$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.4.2**

Lecture : **Eigen and Singular Values**

Topic : **SVD & Pseudoinverse**

Concept : **Importance of SVD**

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Importance of SVD

Why is SVD important?

- It gives us the **dimensions of the fundamental subspaces**
- It lets us **compute various norms**
- It tells about **sensitivity of linear systems**
- It gives us optimal solutions to **least-squares linear systems**
- It gives us the **least-error rank-k decomposition**
- **Every matrix has one**

What does **rank the of a matrix** A tell?

1. Number of non-zero singular values of A.
2. Number of independent rows or columns of A.

THEOREM 9.4.4 Singular Value Decomposition (Expanded Form)

If A is an $m \times n$ matrix of rank k, then A can be factored as

$$A = U\Sigma V^T = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_k \mid \mathbf{u}_{k+1} \quad \cdots \quad \mathbf{u}_m] \left[\begin{array}{cccc|c} \sigma_1 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & \cdots & 0 & 0_{k\times(n-k)} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & \sigma_k & \\ \hline & & 0_{(m-k)\times k} & & 0_{(m-k)\times(n-k)} \end{array}\right] \left[\begin{array}{c} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_k^T \\ \hline \mathbf{v}_{k+1}^T \\ \vdots \\ \mathbf{v}_n^T \end{array}\right]$$

in which U, Σ, and V have sizes $m \times m$, $m \times n$, and $n \times n$, respectively, and in which:

(*a*) $V = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]$ *orthogonally diagonalizes* A^TA.

(*b*) *The nonzero diagonal entries of Σ are $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}, \ldots, \sigma_k = \sqrt{\lambda_k}$, where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the nonzero eigenvalues of A^TA corresponding to the column vectors of V.*

(*c*) *The column vectors of V are ordered so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$.*

(*d*) $\mathbf{u}_i = \dfrac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|} = \dfrac{1}{\sigma_i}A\mathbf{v}_i \quad (i = 1, 2, \ldots, k)$

(*e*) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ *is an orthonormal basis for* $\operatorname{col}(A)$.

(*f*) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_m\}$ *is an extension of* $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ *to an orthonormal basis for* R^m.

Ref: https://www.mpi-inf.mpg.de/fileadmin/inf/d5/teaching/ss17_dmm/lectures/2017-05-08-lin_alg_and_svd.pdf

Anton, Elementary Linear Algebra: Applications Version (11th Edition)

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SVD and the Four Fundamental Subspaces

Singular Value Decomposition

SVD is a decomposition of rectangular $m \times n$ matrix A as

- $A = U \Sigma V^T$ where
- U is an $m \times m$ orthogonal matrix with Eigenvectors of AA^T
- Σ is an diagonal $m \times n$ matrix with Eigenvalues of both $A^T A$ and AA^T
- V is an $n \times n$ orthogonal matrix with Eigenvectors of $A^T A$

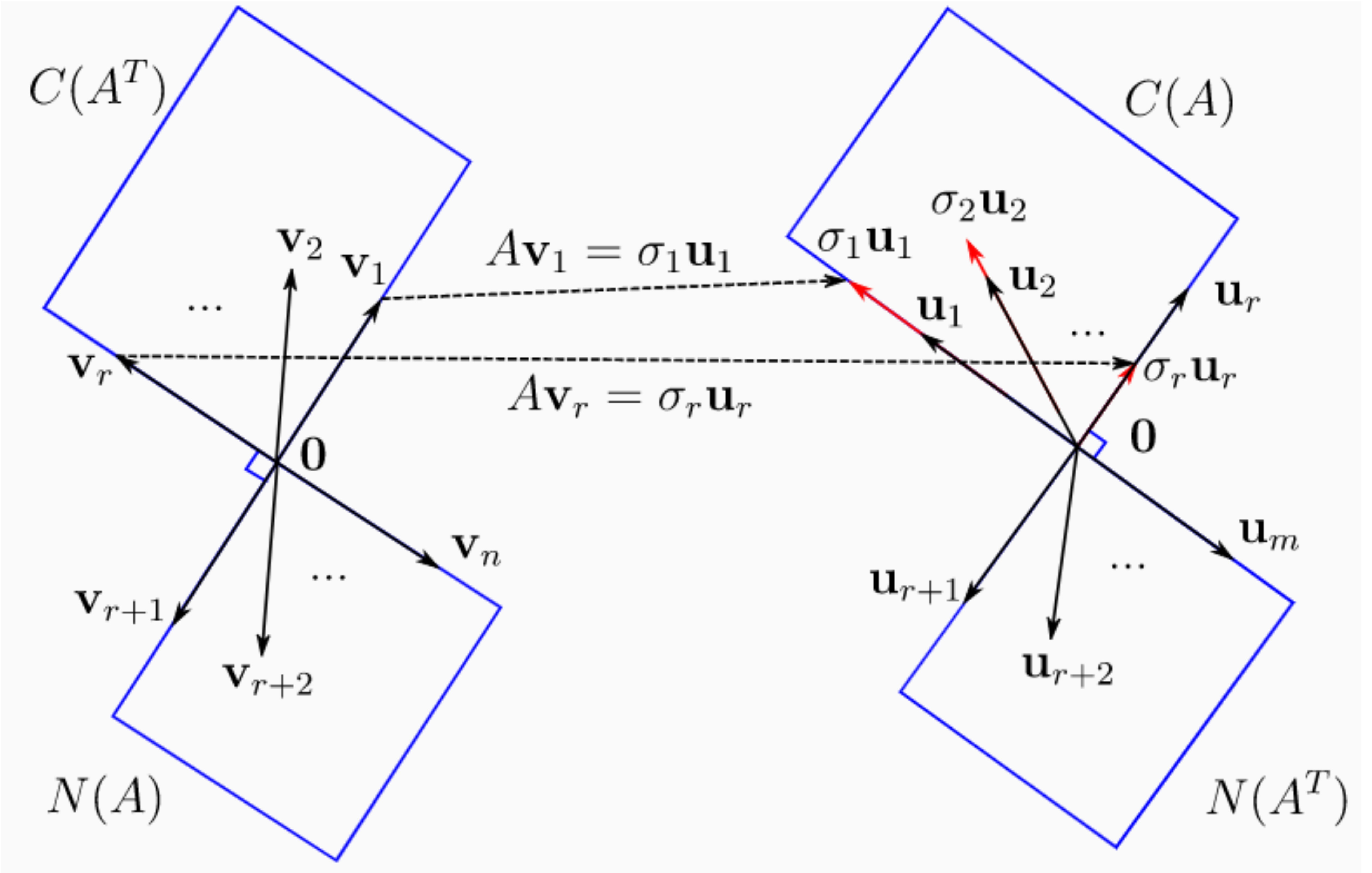
- Singular vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are in the row space of A $= C(A^T)$
- applying A to $\mathbf{v}_i$ gives $A\mathbf{v}_i = \sigma_i \mathbf{u}_i$
- $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are in the column space of A
- Singular values $\sigma_1, \ldots, \sigma_r$ are all positive numbers
- so V and U diagonalize A:
- $A\mathbf{v}_i = \sigma_i \mathbf{u}_i \Rightarrow AV = U\Sigma \implies A = U\Sigma V^{-1} = U\Sigma V^T$
- The singular values σ_i in Σ are arranged in monotonic non-increasing order

Note: As V is an orthogonal matrix, $V^{-1} = V^T$.

Note: Rank of matrix A = r.

Orthogonal Basis for the Four Fundamental Subspaces

But it's not only a decomposition, but a way of finding the bases for the Four Fundamental Subspaces of A:

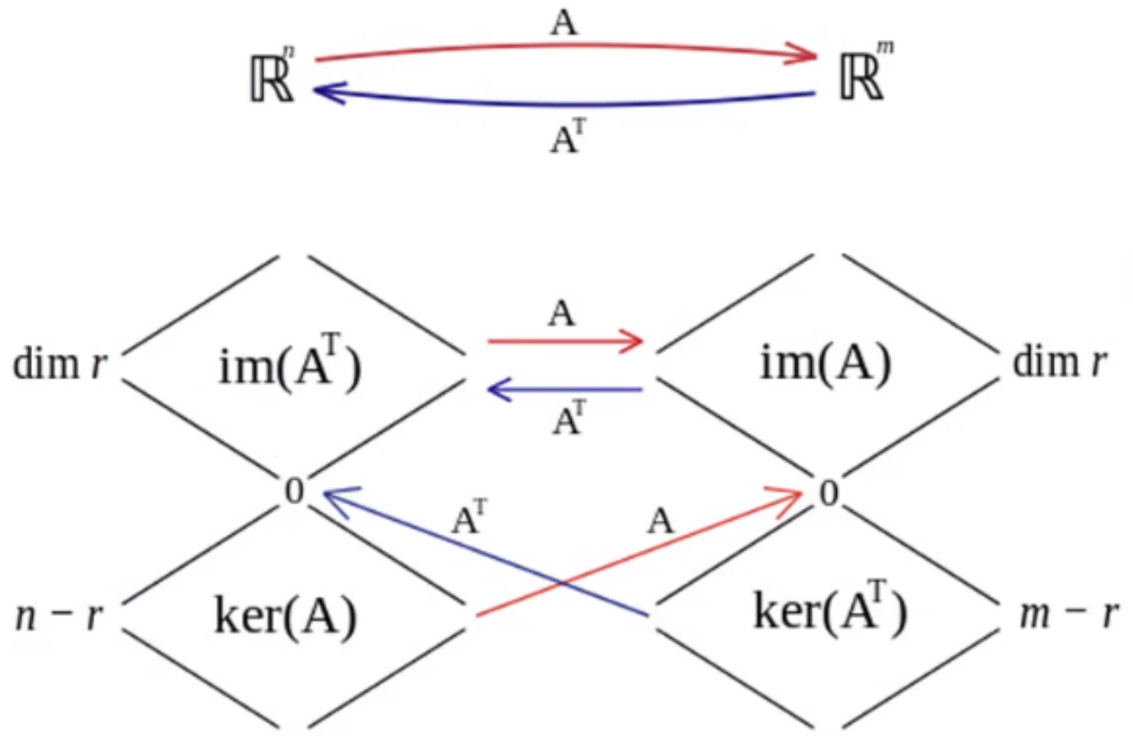


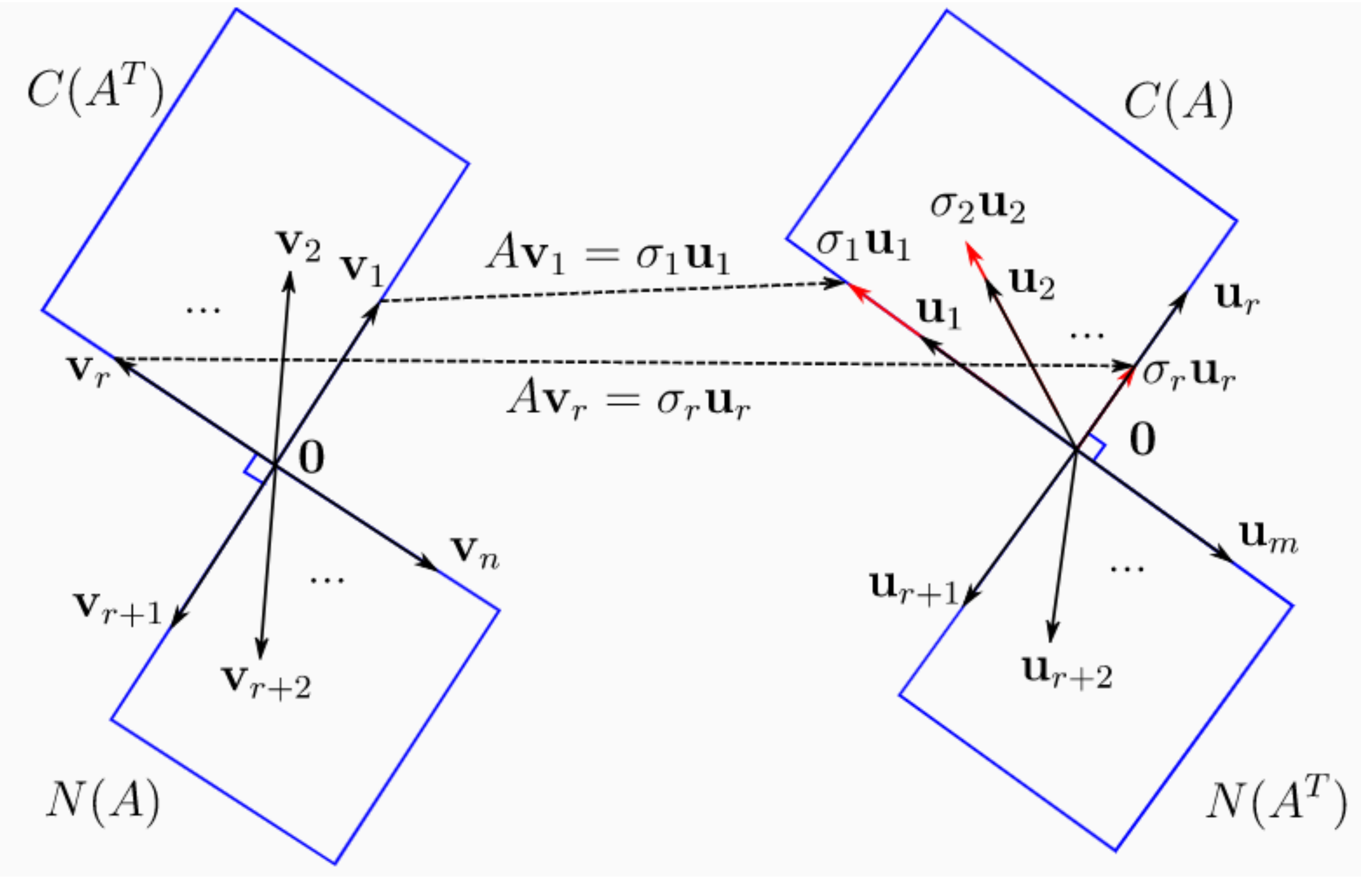
Ref: mlwiki.org/index.php/Singular_Value_Decomposition#Orthogonal_Basis_for_the_Four_Fundamental_Subspaces

SVD and the Four Fundamental Subspaces

Gilbert Strang: Four Fundamental Subspaces of Linear Algebra

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & \ldots & n \end{array} \\
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
a_{31} & a_{32} & \ldots & a_{3n} \\
\vdots & \vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}
\end{array}
$$



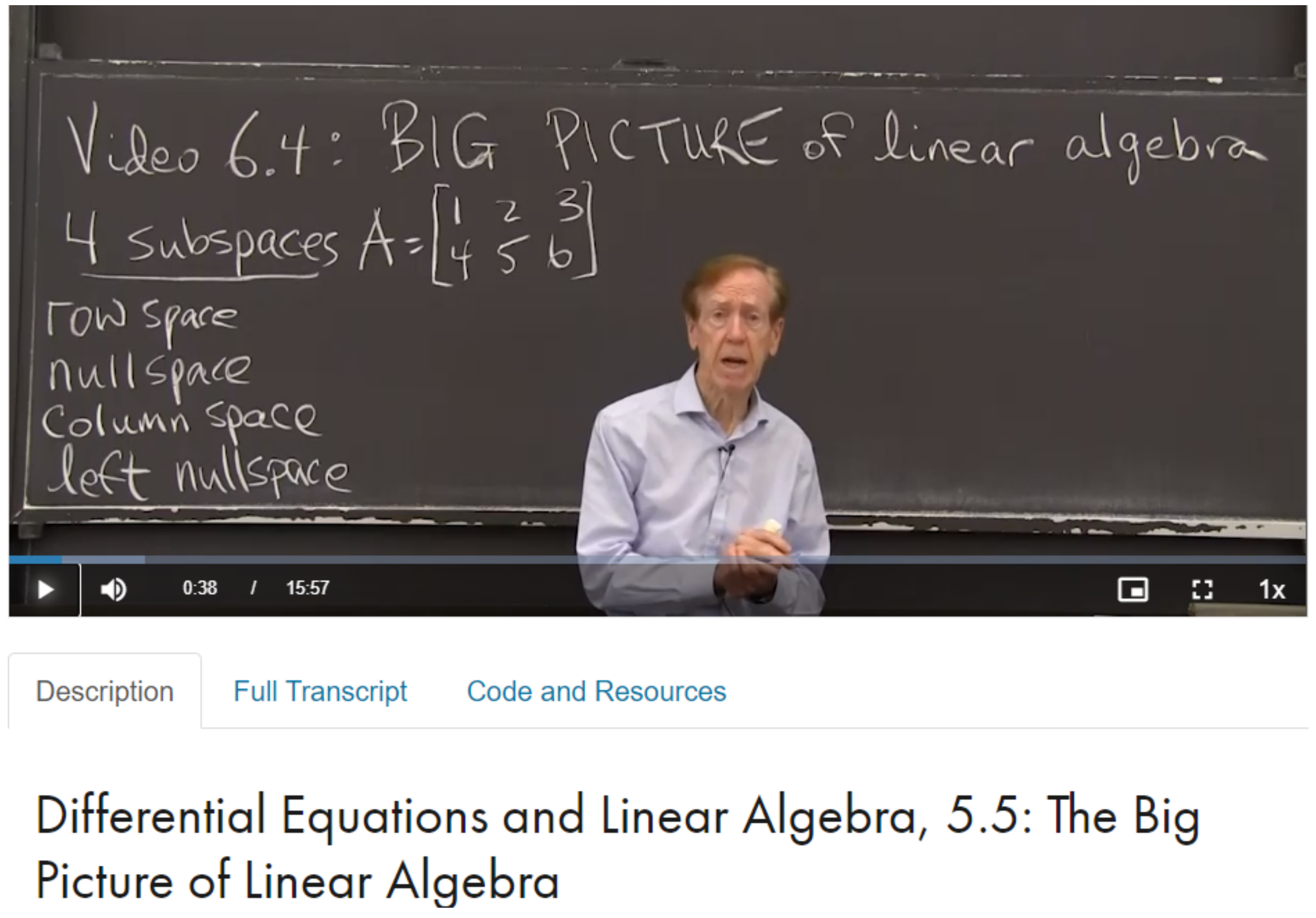


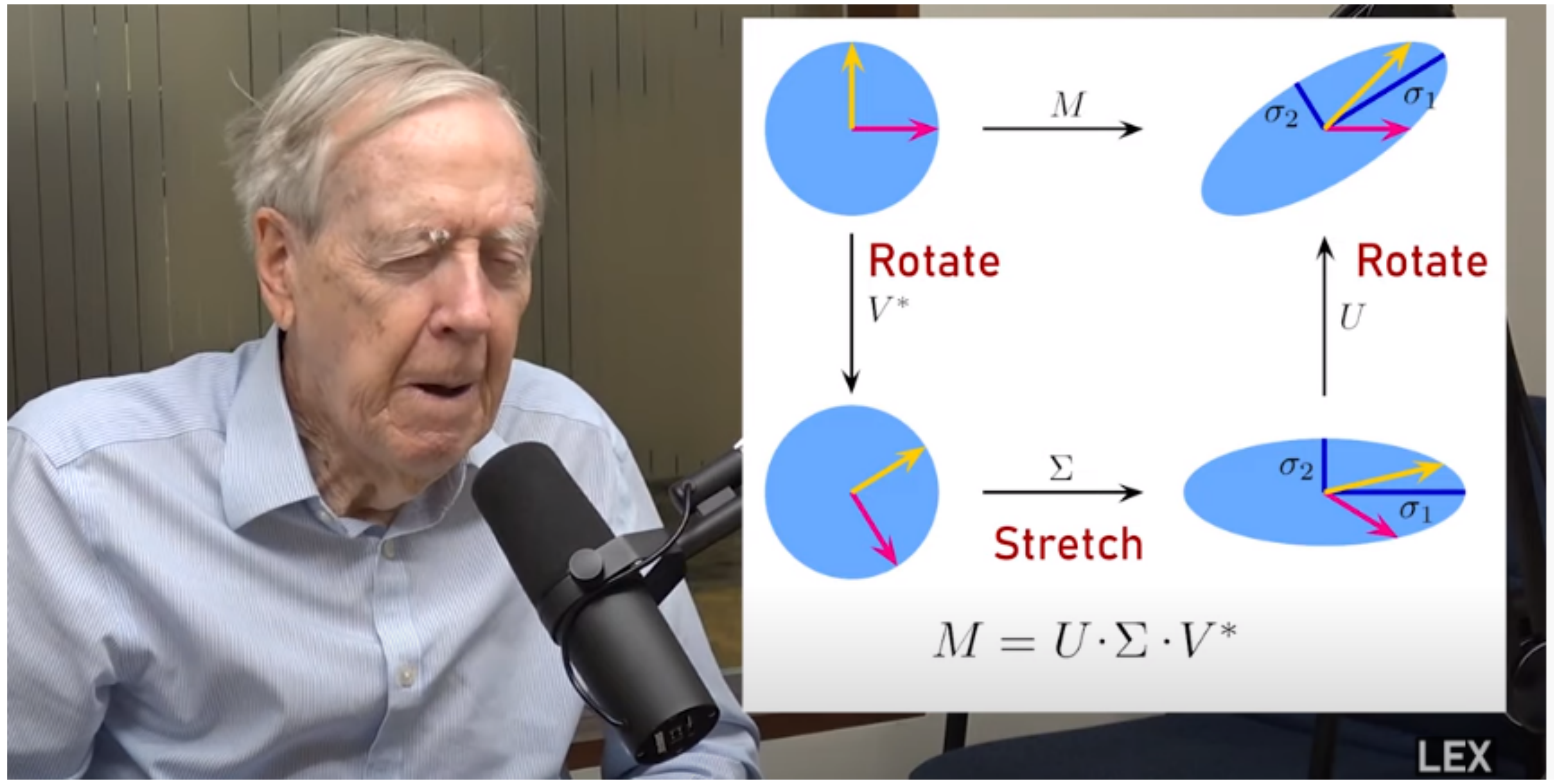
Subspace	Subspace of	Symbol	Dimension	Basis
column space	$\mathbb{R}^m$	$\mathrm{im}(A)$	r = rank	First r columns of U
nullspace (kernel)	$\mathbb{R}^n$	$\ker(A)$	$n-r$	Last $n-r$ columns of V
row space	$\mathbb{R}^n$	$\mathrm{im}(A^T)$	r	First r columns of V
left nullspace (kernel)	$\mathbb{R}^m$	$\ker(A^T)$	$m-r$	Last $m-r$ columns of U

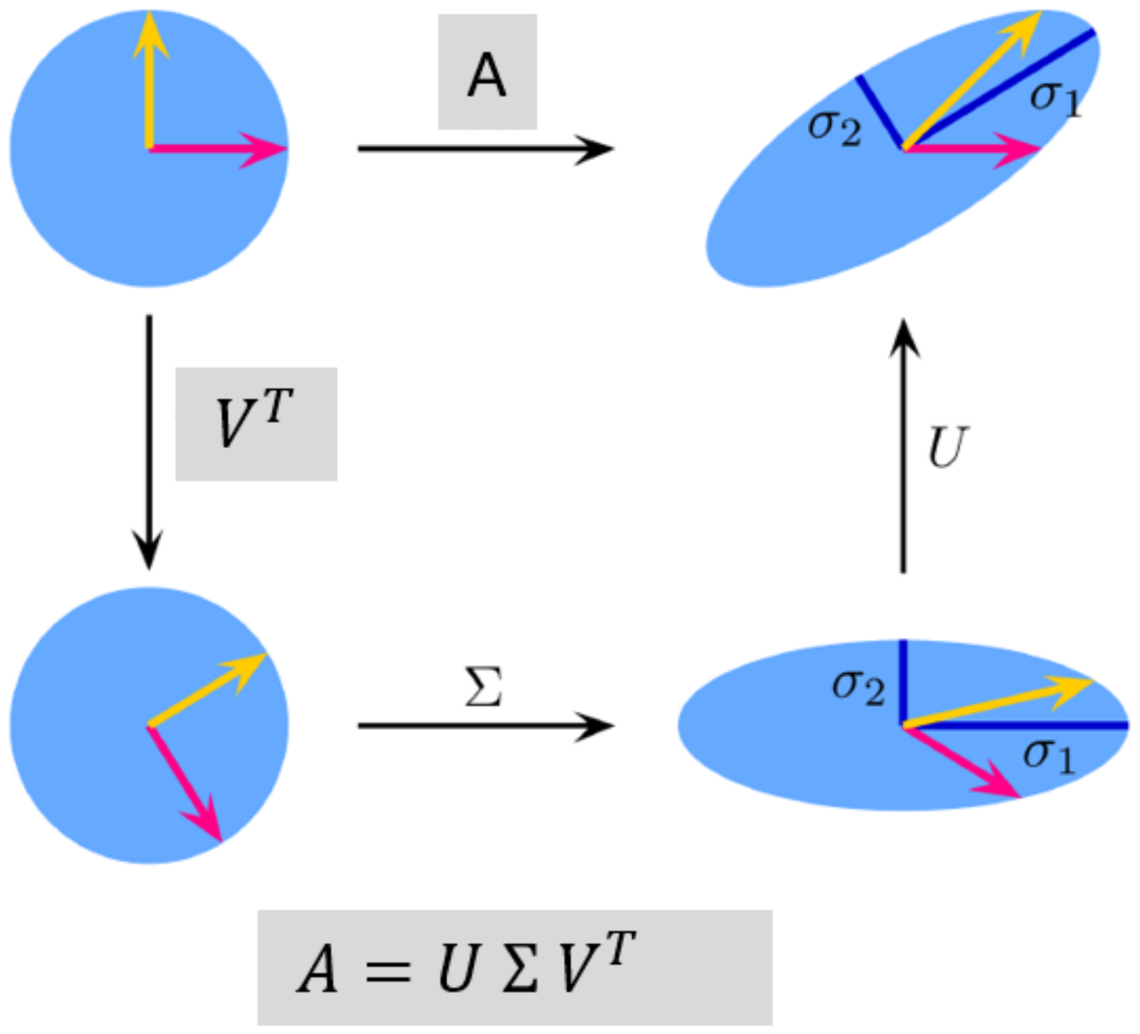
Ref: https://www.youtube.com/watch?v=_G6Sh7P-cK4

SVD and the Four Fundamental Subspaces

Gilbert Strang: Four Fundamental Subspaces of Linear Algebra







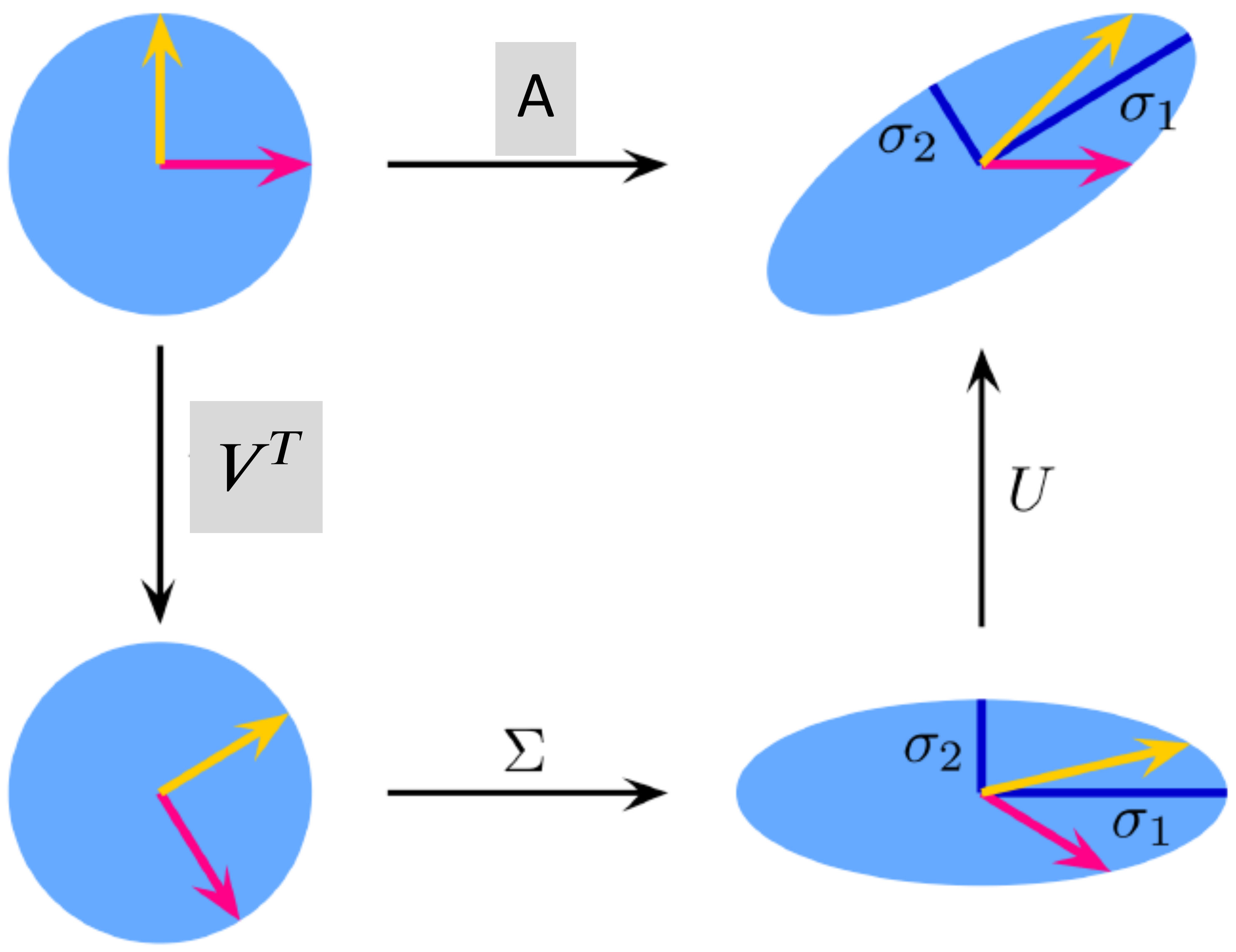
Ref: https://ww2.mathworks.cn/en/videos/differential-equations-and-linear-algebra-55-the-big-picture-of-linear-algebra-117460.html

Ref: https://www.youtube.com/watch?v=YPe5OP7Clv4

Geometric Interpretation of SVD

Geometric interpretation

- Let A = $\boldsymbol{U\Sigma V}^T$
- Any linear mapping $\boldsymbol{y}$= Ax can be expressed as a rotation, stretching, and rotation operation
  - $\boldsymbol{y}_1 = \boldsymbol{V}^T\boldsymbol{x}$ is the first rotation
  - $\boldsymbol{y}_2 = \boldsymbol{\Sigma y}_1$ is the stretching
  - $\boldsymbol{y} = \boldsymbol{U y}_2$ is the final rotation



$$A = U\Sigma V^T$$

Properties of the Matrices U, Σ, V^T

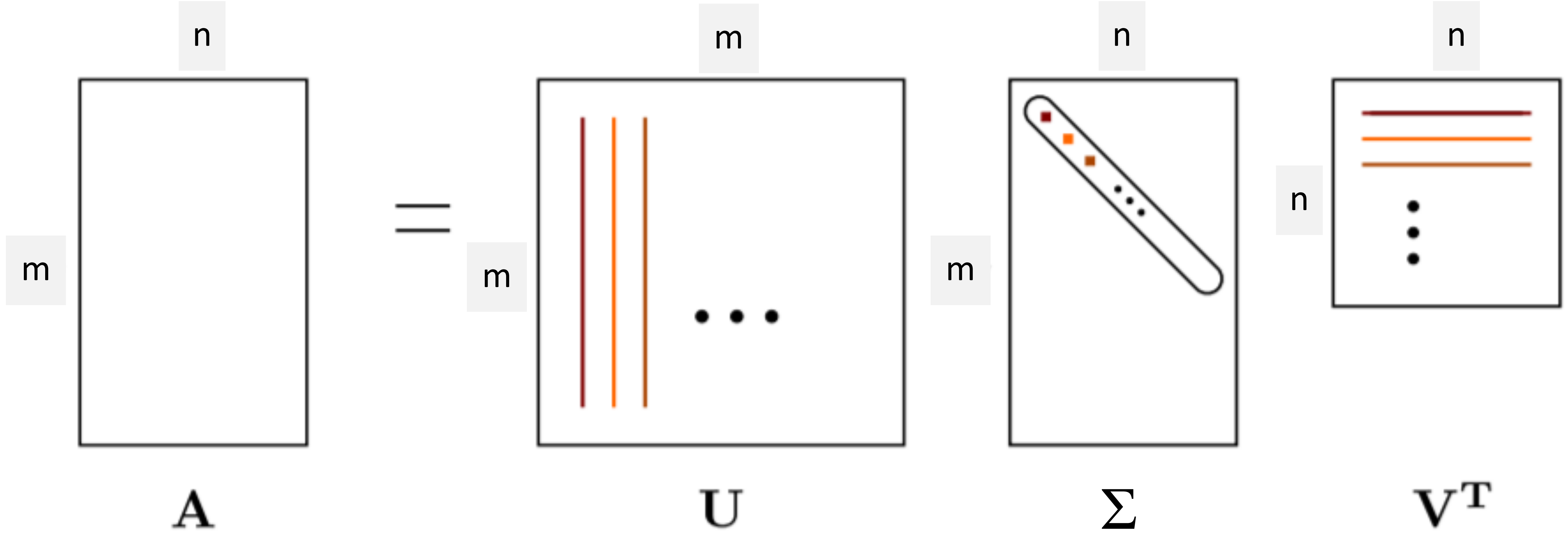


Figure 2: The singular value decomposition (SVD). Each singular value in Σ has an associated left singular vector in $\mathbf{U}$, and right singular vector in $\mathbf{V}$.

Given a **tall and thin** matrix A of rank r:

1. U is a **square matrix**. Its first r columns span the column space of A.
2. W is a diagonal matrix, consisting of the singular values. Assume it has been ordered: $\sigma_1 \geq \sigma_2 \ldots \geq \sigma_r$.
3. V is a **square matrix**. Its first r columns span the row space of A.

Ref: theory.stanford.edu/~tim/s15/l/l9.pdf

NOTE: [U, E, V] = svd(A) % MATLAB Command!

Properties of the Matrices U, Σ, V^T

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

$$A \in R^{m\times n}$$

U, V are orthogonal matrices, i.e.,

- $UU^T = U^TU = I \implies U^{-1} = U^T$
- $VV^T = V^TV = I \implies V^{-1} = V^T$

$$A \in C^{m\times n}$$

U, V are unitary matrices, i.e.,

- $UU^* = U^*U = I \implies U^{-1} = U^*$
- $VV^* = V^*V = I \implies V^{-1} = V^*$

Note: U^* is the conjugate transpose of U.

Ref: https://en.wikipedia.org/wiki/Singular_value_decomposition

Ref: https://en.wikipedia.org/wiki/Conjugate_transpose