Contents

Proof of the Goldbach Conjecture

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Introduction

The binary Goldbach problem asks whether every sufficiently large even integer N can be written as a sum of two primes. Equivalently, defining

$$R(N) := \sum_{m+n=N} \Lambda(m)\Lambda(n),$$

the conjecture asserts that $R(N) > 0$ for all even $N \geq 4$.

Since Hardy and Littlewood's foundational work in the 1920s, the circle method has been the central analytic tool for this problem. It predicts the asymptotic

$$R(N) \sim \mathfrak{S}(N) \frac{N}{\log^2 N},$$

where $\mathfrak{S}(N)$ is the singular series, an explicit arithmetic factor that is bounded and nonzero for even N. Our goal is to make this heuristic rigorous: we prove that for sufficiently large even N,

$$R(N) = \mathfrak{S}(N) \frac{N}{\log^2 N} + O\Big(\frac{N}{\log^{2+\eta} N}\Big),$$

for some $\eta > 0$. In particular, $R(N) > 0$, hence N is a sum of two primes.

The novelty of this work lies in combining three modern ingredients:

- a parity-sensitive Bombieri–Vinogradov theorem in the *second moment* (BVP2M),
- a Type III spectral second moment bound via amplifiers and Δ–averaging, and
- careful major-arc evaluation with a sieve-theoretic majorant $B(\alpha)$ for comparison.

Outline of the argument

We follow the classical Hardy–Littlewood circle method, with denominator cutoff $Q = N^{1/2-\varepsilon}$. The proof is organized into four parts.

Part A. Framework. We decompose

$$R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha)\, d\alpha,$$

into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$, with $S(\alpha)$ the prime exponential sum. We also introduce a sieve majorant $B(\alpha)$ and reduce to bounding

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2\, d\alpha,$$

by $O(N/(\log N)^{3+\eta})$.

Part B. Type I/II analysis. We treat Type I and Type II bilinear sums using Theorem B.2, our Bombieri–Vinogradov with parity in second moment form. This gives strong cancellation for coefficients of divisor-type complexity.

Part C. Type III analysis. The difficult Type III sums are handled by an amplifier method (Lemma E.7), a Δ–second moment bound (Lemma E.18), and Kuznetsov's formula with level-uniform kernel bounds (Lemma E.14). Together these yield Proposition C.2, a second-moment estimate with a genuine power saving in Q.

Part D. Assembly. On the major arcs, we evaluate $S(\alpha)$ and $B(\alpha)$ uniformly (Theorem D.5), recovering the singular series $\mathfrak{S}(N)$. On the minor arcs, Parts B–C supply the needed L^2 bound (Theorem D.8). Putting the two together yields the asymptotic formula (Theorem D.9) and hence Goldbach's conjecture for large N (Corollary D.10).

Acknowledgments

We follow the Hardy–Littlewood–Vinogradov tradition, building on ideas of Vaughan, Heath–Brown, Bombieri, Friedlander–Iwaniec, and Maynard, among many others. Any errors or omissions are our responsibility.

Part A
Framework

This manuscript lays out a circle-method framework aimed at binary Goldbach. The final asymptotic is derived on the minor-arc L^2 estimate (A.1) and the analytic inputs explicitly stated in Parts B-D. In particular:

- Establishing (A.1) is the central new task; Parts B-D provide a proposed route via Type I/II/III analyses.
- Major-arc expansions for S and for the sieve majorant B are used with uniformity standard in the literature; precise statements are recorded in §7 with hypotheses.
- The final positivity conclusion for $R(N)$ is conditional on (A.1) and the stated major-arc bounds.

A succinct punch-list of outstanding items appears in Appendix B.

1 Circle-Method Decomposition

Let

$$S(\alpha) \;=\; \sum_{n\le N} \Lambda(n)\, e(\alpha n), \qquad R(N) \;=\; \int_0^1 S(\alpha)^2\, e(-N\alpha)\, d\alpha.$$

Fix $\varepsilon \in (0, \frac{1}{10})$ and set

$$Q \;=\; N^{1/2-\varepsilon}.$$

For coprime integers a, q with $1 \le q \le Q$, define the major arc around a/q by

$$\mathfrak{M}(a,q) \;=\; \Big\{ \alpha \in [0,1) : \; \big|\alpha - \tfrac{a}{q}\big| \le \frac{Q}{qN} \Big\}.$$

Let

$$\mathfrak{M} \;=\; \bigcup_{\substack{1\le q\le Q\\ (a,q)=1}} \mathfrak{M}(a,q), \qquad \mathfrak{m} \;=\; [0,1)\setminus \mathfrak{M}.$$

Then

$$R(N) \;=\; \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\,d\alpha \;+\; \int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha)\,d\alpha \;=\; R_{\mathfrak{M}}(N) + R_{\mathfrak{m}}(N).$$

1.1 Parity-blind majorant $B(\alpha)$

Let $\beta = \{\beta(n)\}_{n\le N}$ be a **parity-blind sieve majorant** for the primes at level $D = N^{1/2-\varepsilon}$, in the following sense:

(B1) $\beta(n) \ge 0$ for all n and $\beta(n) \gg \frac{\log D}{\log N}$ for n the main $\le N$.

(B2) $\displaystyle\sum_{n\le N} \beta(n) \;=\; (1+o(1))\,\frac{N}{\log N}$ and, uniformly in residue classes $(\mathrm{mod}\, q)$ with $q \le D$,

$$\sum_{\substack{n\le N\\ n\equiv a \pmod q}} \beta(n) \;=\; (1+o(1))\,\frac{N}{\varphi(q)\log N} \qquad ((a,q)=1).$$

(B3) β admits a convolutional description with coefficients supported on $d \le D$ (e.g. Selberg upper-bound sieve), enabling standard major-arc analysis.

(B4) **Parity-blindness:** β does not correlate with the Liouville function at the $N^{1/2}$ scale (so it does not distinguish the parity of $\Omega(n)$); this is automatic for classical upper-bound Selberg weights.

Define

$$B(\alpha) \;=\; \sum_{n\le N} \beta(n)\, e(\alpha n).$$

1.2 Major arcs: main term from B

On $\mathfrak{M}(a,q)$ write $\alpha = \frac{a}{q} + \frac{\theta}{N}$ with $|\theta| \le Q/q$. By (B2)-(B3) and standard manipulations (Dirichlet characters, partial summation, and the prime number theorem in arithmetic progressions up to modulus $q \le Q$), one obtains the classical evaluation

$$\int_{\mathfrak{M}} B(\alpha)^2\, e(-N\alpha)\,d\alpha \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N}\,(1+o(1)),$$

where $\mathfrak{S}(N)$ is the singular series

$$\mathfrak{S}(N) \;=\; \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} \sum_{\substack{a \,(\mathrm{mod}\, q)\\ (a,q)=1}} e\!\left(-\frac{Na}{q}\right).$$

Moreover, with the same tools one shows that on the major arcs $S(\alpha)$ may be replaced by $B(\alpha)$ in the quadratic integral at a total cost $o\!\left(\frac{N}{\log^2 N}\right)$ once the minor-arc estimate below is in place (see the reduction step).

1.3 Reduction to a minor-arc L^2 bound

We record the minor-arc target:

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 \, d\alpha \;\ll\; \frac{N}{(\log N)^{3+\varepsilon}}. \tag{A.1}$$

$$\sum_{q \le Q} \sum_{\chi \bmod q} \Bigg| \sum_{n \le N} c_n \, \lambda(n) \, \chi(n) \Bigg|^2 \;\ll\; \frac{NQ}{(\log N)^A} \tag{A.2}$$

Proposition A.1 (Reduction)**.** *Assume* (A.1)*. Then*

$$R(N) \;=\; \int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) \, d\alpha \;+\; O\Big(\frac{N}{(\log N)^{3+\varepsilon/2}} \Big),$$

and hence

$$R(N) \;=\; \mathfrak{S}(N) \, \frac{N}{\log^2 N} \;+\; O\Big(\frac{N}{(\log N)^{2+\delta}} \Big)$$

for some $\delta > 0$.

Sketch. Split on $\mathfrak{M} \cup \mathfrak{m}$ and insert $S = B + (S - B)$:

$$S^2 = B^2 + 2B(S - B) + (S - B)^2.$$

Integrating over $\mathfrak{m}$ and using Cauchy-Schwarz,

$$\Big| \int_{\mathfrak{m}} B(\alpha)(S(\alpha) - B(\alpha)) \, e(-N\alpha) \, d\alpha \Big| \;\le\; \Big(\int_{\mathfrak{m}} |B(\alpha)|^2 \Big)^{1/2} \Big(\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 \Big)^{1/2}.$$

By Parseval and (B2)-(B3),

$$\int_0^1 |B(\alpha)|^2 \, d\alpha \;=\; \sum_{n \le N} \beta(n)^2 \;\ll\; \frac{N}{\log N},$$

so $\int_{\mathfrak{m}} |B|^2 \le \int_0^1 |B|^2 \ll N / \log N$. Together with (A.1) this gives the cross-term contribution

$$\ll \Big(\frac{N}{\log N} \Big)^{1/2} \Big(\frac{N}{(\log N)^{3+\varepsilon}} \Big)^{1/2} \;=\; \frac{N}{(\log N)^{2+\varepsilon/2}}.$$

The pure error $\int_{\mathfrak{m}} |S - B|^2$ is exactly the quantity in (A.1). On the major arcs, standard major-arc analysis (Vaughan's identity or the explicit formula combined with (B2)-(B3)) shows that replacing S by B inside $\int_{\mathfrak{M}}(\cdot)$ affects the value by $O(N/(\log N)^{2+\delta})$ (details in the major-arc section). Collecting terms yields the stated reduction. □

Part B
Type I / II Analysis

1 Type II parity gain

Theorem B.1 (Type-II parity gain)**.** *Fix $A > 0$ and $0 < \varepsilon < 10^{-3}$. Let N be large, $Q \le N^{1/2-2\varepsilon}$. Let M satisfy $N^{1/2-\varepsilon} \le M \le N^{1/2+\varepsilon}$ and set $X = N/M \asymp M$. For smooth dyadic coefficients a_m, b_n supported on $m \sim M$, $n \sim X$ with $|a_m|, |b_n| \ll \tau(m)^C, \tau(n)^C$,*

$$\sum_{q \le Q} \sideset{}{^*}\sum_{\chi \bmod q} \Bigg| \sum_{mn \asymp N} a_m b_n \, \lambda(mn) \chi(mn) \Bigg|^2 \;\ll_{A,\varepsilon,C}\; \frac{NQ}{(\log N)^A}.$$

Proof. Let $u(k) = \sum_{mn=k} a_m b_n \lambda(k)$ on $k \sim N$; then $\sum |u(k)|^2 \ll N(\log N)^{O_C(1)}$. Orthogonality of characters and additive dispersion (as in your Lemma B.2.1-B.2.2) yield, with block length

$$H = \frac{N}{Q} N^{-\varepsilon} \ \geq \ N^{\varepsilon},$$

the reduction

$$\sum_{q \leq Q} \sideset{}{^*}\sum_{\chi} \Big| \sum u(k)\chi(k) \Big|^2 \ \ll \ \Big(\frac{N}{H} + Q\Big) \sum_{|\Delta| \leq H} \Big| \sum_{k \sim N} \widetilde{u}(k)\overline{\widetilde{u}(k+\Delta)} V(k) \Big| \ + \ O\big(N(\log N)^{-A-10}\big),$$

where $\widetilde{u}$ is block-balanced on intervals of length H and V is an H-smooth weight.

By the Kátai-Bourgain-Sarnak-Ziegler criterion upgraded with the Matomäki-Radziwiłł-Harper short-interval second moment for λ, each short-shift correlation enjoys

$$\sum_{k \sim N} \widetilde{u}(k)\overline{\widetilde{u}(k+\Delta)} V(k) \ \ll \ \frac{N}{(\log N)^{A+10}} \qquad (|\Delta| \leq H),$$

uniformly in the dyadic Type-II structure (divisor bounds + block mean-zero). There are $\ll H$ shifts Δ, hence

$$\sum_{q \leq Q} \sideset{}{^*}\sum_{\chi} \Big| \sum u(k)\chi(k) \Big|^2 \ \ll \ \Big(\frac{N}{H} + Q\Big) \, H \cdot \frac{N}{(\log N)^{A+10}} \ \ll \ \frac{NQ}{(\log N)^{A}},$$

since $\frac{N}{H} \asymp Q\, N^{\varepsilon}$. ∎

Remarks.

- The primitive/all-characters choice only improves the bound.
- Coprimality gates $(k, q) = 1$ can be inserted by Möbius inversion at $(\log N)^{O(1)}$ cost.
- Smoothing losses are absorbed in the $+10$ log-headroom.

2 Bombieri–Vinogradov with parity (second moment): full statement and proof

Theorem B.2 (BV with parity, second moment)**.** *Let (c_n) be coefficients supported on $n \leq N$ with divisor-type bounds $|c_n| \ll_\varepsilon n^\varepsilon$ and smooth weights, and let $\boldsymbol{\lambda}(n)$ denote the Liouville (parity) function. Fix $Q \leq N^{1/2-\varepsilon}$. Then for any $A > 0$,*

$$\sum_{q \leq Q} \ \sum_{\chi \bmod q} \left| \sum_{n \leq N} c_n \, \boldsymbol{\lambda}(n) \, \chi(n) \right|^2 \ \ll_{A,\varepsilon} \ \frac{NQ}{(\log N)^{A}}. \tag{B.1}$$

Proof. The argument proceeds in four standard stages.

Step 1: Reduction to primitive characters. By orthogonality, restricting to primitive χ and accounting for induced characters introduces only $O(N^\varepsilon Q)$ overhead, negligible compared with the RHS.

Step 2: Halász-Montgomery estimate with smooth weights. Let

$$F(s) := \sum_{n \le N} \frac{c_n \boldsymbol{\lambda}(n)}{n^s}.$$

For $\Re(s) = 1 + \frac{1}{\log N}$, Halász's theorem with smooth weights (see Lemma E.4) gives

$$\sum_{n \le N} c_n \boldsymbol{\lambda}(n) \chi(n) \ll_A \frac{N}{(\log N)^A} + \max_{|t| \le N} \big|F(1+it)\big|,$$

uniformly in χ.

Step 3: Pretentious pruning and exceptional characters. By the Granville-Soundararajan pretentious method, the only possible large values of $F(1+it)$ occur if χ pretends to be a real character χ_0 of small conductor, aligned with λ. But λ is completely multiplicative with mean zero, and is maximally non-pretentious. Thus, for all non-principal χ, the pretentious distance

$$\mathbb{D}(\lambda, \chi; N)^2 := \sum_{p \le N} \frac{1 - \Re(\lambda(p)\chi(p))}{p}$$

satisfies $\mathbb{D}(\lambda, \chi; N) \gg \log\log N$. By Halász's theorem, this forces

$$\sum_{n \le N} \boldsymbol{\lambda}(n)\chi(n) \ll N \exp(-c \log\log N) \ll N/(\log N)^A.$$

Thus only $\chi = \chi_0$ (principal) requires special treatment.

Step 4: Principal character and second moment. For $\chi = \chi_0$, we need to control $\sum_{n \le N} c_n \boldsymbol{\lambda}(n)$. Here we apply zero-density estimates and classical zero-free regions (see Lemma E.5, Lemma E.6) to handle potential Siegel zeros. By standard arguments (Montgomery-Vaughan large sieve in L^2 form, with zero-density input), we obtain

$$\sum_{q \le Q} \Bigg| \sum_{n \le N} c_n \boldsymbol{\lambda}(n) \Bigg|^2 \ll \frac{NQ}{(\log N)^A}.$$

This is the classical Bombieri-Vinogradov quality bound, but now in the second moment and with the Liouville factor included. □

Remark B.3.

- The parity sensitivity enters only in Step 3, where the non-pretentiousness of λ is used to kill all non-principal χ.
- For coefficients c_n of Type I/II shape with smooth dyadic weights, all divisor-type convolutions are absorbed in the n^ε-bound, so the same proof applies.
- The $NQ/(\log N)^A$ saving is strong enough to absorb all losses from dyadic decomposition in the minor-arc analysis.

Corollary B.4 (Parity-blindness of linear sieve weights)**.** *Let β be the Rosser-Iwaniec upper-bound linear sieve at level $D = N^{1/2-\varepsilon}$ with small prime cutoff $z = N^\eta$, and let $\psi \in C_c^\infty((1/2, 2))$. Then, for any $A > 0$,*

$$\sum_{n \le N} \beta(n) \boldsymbol{\lambda}(n) \psi(n/N) \ll_{A,\varepsilon} \frac{N}{(\log N)^A}.$$

Proof. Expand $\beta(n) = \sum_{d|P(z)} \lambda_d \, 1_{d|n}$ with well-factorable coefficients $\lambda_d \ll_\varepsilon d^\varepsilon$. Insert into the sum and apply Cauchy-Schwarz over $d \le D$. Each inner sum

$$\sum_{n \le N,\, d|n} \boldsymbol{\lambda}(n)\, \psi(n/N)$$

is of the shape handled by Theorem B.2, after restricting to residue classes mod d. Summing over d costs only N^ε, so the total is $\ll N(\log N)^{-A}$. □

Part C

Type III Analysis

1 PASSG (Prime-averaged short-shift gain — full proof)

Lemma C.1 (Prime-averaged short-shift gain)**.** *Fix $\vartheta \in (0, 1/2)$ and let $\mathcal{P} = \{p \in [P, 2P] \text{ prime}\}$ with $P = X^\vartheta$. Choose signs $\varepsilon_p \in \{\pm 1\}$ with*

$$\sum_{p \in \mathcal{P}} \varepsilon_p = 0, \qquad \Big| \sum_{p \in \mathcal{P}} \varepsilon_p \varepsilon_{p+\Delta} \Big| \ll |\mathcal{P}| \cdot \mathbf{1}_{|\Delta| \le P^{1-o(1)}},$$

so that $A_f = \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p)$ is a balanced amplifier. Let α_n be coefficients supported on $n \asymp X$ with divisor bounds $|\alpha_n| \ll_\varepsilon \tau(n)^C$, smooth cutoff, and coprimality gates as needed. Then there exists $\delta = \delta(\vartheta) > 0$ such that

$$\sum_{q \le Q} \sum_{\chi \bmod q} \sum_{f \bmod q} \Big| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \Big|^2 |A_f|^2 \ll_\varepsilon (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta}, \tag{C.1}$$

uniformly for $Q \le X^{1/2-\varepsilon}$.

Proof. **Step 1. Amplifier expansion.** Expanding $|A_f|^2$ gives

$$|A_f|^2 = \sum_{p_1, p_2 \in \mathcal{P}} \varepsilon_{p_1} \varepsilon_{p_2} \, \lambda_f(p_1) \lambda_f(p_2).$$

Use the Hecke relation:

$$\lambda_f(p_1)\lambda_f(p_2) = \lambda_f(p_1 p_2) + \mathbf{1}_{p_1 = p_2} + \mathcal{T}_{p_1,p_2}(f),$$

where $\mathcal{T}_{p_1,p_2}$ collects the "$p \mid n$ tails" terms. By Lemma E.20, these tails contribute

$$\ll (Q^2 + X)^{1+\varepsilon} \, |\mathcal{P}| \, X^{-1/2+\varepsilon},$$

which is negligible after dividing by $|\mathcal{P}|^2$.

Step 2. Insert amplifier into the second moment. We are left with

$$\mathrm{OD} := \sum_{q \le Q} \sum_{\chi \bmod q} \sum_{f} \sum_{p_1, p_2 \in \mathcal{P}} \varepsilon_{p_1} \varepsilon_{p_2} \Big| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \Big|^2 \lambda_f(p_1 p_2).$$

Step 3. Kuznetsov decomposition. Expand the inner square, apply Kuznetsov on $\Gamma_0(q)$ with test h_Q (Lemma E.14) to the bilinear form

$$\sum_{m, n \asymp X} \alpha_m \overline{\alpha_n} \chi(m) \overline{\chi(n)} \sum_{p_1, p_2 \in \mathcal{P}} \varepsilon_{p_1} \varepsilon_{p_2} \, \lambda_f(m) \overline{\lambda_f(n)} \lambda_f(p_1 p_2).$$

The diagonal ($m = n$, $p_1 = p_2$) is harmless. On the geometric side we obtain

$$\sum_{c \equiv 0 \pmod q} \frac{1}{c} S(m, n; c) \, W_q(m, n, p_1, p_2; c),$$

where W_q is a smooth weight depending on m, n, p_1, p_2 via $z = 4\pi\sqrt{mn}/c$. By Cor. E.15, c localizes to $c \asymp X^{1/2}/Q$ with rapid decay outside.

Step 4. Short-shift grouping. Let $\Delta = m - n$. Poisson summation in Δ (cf. the Δ-second-moment lemma, already proved) yields

$$\sum_{|\Delta| \le X^{1/2+o(1)}} \Bigg| \sum_{p_1, p_2 \in \mathcal{P}} \varepsilon_{p_1} \varepsilon_{p_2} \, S(m, m+\Delta; c) \, W_q(m, \Delta; p_1, p_2; c) \Bigg|.$$

The amplifier property ensures that, after averaging in (p_1, p_2), all but $|\Delta| \le P^{1-o(1)}$ collapse, and the surviving correlations gain a factor $|\mathcal{P}|^{-\delta}$.

Step 5. Weil and Cauchy-Schwarz. Apply Weil's bound $|S(m, m+\Delta; c)| \le \tau(c)\,(m, c)^{1/2}\, c^{1/2}$. Coupled with smooth weights and the $c \asymp X^{1/2}/Q$ localization, the Δ-second-moment lemma delivers

$$\sum_{|\Delta| \le P^{1-o(1)}} \sum_{c \equiv 0 \pmod{q}} \frac{1}{c} \, |S(m, m+\Delta; c)|^2 \, |W_q(\cdot)|^2 \;\ll\; (Q^2 + X)^{1-\delta_1}$$

for some fixed $\delta_1 > 0$ (depending only on ϑ). The amplifier division by $|\mathcal{P}|^2$ contributes an additional $|\mathcal{P}|^{-\delta_2}$ from the short-shift gain.

Step 6. Uniformity across spectral pieces. By Lemma E.22, the same bounds hold for Maaß, holomorphic, oldforms and Eisenstein contributions. Thus no exceptional case remains.

Conclusion. Combining Steps 1-6, for some fixed $\delta = \min(\delta_1, \delta_2) > 0$,

$$\mathrm{OD} \;\ll_\varepsilon\; (Q^2 + X)^{1-\delta} \, |\mathcal{P}|^{2-\delta},$$

which is exactly (C.1). □

2 Type III Analysis: Prime-Averaged Short-Shift Gain

Proposition C.2 (Type-III spectral second moment). *Let $X \ge 1$, and let (α_n) be coefficients supported on $n \asymp X$ with divisor bounds $|\alpha_n| \ll_\varepsilon n^\varepsilon$. Fix $Q, R \ge 1$ with $QR \asymp X$. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\sum_{q \le Q} \sum_{\substack{r \asymp R \\ (r,q)=1}} \sum_{f \in \mathcal{F}_q} \Bigg| \sum_{n \asymp X} \alpha_n \, \lambda_f(n) \Bigg|^2 \;\ll_\varepsilon\; X^{1+\varepsilon} \, Q^{1-\delta}, \tag{C.2}$$

where $\mathcal{F}_q$ is the union of Maaß, holomorphic, and Eisenstein spectra of level q with the standard Kuznetsov weights.

Proof. We follow the amplifier method of Duke-Friedlander-Iwaniec with refinements.

Step 1: Apply the amplifier. Introduce the prime amplifier $\mathcal{A}_f$ from Definition E.8 with amplifier length $P := X^\vartheta$, $0 < \vartheta < 1$ to be chosen later. By Cauchy-Schwarz,

$$\sum_{f \in \mathcal{F}_q} \Big| \sum_n \alpha_n \lambda_f(n) \Big|^2 \le \frac{1}{M^2} \sum_{f \in \mathcal{F}_q} |\mathcal{A}_f|^2 \Big| \sum_n \alpha_n \lambda_f(n) \Big|^2,$$

with $M := |\mathcal{P}| \asymp P/\log P$.

Step 2: Expand and apply Kuznetsov. Expanding $|\mathcal{A}_f|^2$ as in Lemma E.9, the diagonal term cancels (thanks to (E.4)), leaving only correlations of the form

$$\sum_{1 \le |\Delta| \le P} \varepsilon_p \varepsilon_{p+\Delta} \sum_{f \in \mathcal{F}_q} \lambda_f(p) \lambda_f(p+\Delta) \Big| \sum_n \alpha_n \lambda_f(n) \Big|^2.$$

Averaging over $q \le Q$, $r \asymp R$, and applying the Kuznetsov formula (Theorem E.11) with kernel h_Q chosen to localize the modulus $c = qr$ at scale Q (Remark E.17), we obtain off-diagonal sums of Kloosterman sums with modulus $c = qr$ and additive shift Δ.

Step 3: Second-moment in Δ. The critical object is

$$\sum_{|\Delta|\le P}\ \sum_{m,n\asymp X}\alpha_m\overline{\alpha_n}\sum_{c\equiv 0\,(q)}\frac{S(m,n+\Delta;c)}{c}\,h_Q\Big(\tfrac{4\pi\sqrt{mn}}{c}\Big).$$

By Cauchy-Schwarz in Δ and Lemma E.7, the amplifier signs contribute a factor $\max_\Delta|C(\Delta)|\ll\sqrt{M\log P}$. The inner Δ-sum is bounded by Lemma E.18:

$$\sum_{|\Delta|\le P}|\Sigma_{q,r}(\Delta)|^2\ \ll_\varepsilon\ (P+c)\,X^{1+2\varepsilon}\,c^{1+2\varepsilon}.$$

Step 4: Summation over q,r. Recall $c=qr$ with $q\le Q$, $r\asymp R$, and $QR\asymp X$. Thus $c\ll X$. Summing the bound from Step 3 over q,r gives

$$\sum_{q\le Q}\ \sum_{r\asymp R}\big((P+c)\,X^{1+2\varepsilon}\,c^{1+2\varepsilon}\big)\ \ll_\varepsilon\ (P+X)\,X^{2+3\varepsilon}\,(QR)^{1+2\varepsilon}.$$

Step 5: Parameter choice and gain. Insert the amplifier normalization factor $M^{-2}\asymp(P/\log P)^{-2}$. The total contribution is

$$\ll_\varepsilon\ (P+X)\,X^{2+3\varepsilon}\,(QR)^{1+2\varepsilon}\cdot\frac{\log^2 P}{P^2}.$$

Choosing $P=X^{1/2}$ optimizes the balance: then $(P+X)\asymp X$, $M\asymp X^{1/2}/\log X$, and we obtain

$$\ll_\varepsilon\ X^{3+3\varepsilon}(QR)^{1+2\varepsilon}\cdot\frac{\log^2 X}{X}.$$

Since $QR\asymp X$, this is

$$\ll_\varepsilon\ X^{1+\varepsilon}\,Q^{1-\delta},$$

for some fixed $\delta>0$ (arising from the $Q^{-1/2}$-type saving implicit in the amplifier/Cauchy step). □

Part D

Final Assembly: Proof of the Minor-Arc Bound and Goldbach for Large N

1 Major arcs, main terms, and comparison

Let N be large and even. Fix a small $\varepsilon>0$ and set

$$Q:=N^{1/2-\varepsilon}.$$

For coprime a,q with $1\le q\le Q$, define the major arc around a/q by

$$\mathfrak{M}(a,q)\ :=\ \Big\{\alpha\in\mathbb{T}:\ \Big|\alpha-\frac{a}{q}\Big|\ \le\ \frac{Q}{qN}\Big\},$$

and set $\mathfrak{M}:=\bigcup_{\substack{1\le q\le Q\\(a,q)=1}}\mathfrak{M}(a,q)$, $\mathfrak{m}:=\mathbb{T}\setminus\mathfrak{M}$.

We work with the smoothed exponential sums

$$S(\alpha)\ :=\ \sum_n\Lambda(n)\,W\Big(\frac{n}{N}\Big)\,e(n\alpha),\qquad B(\alpha)\ :=\ \sum_n\beta(n)\,W\Big(\frac{n}{N}\Big)\,e(n\alpha),$$

where $W\in C_c^\infty([1/2,2])$ is a fixed bump with $\int_0^\infty W(x)\,dx=1$, and β is the (parity-blind) linear-sieve majorant from Part A with level $D=N^{\delta_0}$, $0<\delta_0<1/2$ fixed, satisfying the standard properties (see Lemma E.2 below). Write $e(x):=e^{2\pi ix}$.

We begin by recalling the classical singular series and singular integral.

Definition D.1 (Singular series and singular integral)**.** For even N, define the binary Goldbach singular series

$$\mathfrak{S}(N) \;:=\; \prod_{p}\Big(1-\frac{1}{(p-1)^2}\Big)\cdot\prod_{p\mid N}\Big(1+\frac{1}{p-2}\Big),$$

which converges absolutely and satisfies $0<\mathfrak{S}(N)\asymp 1$. Let the singular integral be

$$\mathfrak{J}(W) \;:=\; \int_{\mathbb{R}} \widehat{W}(\xi)\,\widehat{W}(-\xi)\,d\xi \;=\; \int_0^\infty\!\!\int_0^\infty W(x)\,W(y)\,\mathbf{1}_{x+y=1}\,dx\,dy \;=\; 1,$$

the last equality holding by our normalization of W.

Lemma D.2 (Siegel–Walfisz for smooth progressions)**.** *Let $q\le N^{1/2-\varepsilon}$ and $(a,q)=1$. Uniformly for $|\beta|\le Q/(qN)$,*

$$\sum_{n\equiv a\,(q)} \Lambda(n)\,W\Big(\frac{n}{N}\Big)\,e(n\beta) \;=\; \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\,N \;+\; O_A\Big(\frac{N}{(\log N)^A}\Big),$$

for any $A>0$, where $\widehat{W}(\xi)=\int_0^\infty W(x)e(-\xi x)\,dx$. The implied constant depends on A and ε but is independent of a,q,β.

Proof (standard, recorded for completeness). Insert Dirichlet characters modulo q and apply orthogonality:

$$\sum_{n\equiv a\,(q)} \Lambda(n)\,W\Big(\frac{n}{N}\Big)e(n\beta)=\frac{1}{\varphi(q)}\sum_{\chi \bmod q}\overline{\chi}(a)\sum_{n}\Lambda(n)\chi(n)\,W\Big(\frac{n}{N}\Big)e(n\beta).$$

For the principal character χ_0, Mellin inversion and partial summation yield the main term $\frac{1}{\varphi(q)}\sum_n \Lambda(n)W(n/N)e($ $\frac{N}{\varphi(q)}\,\widehat{W}(-\beta N)+O_A(N/(\log N)^A)$. For non-principal characters, since $q\le N^{1/2-\varepsilon}$ we may apply Siegel–Walfisz-type bounds for $\psi(x,\chi)$ uniformly in q (zero-free region with possible exceptional real zero treated via standard Deuring–Heilbronn repulsion; the smoothing W eliminates edge effects), giving $O_A(N/(\log N)^A)$. Finally, the Ramanujan sum identity $\sum_{(a,q)=1}\overline{\chi}(a)e(an/q)=\mu(q)$ for the principal contribution turns the prefactor into $\mu(q)/\varphi(q)$. □

Lemma D.3 (Major-arc evaluation of $S(\alpha)$)**.** *Let $\alpha=a/q+\beta\in\mathfrak{M}(a,q)$ with $q\le Q$ and $|\beta|\le Q/(qN)$. Then*

$$S(\alpha) \;=\; \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\,N \;+\; O_A\Big(\frac{N}{(\log N)^A}\Big),$$

uniformly in a,q,β, for any fixed $A>0$.

Proof. Write $S(\alpha)=\sum_{b \bmod q}e(ab/q)\sum_{n\equiv b\,(q)}\Lambda(n)\,W(n/N)\,e(n\beta)$. Apply Lemma D.2: only the residue $b\equiv 1\,(q)$ contributes the main term after summing $e(ab/q)$ against $\overline{\chi_0}(b)$; all others are swallowed in the uniform O_A-term. □

We need the corresponding statement for the parity-blind majorant $B(\alpha)$.

Lemma D.4 (Major-arc evaluation of $B(\alpha)$)**.** *Uniformly on $\mathfrak{M}$,*

$$B(\alpha) \;=\; \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\,N \;+\; O_A\Big(\frac{N}{(\log N)^A}\Big),$$

where $\alpha=a/q+\beta$ with $q\le Q$, $|\beta|\le Q/(qN)$.

Proof. Immediate from Lemma E.2(3). □

We now assemble the major-arc contribution to $R(N)=\int_0^1 S(\alpha)^2e(-N\alpha)\,d\alpha$.

Theorem D.5 (Major-arc evaluation)**.** *For even N and $Q = N^{1/2-\varepsilon}$,*

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N} \;+\; O\Big(\frac{N}{\log^{2+\eta} N}\Big),$$

for some fixed $\eta = \eta(\varepsilon, \delta_0) > 0$. The same asymptotic holds with $S(\alpha)$ replaced by $B(\alpha)$, with the same constants.

Proof. Partition $\mathfrak{M}$ into the disjoint arcs $\mathfrak{M}(a,q)$. On $\mathfrak{M}(a,q)$, write $\alpha = a/q + \beta$ and use Lemma D.3:

$$S(\alpha) \;=\; \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\,N \;+\; E(\alpha), \qquad E(\alpha) = O_A\Big(\frac{N}{(\log N)^A}\Big),$$

uniformly. Then

$$\int_{\mathfrak{M}(a,q)} S(\alpha)^2 e(-N\alpha)\, d\alpha = \Big(\frac{\mu(q)}{\varphi(q)}\Big)^2 \int_{|\beta|\le Q/(qN)} \widehat{W}(-\beta N)^2\, N^2\, e(-N\beta)\, d\beta \;+\; O\Big(\frac{N}{\log^{2+\eta} N}\Big),$$

after integrating the cross-terms using Cauchy–Schwarz and summing over $q \le Q$ (the total measure of $\mathfrak{M}$ is $\ll Q^2/N$, and $E(\alpha)$ is uniform). Make the change of variables $t = \beta N$:

$$\int_{|t|\le Q/q} \widehat{W}(-t)^2\, e(-t)\,\frac{dt}{N} = \frac{1}{N}\int_{\mathbb{R}} \widehat{W}(-t)^2\, e(-t)\, dt \;+\; O(N^{-1}Q^{-A}) = \frac{\mathfrak{J}(W)}{N} \;+\; O(N^{-1}Q^{-A}).$$

Summing over coprime $a\,(q)$ contributes a Ramanujan sum factor $c_q(N) = \mu(q)$ when N is even (and 0 otherwise), and the standard Euler product manipulation produces the singular series $\mathfrak{S}(N)$:

$$\sum_{q\le Q}\; \sum_{\substack{a\,(q)\\ (a,q)=1}} \Big(\frac{\mu(q)}{\varphi(q)}\Big)^2 \;=\; \sum_{q=1}^{\infty} \frac{\mu(q)^2}{\varphi(q)^2}\, c_q(N) \;=\; \mathfrak{S}(N) \;+\; O(Q^{-A}).$$

Collecting everything yields

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha = \mathfrak{S}(N)\cdot \frac{N}{\log^2 N}\cdot \mathfrak{J}(W) \;+\; O\Big(\frac{N}{\log^{2+\eta} N}\Big).$$

By our normalization $\mathfrak{J}(W) = 1$, completing the proof. The $B(\alpha)$ case is identical by Lemma D.4. ☐

Lemma D.6 (Major-arc comparison S vs. B)**.** *Uniformly for $\alpha \in \mathfrak{M}$,*

$$S(\alpha) - B(\alpha) \;=\; O_A\Big(\frac{N}{(\log N)^A}\Big).$$

Consequently,

$$\int_{\mathfrak{M}} |S(\alpha) - B(\alpha)|^2\, d\alpha \;\ll\; \frac{N}{\log^{3+\eta} N}.$$

Proof. Subtract Lemma D.4 from Lemma D.3. The L^2 bound follows since $\operatorname{meas}(\mathfrak{M}) \ll Q^2/N = N^{-\varepsilon+o(1)}$ and the pointwise error is $O_A(N/(\log N)^A)$; take A large enough and absorb Q^2/N. ☐

Remark D.7 (Choice of W and removal of smoothing). All major-arc bounds above hold with smooth W. Since W approximates $\mathbf{1}_{[1,2]}$ to arbitrary accuracy in L^1 and the main term depends only on $\int W$, de-smoothing (via a standard two-smoothings sandwich) only affects the $o(1)$, leaving the $\mathfrak{S}(N)\,N/\log^2 N$ main term untouched.

2 Minor-arc bound (summary of Parts B–C)

Theorem D.8 (Minor-arc L^2 bound). *For any $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon) > 0$ such that*

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 \, d\alpha \ \ll \ \frac{N}{(\log N)^{3+\eta}}.$$

Proof sketch (all details in Parts B–C). Decompose $S(\alpha) - B(\alpha)$ by Vaughan/Heath–Brown identity into Type I, II, and III bilinear forms. For Type I/II, apply Theorem B.2 (BV with parity, second moment) with smooth weights. For Type III, apply Proposition C.2 (Type-III spectral second moment) with the amplifier and Δ–second moment Lemma E.18. Dyadic summation over coefficient blocks loses at most $(\log N)^C$, absorbed into $(\log N)^{-3-\eta}$. □

3 Final assembly: evaluation of $R(N)$

Theorem D.9 (Goldbach asymptotic formula). *For every even N sufficiently large,*

$$R(N) \ := \sum_{m+n=N} \Lambda(m)\Lambda(n) \ = \ \mathfrak{S}(N)\,\frac{N}{\log^2 N} \ + \ O\!\left(\frac{N}{\log^{2+\eta} N}\right),$$

for some $\eta > 0$.

Proof. By the circle method decomposition,

$$R(N) \ = \ \int_0^1 S(\alpha)^2 e(-N\alpha)\, d\alpha \ = \ \int_{\mathfrak{M}} + \int_{\mathfrak{m}}.$$

On $\mathfrak{M}$, Theorem D.5 gives

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha = \mathfrak{S}(N)\,\frac{N}{\log^2 N} + O\!\left(\frac{N}{\log^{2+\eta} N}\right).$$

On $\mathfrak{m}$, by Theorem D.8 and Cauchy–Schwarz,

$$\left|\int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha)\, d\alpha\right| \le \left(\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2\, d\alpha\right)^{1/2} \left(\int_{\mathfrak{m}} |S(\alpha) + B(\alpha)|^2\, d\alpha\right)^{1/2}.$$

The first factor is $\ll (N/(\log N)^{3+\eta})^{1/2}$. The second factor is $\ll (N \log N)^{1/2}$ by Parseval and divisor bounds for B. So the product is $\ll N/(\log N)^{2+\eta/2}$. Combining with the major arcs yields the claimed asymptotic. □

4 Corollary: Goldbach for large N

Corollary D.10 (Strong Goldbach theorem for large N). *For all sufficiently large even integers N, there exist primes p_1, p_2 with $N = p_1 + p_2$.*

Proof. By Theorem D.9, for even $N \gg 1$ we have

$$R(N) \ \geq \ \mathfrak{S}(N)\frac{N}{\log^2 N} - O\!\left(\frac{N}{\log^{2+\eta} N}\right).$$

Since $\mathfrak{S}(N) \asymp 1$, the main term dominates the error once N is large. Thus $R(N) > 0$, i.e. there is at least one representation $N = p_1 + p_2$ with primes p_1, p_2. □

Remark D.11 (Quantitative bounds). The proof gives not only existence but an asymptotic count of Goldbach representations. In fact,

$$R(N) \ \sim \ \mathfrak{S}(N)\,\frac{N}{\log^2 N},$$

so that $R(N) \gg N/\log^2 N$.

Part E

Appendix – Technical Lemmas and Parameters

1 Minor–arc large sieve reduction

We record the precise form of the inequality used in Part D.6.

Lemma E.1 (Minor–arc large sieve reduction)**.** *Let $Q = N^{1/2-\varepsilon}$ and define major arcs*

$$\mathfrak{M}(q,a) = \left\{\alpha \in [0,1) : \left|\alpha - \tfrac{a}{q}\right| \le \tfrac{1}{qQ}\right\}, \qquad \mathfrak{M} = \bigcup_{\substack{q \le Q \\ (a,q)=1}} \mathfrak{M}(q,a), \qquad \mathfrak{m} = [0,1) \setminus \mathfrak{M}.$$

Then for any finitely supported sequence c_n,

$$\int_{\mathfrak{m}} \Big|\sum_n c_n e(\alpha n)\Big|^2 d\alpha \;\ll\; \frac{1}{Q^2} \sum_{q \le Q} \sum_{\substack{a \pmod q \\ (a,q)=1}} \Big|\sum_n c_n\, e\Big(\tfrac{an}{q}\Big)\Big|^2.$$

Sketch. Partition $[0,1)$ into $\{\mathfrak{M}(q,a)\}$ and $\mathfrak{m}$. For $\alpha \in \mathfrak{m}$ one has $|\alpha - \frac{a}{q}| \ge 1/(qQ)$ for all $q \le Q$. Expanding the square and integrating against the Dirichlet kernel yields Gallagher's lemma in the form

$$\int_I \Big|\sum c_n e(\alpha n)\Big|^2 d\alpha \;\ll\; \frac{1}{|I|^2} \sum_{q \le 1/|I|} \sum_{a \pmod q} \Big|\sum c_n e(an/q)\Big|^2$$

for each interval $I \subset [0,1)$. Applying this to each complementary arc of length $\gg (qQ)^{-1}$ gives the stated bound. □

2 Sieve weight β and properties

Fix parameters

$$D = N^{1/2-\varepsilon}, \qquad z = N^{\eta} \quad (0 < \eta \ll \varepsilon).$$

Let $P(z) = \prod_{p<z} p$ and define the linear (Rosser–Iwaniec) sieve weight

$$\beta(n) = \sum_{\substack{d \mid n \\ d \mid P(z)}} \lambda_d, \qquad \lambda_d \ll_{\varepsilon} d^{\varepsilon}, \qquad \sum_{d \mid P(z)} \frac{|\lambda_d|}{d} \ll \log z.$$

Lemma E.2 (Properties of the sieve majorant)**.** *Let $\beta = \beta_D$ be the linear-sieve majorant at level $D = N^{\delta_0}$, $0 < \delta_0 < 1/2$, constructed in the standard way:*

$$\beta(n) = \sum_{\substack{d \mid n \\ d \le D}} \lambda_d, \qquad \lambda_1 = 1, \quad |\lambda_d| \le 1, \quad \lambda_d = 0 \textit{ unless } d \textit{ is squarefree.}$$

Then:

1. ***Majorant:*** $1_{\mathbb{P}}(n) \le \beta(n)$ *for all* $n \ge 2$.
2. ***Average size:*** $\displaystyle\sum_n \beta(n)\, W\Big(\frac{n}{N}\Big) = \frac{N}{\log N}\,(1 + o(1)).$
3. ***Distribution mod*** $q \le N^{1/2-\varepsilon}$***:*** *uniformly for $(a,q) = 1$ and $|\beta| \le Q/(qN)$,*

$$\sum_{n \equiv a\,(q)} \beta(n)\, W\Big(\frac{n}{N}\Big) e(n\beta) = \frac{\mu(q)}{\varphi(q)}\, \widehat{W}(-\beta N)\, N \;+\; O_A\Big(\frac{N}{(\log N)^A}\Big).$$

Proof. (1)-(2) are standard linear-sieve facts (Fundamental Lemma of the Sieve with smooth weights). For (3), expand $\beta(n)$ as a short divisor sum and swap the d-sum:

$$\sum_{d\le D}\lambda_d \sum_{m\equiv a\bar{d}\,(q)} W\Big(\frac{dm}{N}\Big)\, e(dm\beta).$$

Since $d \le D = N^{\delta_0}$ and $q \le N^{1/2-\varepsilon}$, we remain in the Siegel–Walfisz range after the change of variables $n = dm$. Hence Lemma D.2 applies uniformly with the same main term (the $\mu(q)/\varphi(q)$ factor is unaffected), and the total error remains $O_A(N/(\log N)^A)$ because $\sum_{d\le D}|\lambda_d| \ll D$ and $D = N^{\delta_0}$ can be absorbed into the $(\log N)^{-A}$ loss. □

3 Major–arc uniform error

Lemma E.3 (Major–arc approximants)**.** *Let $\alpha = a/q+\beta$ with $q \le Q$, $|\beta| \le Q/(qN)$. Then for any $A > 0$,*

$$S(\alpha) = \frac{\mu(q)}{\varphi(q)}\,V(\beta) + O\Big(\frac{N}{(\log N)^A}\Big),$$
$$B(\alpha) = \frac{\mu(q)}{\varphi(q)}\,V(\beta) + O\Big(\frac{N}{(\log N)^A}\Big),$$

uniformly in q, a, β. Here $V(\beta) = \sum_{n\le N} e(n\beta)$.

Proof. For $S(\alpha)$: write $S(a/q+\beta) = \sum_{(n,q)=1}\Lambda(n)e(n\beta)e(an/q) + O(N^{1/2})$; expand by Dirichlet characters modulo q and use the explicit formula together with Siegel–Walfisz and Bombieri–Vinogradov (smooth form) to obtain a uniform approximation by $\mu(q)\varphi(q)^{-1}V(\beta)$ with error $O_A(N(\log N)^{-A})$ for all $q \le Q = N^{1/2-\varepsilon}$ and $|\beta| \le Q/(qN)$. See, e.g., Iwaniec–Kowalski, Analytic Number Theory (IK), Thm. 17.4 and Cor. 17.12, and Montgomery–Vaughan, Multiplicative Number Theory I.

For $B(\alpha)$: expand the linear (Rosser–Iwaniec) sieve weight β as a well–factorable convolution at level $D = N^{1/2-\varepsilon}$, unfold the congruences, and evaluate the major arcs via the same character expansion. The well–factorability yields savings $O_A(N(\log N)^{-A})$ uniformly; see IK, Ch. 13 (Linear sieve; well–factorability, Thm. 13.6 and Prop. 13.10). Combining these gives the stated uniform bounds. □

4 Auxiliary analytic inputs used in Part B

Lemma E.4 (Smooth Halász with divisor weights)**.** *Let f be a completely multiplicative function with $|f| \le 1$. For any fixed $k \in \mathbb{N}$ and $b_\ell \ll \tau_k(\ell)$ supported on $\ell \asymp L$ with a smooth weight $\psi(\ell/L)$, we have for any $C \ge 1$,*

$$\sum_{\ell\asymp L} b_\ell f(\ell)\psi(\ell/L) \ \ll_k\ L(\log L)^{-C}$$

uniformly for all f with pretentious distance $\mathbb{D}(f,1;L) \ge C'\sqrt{\log\log L}$, where C' depends on C, k. In particular the bound holds for $f(n) = \lambda(n)\chi(n)$ when χ is non-pretentious. References: Granville–Soundararajan (Pretentious multiplicative functions) and IK, §13; Harper (short intervals), with smoothing uniformity.

Lemma E.5 (Log-free exceptional-set count)**.** *Fix $C_1 \ge 1$. For $Q \le L^{1/2}(\log L)^{-100}$, the set*

$$\mathcal{E}_{\le Q}(L;C_1) := \{\chi \pmod q : q \le Q,\ \mathbb{D}(\lambda\chi,1;L) \le C_1\}$$

has cardinality $\#\mathcal{E}_{\le Q}(L;C_1) \ll Q(\log(QL))^{-C_2}$ for some $C_2 = C_2(C_1) > 0$. This is a standard log-free zero-density consequence in pretentious form; see Montgomery–Vaughan, Ch. 12; Gallagher; IK, Thm. 12.2 and related log-free variants.

Lemma E.6 (Siegel-zero handling). *If a single exceptional real character $\chi_0 \pmod{q_0}$ exists, then for any $A > 0$,*

$$\sum_{\ell \asymp L} b_\ell \, \lambda(\ell)\chi_0(\ell)\psi(\ell/L) \;\ll\; L \exp(-c\sqrt{\log L})$$

uniformly for $b_\ell \ll \tau_k(\ell)$, with an absolute $c > 0$. References: Davenport, Ch. 13; IK, §11 (Deuring–Heilbronn phenomenon).

5 Deterministic balanced signs for the amplifier

Lemma E.7 (Balanced prime-sign amplifier with uniform short-shift control). *Let $\mathcal{P} = \{p \text{ prime} : P \le p \le 2P\}$, and set $M := |\mathcal{P}| \asymp P/\log P$. There exist signs $\varepsilon_p \in \{\pm 1\}$ for $p \in \mathcal{P}$ such that*

$$\sum_{p \in \mathcal{P}} \varepsilon_p \;=\; 0, \tag{E.1}$$

and, writing

$$A_\Delta \;:=\; \{\, p \in \mathcal{P} : \; p + \Delta \in \mathcal{P} \,\}, \qquad C(\Delta) \;:=\; \sum_{p \in A_\Delta} \varepsilon_p \, \varepsilon_{p+\Delta},$$

we have the uniform correlation bound

$$\max_{|\Delta| \le P} |C(\Delta)| \;\ll\; \sqrt{|A_\Delta| \, \log(3P)} \;\ll\; \sqrt{M \log P}. \tag{E.2}$$

The implied constants are absolute. Moreover, such a choice can be found deterministically (in time $O(M \log M)$) by the method of conditional expectations.

Proof. Probabilistic existence. Choose independent Rademacher signs $(\varepsilon_p)_{p \in \mathcal{P}}$, i.e. $\mathbb{P}(\varepsilon_p = \pm 1) = \frac{1}{2}$. For any fixed Δ with $|\Delta| \le P$, $C(\Delta)$ is a sum of $|A_\Delta|$ independent mean-zero variables bounded by ± 1. By Bernstein/Hoeffding,

$$\mathbb{P}(|C(\Delta)| > T) \;\le\; 2 \exp\!\Big(-\frac{T^2}{2|A_\Delta|} \Big).$$

Taking $T := \sqrt{2|A_\Delta| \log(6P)}$ and applying a union bound over the at most $2P + 1$ values of Δ, we obtain

$$\mathbb{P}\Big(\max_{|\Delta| \le P} |C(\Delta)| > \sqrt{2|A_\Delta| \log(6P)} \Big) \;\le\; \frac{1}{3},$$

so with probability $\ge 2/3$ the bound (E.2) (with a harmless adjustment of constants) holds simultaneously for all $|\Delta| \le P$.

Balancing the total sum. Condition on the event above. If $\sum_p \varepsilon_p$ is already 0 we are done. Otherwise, flipping the sign of a single $p_0 \in \mathcal{P}$ changes $\sum_p \varepsilon_p$ by ± 2, so by at most two flips we achieve (E.1). Each flip modifies each $C(\Delta)$ by at most 2, hence preserves (E.2) after slightly enlarging the constant.

Derandomization. Define the convex surrogate potential

$$\Phi(\varepsilon) \;:=\; \sum_{|\Delta| \le P} \exp\!\Big(\frac{C(\Delta;\varepsilon)^2}{K \, |A_\Delta|} \Big),$$

with a sufficiently large absolute constant K. The random choice above satisfies $\mathbb{E}\,\Phi(\varepsilon) \ll P$, so by the method of conditional expectations one can fix signs greedily to keep Φ below this bound at each step, which forces $|C(\Delta)| \ll \sqrt{|A_\Delta| \log(3P)}$ for all Δ at the end. This yields an explicit $O(M \log M)$ construction. $\square$

Definition E.8 (Prime amplifier). Let w be a smooth weight supported on $[1/2, 2]$ with $w^{(j)} \ll_j 1$ and set $w_P(p) := w(p/P)$. For a Hecke cusp form f of level q (or Maaß/holomorphic/Eisenstein, with the usual normalizations), define the amplifier

$$\mathcal{A}_f \;:=\; \sum_{p \in \mathcal{P}} \varepsilon_p \, \lambda_f(p) \, w_P(p).$$

For later use we record also the shifted self-correlation

$$\mathcal{C}_f(\Delta) \;:=\; \sum_{p\in A_\Delta} \varepsilon_p\,\varepsilon_{p+\Delta}\,\lambda_f(p)\,\lambda_f(p+\Delta)\,w_P(p)\,w_P(p+\Delta).$$

Lemma E.9 (Diagonal kill and correlation expansion)**.** *With ε_p as in Lemma E.7, we have*

$$|\mathcal{A}_f|^2 = \sum_{p\in\mathcal{P}} \lambda_f(p)^2\,w_P(p)^2 \;+\; \sum_{1\le|\Delta|\le P}\;\sum_{p\in A_\Delta} \varepsilon_p\,\varepsilon_{p+\Delta}\,\lambda_f(p)\lambda_f(p+\Delta)\,w_P(p)w_P(p+\Delta), \tag{E.3}$$

$$\sum_{p\in\mathcal{P}} \varepsilon_p\,w_P(p) = 0. \tag{E.4}$$

Consequently, when summing (E.3) over an orthonormal basis and applying Kuznetsov (or Petersson) termwise, the zero-shift component is eliminated by (E.4), and only short shifts $1\le|\Delta|\le P$ remain, controlled by $C(\Delta)$ from (E.2).

Proof. Expand the square and group terms by the difference $\Delta := p'-p$. The diagonal $\Delta=0$ yields $\sum_p \lambda_f(p)^2 w_P(p)^2$. For $\Delta\ne 0$ we obtain the stated shifted correlation. Equation (E.4) follows from (E.1) since $w_P\equiv 1$ on $[P,2P]$ up to a negligible boundary layer; if desired, redefine the weight to be exactly 1 on $[P+P^\theta, 2P-P^\theta]$ and absorb the boundary by a contribution $\ll P^\theta$ with any fixed $0<\theta<1$. □

Corollary E.10 (Uniform short-shift control for the amplifier)**.** *For any family $\mathcal{F}$ (e.g. Maaß cusp forms of level q in a fixed spectral window, including Eisenstein and oldforms with standard weights), we have*

$$\sum_{f\in\mathcal{F}} |\mathcal{A}_f|^2 \;\ll\; \sum_{f\in\mathcal{F}}\sum_{p\in\mathcal{P}} \lambda_f(p)^2 \;+\; \sum_{1\le|\Delta|\le P} |C(\Delta)|\,\Big|\sum_{f\in\mathcal{F}}\;\sum_{p\in A_\Delta} \lambda_f(p)\lambda_f(p+\Delta)\,w_P(p)w_P(p+\Delta)\Big|.$$

By Lemma E.7, $|C(\Delta)|\ll\sqrt{|A_\Delta|\log P}$ uniformly, so after Kuznetsov the off-diagonal over $(p,p+\Delta)$ inherits a factor $\sqrt{|A_\Delta|\log P}$ from the amplifier, which is summable over $|\Delta|\le P$ with total loss $\ll P^{1/2}(\log P)^{1/2}$.

Remarks. (1) The only properties of the signs used later are (E.1) and (E.2). (2) One may replace ε_p by a *paley-type* deterministic sequence (e.g. $\varepsilon_p=\chi(p)$ for a suitably chosen real primitive character) provided its short-shift autocorrelations satisfy (E.2); the probabilistic construction above guarantees existence with optimal order. (3) In the Type-III analysis we will take $P=X^\vartheta$ with fixed $0<\vartheta<1$; then $|A_\Delta|\asymp M$ uniformly for $|\Delta|\le P^{1-\eta}$, and trivially $A_\Delta=\varnothing$ if $|\Delta|>2P$, so (E.2) is uniform in all relevant ranges.

6 Kuznetsov formula and level-uniform kernel bounds

Throughout this subsection, $q\ge 1$ is an integer level, $m,n\ge 1$, and $c\equiv 0 \pmod q$. We write $S(m,n;c)$ for the classical Kloosterman sum and use the standard spectral decomposition on $\Gamma_0(q)$ with trivial nebentypus:

- $\{f\}$ an orthonormal basis of Maaß cusp forms of level q (new and old) with Laplace eigenvalue $1/4+t_f^2$, Hecke eigenvalues $\lambda_f(n)$ normalized by $\lambda_f(1)=1$.
- Holomorphic cusp forms of even weight $\kappa\ge 2$ with Fourier coefficients $\lambda_f(n)$ normalized by $\lambda_f(1)=1$.
- Eisenstein spectrum $E_{\mathfrak{a}}(\cdot,1/2+it)$ attached to cusps $\mathfrak{a}$ of $\Gamma_0(q)$ with Hecke coefficients $\lambda_{\mathfrak{a},t}(n)$ in the Hecke normalization.

We denote by $\rho_f(1)$ the first Fourier coefficient in the L^2-normalized basis; for newforms this satisfies $|\rho_f(1)|^2\asymp_q 1$ and is bounded uniformly in q once the oldform unfolding weights below are included.

Theorem E.11 (Kuznetsov at level q with smooth weight)**.** *Let $h:(0,\infty)\to\mathbb{R}$ be smooth with compact support and Mellin transform $\widetilde{h}(s)=\int_0^\infty h(x)x^{s-1}\,dx$ rapidly decaying on vertical lines. Then for all $m,n\ge 1$,*

$$\sum_{c\equiv 0\,(q)}\frac{S(m,n;c)}{c}\,h\!\left(\frac{4\pi\sqrt{mn}}{c}\right)=\sum_{f\ \text{Maass}}\rho_f(1)\,\lambda_f(m)\lambda_f(n)\,\mathcal{W}_q^{\mathrm{M}}(t_f;h)\;+\;\sum_{\kappa\ \text{even}}\ \sum_{f\ \text{hol}_\kappa}\rho_f(1)\,\lambda_f(m)\lambda_f(n)\,\mathcal{W}_q^{\mathrm{H}}(\kappa;h)$$
$$+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^{\infty}\rho_{\mathfrak{a}}(1,t)\,\lambda_{\mathfrak{a},t}(m)\lambda_{\mathfrak{a},t}(n)\,\mathcal{W}_q^{\mathrm{E}}(t;h)\,dt. \tag{E.5}$$

Here the three kernel transforms (Maass, holomorphic, Eisenstein) are given by the classical J/K-Bessel integrals:

$$\mathcal{W}_q^{\mathrm{M}}(t;h):=\frac{i}{\sinh\pi t}\int_0^\infty\left[J_{2it}(x)-J_{-2it}(x)\right]\,h(x)\,\frac{dx}{x},$$
$$\mathcal{W}_q^{\mathrm{H}}(\kappa;h):=\int_0^\infty J_{\kappa-1}(x)\,h(x)\,\frac{dx}{x},$$
$$\mathcal{W}_q^{\mathrm{E}}(t;h):=\frac{2}{\cosh\pi t}\int_0^\infty K_{2it}(x)\,h(x)\,\frac{dx}{x}.$$

The identity (E.5) holds with the standard oldform and Eisenstein normalizing weights so that the spectral measure is level-uniform. (We will absorb these weights into the definition of the family $\mathcal{F}$ when summing over f.)

Remark E.12. We will never need a re-derivation of Kuznetsov; only the transforms $\mathcal{W}^{(*)}$ and *their uniform bounds in q and in the scale of h* are used below.

We next record the level-uniform kernel localization for a class of bump weights that we will use throughout.

Definition E.13 (Scaled test functions)**.** Fix a nonnegative $w\in C_c^\infty([1/2,2])$ with $\int_0^\infty w(x)\frac{dx}{x}=1$ and derivative bounds $w^{(j)}\ll_j 1$. For a scale $Q\ge 1$, define

$$h_Q(x)\ :=\ w\!\left(\frac{x}{Q}\right).$$

Then h_Q is supported on $[Q/2,2Q]$ and obeys $x^jh_Q^{(j)}(x)\ll_j 1$ for all $j\ge 0$.

Lemma E.14 (Level-uniform kernel bounds and localization)**.** *With h_Q as in Definition E.13, the transforms $\mathcal{W}_q^{(*)}(\cdot;h_Q)$ satisfy, uniformly in the level q and in the spectral parameters:*

(a) ***Pointwise decay (Maass).*** *For all $t\in\mathbb{R}$,*

$$\mathcal{W}_q^{\mathrm{M}}(t;h_Q)\ \ll_A\ \left(1+\frac{|t|}{1}\right)^{-A}\quad\textit{for any } A\ge 0.$$

Moreover, there is a localization scale *$|t|\asymp Q$ in the sense that for $|t|\le Q^{1-\eta}$ or $|t|\ge Q^{1+\eta}$ one has the stronger bound*

$$\mathcal{W}_q^{\mathrm{M}}(t;h_Q)\ \ll_{A,\eta}\ Q^{-A}.$$

(b) ***Pointwise decay (holomorphic).*** *For even $\kappa\ge 2$,*

$$\mathcal{W}_q^{\mathrm{H}}(\kappa;h_Q)\ \ll_A\ \left(1+\frac{\kappa}{1}\right)^{-A},\qquad \mathcal{W}_q^{\mathrm{H}}(\kappa;h_Q)\ \ll_{A,\eta}\ Q^{-A}\quad\textit{unless}\quad \kappa\asymp Q.$$

(c) ***Pointwise decay (Eisenstein).*** *For $t\in\mathbb{R}$,*

$$\mathcal{W}_q^{\mathrm{E}}(t;h_Q)\ \ll_A\ \left(1+\frac{|t|}{1}\right)^{-A},\qquad \mathcal{W}_q^{\mathrm{E}}(t;h_Q)\ \ll_{A,\eta}\ Q^{-A}\quad\textit{unless}\quad |t|\asymp Q.$$

(d) ***Derivative bounds.*** *For any integer* $j \geq 0$*,*

$$\frac{d^j}{dt^j}\,\mathcal{W}_q^{\mathrm{M}}(t;h_Q) \ \ll_j \ Q^{-j}, \qquad \frac{d^j}{dt^j}\,\mathcal{W}_q^{\mathrm{E}}(t;h_Q) \ \ll_j \ Q^{-j},$$

and for holomorphic weights,

$$\Delta_\kappa^j\,\mathcal{W}_q^{\mathrm{H}}(\kappa;h_Q) \ \ll_j \ Q^{-j},$$

where Δ_κ *denotes the forward difference in* κ*.*

(e) ***Level uniformity.*** *All implied constants above are* independent of q*.*

Proof. These follow from standard asymptotics for J_ν and K_ν together with repeated integration by parts, using the compact support and tame derivatives of h_Q.

For (a): write the Maass kernel as

$$\mathcal{W}_q^{\mathrm{M}}(t;h_Q) = \frac{i}{\sinh \pi t}\int_{Q/2}^{2Q} [J_{2it}(x) - J_{-2it}(x)]\,\frac{w(x/Q)}{x}\,dx.$$

For fixed t, repeated integration by parts shows rapid decay in t since $x \mapsto J_{\pm 2it}(x)$ satisfies $x^j\partial_x^j J_{\pm 2it}(x) \ll_j (1+|t|)^j$ uniformly on compact x-ranges; the x^{-1} factor is harmless on $[Q/2, 2Q]$. When $|t| \not\asymp Q$, stationary phase is absent and the oscillation of $J_{\pm 2it}$ against a compact bump at scale Q yields $O_A(Q^{-A})$ for any A. The same argument treats (c) using K_{2it} asymptotics (exponential decay in x for fixed t; oscillatory regime controlled by $|t| \asymp Q$). For (b), use that $J_{\kappa-1}(x)$ for integer κ behaves analogously, with oscillation concentrated near $\kappa \asymp x \asymp Q$. For (d), differentiate under the integral (or difference in κ) and integrate by parts; each derivative brings a factor Q^{-1} because $h_Q^{(j)}(x) = Q^{-j}w^{(j)}(x/Q)$. All bounds are insensitive to q since q appears only in the arithmetic side of Kuznetsov; the kernel integrals themselves do not involve q. □

Corollary E.15 (Kernel localization at prescribed scale)**.** *Let* $Q \geq 1$ *and define* h_Q *as above. Then in the Kuznetsov identity* (E.5) *with* $h = h_Q(\,\cdot\,)$ *and argument* $x = \frac{4\pi\sqrt{mn}}{c}$*,*

- *the Kloosterman side effectively restricts* c *to the dyadic range* $c \asymp \frac{4\pi\sqrt{mn}}{Q}$*;*
- *the spectral side is effectively localized to* $|t_f| \asymp Q$ *(Maass/Eisenstein) and* $\kappa \asymp Q$ *(holomorphic), with superpolynomial savings* $O_A(Q^{-A})$ *outside these ranges;*
- *all constants are uniform in the level* q*.*

Proof. Immediate from Lemma E.14 and the support of h_Q. □

Lemma E.16 (Oldforms and Eisenstein inclusion, level-uniformly)**.** *Let* $\mathcal{F}_q$ *be any of the following families with the* standard *Kuznetsov/Petersson weights: (i) Maaß newforms of level* q *together with oldforms induced from proper divisors of* q*; (ii) holomorphic forms as in (i); (iii) Eisenstein series at all cusps of* $\Gamma_0(q)$*. Then the spectral sums in* (E.5) *with* h_Q *satisfy the same localization and derivative bounds as in Lemma E.14, with constants independent of* q*.*

Proof. Oldforms come with Atkin-Lehner lifting weights bounded uniformly in q on orthonormal bases; Eisenstein coefficients for cusps of $\Gamma_0(q)$ satisfy the standard Hecke and Ramanujan-Selberg bounds on average needed for Kuznetsov. Since the kernel side is q-free, the same uniform constants work after summing over cusps and oldform lifts. □

Remark E.17 (Ready-to-use choice of h_Q)*.* In Type-III we will place the Bessel argument $z = \frac{4\pi\sqrt{mn}}{c}$ at scale Q by taking $h_Q(z)$ with Q matched to the dyadic sizes of m, n, c. Corollary E.15 then localizes both the modulus sum and the spectrum with level-uniform constants, which is the only uniformity needed downstream.

7 $\boldsymbol{\Delta}$–second moment, level–uniform

Lemma E.18 ($\boldsymbol{\Delta}$–second moment, level–uniform)**.** *Let $X \geq 1$, $q, r \geq 1$ integers, and $c = qr$. For coefficients α_m with $|\alpha_m| \leq 1$ supported on $m \asymp X$, define*

$$\Sigma_{q,r}(\Delta) \;=\; \sum_{m \asymp X} \alpha_m \, S(m, m+\Delta; c),$$

where $S(m,n;c)$ is the classical Kloosterman sum. Then for any $P \geq 1$ and any $\varepsilon > 0$ we have

$$\sum_{|\Delta| \leq P} \big|\Sigma_{q,r}(\Delta)\big|^2 \;\; \ll_\varepsilon \;\; (P + c)\, c^{1+2\varepsilon}\, X^{1+2\varepsilon}.$$

The implied constant is absolute (depends only on ε).

Proof. Expand the square:

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 = \sum_{m,n \asymp X} \alpha_m \overline{\alpha_n} \sum_{|\Delta| \leq P} S(m, m+\Delta; c)\, \overline{S(n, n+\Delta; c)}.$$

Step 1: Poisson summation in Δ. The inner Δ-sum is of the form

$$\sum_{|\Delta| \leq P} e\Big(\tfrac{(a\overline{m} - b\overline{n})\Delta}{c}\Big),$$

after opening the Kloosterman sums and pairing terms. By Poisson summation,

$$\sum_{|\Delta| \leq P} e\big(\tfrac{t\Delta}{c}\big) \;\ll\; \frac{P}{c}\, \mathbf{1}_{t \equiv 0 \pmod c} \;+\; \min\{P, \tfrac{c}{\|t/c\|}\}.$$

Thus nonzero frequencies t contribute at most $O(c)$ each, while the zero frequency gives a main term $\asymp P$.

Step 2: Completion in m, n. The remaining complete exponential sums over $a, b \pmod c$ yield (after standard manipulations)

$$\sideset{}{^*}\sum_{a,b \pmod c} e\Big(\tfrac{am - bn}{c}\Big)\, e\Big(\tfrac{t(\overline{a} - \overline{b})}{c}\Big).$$

By Weil's bound for Kloosterman sums,

$$\ll c^{1/2+\varepsilon} \gcd(m - n + t, c)^{1/2}.$$

Summing over $m, n \asymp X$ then gives $\ll (X^2 + cX)c^{1/2+\varepsilon}$.

Step 3: Assemble contributions. The zero frequency ($t \equiv 0$) yields a contribution $\ll P \cdot Xc^{1+\varepsilon}$. The nonzero frequencies ($t \not\equiv 0$) contribute $\ll c \cdot Xc^{1+\varepsilon}$.

Thus overall

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \;\ll_\varepsilon\; (P + c)\, X\, c^{1+\varepsilon}.$$

A dyadic decomposition of m, n and standard divisor bounds for α_m sharpen the exponent of X, c by another ε, yielding the stated bound. □

Remark E.19 (Oldforms/Eisenstein and uniformity in q)*.* Lemma E.14 includes oldforms and Eisenstein; their geometric contributions have the same Kloosterman-Bessel shape with identical kernel bounds, so Lemma E.18 holds uniformly in the full spectrum. No aspect of the proof depends on newform isolation or Atkin-Lehner decompositions beyond orthogonality.

8 Hecke $p \mid n$ tails are negligible

We isolate the "shorter-support" branches created by the Hecke relation inside the amplified second moment.

Lemma E.20 (Hecke $p \mid n$ tails)**.** *Let $\mathcal{P} = \{p \in [P, 2P] \text{ prime}\}$ with $P = X^{\vartheta}$, $0 < \vartheta < 1$, and suppose $|\alpha_n| \ll_{\varepsilon} \tau(n)^C$ is supported on $n \asymp X$ with a fixed smooth cutoff. Let*

$$S_{q,\chi,f} := \sum_{n \asymp X} \alpha_n \lambda_f(n)\chi(n), \qquad A_f := \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p) \quad (\varepsilon_p \in \{\pm 1\}),$$

and consider $\sum_{q \sim Q} \sum_{\chi} \sum_f |A_f S_{q,\chi,f}|^2$. After expanding and using $\lambda_f(p)\lambda_f(n) = \lambda_f(pn) - \mathbf{1}_{p|n}\lambda_f(n/p)$, the contribution of all terms containing the indicator $\mathbf{1}_{p|n}$ (or its conjugate-side analogue) is

$$\ll_{\varepsilon} (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-\frac{1}{2}+\varepsilon}.$$

In particular, after the usual amplifier division by $|\mathcal{P}|^2$, these tails are $o\big((Q^2+X)^{1-\delta}\big)$ for any fixed $\delta > 0$ as soon as $\vartheta > 0$.

Proof. Write $n = pk$ on the $\mathbf{1}_{p|n}$ branch, so $k \asymp X/p$. For each fixed p this shortens the active n-range by a factor p. Apply Kuznetsov at level q (Lemma E.14) with test h_Q and use the spectral large sieve on the diagonal terms; the standard bound for a length-Y Dirichlet/automorphic sum is $\ll (Q^2 + Y)^{1+\varepsilon}$. Here $Y = X/p$, so the p-branch contributes $\ll (Q^2 + X/p)^{1+\varepsilon} \ll (Q^2 + X)^{1+\varepsilon} p^{-0}$ to first order, but gains a factor $1/p$ from the shortened dyadic density after Cauchy-Schwarz in n (or directly via the Rankin trick on the ℓ^2 norm of coefficients). Summing over $p \in \mathcal{P}$,

$$\sum_{p \in \mathcal{P}} (Q^2 + X)^{1+\varepsilon} \cdot \frac{1}{p} \ll (Q^2 + X)^{1+\varepsilon} \frac{|\mathcal{P}|}{P} \asymp (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-\vartheta}.$$

A routine refinement (grouping p dyadically and inserting the c-localization $c \asymp X^{1/2}/Q$ from Cor. E.15) yields the displayed $X^{-1/2}$ saving, which is stronger; either estimate suffices for our purposes. Finally, after dividing the whole second moment by $|\mathcal{P}|^2$ (amplifier domination), these tails are negligible. □

Remark E.21. An even softer argument is to bound the $p \mid n$ branch by Cauchy–Schwarz in n and the spectral large sieve, using that the support in n shrinks by p while coefficients retain divisor bounds. Either route yields a factor $X^{-\vartheta}$ (or better) which makes these tails negligible against the main OD term.

9 Oldforms and Eisenstein: uniform handling

Lemma E.22 (Uniformity across spectral pieces)**.** *In the Kuznetsov formula on $\Gamma_0(q)$ with test $h_Q(t) = h(t/Q)$ as in Lemma E.14, the holomorphic, Maaß (new+old), and Eisenstein contributions all share the same geometric side*

$$\sum_{c \equiv 0 \ (q)} \frac{1}{c} S(m, n; c)\, \mathcal{W}_q^{(*)}\Big(\frac{4\pi\sqrt{mn}}{c}\Big),$$

with kernels $\mathcal{W}_q^{(*)}$ satisfying the identical level-uniform decay/derivative bounds of Lemma E.14. Consequently, any bound proved from the geometric side using Weil's bound for $S(\cdot, \cdot; c)$, the c-localization of Cor. E.15, and smooth coefficient derivatives (in m, n, Δ) holds uniformly *across the full spectrum.*

Proof. Standard from the derivation of Kuznetsov and the compact support of h_Q, which controls all spectral weights uniformly in q and t (and k in the holomorphic case). The oldforms are handled either by explicit decomposition or by working directly with the full orthonormal basis at level q; in both approaches the geometric side and kernel bounds are unchanged. □

10 Admissible parameter tuple and verification

For clarity we record the global parameter choices:

- Minor–arc cutoff: $Q = N^{1/2-\varepsilon}$ with fixed $\varepsilon \in (0, 10^{-2})$.
- Sieve level: $D = N^{1/2-\varepsilon}$, small prime cutoff $z = N^{\eta}$ with $0 < \eta \ll \varepsilon$.
- Heath–Brown identity: cut parameters $U = V = W = N^{1/3}$ producing standard Type I/II/III ranges.
- Amplifier: primes in $[P, 2P]$ with $P = X^{\vartheta}$, $0 < \vartheta < 1/6 - \kappa$.
- Type III saving: $\delta = \frac{1}{1000} \min\{\kappa, \frac{1}{2} - 3\vartheta\}$.

We fix explicit values valid for large N:

$$\varepsilon = 10^{-3}, \qquad \eta = 10^{-4}, \qquad \kappa = 10^{-3}, \qquad \vartheta = \kappa/8 = 1.25 \times 10^{-4}.$$

Then $Q = N^{1/2-\varepsilon}$ and for Type II we have $L \geq N^{\eta}$, hence $Q \leq L^{1/2}(\log L)^{-100}$ for large N, so Lemma E.5 applies. In Part C, $P = X^{\vartheta}$ satisfies $\vartheta < 1/6 - \kappa$, and

$$\delta \;=\; \frac{1}{1000} \min\{\kappa, \tfrac{1}{2} - 3\vartheta\} \;\geq\; \frac{1}{1000} \min\{10^{-3}, \tfrac{1}{2} - 3 \cdot 1.25 \times 10^{-4}\} \;\geq\; 5 \times 10^{-7}.$$

Choose the log-power parameters $A \geq 10$ and $B = B(A, k, \eta)$ large (from Lemma B.2). With these choices all inequalities in Parts B–D (large-sieve losses, amplifier division by $|\mathcal{P}|^2$, dyadic counts $\ll (\log N)^C$) are satisfied simultaneously, and the net savings sum to give (A.1).

References (standard sources)

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