$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **7.1.4**

Lecture : **Least Squares**

Topic : **Least Squares**

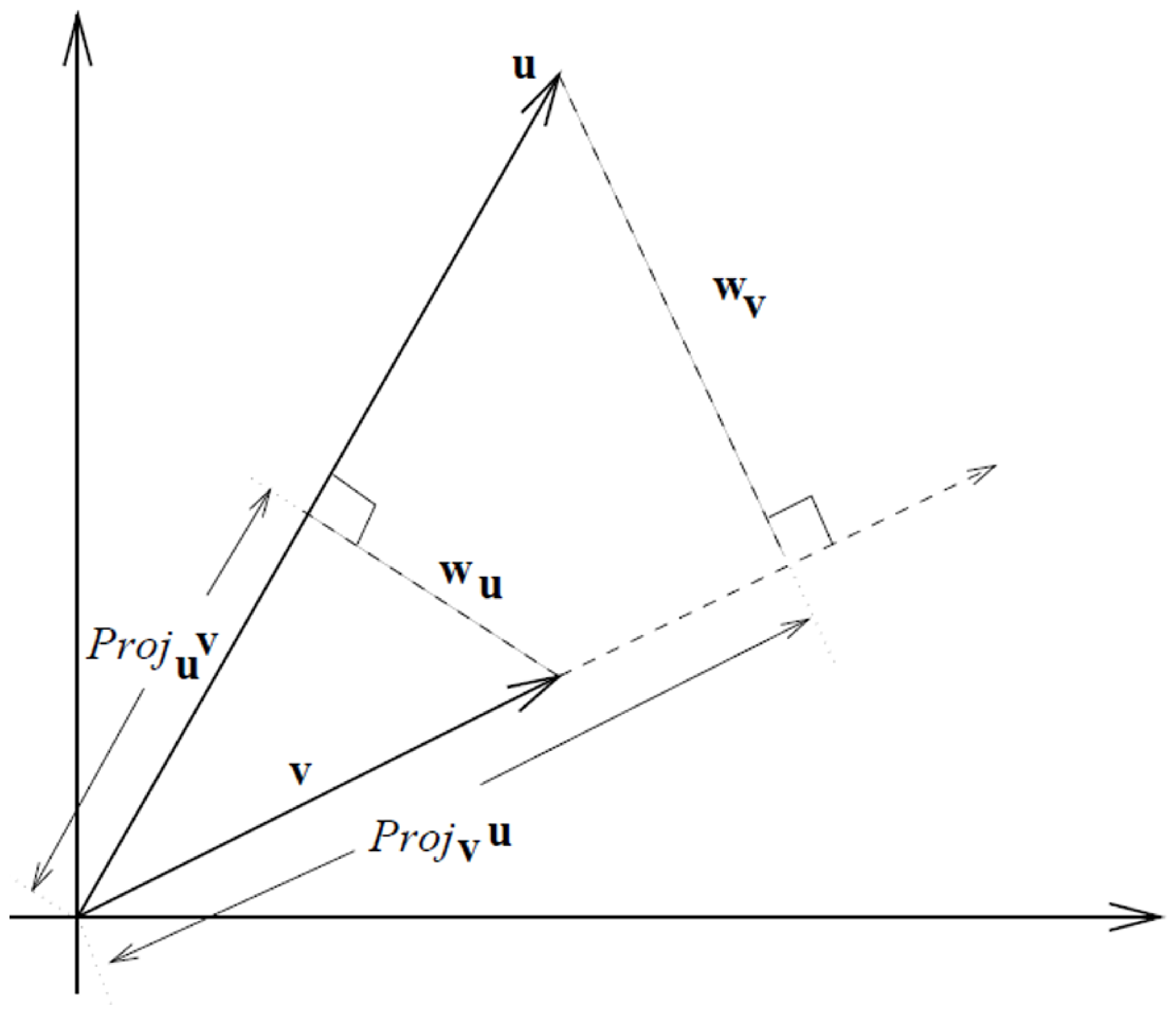
Concept : **Projection Matrix and its Properties**

Instructor: **A/P Chng Eng Siong**

TAs: **Zhang Su, Vishal Choudhari**

Rev: 2nd July 2020

Projection matrix for a space spanned by a single vector v



$$Proj_{\mathbf{v}}\mathbf{u} = \mathbf{v}\left(\frac{\mathbf{u}^T\mathbf{v}}{||\mathbf{v}||^2}\right),$$

Ref: 6.2.2_Orthogonal_Projections

$$\begin{aligned} proj_{\mathbf{v}}\mathbf{u} &= \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v} \\ &= \mathbf{v}\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}} \end{aligned}$$

Next, use the transpose definition of the inner product followed by the associative property of multiplication. Remember, when performing the dot product, a scalar multiplier may be placed anywhere you wish.

$$\begin{aligned} proj_{\mathbf{v}}\mathbf{u} &= \frac{1}{\mathbf{v}^T\mathbf{v}}\,\mathbf{v}(\mathbf{v}^T\mathbf{u}) \\ &= \frac{1}{\mathbf{v}^T\mathbf{v}}\,(\mathbf{v}\mathbf{v}^T)\mathbf{u} \\ &= \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\,\mathbf{u} \end{aligned}$$

The expression $\mathbf{v}\mathbf{v}^T$ is called an **outer product** (the transpose operator is outside the product versus its inside position in the inner product). If we define $P = \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}$, then the projection formula becomes

$$proj_{\mathbf{v}}\mathbf{u} = P\mathbf{u}, \text{ where } P = \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}.$$

The matrix P is called the projection matrix. You can project any vector onto the vector $\mathbf{v}$ by multiplying by the matrix P.

Ref: https://mse.redwoods.edu/darnold/math45/Activities/ProjectionMatrix/proj.pdf

Projection Matrix for col(A) and the Least Squares Solution

Consider solving the system of equations: $Ax = b$

- If b is not in the col(A), we find the least squares solution $\hat{x}$.

$\hat{x}$ satisfies: $A^T A\hat{x} = A^T b$

Thus, $\hat{x} = (A^T A)^{-1} A^T b$

Qn: **What is projection of b in the column space of A?**

Ans: $\hat{b} = proj_{Col(A)} b = A\hat{x}$

NOTE:

$$\hat{b} = proj_{Col(A)} b = A\hat{x}$$

$$\Longrightarrow \hat{b} = proj_{Col(A)} b = A(A^T A)^{-1} A^T\, b$$

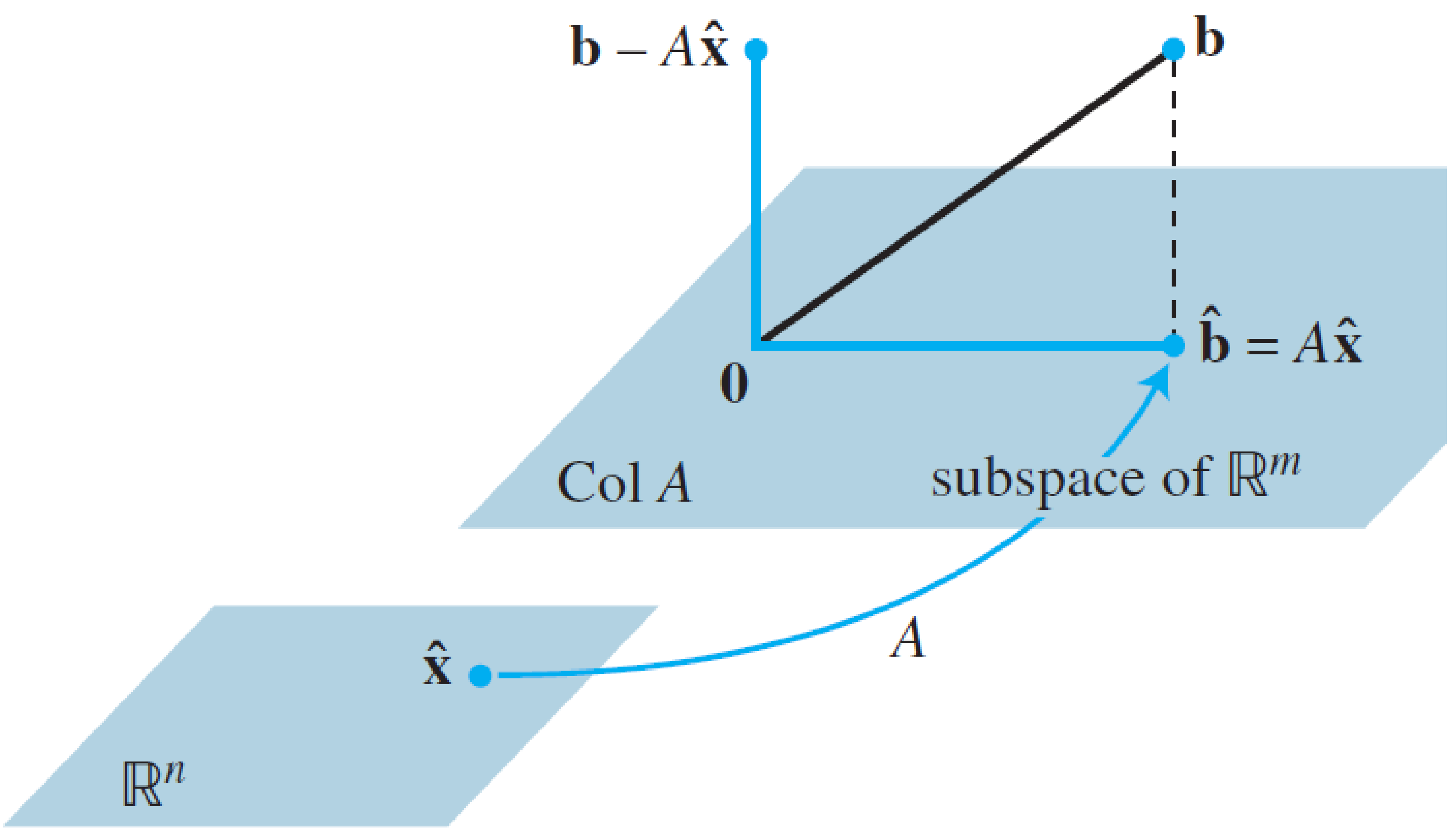


FIGURE 2 The least-squares solution $\hat{\mathbf{x}}$ is in $\mathbb{R}^n$.

$P = A(A^T A)^{-1} A^T$ is the **projection** matrix

The **vector** b can be **projected** into **column space of** A by **multiplying** it with **projection** matrix P.

Projection matrix P maps the actual response values b with predicted values $\hat{b}$.

Ref: https://en.wikipedia.org/wiki/Projection_matrix

Lay, Linear Algebra and its Applications (4th Edition)

360 CHAPTER 6 Orthogonality and Least Squares

Properties of Projection Matrix

When P is multiplied by vector b, the resulting vector $\hat{b} = Pb = A\hat{x}$ is the least squares (nearest) solution in the column space of A.

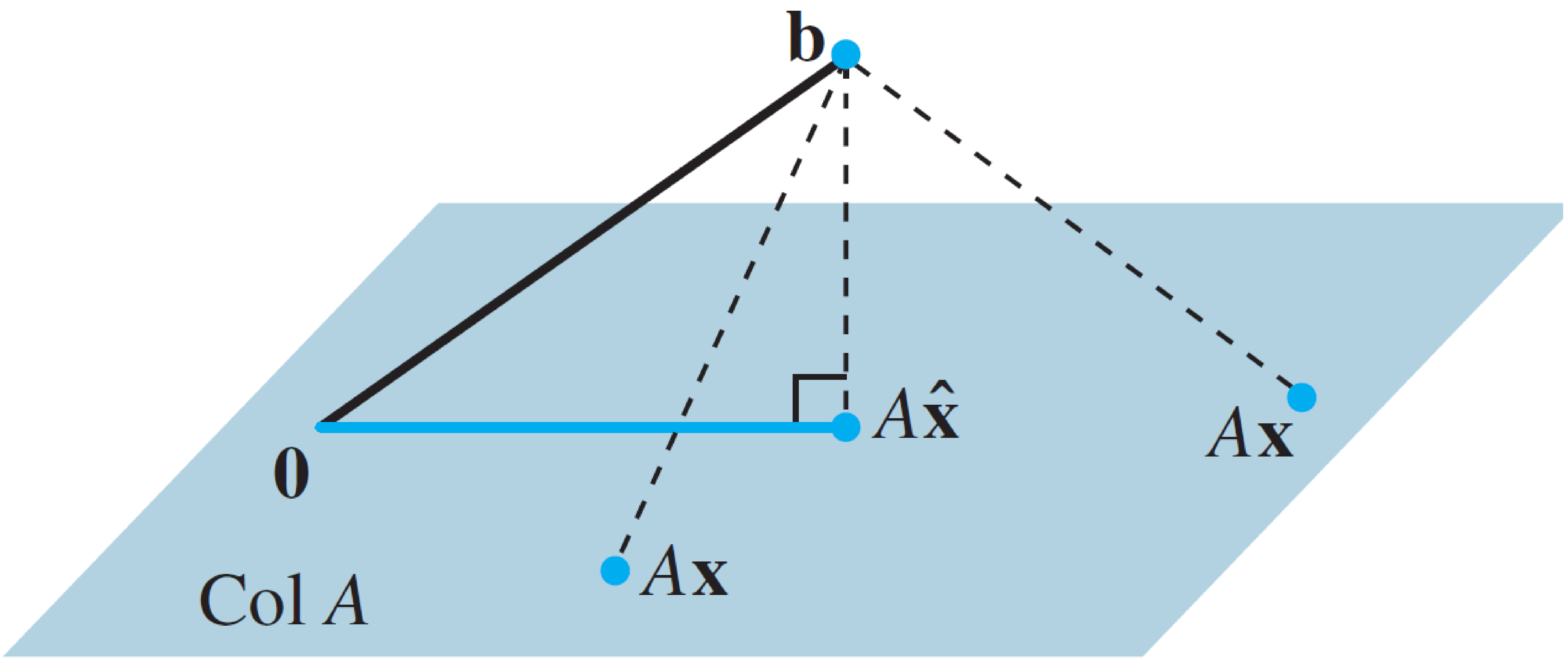


FIGURE 1 The vector **b** is closer to $A\hat{\mathbf{x}}$ than to $A\mathbf{x}$ for other **x**.

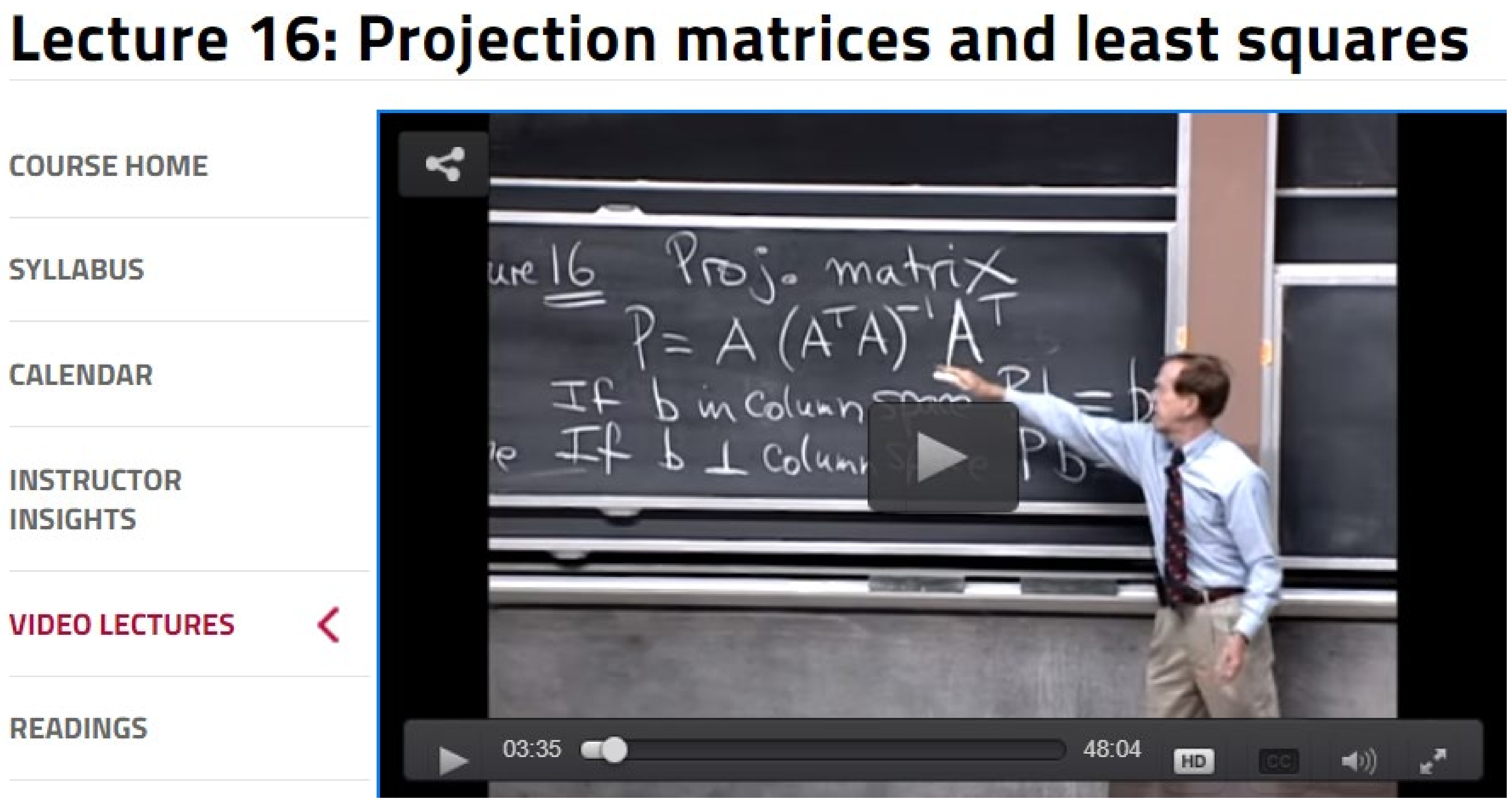
Lay, Linear Algebra and its Applications (4th Edition)

360 CHAPTER 6 Orthogonality and Least Squares

Properties of Projection Matrix:

1. $P^T = P$
2. $P^N = P$ [Idempotent Property]

Properties of Projection Matrix



Ref: https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/lecture-16-projection-matrices-and-least-squares/

Properties of Projection Matrix:

1. $P^T = P$
2. $P^N = P$ [Idempotent Property]

Example 2: Projection Matrix Properties

4.3 *Projection matrices.* A matrix $P \in \mathbf{R}^{n\times n}$ is called a *projection matrix* if $P = P^T$ and $P^2 = P$.

(a) Show that if P is a projection matrix then so is $I - P$.

(b) Suppose that the columns of $U \in \mathbf{R}^{n\times k}$ are orthonormal. Show that UU^T is a projection matrix. (Later we will show that the converse is true: every projection matrix can be expressed as UU^T for some U with orthonormal columns.)

(c) Suppose $A \in \mathbf{R}^{n\times k}$ is full rank, with $k \le n$. Show that $A(A^TA)^{-1}A^T$ is a projection matrix.

(d) If $S \subseteq \mathbf{R}^n$ and $x \in \mathbf{R}^n$, the point y in S closest to x is called the *projection of x on S*. Show that if P is a projection matrix, then $y = Px$ is the projection of x on $\mathcal{R}(P)$. (Which is why such matrices are called projection matrices ...)

Ref: Boyd EE263 Homework 3, Q4.3:
https://web.stanford.edu/class/archive/ee/ee263/ee263.1082/hw/hw3sol.pdf

Example 1

Solution:

(a) To show that $I-P$ is a projection matrix we need to check two properties:

i. $I-P=(I-P)^T$

ii. $(I-P)^2=I-P$.

The first one is easy: $(I-P)^T=I-P^T=I-P$ because $P=P^T$ (P is a projection matrix.) The show the second property we have

$$\begin{aligned}(I-P)^2 &= I-2P+P^2\\ &= I-2P+P \qquad \text{(since } P=P^2\text{)}\\ &= I-P\end{aligned}$$

and we are done.

(b) Since the columns of U are orthonormal we have $U^TU=I$. Using this fact it is easy to prove that UU^T is a projection matrix, *i.e.*, $(UU^T)^T=UU^T$ and $(UU^T)^2=UU^T$. Clearly, $(UU^T)^T=(U^T)^TU^T=UU^T$ and

$$\begin{aligned}(UU^T)^2 &= (UU^T)(UU^T)\\ &= U(U^TU)U^T\\ &= UU^T \qquad \text{(since } U^TU=I\text{)}.\end{aligned}$$

(c) First note that $\left(A(A^TA)^{-1}A^T\right)^T=A(A^TA)^{-1}A^T$ because

$$\begin{aligned}\left(A(A^TA)^{-1}A^T\right)^T &= (A^T)^T\left((A^TA)^{-1}\right)^TA^T\\ &= A\left((A^TA)^T\right)^{-1}A^T\\ &= A(A^TA)^{-1}A^T.\end{aligned}$$

Also $\left(A(A^TA)^{-1}A^T\right)^2=A(A^TA)^{-1}A^T$ because

$$\begin{aligned}\left(A(A^TA)^{-1}A^T\right)^2 &= \left(A(A^TA)^{-1}A^T\right)\left(A(A^TA)^{-1}A^T\right)\\ &= A\left((A^TA)^{-1}A^TA\right)(A^TA)^{-1}A^T\\ &= A(A^TA)^{-1}A^T \qquad \text{(since } (A^TA)^{-1}A^TA=I\text{)}.\end{aligned}$$

(d) To show that Px is the projection of x on $\mathcal{R}(P)$ we verify that the "error" $x-Px$ is orthogonal to *any* vector in $\mathcal{R}(P)$. Since $\mathcal{R}(P)$ is nothing but the span of the columns of P we only need to show that $x-Px$ is orthogonal to the columns of P, or in other words, $P^T(x-Px)=0$. But

$$\begin{aligned}P^T(x-Px) &= P(x-Px) \qquad \text{(since } P=P^T\text{)}\\ &= Px-P^2x\\ &= 0 \qquad \text{(since } P^2=P\text{)}\end{aligned}$$

and we are done.

Ref: Boyd EE263 Homework 3, Q4.3:
https://web.stanford.edu/class/archive/ee/ee263/ee263.1082/hw/hw3sol.pdf