$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **7.1.1**

Lecture : **Least Squares**

Topic : **Introduction**

Concept : **Consistency in a System of Equations**

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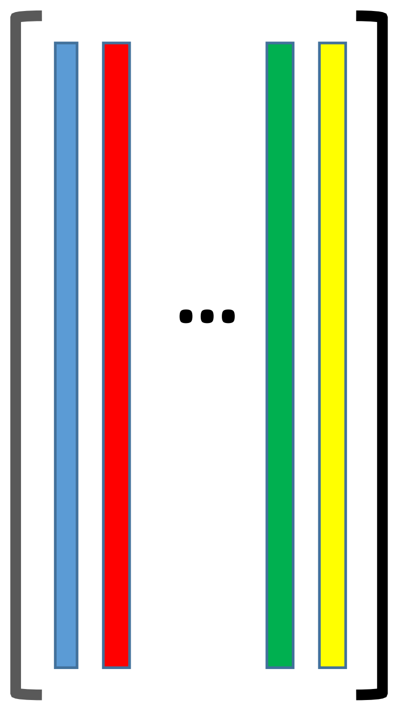
Consistency in a System of Equations

Consider solving the system of equations: $Ax = b$

Note:

- Matrix $A \in R^{M \times N}$, where
  - M denotes no. of rows/equations
  - N denotes no. of columns/unknowns
- $x \in R^N$
- $b \in R^M$
- The above system of equations can either be
  1. consistent (or)
  2. inconsistent.

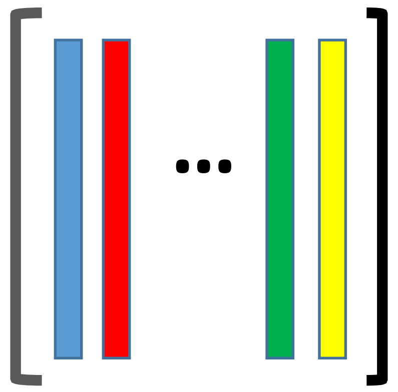
Based on M & N, there exist three cases:



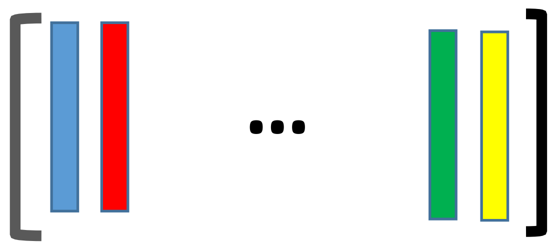
$M \gg N$

More equations,
less unknowns.

Hence, **over-determined**!



$M \approx N$



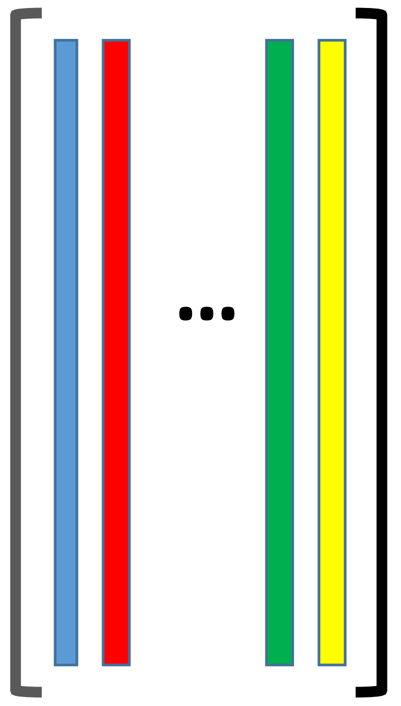
$M \ll N$

Less equations,
more unknowns.

Hence, **under-determined**!

Consistency in a System of Equations

$$Ax = b$$



$$M \gg N$$

Consistent (solutions exist)

- b is in column space of A, i.e,
  b is formed by linear combinations of A's columns.
- Rank (A) = Rank ($A \mid b$), i.e,
  rank of A is same as that of the augmented matrix.

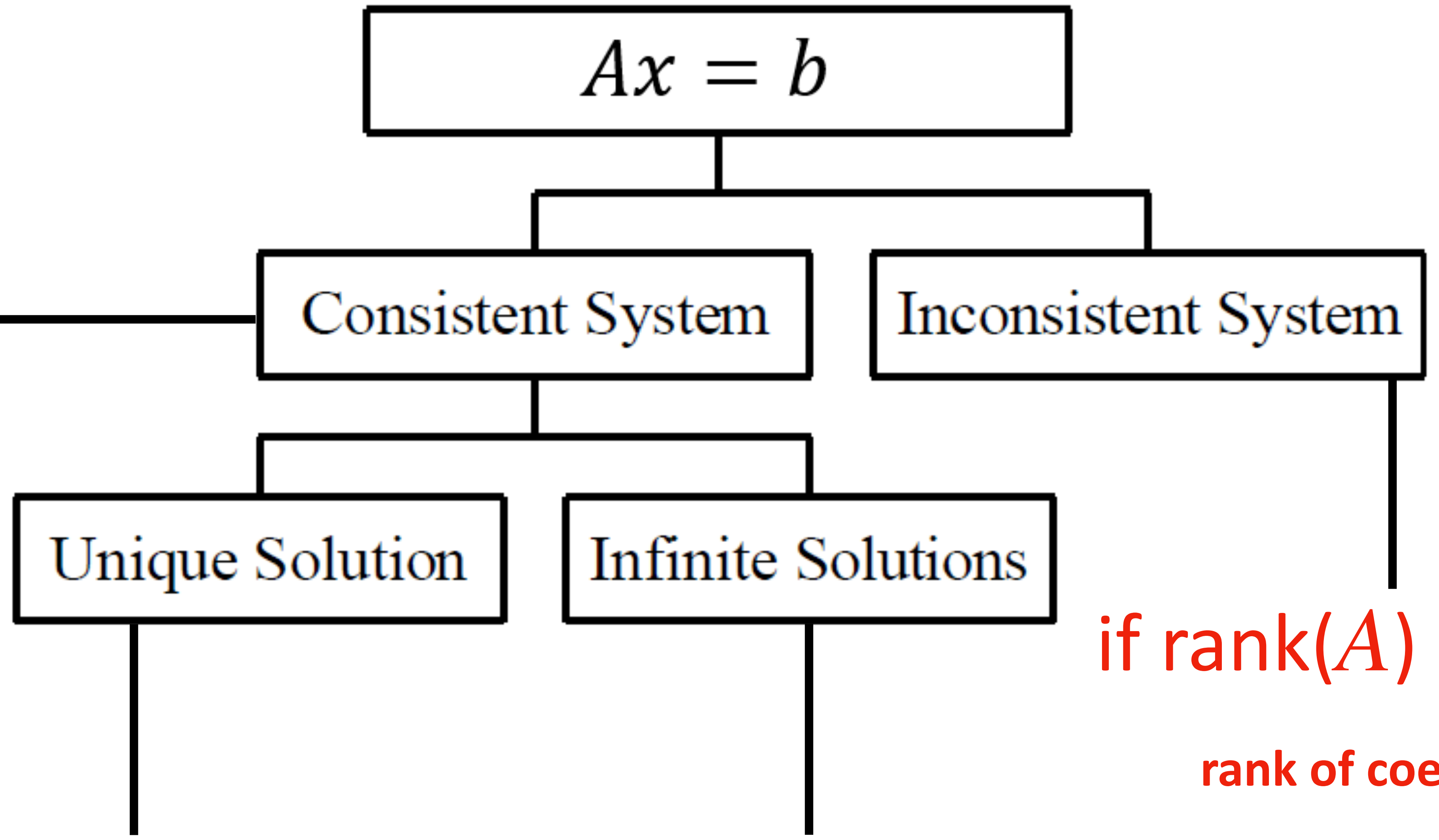
Inconsistent (solutions don't exist)

- b is NOT in column space of A, i.e,
  b is NOT formed by linear combinations of A's columns.
- Occurs when $M \gg N$ (**over-determined**), i.e,
  there exist more equations than unknowns.
- The rows of A are dependent but,
  their corresponding b values are not consistent.
- Rank (A) < Rank ($A \mid b$), i.e,
  rank of A is less than that of the augmented matrix.

Consistency in a System of Equations

A system of equations can be consistent or inconsistent. What does that mean?

A system of equations $Ax = b$ is consistent if there is a solution, and it is inconsistent if there is no solution. However, consistent system of equations does not mean a unique solution, that is, a consistent system of equation may have a unique solution or infinite solutions.



if rank(A) = rank(A | b)

rank of coefficient matrix
=
rank of augmented matrix

if rank(A) < rank(A | b)

rank of coefficient matrix
<
rank of augmented matrix

if rank(A) = N

rank of coefficient matrix
=
rank of augmented matrix
=
no. of unknowns

if rank(A) < N

rank of coefficient matrix
=
rank of augmented matrix
<
no. of unknowns

NOTE: Rank (A) is the maximum number of independent rows or columns of A.

You can find number of independent row or columns by:
1. row reduction process
2. rank(A) in MATLAB

Ref: https://en.wikipedia.org/wiki/Augmented_matrix
Ref: https://www.mathsisfun.com/algebra/matrix-rank.html

Note: rank (A) > rank (A | b) is never possible. Why?

Examples

$\text{rank}(A) = \text{rank}(A\,|\,b) = N$

Consistent and Unique Solution

a) The system of equations

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

is a consistent system of equations as it has a unique solution, that is,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

```
>> A_b = [ 2 4 6; 1 3 4]

A_b =

     2     4     6
     1     3     4

>> rank(A_b)

ans =

     2
```

Inconsistent and No solutions Exist

c) The system of equations

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$\text{rank}(A) < \text{rank}(A\,|\,b)$

```
>> A = [2 4; 1 2]

A =

     2     4
     1     2

>> rank(A)

ans =

     1
```

```
>> A_b = [ 2 4 6; 1 2 4]

A_b =

     2     4     6
     1     2     4

>> rank(A_b)

ans =

     2
```

Consistent and Having Infinite Solutions

b) The system of equations

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

is also a consistent system of equations but it has infinite solutions as given as follows.

Expanding the above set of equations,

$$2x + 4y = 6$$
$$x + 2y = 3$$

you can see that they are the same equation. Hence any combination of (x, y) that satisfies

$$2x + 4y = 6$$

is a solution. For example $(x, y) = (1,1)$ is a solution and other solutions include $(x, y) = (0.5,1.25)$, $(x, y) = (0,\ 1.5)$ and so on.

```
>> A_b = [ 2 4 6; 1 2 3]

A_b =

     2     4     6
     1     2     3

>> rank(A_b)

ans =

     1
```

$\text{rank}(A) = \text{rank}(A\,|\,b) < N$

Ref: Introduction to Matrix Algebra, Autar Kaw (pg 68)