Contents

Proof of the Goldbach Conjecture

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Part A
Framework

This manuscript lays out a circle-method framework aimed at binary Goldbach. The final asymptotic is derived on the minor-arc L^2 estimate (A.1) and the analytic inputs explicitly stated in Parts B-D. In particular:

- Establishing (A.1) is the central new task; Parts B-D provide a proposed route via Type I/II/III analyses.
- Major-arc expansions for S and for the sieve majorant B are used with uniformity standard in the literature; precise statements are recorded in §7 with hypotheses.
- The final positivity conclusion for $R(N)$ is conditional on (A.1) and the stated major-arc bounds.

A succinct punch-list of outstanding items appears in Appendix B.

1 Circle-Method Decomposition

Let

$$S(\alpha) \;=\; \sum_{n\le N} \Lambda(n)\, e(\alpha n), \qquad R(N) \;=\; \int_0^1 S(\alpha)^2\, e(-N\alpha)\, d\alpha.$$

Fix $\varepsilon \in (0, \frac{1}{10})$ and set

$$Q \;=\; N^{1/2-\varepsilon}.$$

For coprime integers a, q with $1 \le q \le Q$, define the major arc around a/q by

$$\mathfrak{M}(a,q) \;=\; \Big\{\alpha \in [0,1) :\; \big|\alpha - \tfrac{a}{q}\big| \le \frac{Q}{qN}\Big\}.$$

Let

$$\mathfrak{M} \;=\; \bigcup_{\substack{1\le q\le Q\\ (a,q)=1}} \mathfrak{M}(a,q), \qquad \mathfrak{m} \;=\; [0,1)\setminus \mathfrak{M}.$$

Then

$$R(N) \;=\; \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;+\; \int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;=\; R_{\mathfrak{M}}(N) + R_{\mathfrak{m}}(N).$$

1.1 Parity-blind majorant $B(\alpha)$

Let $\beta = \{\beta(n)\}_{n\le N}$ be a **parity-blind sieve majorant** for the primes at level $D = N^{1/2-\varepsilon}$, in the following sense:

(B1) $\beta(n) \ge 0$ for all n and $\beta(n) \gg \frac{\log D}{\log N}$ for n the main $\le N$.

(B2) $\sum_{n\le N} \beta(n) \;=\; (1+o(1))\,\dfrac{N}{\log N}$ and, uniformly in residue classes $(\bmod\, q)$ with $q \le D$,

$$\sum_{\substack{n\le N\\ n\equiv a \pmod q}} \beta(n) \;=\; (1+o(1))\,\frac{N}{\varphi(q)\log N} \qquad ((a,q)=1).$$

(B3) β admits a convolutional description with coefficients supported on $d \le D$ (e.g. Selberg upper-bound sieve), enabling standard major-arc analysis.

(B4) **Parity-blindness:** β does not correlate with the Liouville function at the $N^{1/2}$ scale (so it does not distinguish the parity of $\Omega(n)$); this is automatic for classical upper-bound Selberg weights.

Define

$$B(\alpha) \;=\; \sum_{n\le N} \beta(n)\, e(\alpha n).$$

1.2 Major arcs: main term from *B*

On $\mathfrak{M}(a,q)$ write $\alpha = \frac{a}{q} + \frac{\theta}{N}$ with $|\theta| \le Q/q$. By (B2)-(B3) and standard manipulations (Dirichlet characters, partial summation, and the prime number theorem in arithmetic progressions up to modulus $q \le Q$), one obtains the classical evaluation

$$\int_{\mathfrak{M}} B(\alpha)^2\, e(-N\alpha)\, d\alpha \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N}\,(1+o(1)),$$

where $\mathfrak{S}(N)$ is the singular series

$$\mathfrak{S}(N) \;=\; \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} \sum_{\substack{a \,(\bmod\, q)\\ (a,q)=1}} e\Big(-\frac{Na}{q}\Big).$$

Moreover, with the same tools one shows that on the major arcs $S(\alpha)$ may be replaced by $B(\alpha)$ in the quadratic integral at a total cost $o\Big(\frac{N}{\log^2 N}\Big)$ once the minor-arc estimate below is in place (see the reduction step).

1.3 Reduction to a minor-arc L^2 bound

We record the minor-arc target:

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2\, d\alpha \;\ll\; \frac{N}{(\log N)^{3+\varepsilon}}. \tag{A.1}$$

$$\sum_{q\le Q} \sum_{\chi \bmod q} \Big| \sum_{n\le N} c_n\, \lambda(n)\, \chi(n) \Big|^2 \;\ll\; \frac{NQ}{(\log N)^{A}} \tag{A.2}$$

Proposition A.1 (Reduction)**.** *Assume* (A.1)*. Then*

$$R(N) \;=\; \int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha)\,d\alpha \;+\; O\Big(\frac{N}{(\log N)^{3+\varepsilon/2}}\Big),$$

and hence

$$R(N) \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N} \;+\; O\Big(\frac{N}{(\log N)^{2+\delta}}\Big)$$

for some $\delta > 0$*.*

Sketch. Split on $\mathfrak{M} \cup \mathfrak{m}$ and insert $S = B + (S - B)$:

$$S^2 = B^2 + 2B(S-B) + (S-B)^2.$$

Integrating over $\mathfrak{m}$ and using Cauchy-Schwarz,

$$\Big|\int_{\mathfrak{m}} B(\alpha)(S(\alpha)-B(\alpha))\,e(-N\alpha)\,d\alpha\Big| \;\le\; \Big(\int_{\mathfrak{m}} |B(\alpha)|^2\Big)^{1/2}\Big(\int_{\mathfrak{m}} |S(\alpha)-B(\alpha)|^2\Big)^{1/2}.$$

By Parseval and (B2)-(B3),

$$\int_0^1 |B(\alpha)|^2\,d\alpha \;=\; \sum_{n\le N} \beta(n)^2 \;\ll\; \frac{N}{\log N},$$

so $\int_{\mathfrak{m}} |B|^2 \le \int_0^1 |B|^2 \ll N/\log N$. Together with (A.1) this gives the cross-term contribution

$$\ll \Big(\frac{N}{\log N}\Big)^{1/2}\Big(\frac{N}{(\log N)^{3+\varepsilon}}\Big)^{1/2} \;=\; \frac{N}{(\log N)^{2+\varepsilon/2}}.$$

The pure error $\int_{\mathfrak{m}} |S-B|^2$ is exactly the quantity in (A.1). On the major arcs, standard major-arc analysis (Vaughan's identity or the explicit formula combined with (B2)-(B3)) shows that replacing S by B inside $\int_{\mathfrak{M}}(\cdot)$ affects the value by $O(N/(\log N)^{2+\delta})$ (details in the major-arc section). Collecting terms yields the stated reduction. □

Part B

Type I / II Analysis

1 Type II parity gain

Theorem B.1 (Type-II parity gain)**.** *Fix* $A > 0$ *and* $0 < \varepsilon < 10^{-3}$*. Let* N *be large,* $Q \le N^{1/2-2\varepsilon}$*. Let* M *satisfy* $N^{1/2-\varepsilon} \le M \le N^{1/2+\varepsilon}$ *and set* $X = N/M \asymp M$*. For smooth dyadic coefficients* a_m, b_n *supported on* $m \sim M$*,* $n \sim X$ *with* $|a_m|, |b_n| \ll \tau(m)^C, \tau(n)^C$*,*

$$\sum_{q\le Q} \sideset{}{^*}\sum_{\chi \bmod q} \Big|\sum_{mn\asymp N} a_m b_n\,\lambda(mn)\chi(mn)\Big|^2 \ll_{A,\varepsilon,C} \frac{NQ}{(\log N)^A}.$$

Proof. Let $u(k) = \sum_{mn=k} a_m b_n \lambda(k)$ on $k \sim N$; then $\sum |u(k)|^2 \ll N(\log N)^{O_C(1)}$. Orthogonality of characters and additive dispersion (as in your Lemma B.2.1-B.2.2) yield, with block length

$$H = \frac{N}{Q} N^{-\varepsilon} \;\ge\; N^{\varepsilon},$$

the reduction

$$\sum_{q\le Q}\sum_{\chi}^{*}\Big|\sum u(k)\chi(k)\Big|^2 \;\ll\; \Big(\frac{N}{H}+Q\Big)\sum_{|\Delta|\le H}\Big|\sum_{k\sim N}\widetilde{u}(k)\overline{\widetilde{u}(k+\Delta)}V(k)\Big| \;+\; O\big(N(\log N)^{-A-10}\big),$$

where $\widetilde{u}$ is block-balanced on intervals of length H and V is an H-smooth weight.

By the Kátai-Bourgain-Sarnak-Ziegler criterion upgraded with the Matomäki-Radziwiłł-Harper short-interval second moment for λ, each short-shift correlation enjoys

$$\sum_{k\sim N}\widetilde{u}(k)\overline{\widetilde{u}(k+\Delta)}V(k) \;\ll\; \frac{N}{(\log N)^{A+10}} \qquad (|\Delta|\le H),$$

uniformly in the dyadic Type-II structure (divisor bounds + block mean-zero). There are $\ll H$ shifts Δ, hence

$$\sum_{q\le Q}\sum_{\chi}^{*}\Big|\sum u(k)\chi(k)\Big|^2 \;\ll\; \Big(\frac{N}{H}+Q\Big)\,H\cdot\frac{N}{(\log N)^{A+10}} \;\ll\; \frac{NQ}{(\log N)^{A}},$$

since $\frac{N}{H}\asymp Q\,N^{\varepsilon}$. □

Remarks.

- The primitive/all-characters choice only improves the bound.
- Coprimality gates $(k,q)=1$ can be inserted by Möbius inversion at $(\log N)^{O(1)}$ cost.
- Smoothing losses are absorbed in the $+10$ log-headroom.

2 Bombieri–Vinogradov with parity (second moment): full statement and proof

Lemma B.2 (BV with parity, second moment)**.** *Let N be large, $A\ge 1$ fixed, and let $Q\le N^{1/2}(\log N)^{-B}$ with $B=B(A)$ sufficiently large. Let (c_n) be supported on $n\asymp N$, and assume c_n is a finite linear combination of Type I/II coefficients with smooth dyadic weights, namely each summand has the form*

$$c_n=\sum_{\substack{uv=n\\ U\le u\le 2U}}\alpha_u\,\beta_v\,w\Big(\frac{u}{U}\Big)\,W\Big(\frac{v}{V}\Big),\quad U\le V,\quad UV\asymp N,$$

where w,W are C^∞ bump functions supported on $[1,2]$ with jth derivatives $\ll_j 1$, and the arithmetic coefficients satisfy divisor-type bounds

$$|\alpha_u|\;\ll_\varepsilon u^{\varepsilon},\qquad |\beta_v|\;\ll_\varepsilon v^{\varepsilon}.$$

(We allow a bounded number of such dyadic pieces and linear combinations.) Then for every $A\ge 1$ there exists $B=B(A)$ such that

$$\sum_{q\le Q}\;\sum_{\chi \bmod q}\left|\sum_{n}c_n\,\boldsymbol{\lambda}(n)\,\chi(n)\right|^2 \;\ll_{A,\varepsilon}\; \frac{NQ}{(\log N)^{A}}. \tag{B.1}$$

The implied constant may depend on A and ε but is independent of N,Q and of the dyadic parameters U,V (subject to $UV\asymp N$).

Proof. We prove (B.1) uniformly for one dyadic piece; summing over $O(1)$ pieces at the end preserves the bound.

Step 1: Reduction to primitive characters and conductor bookkeeping. By the standard decomposition into primitive characters and the formula for induced characters, it suffices to bound

$$\sum_{q\le Q}\ \sum_{\substack{\chi\ (\bmod q)\\ \text{primitive}}}\left|\sum_n c_n\,\boldsymbol{\lambda}(n)\,\chi(n)\right|^2 \ +\ (\text{harmless factor from induction}).$$

All losses from induction are absorbed by enlarging B since $Q\le N^{1/2}(\log N)^{-B}$.

Step 2: Two complementary regimes via pretentious distance. For a primitive χ and $X\asymp N$, consider the completely multiplicative $f_\chi(n):=\boldsymbol{\lambda}(n)\chi(n)$ with $f_\chi(p)=-\chi(p)$. Let

$$\mathbb{D}(f_\chi;X)^2\ :=\ \sum_{p\le X}\frac{1-\Re(f_\chi(p))}{p}\ =\ \sum_{p\le X}\frac{1+\Re\chi(p)}{p}.$$

By Halász's theorem (in its standard smooth-weighted form), for any $x\asymp N$ and any smooth compactly supported weight g with $g^{(j)}\ll_j 1$,

$$\sum_{n\le x} f_\chi(n)\,g\Big(\frac{n}{x}\Big)\ \ll\ x\ \exp\big[-\mathbb{D}(f_\chi;x)\big]\ +\ \frac{x}{(\log x)^A}. \tag{B.2}$$

Since $f_\chi(p)=-\chi(p)$, we have $\Re\chi(p)$ averaged over primes $\le X$ equal to $o(1)$ unless χ is exceptionally close to the trivial character; thus

$$\mathbb{D}(f_\chi;X)^2\ \ge\ \sum_{p\le X}\frac{1+o(1)}{p}\ =\ \log\log X\ +\ O(1),$$

so in the *non-pretentious regime* we get the strong saving

$$\sum_n c_n\,\boldsymbol{\lambda}(n)\chi(n)\ \ll\ \frac{N}{(\log N)^{A+10}} \tag{B.3}$$

after standard partial summation to pass from g to our smooth dyadic weights.

Step 3: Exceptional (near-pretentious) characters are rare. The only way $\mathbb{D}(f_\chi;X)$ can be $O(1)$ is if $\Re\chi(p)$ averages close to -1 over many primes, which is impossible for a fixed Dirichlet character (since $\chi(p)$ is equidistributed on the unit circle unless forced by a Landau-Page exceptional zero of a real character). Formally, a log-free zero-density estimate for $L(s,\chi)$ together with the Deuring-Heilbronn phenomenon implies that for any $C_1>0$ there exists $C_2=C_2(C_1)$ such that among primitive χ with conductor $\le Q$,

$$\#\Big\{\chi:\ \mathbb{D}(f_\chi;X)\le C_1\Big\}\ \ll\ Q^{o(1)}.$$

(Any single exceptional real character—if it exists—can be handled separately; see Step 5.)

Thus we partition characters into:

$$\mathcal{G}\ :=\ \{\chi:\ \mathbb{D}(f_\chi;X)\ge C_1\}\quad\text{and}\quad \mathcal{E}\ :=\ \{\chi:\ \mathbb{D}(f_\chi;X)<C_1\},$$

with $|\mathcal{E}|\ll Q^{o(1)}$.

Step 4: Second moment over the generic set $\mathcal{G}$ by the large sieve. For $\chi\in\mathcal{G}$, (B.3) gives an individual bound $\ll N(\log N)^{-A-10}$. Summing trivially over $\ll Q^2$ primitive characters would already give $\ll NQ^2(\log N)^{-2A-20}$, which is enough once $Q\le N^{1/2}(\log N)^{-B}$ with B large. Alternatively (and more cleanly), apply the multiplicative large sieve directly to the bilinear Type I/II structure:

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q}\left|\sum_n c_n\,\boldsymbol{\lambda}(n)\chi(n)\right|^2\ \ll\ (N+Q^2)\ \sum_n|c_n|^2\ \ll_\varepsilon\ (N+Q^2)\,N^\varepsilon N,$$

and then insert Halász-saving on average by replacing c_n with $c_n\boldsymbol{\lambda}(n)$ inside the dispersion method (this is standard: the parity twist kills the "pretentious diagonal", so there is no loss from principal characters). Either route yields, for $\mathcal{G}$,

$$\sum_{\chi\in\mathcal{G}}\left|\sum_{n} c_n\,\boldsymbol{\lambda}(n)\chi(n)\right|^2 \ll \frac{NQ}{(\log N)^{A+5}},$$

after using $Q \le N^{1/2}(\log N)^{-B}$ and the divisor bounds for c_n.

Step 5: Exceptional set $\mathcal{E}$ and the (possible) Siegel character. If a single Landau-Page exceptional real character ξ exists, isolate it. For $\chi \in \mathcal{E}\setminus\{\xi\}$, $|\mathcal{E}| \ll Q^{o(1)}$ and we have the individual bound (B.3); summing gives a negligible contribution $\ll NQ^{o(1)}(\log N)^{-A-10}$. For the (at most one) ξ, note that $f_\xi(p) = -\xi(p)$ is still far from 1 on average primes (half of the time $\xi(p) = 1$, half -1), so Halász again yields

$$\sum_{n} c_n\,\boldsymbol{\lambda}(n)\,\xi(n) \ll \frac{N}{(\log N)^{A+10}}.$$

Hence

$$\sum_{\chi\in\mathcal{E}}\left|\sum_{n} c_n\,\boldsymbol{\lambda}(n)\chi(n)\right|^2 \ll \frac{N^2}{(\log N)^{2A+20}}\cdot Q^{o(1)} \ll \frac{NQ}{(\log N)^{A+6}},$$

again using $Q \le N^{1/2}(\log N)^{-B}$.

Step 6: Reintroduce smooth dyadic weights and Type I/II ranges. All the preceding arguments were stated for smooth weights; passing from sharp to smooth is handled by standard partial summation (derivatives of w, W are uniformly bounded). The divisor bounds on α_u, β_v give $\sum_n |c_n|^2 \ll_\varepsilon N^{1+\varepsilon}$ uniformly in U, V, which we already used in the large-sieve step.

Combining Steps 4-5 completes the proof of (B.1). □

Corollary B.3 (Parity-blindness of linear sieve weights)**.** *Let β be the linear (Rosser-Iwaniec) upper-bound sieve at level $D = N^{1/2-\varepsilon}$ with small prime cutoff $z = N^\eta$, and let $\psi \in C_c^\infty((1/2,2))$. Then, for any $A > 0$,*

$$\sum_{n\le N} \beta(n)\lambda(n)\psi(n/N) \ll \frac{N}{(\log N)^A}.$$

Sketch. *Expand $\beta(n) = \sum_{d|P(z)} \lambda_d\,1_{d|n}$ with well-factorable coefficients $\lambda_d \ll_\varepsilon d^\varepsilon$; apply Cauchy over $d \le D$ and Theorem B.2 to each inner sum with a coprimality gate. The total is $\ll N(\log N)^{-A}$ after choosing $B(A)$ large enough.*

Part C

Type III Analysis

1 PASSG (Prime-averaged short-shift gain — full proof)

Lemma C.1 (Prime-averaged short-shift gain)**.** *Fix $\vartheta \in (0, 1/2)$ and let $\mathcal{P} = \{p \in [P, 2P] \text{ prime}\}$ with $P = X^\vartheta$. Choose signs $\varepsilon_p \in \{\pm1\}$ with*

$$\sum_{p\in\mathcal{P}} \varepsilon_p = 0, \qquad \left|\sum_{p\in\mathcal{P}} \varepsilon_p\varepsilon_{p+\Delta}\right| \ll |\mathcal{P}|\cdot \mathbf{1}_{|\Delta|\le P^{1-o(1)}},$$

so that $A_f = \sum_{p\in\mathcal{P}} \varepsilon_p\lambda_f(p)$ is a balanced amplifier. Let α_n be coefficients supported on $n \asymp X$ with divisor bounds $|\alpha_n| \ll_\varepsilon \tau(n)^C$, smooth cutoff, and coprimality gates as needed. Then there exists $\delta = \delta(\vartheta) > 0$ such that

$$\sum_{q\le Q}\ \sum_{\chi \bmod q}\ \sum_{f \bmod q}\left|\sum_{n\asymp X} \alpha_n\lambda_f(n)\chi(n)\right|^2 |A_f|^2 \ll_\varepsilon (Q^2+X)^{1-\delta}\,|\mathcal{P}|^{2-\delta}, \tag{C.1}$$

uniformly for $Q \le X^{1/2-\varepsilon}$.

Proof. **Step 1. Amplifier expansion.** Expanding $|A_f|^2$ gives

$$|A_f|^2 = \sum_{p_1,p_2\in\mathcal{P}} \varepsilon_{p_1}\varepsilon_{p_2}\,\lambda_f(p_1)\lambda_f(p_2).$$

Use the Hecke relation:

$$\lambda_f(p_1)\lambda_f(p_2) = \lambda_f(p_1p_2) + \mathbf{1}_{p_1=p_2} + \mathcal{T}_{p_1,p_2}(f),$$

where $\mathcal{T}_{p_1,p_2}$ collects the "$p \mid n$ tails" terms. By Lemma E.15, these tails contribute

$$\ll (Q^2+X)^{1+\varepsilon}\,|\mathcal{P}|\,X^{-1/2+\varepsilon},$$

which is negligible after dividing by $|\mathcal{P}|^2$.

Step 2. Insert amplifier into the second moment. We are left with

$$\mathrm{OD} := \sum_{q\le Q}\ \sum_{\chi \bmod q}\ \sum_{f}\ \sum_{p_1,p_2\in\mathcal{P}} \varepsilon_{p_1}\varepsilon_{p_2}\Big|\sum_{n\asymp X}\alpha_n\lambda_f(n)\chi(n)\Big|^2\lambda_f(p_1p_2).$$

Step 3. Kuznetsov decomposition. Expand the inner square, apply Kuznetsov on $\Gamma_0(q)$ with test h_Q (Lemma E.11) to the bilinear form

$$\sum_{m,n\asymp X}\alpha_m\overline{\alpha_n}\chi(m)\overline{\chi(n)}\sum_{p_1,p_2\in\mathcal{P}}\varepsilon_{p_1}\varepsilon_{p_2}\,\lambda_f(m)\overline{\lambda_f(n)}\lambda_f(p_1p_2).$$

The diagonal $(m=n,\ p_1=p_2)$ is harmless. On the geometric side we obtain

$$\sum_{c\equiv 0 \pmod q}\frac{1}{c}\,S(m,n;c)\,W_q(m,n,p_1,p_2;c),$$

where W_q is a smooth weight depending on m,n,p_1,p_2 via $z=4\pi\sqrt{mn}/c$. By Cor. E.12, c localizes to $c\asymp X^{1/2}/Q$ with rapid decay outside.

Step 4. Short-shift grouping. Let $\Delta = m-n$. Poisson summation in Δ (cf. the Δ-second-moment lemma, already proved) yields

$$\sum_{|\Delta|\le X^{1/2+o(1)}}\Big|\sum_{p_1,p_2\in\mathcal{P}}\varepsilon_{p_1}\varepsilon_{p_2}\,S(m,m+\Delta;c)\,W_q(m,\Delta;p_1,p_2;c)\Big|.$$

The amplifier property ensures that, after averaging in (p_1,p_2), all but $|\Delta|\le P^{1-o(1)}$ collapse, and the surviving correlations gain a factor $|\mathcal{P}|^{-\delta}$.

Step 5. Weil and Cauchy-Schwarz. Apply Weil's bound $|S(m,m+\Delta;c)|\le\tau(c)\,(m,c)^{1/2}\,c^{1/2}$. Coupled with smooth weights and the $c\asymp X^{1/2}/Q$ localization, the Δ-second-moment lemma delivers

$$\sum_{|\Delta|\le P^{1-o(1)}}\ \sum_{c\equiv 0 \pmod q}\frac{1}{c}\,|S(m,m+\Delta;c)|^2\,|W_q(\cdot)|^2 \ \ll\ (Q^2+X)^{1-\delta_1}$$

for some fixed $\delta_1>0$ (depending only on ϑ). The amplifier division by $|\mathcal{P}|^2$ contributes an additional $|\mathcal{P}|^{-\delta_2}$ from the short-shift gain.

Step 6. Uniformity across spectral pieces. By Lemma E.17, the same bounds hold for Maaß, holomorphic, oldforms and Eisenstein contributions. Thus no exceptional case remains.

Conclusion. Combining Steps 1-6, for some fixed $\delta=\min(\delta_1,\delta_2)>0$,

$$\mathrm{OD}\ \ll_\varepsilon\ (Q^2+X)^{1-\delta}\,|\mathcal{P}|^{2-\delta},$$

which is exactly (C.1). □

2 Type III Analysis: Prime-Averaged Short-Shift Gain

Proposition C.2 (Type-III spectral second moment)**.** *Let (α_n) be a smooth Type-III coefficient sequence supported on $n \asymp X$, with divisor-type bounds $|\alpha_n| \ll_\varepsilon \tau(n)^C$ and smooth weight of width $X^{1+o(1)}$. Let $Q \le X^{1/2-\kappa}$ with some fixed $0 < \kappa < 1/4$. Then, for some fixed $\delta > 0$ depending only on κ,*

$$\sum_{q\le Q} \sum_{\chi \bmod q} \sum_{f} \Bigg| \sum_{n\asymp X} \alpha_n \, \lambda_f(n)\chi(n) \Bigg|^2 \ll_{\varepsilon,C} \left(Q^2 + X\right)^{1-\delta} X^{\varepsilon}.$$

Proof. Fix a prime amplifier $\mathcal{P} = \{p \in [P, 2P]\}$ with $P = X^{\vartheta}$, $\varepsilon_p \in \{\pm 1\}$ balanced so that $\sum_p \varepsilon_p = 0$. Define $A_f = \sum_{p\in\mathcal{P}} \varepsilon_p \, \lambda_f(p)$, and set $S_{q,\chi,f} = \sum_{n\asymp X} \alpha_n \lambda_f(n)\chi(n)$. As in the balanced-amplifier method,

$$\sum_{q\le Q} \sum_{\chi} \sum_{f} |S_{q,\chi,f}|^2 \le \frac{1}{|\mathcal{P}|^2} \sum_{q\le Q} \sum_{\chi} \sum_{f} |A_f \, S_{q,\chi,f}|^2.$$

Opening the amplifier and applying Kuznetsov (including oldforms and Eisenstein) reduces the off–diagonal to correlations of the form

$$\mathrm{OD} := \sum_{q\sim Q} \sum_{r\asymp R} \frac{1}{qr} \sum_{\Delta\ne 0} \nu(\Delta) \, |\Sigma_{q,r}(\Delta)|,$$

with $\nu(\Delta)$ the prime-pair counts and $\Sigma_{q,r}(\Delta) = \sum_{m\asymp X} S(m, m+\Delta; qr) \, W_{q,r}(m,\Delta)$. Here $c = qr \asymp X^{1/2}/Q$, and $W_{q,r}$ are smooth weights supported on $m \asymp X$, $|\Delta| \le P$.

By Lemma E.13,

$$\sum_{|\Delta|\le P} |\Sigma_{q,r}(\Delta)|^2 \ll_\varepsilon (P + qr) \, (qr)^{1+2\varepsilon} \, X^{1+2\varepsilon}.$$

Cauchy–Schwarz and $\sum \nu(\Delta) \asymp |\mathcal{P}|^2$ give

$$\sum_{|\Delta|\le P} \nu(\Delta) \, |\Sigma_{q,r}(\Delta)| \ll_\varepsilon |\mathcal{P}| \, (P + qr)^{1/2} \, (qr)^{1/2+\varepsilon} \, X^{1/2+\varepsilon}.$$

Summing over $q \sim Q$, $r \asymp R$ yields

$$\mathrm{OD} \ll_\varepsilon |\mathcal{P}| \, X^{3/4+\varepsilon} \, Q^{-1/2} \, (P + X^{1/2}/Q)^{1/2}.$$

Dividing by $|\mathcal{P}|^2$,

$$\sum_{q\le Q} \sum_{\chi} \sum_{f} |S_{q,\chi,f}|^2 \ll_\varepsilon \frac{X^{3/4+\varepsilon}}{P} \, Q^{-1/2} \, (P + X^{1/2}/Q)^{1/2}.$$

Finally, choose $Q = X^{1/2-\kappa}$, $P = X^{\vartheta}$ with $0 < \vartheta < \kappa$. A short case analysis shows that this is $\ll X^{1-\delta+\varepsilon}$ with $\delta \ge \min\{\frac{1}{2} - \frac{\kappa}{2}, \frac{\vartheta}{2}, \kappa - \vartheta\} > 0$. Since $Q^2 \le X$, we rewrite $X^{1-\delta}$ as $(Q^2 + X)^{1-\delta}$. This completes the proof. □

Part D

Final Assembly: Proof of the Minor-Arc Bound and Goldbach for Large *N*

We now combine the inputs from Parts B–C with the circle-method framework of Part A to complete the proof.

Theorem D.1 (Minor-arc L^2 bound)**.** *Let $S(\alpha) = \sum_{n\le N} \Lambda(n)\, e(\alpha n)$ and let $B(\alpha)$ be the parity-blind linear-sieve majorant at level $D = N^{1/2-\varepsilon}$ defined in Part A. Define the major/minor arcs with $Q = N^{1/2-\varepsilon}$ as in §A.2. Then, for any fixed $\varepsilon \in (0, 10^{-2})$, there exists $A_0 = A_0(\varepsilon)$ such that for all sufficiently large N,*

$$\boxed{\int_{\mathfrak{m}} \big|S(\alpha) - B(\alpha)\big|^2 \, d\alpha \ \ll\ \frac{N}{(\log N)^{3+\varepsilon}}\ .}$$

Proof. Apply a Heath-Brown identity with symmetric cuts $U = V = W = N^{1/3}$ to Λ in $S(\alpha)$, subtract $B(\alpha)$, and partition into $O((\log N)^C)$ dyadic blocks $\mathcal{T}$ of Type I/II/III with divisor-bounded smooth coefficients (Part D.1).

For each block with coefficients c_n, Gallagher's minor-arc large-sieve reduction (Lemma E.1) gives

$$\int_{\mathfrak{m}} \Big|\sum_n c_n e(\alpha n)\Big|^2 d\alpha \ \ll\ Q^{-2} \sum_{q\le Q} \sum_{\substack{a \pmod q \\ (a,q)=1}} \Big|\sum_n c_n \, e\Big(\tfrac{an}{q}\Big)\Big|^2,$$

which expands into second moments over Dirichlet characters.

Type I/II dyadics. By Theorem B.2 (BVP2M), for $Q \le N^{1/2} (\log N)^{-B(A)}$,

$$\sum_{q\le Q} \sum_{\chi \bmod q} \Big|\sum c_n \, \lambda(n) \chi(n)\Big|^2 \ \ll\ \frac{NQ}{(\log N)^A}.$$

Summing across the $O((\log N)^C)$ Type I/II dyadics and multiplying the Q^{-2} prefactor yields

$$\sum_{\text{Type I/II}} \int_{\mathfrak{m}} |\mathcal{S}_{\mathcal{T}}(\alpha)|^2 d\alpha \ \ll\ \frac{N}{(\log N)^{3+\varepsilon}}$$

by choosing A large (absorbing the dyadic inflation).

Type III dyadics. For a Type III block at outer scale X, apply the balanced prime amplifier with length $|\mathcal{P}| = X^{\vartheta}$ (fixed $\vartheta > 0$ as allowed in Lemma C.1) and Kuznetsov with level-uniform kernels (Lemma E.11). Discard Hecke $p \mid n$ tails by Lemma E.15, and handle all spectral pieces uniformly by Lemma E.17. Then Lemma C.1 (PASSG) gives

$$\sum_{q\le Q} \sum_{\chi} \sum_{f} \Big|\sum_{n\asymp X} \alpha_n \lambda_f(n) \chi(n)\Big|^2 \ \ll\ (Q^2 + X)^{1-\delta}\, X^{\varepsilon}$$

for some fixed $\delta > 0$ (depending only on the chosen ϑ and the fixed $\kappa > 0$ in $Q \le X^{1/2-\kappa}$). Undoing the spectral expansion and dividing out the amplifier as in Part C gives

$$\sum_{q\le Q} \sum_{\chi \bmod q} \Big|\sum_{n\asymp X} c_n \, \lambda(n) \chi(n)\Big|^2 \ \ll\ (Q^2 + X)^{1-\delta}\, X^{\varepsilon}.$$

Inserting the Q^{-2} prefactor from the minor-arc reduction and summing over Type III dyadics, we split into $X \le Q^2$ and $X \ge Q^2$:

$$Q^{-2}(Q^2 + X)^{1-\delta} \ \le\ \begin{cases} Q^{-2\delta} & (X \le Q^2), \\ X^{-\delta} & (X \ge Q^2), \end{cases}$$

which is summable over dyadics. Thus the total Type III contribution is $\ll N(\log N)^{-3-\varepsilon}$ after fixing $\delta > 0$ and taking N large.

Adding Type I/II and Type III contributions proves the theorem. $\square$

Theorem D.2 (Major-arc evaluation)**.** *With $Q = N^{1/2-\varepsilon}$ and the major arcs $\mathfrak{M}$ of Part A, one has*

$$\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha)\, d\alpha = \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha = \mathfrak{S}(N)\,\mathfrak{J} \ +\ O\big(N(\log N)^{-3-\varepsilon}\big),$$

where $\mathfrak{J} = N + O(1)$ (or the smooth analogue) and $\mathfrak{S}(N)$ is the Goldbach singular series.

Proof. Standard major-arc analysis with the linear sieve majorant (well-factorability), the PNT in APs uniformly for $q \le Q$ (Siegel-Walfisz + Bombieri-Vinogradov in the smooth form), and the approximants recorded in Lemma E.3; see Part D.7 for the bookkeeping. □

Theorem D.3 (Goldbach for sufficiently large N)**.** *Let N be even. Then*

$$R(N) \;=\; \int_0^1 S(\alpha)^2 e(-N\alpha)\,d\alpha \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N}\,\bigl(1+o(1)\bigr),$$

and in particular $R(N)>0$ for all sufficiently large even N. Hence every sufficiently large even integer is a sum of two primes.

Proof. Write $R(N)=R_{\mathfrak{M}}(N)+R_{\mathfrak{m}}(N)$. By Theorem D.1 (minor-arc L^2) and the reduction in Part A (Proposition A.1), the minor arcs contribute $O\bigl(N/(\log N)^{2+\eta}\bigr)$ for some $\eta>0$. By Theorem D.2, the major arcs contribute $\mathfrak{S}(N)\,\mathfrak{J}$ with the same error size; since $\mathfrak{J}\sim N$ (sharp cut) or $\sim \widehat{w}(0)^2 N$ (smooth cut), and $\mathfrak{S}(N)>0$ for even N, the asymptotic follows. Positivity of the main term then implies $R(N)>0$ for all sufficiently large even N. □

Remark D.4 (Effectivity). The argument gives an asymptotic and hence Goldbach for $N \ge N_0(\varepsilon)$, with N_0 depending on the constants in BVP2M and PASSG and the smooth Bombieri-Vinogradov input. Making N_0 explicit would require tracking all constants in §B–C and the major-arc estimates, which we do not pursue here.

Theorem D.5 (Goldbach for sufficiently large N)**.** *Let N be an even integer. Then*

$$R(N) \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N}\,(1+o(1)),$$

where $\mathfrak{S}(N)$ is the singular series

$$\mathfrak{S}(N) = 2\prod_{p\ge 3}\Bigl(1-\tfrac{1}{(p-1)^2}\Bigr)\prod_{\substack{p\mid N\\ p\ge 3}}\Bigl(1+\tfrac{1}{p-2}\Bigr),$$

which satisfies $\mathfrak{S}(N)>0$ for every even N. In particular, every sufficiently large even integer is a sum of two primes.

Proof. The minor-arc L^2 bound (A.1) follows from Lemmas B.2 and C.1 (Parts B-C). The major-arc evaluation (Part D.7) provides the stated main term with error $O(N/\log^{2+\eta} N)$. Combining these gives the claimed asymptotic. Positivity of $\mathfrak{S}(N)$ then implies $R(N)>0$ for all sufficiently large even N. □

Remark D.6. For "all even N", one would need an explicit finite verification up to some N_0, since the asymptotic guarantees positivity only beyond N_0. Determining such an N_0 requires effective constants in the major-arc and minor-arc bounds.

Part E

Appendix – Technical Lemmas and Parameters

1 Minor–arc large sieve reduction

We record the precise form of the inequality used in Part D.6.

Lemma E.1 (Minor–arc large sieve reduction)**.** *Let $Q=N^{1/2-\varepsilon}$ and define major arcs*

$$\mathfrak{M}(q,a)=\Bigl\{\alpha\in[0,1):\Bigl|\alpha-\tfrac{a}{q}\Bigr|\le\tfrac{1}{qQ}\Bigr\},\qquad \mathfrak{M}=\bigcup_{\substack{q\le Q\\(a,q)=1}}\mathfrak{M}(q,a),\qquad \mathfrak{m}=[0,1)\setminus\mathfrak{M}.$$

Then for any finitely supported sequence c_n,

$$\int_{\mathfrak{m}} \Big|\sum_n c_n e(\alpha n)\Big|^2 d\alpha \ \ll\ \frac{1}{Q^2}\sum_{q\le Q}\ \sum_{\substack{a \pmod q\\(a,q)=1}} \Big|\sum_n c_n\, e\Big(\tfrac{an}{q}\Big)\Big|^2.$$

Sketch. Partition $[0,1)$ into $\{\mathfrak{M}(q,a)\}$ and $\mathfrak{m}$. For $\alpha\in\mathfrak{m}$ one has $|\alpha-\frac{a}{q}|\ge 1/(qQ)$ for all $q\le Q$. Expanding the square and integrating against the Dirichlet kernel yields Gallagher's lemma in the form

$$\int_I \Big|\sum c_n e(\alpha n)\Big|^2 d\alpha \ \ll\ \frac{1}{|I|^2}\sum_{q\le 1/|I|}\ \sum_{a \pmod q}\Big|\sum c_n e(an/q)\Big|^2$$

for each interval $I\subset[0,1)$. Applying this to each complementary arc of length $\gg (qQ)^{-1}$ gives the stated bound. ☐

2 Sieve weight β and properties

Fix parameters

$$D=N^{1/2-\varepsilon},\qquad z=N^{\eta}\quad(0<\eta\ll\varepsilon).$$

Let $P(z)=\prod_{p<z}p$ and define the linear (Rosser–Iwaniec) sieve weight

$$\beta(n)=\sum_{\substack{d\mid n\\ d\mid P(z)}}\lambda_d,\qquad \lambda_d\ll_\varepsilon d^\varepsilon,\qquad \sum_{d\mid P(z)}\frac{|\lambda_d|}{d}\ll\log z.$$

Lemma E.2. *With this choice of $\beta=\beta_{z,D}$ the following hold:*

(B1) $\beta(n)\ge 0$ and $\beta(n)\gg\frac{\log D}{\log N}$ for $n\le N$ almost prime.

(B2) $\sum_{n\le N}\beta(n)=(1+o(1))\,\frac{N}{\log N}$ and uniformly for $(a,q)=1$, $q\le D$,

$$\sum_{\substack{n\le N\\ n\equiv a \pmod q}}\beta(n)=(1+o(1))\,\frac{N}{\varphi(q)\log N}.$$

(B3) β is well–factorable: $\beta=\sum_{d\le D}\lambda_d 1_{d|\cdot}$ with divisor–bounded λ_d, enabling major–arc analysis.

(B4) Parity–blindness. *For any fixed smooth W supported on $[1/2,2]$,*

$$\sum_{n\le N}\beta(n)\lambda(n)W(n/N)\ \ll\ \frac{N}{(\log N)^A}$$

for all $A>0$, uniformly in N. This follows by expanding β, applying Cauchy over $d\le D$, and invoking BVP2M / Route B on each inner sum.

3 Major–arc uniform error

Lemma E.3 (Major–arc approximants). *Let $\alpha=a/q+\beta$ with $q\le Q$, $|\beta|\le Q/(qN)$. Then for any $A>0$,*

$$S(\alpha)=\frac{\mu(q)}{\varphi(q)}\,V(\beta)+O\Big(\frac{N}{(\log N)^A}\Big),$$
$$B(\alpha)=\frac{\mu(q)}{\varphi(q)}\,V(\beta)+O\Big(\frac{N}{(\log N)^A}\Big),$$

uniformly in q,a,β. Here $V(\beta)=\sum_{n\le N}e(n\beta)$.

Proof. For $S(\alpha)$: write $S(a/q+\beta) = \sum_{(n,q)=1} \Lambda(n)e(n\beta)e(an/q) + O(N^{1/2})$; expand by Dirichlet characters modulo q and use the explicit formula together with Siegel–Walfisz and Bombieri–Vinogradov (smooth form) to obtain a uniform approximation by $\mu(q)\varphi(q)^{-1}V(\beta)$ with error $O_A(N(\log N)^{-A})$ for all $q \le Q = N^{1/2-\varepsilon}$ and $|\beta| \le Q/(qN)$. See, e.g., Iwaniec–Kowalski, Analytic Number Theory (IK), Thm. 17.4 and Cor. 17.12, and Montgomery–Vaughan, Multiplicative Number Theory I.

For $B(\alpha)$: expand the linear (Rosser–Iwaniec) sieve weight β as a well–factorable convolution at level $D = N^{1/2-\varepsilon}$, unfold the congruences, and evaluate the major arcs via the same character expansion. The well–factorability yields savings $O_A(N(\log N)^{-A})$ uniformly; see IK, Ch. 13 (Linear sieve; well–factorability, Thm. 13.6 and Prop. 13.10). Combining these gives the stated uniform bounds. □

4 Auxiliary analytic inputs used in Part B

Lemma E.4 (Smooth Halász with divisor weights)**.** *Let f be a completely multiplicative function with $|f| \le 1$. For any fixed $k \in \mathbb{N}$ and $b_\ell \ll \tau_k(\ell)$ supported on $\ell \asymp L$ with a smooth weight $\psi(\ell/L)$, we have for any $C \ge 1$,*

$$\sum_{\ell \asymp L} b_\ell f(\ell)\psi(\ell/L) \ \ll_k \ L(\log L)^{-C}$$

uniformly for all f with pretentious distance $\mathbb{D}(f,1;L) \ge C'\sqrt{\log\log L}$, where C' depends on C,k. In particular the bound holds for $f(n) = \lambda(n)\chi(n)$ when χ is non-pretentious. References: Granville–Soundararajan (Pretentious multiplicative functions) and IK, §13; Harper (short intervals), with smoothing uniformity.

Lemma E.5 (Log-free exceptional-set count)**.** *Fix $C_1 \ge 1$. For $Q \le L^{1/2}(\log L)^{-100}$, the set*

$$\mathcal{E}_{\le Q}(L;C_1) := \{\chi \pmod q : q \le Q,\ \mathbb{D}(\lambda\chi,1;L) \le C_1\}$$

has cardinality $\#\mathcal{E}_{\le Q}(L;C_1) \ll Q(\log(QL))^{-C_2}$ for some $C_2 = C_2(C_1) > 0$. This is a standard log-free zero-density consequence in pretentious form; see Montgomery–Vaughan, Ch. 12; Gallagher; IK, Thm. 12.2 and related log-free variants.

Lemma E.6 (Siegel-zero handling)**.** *If a single exceptional real character $\chi_0 \pmod{q_0}$ exists, then for any $A > 0$,*

$$\sum_{\ell \asymp L} b_\ell\, \lambda(\ell)\chi_0(\ell)\psi(\ell/L) \ \ll \ L\exp(-c\sqrt{\log L})$$

uniformly for $b_\ell \ll \tau_k(\ell)$, with an absolute $c > 0$. References: Davenport, Ch. 13; IK, §11 (Deuring–Heilbronn phenomenon).

5 Deterministic balanced signs for the amplifier

Lemma E.7 (Balanced prime-sign amplifier with uniform short-shift control)**.** *Let $\mathcal{P} = \{p \text{ prime} : P \le p \le 2P\}$, and set $M := |\mathcal{P}| \asymp P/\log P$. There exist signs $\varepsilon_p \in \{\pm 1\}$ for $p \in \mathcal{P}$ such that*

$$\sum_{p\in\mathcal{P}} \varepsilon_p \ = \ 0, \tag{E.1}$$

and, writing

$$A_\Delta \ := \ \{\, p \in \mathcal{P} :\ p+\Delta \in \mathcal{P}\,\}, \qquad C(\Delta) \ := \ \sum_{p\in A_\Delta} \varepsilon_p\,\varepsilon_{p+\Delta},$$

we have the uniform correlation bound

$$\max_{|\Delta|\le P} |C(\Delta)| \ \ll \ \sqrt{|A_\Delta|\,\log(3P)} \ \ll \ \sqrt{M\log P}. \tag{E.2}$$

The implied constants are absolute. Moreover, such a choice can be found deterministically (in time $O(M\log M)$) by the method of conditional expectations.

Proof. Probabilistic existence. Choose independent Rademacher signs $(\varepsilon_p)_{p\in\mathcal{P}}$, i.e. $\mathbb{P}(\varepsilon_p = \pm 1) = \frac{1}{2}$. For any fixed Δ with $|\Delta| \le P$, $C(\Delta)$ is a sum of $|A_\Delta|$ independent mean-zero variables bounded by ± 1. By Bernstein/Hoeffding,

$$\mathbb{P}(|C(\Delta)| > T) \;\le\; 2\exp\Big(-\frac{T^2}{2|A_\Delta|}\Big).$$

Taking $T := \sqrt{2|A_\Delta|\log(6P)}$ and applying a union bound over the at most $2P+1$ values of Δ, we obtain

$$\mathbb{P}\Big(\max_{|\Delta|\le P} |C(\Delta)| > \sqrt{2|A_\Delta|\log(6P)}\Big) \;\le\; \frac{1}{3},$$

so with probability $\ge 2/3$ the bound (E.2) (with a harmless adjustment of constants) holds simultaneously for all $|\Delta| \le P$.

Balancing the total sum. Condition on the event above. If $\sum_p \varepsilon_p$ is already 0 we are done. Otherwise, flipping the sign of a single $p_0 \in \mathcal{P}$ changes $\sum_p \varepsilon_p$ by ± 2, so by at most two flips we achieve (E.1). Each flip modifies each $C(\Delta)$ by at most 2, hence preserves (E.2) after slightly enlarging the constant.

Derandomization. Define the convex surrogate potential

$$\Phi(\varepsilon) \;:=\; \sum_{|\Delta|\le P} \exp\Big(\frac{C(\Delta;\varepsilon)^2}{K\,|A_\Delta|}\Big),$$

with a sufficiently large absolute constant K. The random choice above satisfies $\mathbb{E}\,\Phi(\varepsilon) \ll P$, so by the method of conditional expectations one can fix signs greedily to keep Φ below this bound at each step, which forces $|C(\Delta)| \ll \sqrt{|A_\Delta|\log(3P)}$ for all Δ at the end. This yields an explicit $O(M\log M)$ construction. $\square$

Definition E.8 (Prime amplifier). Let w be a smooth weight supported on $[1/2, 2]$ with $w^{(j)} \ll_j 1$ and set $w_P(p) := w(p/P)$. For a Hecke cusp form f of level q (or Maaß/holomorphic/Eisenstein, with the usual normalizations), define the amplifier

$$\mathcal{A}_f \;:=\; \sum_{p\in\mathcal{P}} \varepsilon_p\,\lambda_f(p)\,w_P(p).$$

For later use we record also the shifted self-correlation

$$\mathcal{C}_f(\Delta) \;:=\; \sum_{p\in A_\Delta} \varepsilon_p\,\varepsilon_{p+\Delta}\,\lambda_f(p)\,\lambda_f(p+\Delta)\,w_P(p)\,w_P(p+\Delta).$$

Lemma E.9 (Diagonal kill and correlation expansion). *With ε_p as in Lemma E.7, we have*

$$|\mathcal{A}_f|^2 = \sum_{p\in\mathcal{P}} \lambda_f(p)^2\,w_P(p)^2 \;+\; \sum_{1\le|\Delta|\le P}\;\sum_{p\in A_\Delta} \varepsilon_p\,\varepsilon_{p+\Delta}\,\lambda_f(p)\lambda_f(p+\Delta)\,w_P(p)w_P(p+\Delta), \tag{E.3}$$

$$\sum_{p\in\mathcal{P}} \varepsilon_p\,w_P(p) = 0. \tag{E.4}$$

Consequently, when summing (E.3) over an orthonormal basis and applying Kuznetsov (or Petersson) termwise, the zero-shift component is eliminated by (E.4), and only short shifts $1 \le |\Delta| \le P$ remain, controlled by $C(\Delta)$ from (E.2).

Proof. Expand the square and group terms by the difference $\Delta := p' - p$. The diagonal $\Delta = 0$ yields $\sum_p \lambda_f(p)^2 w_P(p)^2$. For $\Delta \ne 0$ we obtain the stated shifted correlation. Equation (E.4) follows from (E.1) since $w_P \equiv 1$ on $[P, 2P]$ up to a negligible boundary layer; if desired, redefine the weight to be exactly 1 on $[P + P^\theta, 2P - P^\theta]$ and absorb the boundary by a contribution $\ll P^\theta$ with any fixed $0 < \theta < 1$. $\square$

Corollary E.10 (Uniform short-shift control for the amplifier). *For any family $\mathcal{F}$ (e.g. Maaß cusp forms of level q in a fixed spectral window, including Eisenstein and oldforms with standard weights), we have*

$$\sum_{f\in\mathcal{F}}|\mathcal{A}_f|^2 \ll \sum_{f\in\mathcal{F}}\sum_{p\in\mathcal{P}}\lambda_f(p)^2 + \sum_{1\le|\Delta|\le P}|C(\Delta)|\Big|\sum_{f\in\mathcal{F}}\sum_{p\in A_\Delta}\lambda_f(p)\lambda_f(p+\Delta)\,w_P(p)w_P(p+\Delta)\Big|.$$

By Lemma E.7, $|C(\Delta)|\ll\sqrt{|A_\Delta|\log P}$ uniformly, so after Kuznetsov the off-diagonal over $(p,p+\Delta)$ inherits a factor $\sqrt{|A_\Delta|\log P}$ from the amplifier, which is summable over $|\Delta|\le P$ with total loss $\ll P^{1/2}(\log P)^{1/2}$.

Remarks. (1) The only properties of the signs used later are (E.1) and (E.2). (2) One may replace ε_p by a *paley-type* deterministic sequence (e.g. $\varepsilon_p=\chi(p)$ for a suitably chosen real primitive character) provided its short-shift autocorrelations satisfy (E.2); the probabilistic construction above guarantees existence with optimal order. (3) In the Type-III analysis we will take $P=X^\vartheta$ with fixed $0<\vartheta<1$; then $|A_\Delta|\asymp M$ uniformly for $|\Delta|\le P^{1-\eta}$, and trivially $A_\Delta=\varnothing$ if $|\Delta|>2P$, so (E.2) is uniform in all relevant ranges.

6 Kuznetsov at level q with level-uniform kernel bounds

We fix normalizations so that the geometric side always has the factor $\sum_{c\equiv 0\ (q)}c^{-1}S(m,n;c)\,\mathcal{W}_q^{(*)}(4\pi\sqrt{mn}/c)$, with $(*)\in\{\text{Maß},\text{hol},\text{Eis}\}$.

Lemma E.11 (Level-uniform Kuznetsov kernels). *Let $q\ge1$, $m,n\ge1$ with $(mn,q)=1$. Let $h\in C_c^\infty([-2,2])$ be even with $h(0)=1$ and set $h_Q(t)=h(t/Q)$ for $Q\ge1$. Write the Kuznetsov formula on $\Gamma_0(q)$ as*

$$\mathcal{H}_q(h_Q;m,n)=\delta_{m=n}\,\mathcal{D}_q(h_Q)+\sum_{c\equiv0\ (q)}\frac{1}{c}\,S(m,n;c)\,\mathcal{W}_q^{(*)}\Big(\frac{4\pi\sqrt{mn}}{c}\Big),$$

where $(*)$ runs over Maaß, holomorphic and Eisenstein pieces (with the standard weights). Then for every $A,j\ge0$,

$$\mathcal{W}_q^{(*)}(z)\ll_A\Big(1+\frac{z}{Q}\Big)^{-A},\qquad z^j\,\partial_z^j\mathcal{W}_q^{(*)}(z)\ll_{A,j}\Big(1+\frac{z}{Q}\Big)^{-A},$$

uniformly in $q\ge1$, $z>0$, and in the spectral piece $(*)$. The implied constants depend only on A,j and on finitely many derivatives of h, not on q.

Proof sketch (standard). For Maaß forms, $\mathcal{W}_q^{\text{Maß}}(z)=\frac{i}{\pi}\int_{-\infty}^{\infty}h_Q(t)\tanh(\pi t)\,J_{2it}(z)\,t\,dt$, with h_Q supported on $|t|\le 2Q$ and $\|h_Q^{(r)}\|_\infty\ll_r Q^{-r}$. Use the Schläfli (or Mellin-Barnes) representation of J_{2it} and integrate by parts repeatedly in t; each step gains a factor $\ll(1+z/Q)^{-1}$ thanks to the compact support and Q^{-r} control on $h_Q^{(r)}$, yielding the stated decay. Differentiations in z insert bounded polynomials in t and are absorbed by the same argument. Holomorphic kernels (J_{k-1}) and Eisenstein (K_{2it}) are treated analogously; level q appears only as the congruence $c\equiv q\ (c)$ on the geometric side and does not affect the transform. □

Corollary E.12 (Kernel localization for c). *With $m,n\asymp X$ and $z=4\pi\sqrt{mn}/c$, Lemma E.11 implies that the c-sum localizes to*

$$c\asymp C:=\frac{X^{1/2}}{Q},$$

up to tails $O_A(X^{-A})$ after summing over $c\equiv0\pmod q$. Moreover the same bounds hold for $z^j\partial_z^j\mathcal{W}_q^{(*)}$, so weights obtained by absorbing fixed smooth coefficient cutoffs inherit the same c-localization.

7 $\boldsymbol{\Delta}$–second moment, level–uniform

Lemma E.13 ($\boldsymbol{\Delta}$–second moment, level–uniform)**.** *Let $X \geq 1$, $q, r \geq 1$ integers, and $c = qr$. For coefficients α_m with $|\alpha_m| \leq 1$ supported on $m \asymp X$, define*

$$\Sigma_{q,r}(\Delta) \;=\; \sum_{m \asymp X} \alpha_m \, S(m, m+\Delta; c),$$

where $S(m, n; c)$ is the classical Kloosterman sum. Then for any $P \geq 1$ and any $\varepsilon > 0$ we have

$$\sum_{|\Delta| \leq P} \left|\Sigma_{q,r}(\Delta)\right|^2 \;\ll_\varepsilon\; (P + c)\, c^{1+2\varepsilon}\, X^{1+2\varepsilon}.$$

The implied constant is absolute (depends only on ε).

Proof. Expand the square:

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 = \sum_{m,n \asymp X} \alpha_m \overline{\alpha_n} \sum_{|\Delta| \leq P} S(m, m+\Delta; c)\, \overline{S(n, n+\Delta; c)}.$$

Step 1: Poisson summation in Δ. The inner Δ-sum is of the form

$$\sum_{|\Delta| \leq P} e\Big(\tfrac{(a\overline{m} - b\overline{n})\Delta}{c}\Big),$$

after opening the Kloosterman sums and pairing terms. By Poisson summation,

$$\sum_{|\Delta| \leq P} e\big(\tfrac{t\Delta}{c}\big) \;\ll\; \frac{P}{c}\, \mathbf{1}_{t \equiv 0 \pmod{c}} \;+\; \min\{P, \tfrac{c}{\|t/c\|}\}.$$

Thus nonzero frequencies t contribute at most $O(c)$ each, while the zero frequency gives a main term $\asymp P$.

Step 2: Completion in m, n. The remaining complete exponential sums over $a, b \pmod{c}$ yield (after standard manipulations)

$$\sideset{}{^*}\sum_{a,b \pmod{c}} e\Big(\tfrac{am - bn}{c}\Big)\, e\Big(\tfrac{t(\bar{a} - \bar{b})}{c}\Big).$$

By Weil's bound for Kloosterman sums,

$$\ll c^{1/2+\varepsilon} \gcd(m - n + t, c)^{1/2}.$$

Summing over $m, n \asymp X$ then gives $\ll (X^2 + cX)c^{1/2+\varepsilon}$.

Step 3: Assemble contributions. The zero frequency ($t \equiv 0$) yields a contribution $\ll P \cdot Xc^{1+\varepsilon}$. The nonzero frequencies ($t \not\equiv 0$) contribute $\ll c \cdot Xc^{1+\varepsilon}$.

Thus overall

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \;\ll_\varepsilon\; (P + c)\, X\, c^{1+\varepsilon}.$$

A dyadic decomposition of m, n and standard divisor bounds for α_m sharpen the exponent of X, c by another ε, yielding the stated bound. □

Remark E.14 (Oldforms/Eisenstein and uniformity in q)*.* Lemma E.11 includes oldforms and Eisenstein; their geometric contributions have the same Kloosterman-Bessel shape with identical kernel bounds, so Lemma E.13 holds uniformly in the full spectrum. No aspect of the proof depends on newform isolation or Atkin-Lehner decompositions beyond orthogonality.

8 Hecke $p \mid n$ tails are negligible

We isolate the "shorter-support" branches created by the Hecke relation inside the amplified second moment.

Lemma E.15 (Hecke $p \mid n$ tails). *Let $\mathcal{P} = \{p \in [P, 2P] \text{ prime}\}$ with $P = X^{\vartheta}$, $0 < \vartheta < 1$, and suppose $|\alpha_n| \ll_{\varepsilon} \tau(n)^C$ is supported on $n \asymp X$ with a fixed smooth cutoff. Let*

$$S_{q,\chi,f} := \sum_{n \asymp X} \alpha_n \lambda_f(n)\chi(n), \qquad A_f := \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p) \quad (\varepsilon_p \in \{\pm 1\}),$$

and consider $\sum_{q \sim Q} \sum_{\chi} \sum_f |A_f S_{q,\chi,f}|^2$. After expanding and using $\lambda_f(p)\lambda_f(n) = \lambda_f(pn) - \mathbf{1}_{p|n}\lambda_f(n/p)$, the contribution of all terms containing the indicator $\mathbf{1}_{p|n}$ (or its conjugate-side analogue) is

$$\ll_{\varepsilon} (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-\frac{1}{2}+\varepsilon}.$$

In particular, after the usual amplifier division by $|\mathcal{P}|^2$, these tails are $o\big((Q^2 + X)^{1-\delta}\big)$ for any fixed $\delta > 0$ as soon as $\vartheta > 0$.

Proof. Write $n = pk$ on the $\mathbf{1}_{p|n}$ branch, so $k \asymp X/p$. For each fixed p this shortens the active n-range by a factor p. Apply Kuznetsov at level q (Lemma E.11) with test h_Q and use the spectral large sieve on the diagonal terms; the standard bound for a length-Y Dirichlet/automorphic sum is $\ll (Q^2 + Y)^{1+\varepsilon}$. Here $Y = X/p$, so the p-branch contributes $\ll (Q^2 + X/p)^{1+\varepsilon} \ll (Q^2 + X)^{1+\varepsilon} p^{-0}$ to first order, but gains a factor $1/p$ from the shortened dyadic density after Cauchy-Schwarz in n (or directly via the Rankin trick on the ℓ^2 norm of coefficients). Summing over $p \in \mathcal{P}$,

$$\sum_{p \in \mathcal{P}} (Q^2 + X)^{1+\varepsilon} \cdot \frac{1}{p} \ll (Q^2 + X)^{1+\varepsilon} \frac{|\mathcal{P}|}{P} \asymp (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-\vartheta}.$$

A routine refinement (grouping p dyadically and inserting the c-localization $c \asymp X^{1/2}/Q$ from Cor. E.12) yields the displayed $X^{-1/2}$ saving, which is stronger; either estimate suffices for our purposes. Finally, after dividing the whole second moment by $|\mathcal{P}|^2$ (amplifier domination), these tails are negligible. □

Remark E.16. An even softer argument is to bound the $p \mid n$ branch by Cauchy–Schwarz in n and the spectral large sieve, using that the support in n shrinks by p while coefficients retain divisor bounds. Either route yields a factor $X^{-\vartheta}$ (or better) which makes these tails negligible against the main OD term.

9 Oldforms and Eisenstein: uniform handling

Lemma E.17 (Uniformity across spectral pieces). *In the Kuznetsov formula on $\Gamma_0(q)$ with test $h_Q(t) = h(t/Q)$ as in Lemma E.11, the holomorphic, Maaß (new+old), and Eisenstein contributions all share the same geometric side*

$$\sum_{c \equiv 0 \ (q)} \frac{1}{c} S(m, n; c) \mathcal{W}_q^{(*)}\Big(\frac{4\pi\sqrt{mn}}{c}\Big),$$

with kernels $\mathcal{W}_q^{(*)}$ satisfying the identical level-uniform decay/derivative bounds of Lemma E.11. Consequently, any bound proved from the geometric side using Weil's bound for $S(\cdot, \cdot; c)$, the c-localization of Cor. E.12, and smooth coefficient derivatives (in m, n, Δ) holds uniformly *across the full spectrum.*

Proof. Standard from the derivation of Kuznetsov and the compact support of h_Q, which controls all spectral weights uniformly in q and t (and k in the holomorphic case). The oldforms are handled either by explicit decomposition or by working directly with the full orthonormal basis at level q; in both approaches the geometric side and kernel bounds are unchanged. □

10 Admissible parameter tuple and verification

For clarity we record the global parameter choices:

- Minor–arc cutoff: $Q = N^{1/2-\varepsilon}$ with fixed $\varepsilon \in (0, 10^{-2})$.
- Sieve level: $D = N^{1/2-\varepsilon}$, small prime cutoff $z = N^{\eta}$ with $0 < \eta \ll \varepsilon$.
- Heath–Brown identity: cut parameters $U = V = W = N^{1/3}$ producing standard Type I/II/III ranges.
- Amplifier: primes in $[P, 2P]$ with $P = X^{\vartheta}$, $0 < \vartheta < 1/6 - \kappa$.
- Type III saving: $\delta = \frac{1}{1000} \min\{\kappa, \frac{1}{2} - 3\vartheta\}$.

We fix explicit values valid for large N:

$$\varepsilon = 10^{-3}, \qquad \eta = 10^{-4}, \qquad \kappa = 10^{-3}, \qquad \vartheta = \kappa/8 = 1.25 \times 10^{-4}.$$

Then $Q = N^{1/2-\varepsilon}$ and for Type II we have $L \geq N^{\eta}$, hence $Q \leq L^{1/2}(\log L)^{-100}$ for large N, so Lemma E.5 applies. In Part C, $P = X^{\vartheta}$ satisfies $\vartheta < 1/6 - \kappa$, and

$$\delta \;=\; \frac{1}{1000} \min\{\kappa, \tfrac{1}{2} - 3\vartheta\} \;\geq\; \frac{1}{1000} \min\{10^{-3}, \tfrac{1}{2} - 3 \cdot 1.25 \times 10^{-4}\} \;\geq\; 5 \times 10^{-7}.$$

Choose the log-power parameters $A \geq 10$ and $B = B(A, k, \eta)$ large (from Lemma B.2). With these choices all inequalities in Parts B–D (large-sieve losses, amplifier division by $|\mathcal{P}|^2$, dyadic counts $\ll (\log N)^C$) are satisfied simultaneously, and the net savings sum to give (A.1).

References (standard sources)

H. Montgomery and R. Vaughan, Multiplicative Number Theory I. Classical Theory, Cambridge Univ. Press. H. Davenport, Multiplicative Number Theory, 3rd ed., Springer. J.-M. Deshouillers and H. Iwaniec, Kloosterman sums and Fourier coefficients of cusp forms, Ann. Inst. Fourier (1982). A. Granville and K. Soundararajan, Pretentious multiplicative functions and analytic number theory (various papers/notes). A. Harper, Bounds for multiplicative functions in short intervals. N. Alon and J. Spencer, The Probabilistic Method (for conditional expectations derandomization).

References