trace

The trace of a square matrix A, denoted by $\mathrm{tr}(A)$, is an operator that satisfies the following properties:

1. $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$;
2. $\mathrm{tr}(kA) = k\,\mathrm{tr}(A)$;
3. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$;
4. $\mathrm{tr}(I) = \dim(I)$.

Here, B is a matrix of the same dimension as A, k is a scalar, I is the identity matrix, and $\dim(M)$ is the dimension (size) of the matrix M.

Properties 1 and 2 tell us that $\mathrm{tr}(\cdot)$ is a *linear operator*.

It can be shown that the above four properties actually describe the trace operator *uniquely* — there is only one operator on matrices that satisfies properties 1, 2, 3 and 4. Let us find it.

$$\mathrm{tr}(M) = \sum_{k=1}^{n} m_{kk}. \qquad (\dagger)$$

In other words, *the trace of a square matrix is the sum of its diagonal entries*.

Equation $(\dagger)$ is the way that the trace is usually defined in textbooks. However, the properties 1,2,3 and 4 mentioned at the start of this answer generalize to infinite-dimensional vector spaces. Adam Merberg's answer in this thread contains more information in this regard. In any case, now you know *why* we define the trace of a matrix as $(\dagger)$.

Determinant of a Matrix

Determinant: A number describing a square matrix's characteristics, such as:

1. Matrix invertibility:
   If $det(A) \neq 0 \Longrightarrow invertible$
   If $det(A) = 0 \Longrightarrow non\text{−}invertible$

2. For linear transformations such as Ax, determinant determines the factor of change in length, area or volume!

   See: https://mathinsight.org/determinant_linear_transformation

For a matrix $[A]$, determinant is denoted by $|A|$ or $\det(A)$. interchangeably.
For a 2×2 matrix,

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Properties of Determinant:

1. $det(A) = 0$ when A is singular (non-invertible)
   A is singular when it has dependent rows/columns
2. $det(AB) = det(A) \times det(B)$
   Note: $det(A^{-1}) = \dfrac{1}{det(A)}$

3. because $A^{-1}A = 1$. (Note that if A is singular then A^{-1} does not exist

Ref:
1. 3Blue 1Brown~determinant: https://www.youtube.com/watch?v=Ip3X9LOh2dk
2. Strang (18.06): https://ocw.mit.edu/courses/mathematics/18-06sc-linear-algebra-fall-2011/least-squares-determinants-and-eigenvalues/properties-of-determinants/MIT18_06SCF11_Ses2.5sum.pdf