$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.4.5**

Lecture : **Eigen and Singular Values**

Topic : **SVD & Pseudoinverse**

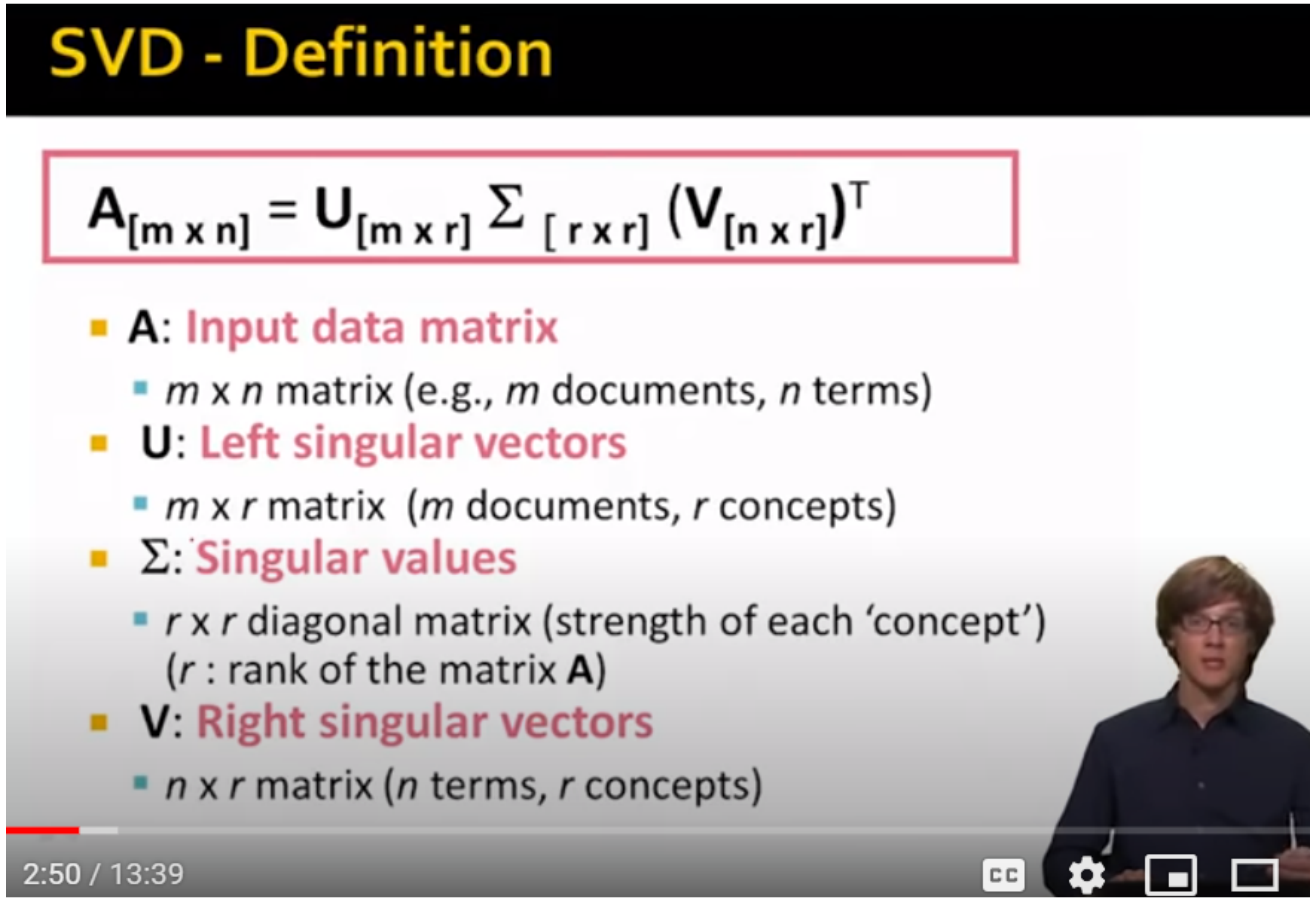
Concept : **SVD Applications and References**

Instructor: **A/P Chng Eng Siong**

TAs: **Zhang Su, Vishal Choudhari**

Rev: 27th July 2020

SVD Applications

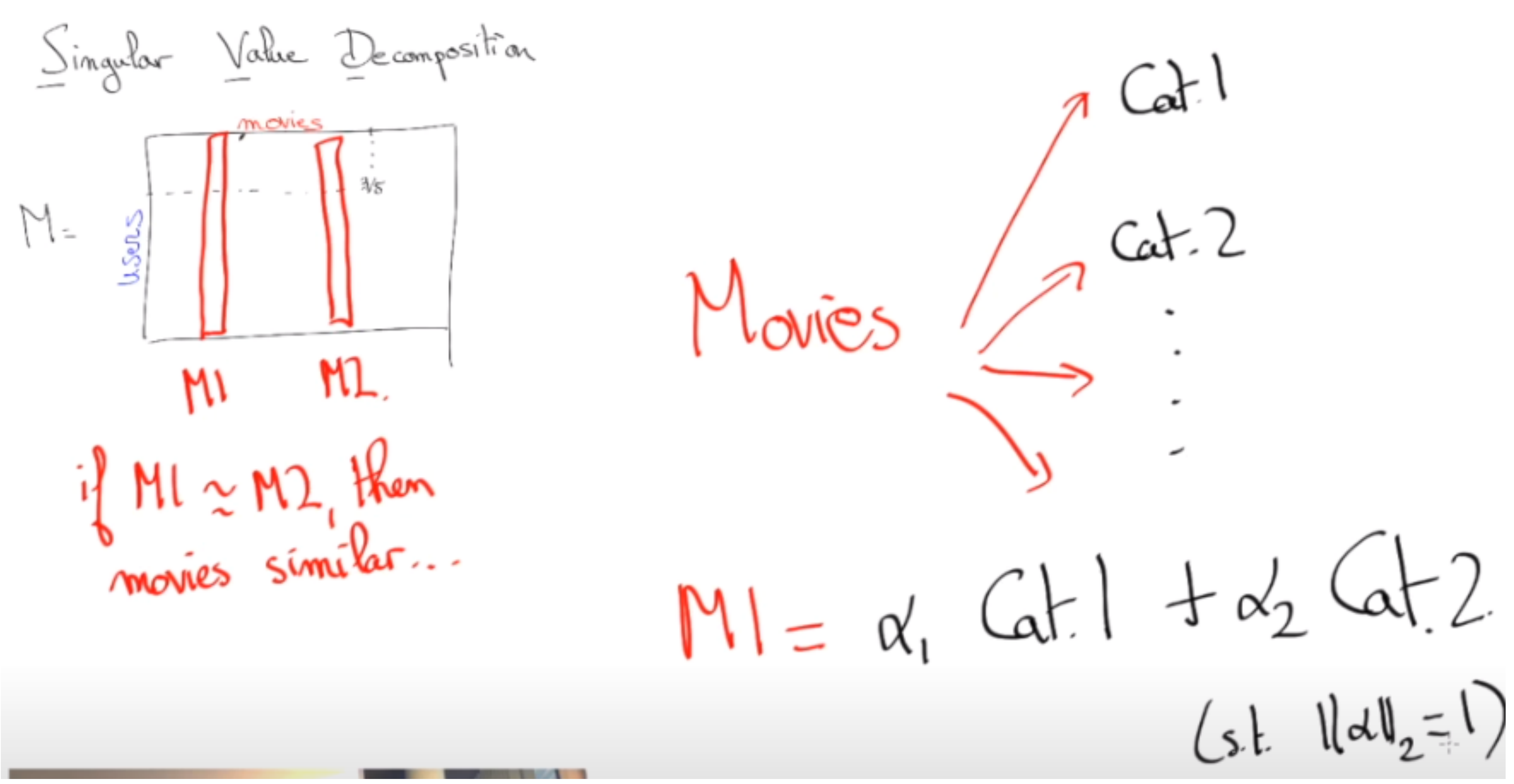


Stanford University: Lecture 47 - SVD

Ref: https://www.youtube.com/watch?v=P5mlg91as1c

Image Compression

Ref: https://www.youtube.com/watch?v=H7qMMudo3e8

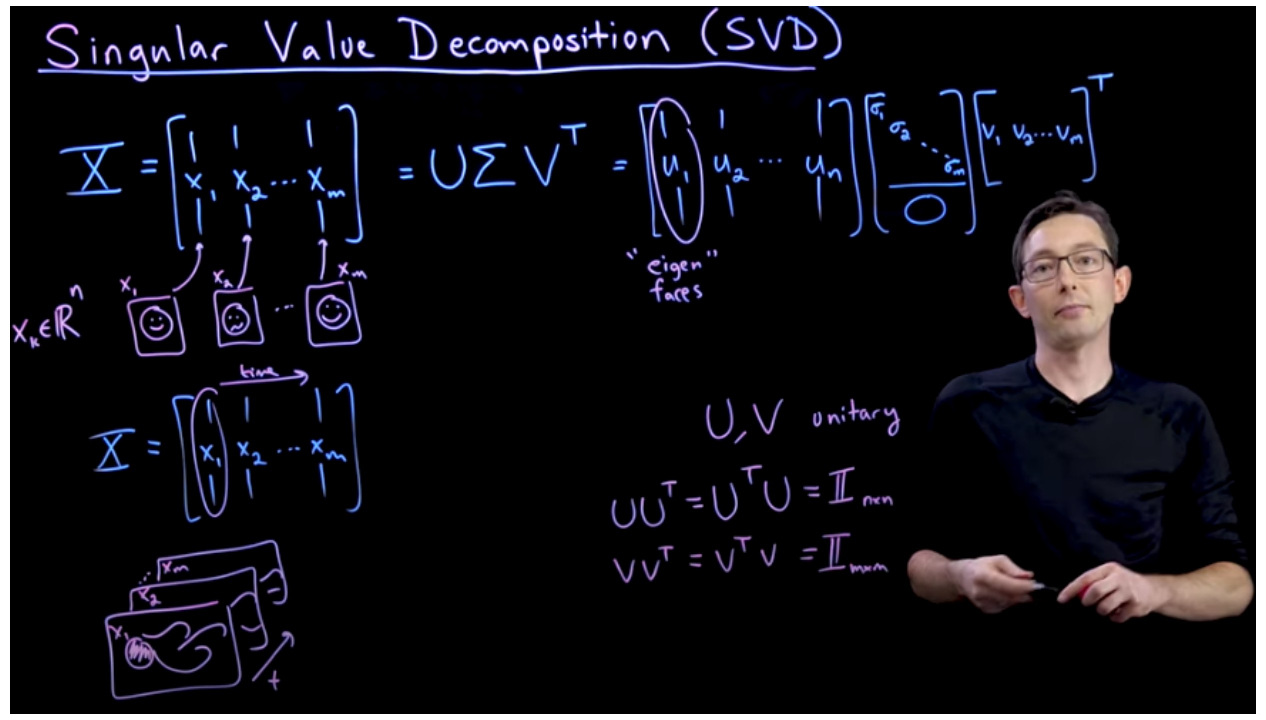


EPFL: Classification of Movies

Ref: https://www.youtube.com/watch?v=CQbbsKK1kus

SVD References

Steve Brunton: SVD - Math Overview

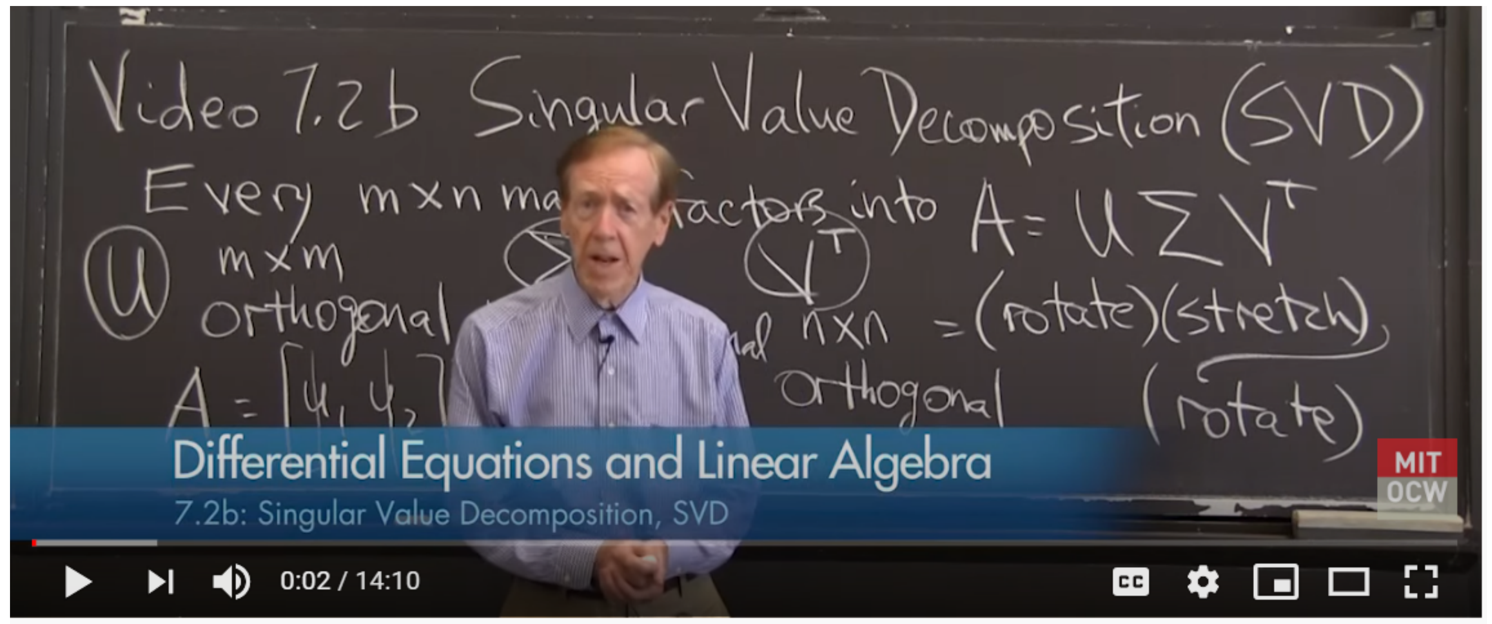


Ref: https://www.youtube.com/watch?v=nbBvuuNVfco

Steve Brunton: SVD - Matrix Approximation

Ref: https://www.youtube.com/watch?v=02QCtHM1qb4

Gilbert Strang: SVD



Singular Value Decomposition (the SVD)

Ref: https://www.youtube.com/watch?v=mBcLRGuAFUk

More References

1. **Cornell:**
    a. https://www.cs.cornell.edu/courses/cs322/2008sp/stuff/TrefethenBau_Lec4_SVD.pdf
    b. https://www.cs.cornell.edu/courses/cs3220/2010sp/notes/svd.pdf

2. **Stanford:**
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    b. citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.725.8741

3. **Dan Kalman:**
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4. **Max Planck:**
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Appendix: SVD and $A^T A$

Singular Values

Since matrix products of the form A^TA will play an important role in our work, we will begin with two basic theorems about them.

THEOREM 9.4.1 *If A is an $m \times n$ matrix, then:*

(*a*) *A and A^TA have the same null space.*

(*b*) *A and A^TA have the same row space.*

(*c*) *A^T and A^TA have the same column space.*

(*d*) *A and A^TA have the same rank.*

We will prove part (*a*) and leave the remaining proofs for the exercises.

Proof (*a*) We must show that every solution of $A\mathbf{x} = \mathbf{0}$ is a solution of $A^TA\mathbf{x} = \mathbf{0}$, and conversely. If $\mathbf{x}_0$ is any solution of $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}_0$ is also a solution of $A^TA\mathbf{x} = \mathbf{0}$ since

$$A^TA\mathbf{x}_0 = A^T(A\mathbf{x}_0) = A^T\mathbf{0} = \mathbf{0}$$

Conversely, if $\mathbf{x}_0$ is any solution of $A^TA\mathbf{x} = \mathbf{0}$, then $\mathbf{x}_0$ is in the null space of A^TA and hence is orthogonal to all vectors in the row space of A^TA by part (*q*) of Theorem 4.8.8. However, A^TA is symmetric, so $\mathbf{x}_0$ is also orthogonal to every vector in the column space of A^TA. In particular, $\mathbf{x}_0$ must be orthogonal to the vector $(A^TA)\mathbf{x}_0$; that is,

$$\mathbf{x}_0 \cdot (A^TA)\mathbf{x}_0 = 0$$

Using the first formula in Table 1 of Section 3.2 and properties of the transpose operation we can rewrite this as

$$\mathbf{x}_0^T(A^TA)\mathbf{x}_0 = (A\mathbf{x}_0)^T(A\mathbf{x}_0) = (A\mathbf{x}_0) \cdot (A\mathbf{x}_0) = \|A\mathbf{x}_0\|^2 = 0$$

which implies that $A\mathbf{x}_0 = \mathbf{0}$, thereby proving that $\mathbf{x}_0$ is a solution of $A\mathbf{x}_0 = \mathbf{0}$. ◀

Anton, Elementary Linear Algebra: Applications Version (11th Edition)

THEOREM 9.4.2 *If A is an $m \times n$ matrix, then:*

(*a*) *A^TA is orthogonally diagonalizable.*

(*b*) *The eigenvalues of A^TA are nonnegative.*

Proof (*a*) The matrix A^TA, being symmetric, is orthogonally diagonalizable by Theorem 7.2.1.

Proof (*b*) Since A^TA is orthogonally diagonalizable, there is an orthonormal basis for R^n consisting of eigenvectors of A^TA, say $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. If we let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues, then for $1 \le i \le n$ we have

$$\|A\mathbf{v}_i\|^2 = A\mathbf{v}_i \cdot A\mathbf{v}_i = \mathbf{v}_i \cdot A^TA\mathbf{v}_i \qquad \text{[Formula (26) of Section 3.2]}$$
$$= \mathbf{v}_i \cdot \lambda_i\mathbf{v}_i = \lambda_i(\mathbf{v}_i \cdot \mathbf{v}_i) = \lambda_i\|\mathbf{v}_i\|^2 = \lambda_i$$

It follows from this relationship that $\lambda_i \ge 0$. ◀

Appendix: SVD and $A^T A$

THEOREM 7.2.1 *If A is an $n \times n$ matrix with real entries, then the following are equivalent.*

(a) *A is orthogonally diagonalizable.*

(b) *A has an orthonormal set of n eigenvectors.*

(c) *A is symmetric.*

Proof **(a) $\Rightarrow$ (b)** Since A is orthogonally diagonalizable, there is an orthogonal matrix P such that $P^{-1}AP$ is diagonal. As shown in Formula (2) in the proof of Theorem 5.2.1, the n column vectors of P are eigenvectors of A. Since P is orthogonal, these column vectors are orthonormal, so A has n orthonormal eigenvectors.

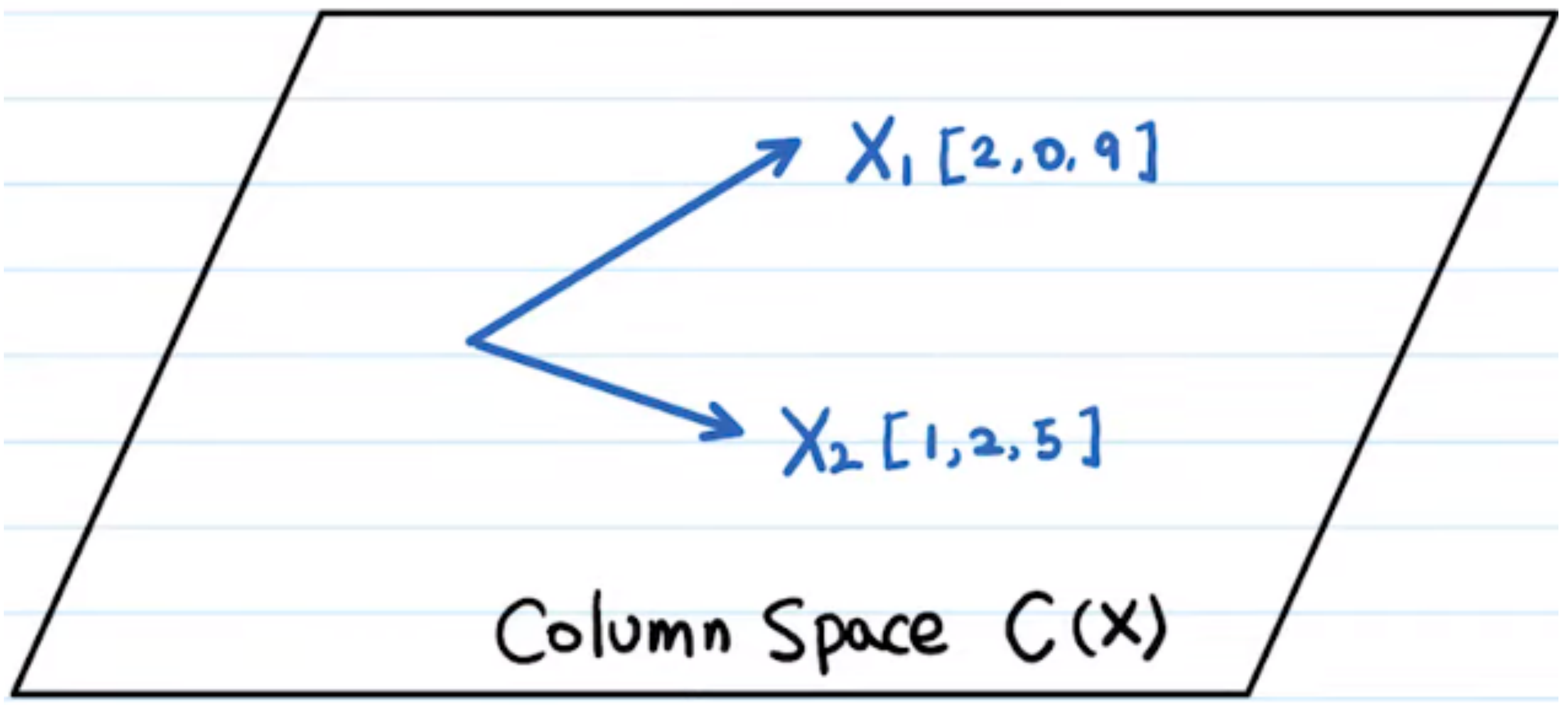
(b) $\Rightarrow$ (a) Assume that A has an orthonormal set of n eigenvectors $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$. As shown in the proof of Theorem 5.2.1, the matrix P with these eigenvectors as columns diagonalizes A. Since these eigenvectors are orthonormal, P is orthogonal and thus orthogonally diagonalizes A.

(a) $\Rightarrow$ (c) In the proof that (a) $\Rightarrow$ (b) we showed that an orthogonally diagonalizable $n \times n$ matrix A is orthogonally diagonalized by an $n \times n$ matrix P whose columns form an orthonormal set of eigenvectors of A. Let D be the diagonal matrix

$$D = P^T A P$$

Anton, Elementary Linear Algebra: Applications Version (11th Edition)

Appendix: Row and Column Spaces



Ref: https://en.wikipedia.org/wiki/Row_and_column_spaces

Row and column spaces

From Wikipedia, the free encyclopedia

In linear algebra, the **column space** (also called the **range** or **image**) of a matrix *A* is the span (set of all possible linear combinations) of its column vectors. The column space of a matrix is the image or range of the corresponding matrix transformation.

Let $\mathbb{F}$ be a field. The column space of an $m \times n$ matrix with components from $\mathbb{F}$ is a linear subspace of the m-space $\mathbb{F}^m$. The dimension of the column space is called the rank of the matrix and is at most min(m, n).[1] A definition for matrices over a ring $\mathbb{K}$ is also possible.

The **row space** is defined similarly.

This article considers matrices of real numbers. The row and column spaces are subspaces of the real spaces $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively.[2]

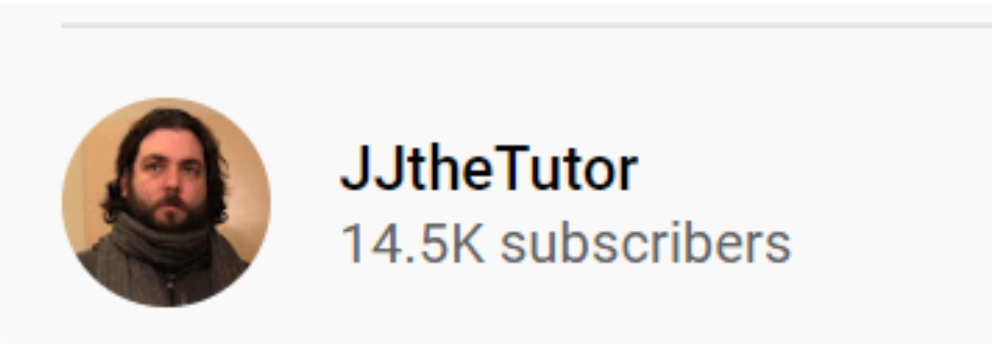
$$\begin{bmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{bmatrix}$$

The row vectors of a matrix. The row space of this matrix is the vector space generated by linear combinations of the row vectors.

$$\begin{bmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{bmatrix}$$

Appendix: Mechanics of Computing SVD

Finding the SVD of a 3×2 matrix!



$$A = U\Sigma V^T, \qquad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$V^T = eigenvectors(A^T A)^T = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Ref: https://www.youtube.com/watch?v=Ls2TgGFfZnU

NOTE: We are not keen on the mechanics!