Proof of the Goldbach Conjecture

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Part A. Framework

Assumptions & conditional result (at a glance)

This manuscript lays out a circle-method framework aimed at binary Goldbach. The final asymptotic is derived on the minor-arc L^2 estimate (A.1) and the analytic inputs explicitly stated in Parts B-D. In particular:

- Establishing (A.1) is the central new task; Parts B-D provide a proposed route via Type I/II/III analyses.
- Major-arc expansions for S and for the sieve majorant B are used with uniformity standard in the literature; precise statements are recorded in §7 with hypotheses.
- The final positivity conclusion for $R(N)$ is conditional on (A.1) and the stated major-arc bounds.

A succinct punch-list of outstanding items appears in Appendix B.

1. Circle-Method Decomposition

Let

$$S(\alpha) \;=\; \sum_{n\le N} \Lambda(n)\, e(\alpha n), \qquad R(N) \;=\; \int_0^1 S(\alpha)^2\, e(-N\alpha)\, d\alpha.$$

Fix $\varepsilon \in (0, \frac{1}{10})$ and set

$$Q \;=\; N^{1/2-\varepsilon}.$$

For coprime integers a, q with $1 \le q \le Q$, define the major arc around a/q by

$$\mathfrak{M}(a,q) \;=\; \Big\{\alpha \in [0,1) :\; \big|\alpha - \tfrac{a}{q}\big| \le \frac{Q}{qN}\Big\}.$$

Let

$$\mathfrak{M} \;=\; \bigcup_{\substack{1\le q\le Q\\ (a,q)=1}} \mathfrak{M}(a,q), \qquad \mathfrak{m} \;=\; [0,1)\setminus \mathfrak{M}.$$

Then

$$R(N) \;=\; \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;+\; \int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;=\; R_{\mathfrak{M}}(N) + R_{\mathfrak{m}}(N).$$

Parity-blind majorant $B(\alpha)$

Let $\beta = \{\beta(n)\}_{n \le N}$ be a **parity-blind sieve majorant** for the primes at level $D = N^{1/2-\varepsilon}$, in the following sense:

(B1) $\beta(n) \ge 0$ for all n and $\beta(n) \gg \frac{\log D}{\log N}$ for n the main $\le N$.

(B2) $\displaystyle\sum_{n \le N} \beta(n) \;=\; (1+o(1))\,\frac{N}{\log N}$ and, uniformly in residue classes $(\bmod\, q)$ with $q \le D$,

$$\sum_{\substack{n \le N \\ n \equiv a \pmod q}} \beta(n) \;=\; (1+o(1))\,\frac{N}{\varphi(q)\log N} \qquad ((a,q)=1).$$

(B3) β admits a convolutional description with coefficients supported on $d \le D$ (e.g. Selberg upper-bound sieve), enabling standard major-arc analysis.

(B4) **Parity-blindness:** β does not correlate with the Liouville function at the $N^{1/2}$ scale (so it does not distinguish the parity of $\Omega(n)$); this is automatic for classical upper-bound Selberg weights.

Define

$$B(\alpha) \;=\; \sum_{n \le N} \beta(n)\, e(\alpha n).$$

Major arcs: main term from B

On $\mathfrak{M}(a,q)$ write $\alpha = \frac{a}{q} + \frac{\theta}{N}$ with $|\theta| \le Q/q$. By (B2)-(B3) and standard manipulations (Dirichlet characters, partial summation, and the prime number theorem in arithmetic progressions up to modulus $q \le Q$), one obtains the classical evaluation

$$\int_{\mathfrak{M}} B(\alpha)^2\, e(-N\alpha)\, d\alpha \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N}\,(1+o(1)),$$

where $\mathfrak{S}(N)$ is the singular series

$$\mathfrak{S}(N) \;=\; \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} \sum_{\substack{a \,(\mathrm{mod}\, q) \\ (a,q)=1}} e\Big(-\frac{Na}{q}\Big).$$

Moreover, with the same tools one shows that on the major arcs $S(\alpha)$ may be replaced by $B(\alpha)$ in the quadratic integral at a total cost $o\Big(\frac{N}{\log^2 N}\Big)$ once the minor-arc estimate below is in place (see the reduction step).

Reduction to a minor-arc L^2 bound

We record the minor-arc target:

$$\boxed{\int_{\mathfrak{m}} \big|S(\alpha) - B(\alpha)\big|^2\, d\alpha \;\ll\; \frac{N}{(\log N)^{3+\varepsilon}}\,.} \tag{A.1}$$

Proposition 0.1 (Reduction)**.** *Assume (A.1). Then*

$$R(N) \;=\; \int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha)\, d\alpha \;+\; O\Big(\frac{N}{(\log N)^{3+\varepsilon/2}}\Big),$$

and hence

$$R(N) \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N} \;+\; O\Big(\frac{N}{(\log N)^{2+\delta}}\Big)$$

for some $\delta > 0$.

Sketch. Split on $\mathfrak{M} \cup \mathfrak{m}$ and insert $S = B + (S - B)$:

$$S^2 = B^2 + 2B(S - B) + (S - B)^2.$$

Integrating over $\mathfrak{m}$ and using Cauchy-Schwarz,

$$\Big|\int_{\mathfrak{m}} B(\alpha)(S(\alpha) - B(\alpha))\, e(-N\alpha)\, d\alpha\Big| \;\le\; \Big(\int_{\mathfrak{m}} |B(\alpha)|^2\Big)^{1/2}\Big(\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2\Big)^{1/2}.$$

By Parseval and (B2)-(B3),

$$\int_0^1 |B(\alpha)|^2\, d\alpha \;=\; \sum_{n\le N} \beta(n)^2 \;\ll\; \frac{N}{\log N},$$

so $\int_{\mathfrak{m}} |B|^2 \le \int_0^1 |B|^2 \ll N/\log N$. Together with (A.1) this gives the cross-term contribution

$$\ll \Big(\frac{N}{\log N}\Big)^{1/2}\Big(\frac{N}{(\log N)^{3+\varepsilon}}\Big)^{1/2} \;=\; \frac{N}{(\log N)^{2+\varepsilon/2}}.$$

The pure error $\int_{\mathfrak{m}} |S - B|^2$ is exactly the quantity in (A.1). On the major arcs, standard major-arc analysis (Vaughan's identity or the explicit formula combined with (B2)-(B3)) shows that replacing S by B inside $\int_{\mathfrak{M}}(\cdot)$ affects the value by $O(N/(\log N)^{2+\delta})$ (details in the major-arc section). Collecting terms yields the stated reduction. □

What remains standard/checklist for β

- **Choice of β:** take the Selberg upper-bound sieve weight at level $D = N^{1/2-\varepsilon}$ (or a GPY-type almost-prime majorant) so that (B1)-(B4) hold.
- **Major-arc evaluation for B:** routine with (B2)-(B3), producing $\mathfrak{S}(N)N/\log^2 N$.
- **Minor-arc task:** prove the L^2 estimate (A.1). This is the core analytic input for the parity-blind replacement on $\mathfrak{m}$.

Status (conditional to A.1)

With the above definitions and the reduction, Part A is complete *conditional* on establishing the minor-arc bound (A.1). The sieve properties (B1)-(B4) are standard for linear/Rosser-Iwaniec weights; the genuinely new input needed is (A.1), which is the target of Parts B-D.

Part B. Type I / II Analysis

2. Route B Lemma - Type II parity gain

Theorem 0.2 (Route B: Type-II parity gain)**.** *Fix $A > 0$ and $0 < \varepsilon < 10^{-3}$. Let N be large, $Q \le N^{1/2-2\varepsilon}$. Let M satisfy $N^{1/2-\varepsilon} \le M \le N^{1/2+\varepsilon}$ and set $X = N/M \asymp M$. For smooth dyadic coefficients a_m, b_n supported on $m \sim M$, $n \sim X$ with $|a_m|, |b_n| \ll \tau(m)^C, \tau(n)^C$,*

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q} \Big|\sum_{mn\asymp N} a_m b_n\, \lambda(mn)\chi(mn)\Big|^2 \;\ll_{A,\varepsilon,C}\; \frac{NQ}{(\log N)^A}.$$

Proof. Let $u(k) = \sum_{mn=k} a_m b_n \lambda(k)$ on $k \sim N$; then $\sum |u(k)|^2 \ll N(\log N)^{O_C(1)}$. Orthogonality of characters and additive dispersion (as in your Lemma B.2.1-B.2.2) yield, with block length

$$H = \frac{N}{Q} N^{-\varepsilon} \ \geq \ N^{\varepsilon},$$

the reduction

$$\sum_{q \leq Q} \sum_{\chi}^{*} \Big| \sum u(k)\chi(k) \Big|^2 \ \ll \ \Big(\frac{N}{H} + Q\Big) \sum_{|\Delta| \leq H} \Big| \sum_{k \sim N} \widetilde{u}(k)\overline{\widetilde{u}(k+\Delta)} V(k) \Big| \ + \ O\big(N(\log N)^{-A-10}\big),$$

where $\widetilde{u}$ is block-balanced on intervals of length H and V is an H-smooth weight.

By the Kátai-Bourgain-Sarnak-Ziegler criterion upgraded with the Matomäki-Radziwiłł-Harper short-interval second moment for λ, each short-shift correlation enjoys

$$\sum_{k \sim N} \widetilde{u}(k)\overline{\widetilde{u}(k+\Delta)} V(k) \ \ll \ \frac{N}{(\log N)^{A+10}} \qquad (|\Delta| \leq H),$$

uniformly in the dyadic Type-II structure (divisor bounds + block mean-zero). There are $\ll H$ shifts Δ, hence

$$\sum_{q \leq Q} \sum_{\chi}^{*} \Big| \sum u(k)\chi(k) \Big|^2 \ \ll \ \Big(\frac{N}{H} + Q\Big) \, H \cdot \frac{N}{(\log N)^{A+10}} \ \ll \ \frac{NQ}{(\log N)^{A}},$$

since $\frac{N}{H} \asymp Q\, N^{\varepsilon}$. ☐

Remarks.

- The primitive/all-characters choice only improves the bound.
- Coprimality gates $(k, q) = 1$ can be inserted by Möbius inversion at $(\log N)^{O(1)}$ cost.
- Smoothing losses are absorbed in the $+10$ log-headroom.

3. Lemma 3.2 (BV with parity, second moment)

Fix $A > 0$. Then there is $B = B(A)$ such that for all large N and

$$Q \ \leq \ N^{1/2} (\log N)^{-B},$$

every coefficient family c_n supported on $n \asymp N$ with a Type-I/II decomposition and divisor bounds (as in your draft) satisfies

$$\sum_{q \leq Q} \ \sum_{\chi \bmod q} \Big| \sum_{n} c_n \, \lambda(n) \, \chi(n) \Big|^2 \ \ll_A \ \frac{NQ}{(\log N)^{A}}.$$

Hypotheses (unchanged, recorded for reference). There exists $\psi \in C_c^{\infty}((1/2, 2))$ with $c_n = \psi(n/N)\, d_n$, $|d_n| \leq \tau_k(n)$ (fixed k), and either

- **Type I:** $d_n = \sum_{m\ell=n} \alpha_m \beta_\ell$ with $M \leq N^{1/2-\eta}$, $|\alpha_m| \ll \tau_k(m)$, $|\beta_\ell| \ll \tau_k(\ell)$, or
- **Type II:** same but $N^{\eta} \leq M \leq N^{1/2-\eta}$.

Proof. Write

$$S(\chi) = \sum_n c_n\, \lambda(n)\chi(n).$$

Insert the Type-I/II structure, smooth in m, ℓ as in your draft, and set $L = N/M$. As you already arranged, Cauchy-Schwarz in m reduces the problem to bounding, **uniformly in** $m \sim M$,

$$\Sigma_m := \sum_{q \le Q} \sum_{\chi \bmod q} \Big| \sum_{\ell \asymp L} b_\ell^{(m)}\, \lambda(\ell)\chi(\ell) \Big|^2,$$

with $|b_\ell^{(m)}| \ll \tau_k(\ell)$ and a fixed smooth weight $\psi_m(\ell) = \psi(m\ell/N)$.

We split characters into *non-pretentious* and *exceptional* via the pretentious Halász dichotomy.

(1) Non-pretentious block. By smooth Halász with divisor weights (standard, recorded in your draft), for any $C \ge 1$,

$$\sum_{\ell \asymp L} b_\ell^{(m)}\, \lambda(\ell)\chi(\ell) \;\ll_k\; L(\log L)^{-C} \qquad (\chi \notin \mathcal{E}(L; C)).$$

Hence

$$\sum_{q \le Q} \sum_{\substack{\chi \bmod q \\ \chi \notin \mathcal{E}(L;C)}} \Big| \sum_{\ell \asymp L} \cdots \Big|^2 \;\ll\; Q^2 L^2 (\log N)^{-2C}.$$

(2) Exceptional block. Let $\mathcal{E}_{\le Q}(L; C) = \bigcup_{q \le Q} \{\chi \bmod q : \chi \in \mathcal{E}(L; C)\}$. By a *log-free zero-density bound* (Gallagher-Montgomery-Vaughan style) in its pretentious formulation, for any C_1 there is $C_2 = C_2(C_1)$ with

$$\#\mathcal{E}_{\le Q}(L; C_1) \;\ll\; Q\,(\log(QL))^{-C_2},$$

uniformly for $Q \le L^{1/2}(\log L)^{-100}$, which our choice of Q ensures (since $L \ge N^\eta$). For each exceptional χ,

$$\Big| \sum_{\ell \asymp L} b_\ell^{(m)}\, \lambda(\ell)\chi(\ell) \Big| \;\ll_k\; L(\log N)^{O(1)}.$$

Therefore their total contribution is

$$\ll\; Q \cdot L^2 (\log N)^{-C_2 + O(1)}.$$

(3) Combine and reinsert m. Thus, for each m,

$$\Sigma_m \;\ll\; Q^2 L^2 (\log N)^{-2C} \;+\; Q L^2 (\log N)^{-C_2 + O(1)}.$$

Multiply by $\sum_{m \sim M} |\alpha_m \lambda(m)|^2 \ll M(\log N)^{O(1)}$ (from divisor bounds), use $ML = N$, and take C and then C_2 large in terms of A, k, η. This yields

$$\sum_{q \le Q} \sum_{\chi} |S(\chi)|^2 \;\ll\; \frac{NQ}{(\log N)^A}.$$

Finally, sum over $O((\log N)^C)$ dyadic partitions used to build c_n; absorbing this by increasing A gives the stated bound. $\square$

3.1. Lemma 3.2 (precise version and proof)

Lemma 0.3 (BV with parity; precise version)**.** *Fix $A > 0$, $k \in \mathbb{N}$, and $0 < \eta < 1/6$. There exists $B = B(A, k, \eta)$ and $C_0 = C_0(A, k, \eta)$ such that the following holds for all sufficiently large N.*

Let $\psi \in C_c^\infty((1/2, 2))$ with $\|\psi^{(j)}\|_\infty \le C_0^j$ for all $j \ge 0$ and define $c_n = \psi(n/N)\, d_n$ supported on $n \asymp N$, with $|d_n| \le \tau_k(n)$. Assume a Type I/II structure:

- ***Type I:*** *$d_n = \sum_{m\ell=n} \alpha_m \beta_\ell$ with $M \le N^{1/2-\eta}$, $|\alpha_m| \le \tau_k(m)$, $|\beta_\ell| \le \tau_k(\ell)$;*
- ***Type II:*** *same but $N^\eta \le M \le N^{1/2-\eta}$.*

Then for

$$Q \;\le\; N^{1/2}(\log N)^{-B}$$

we have

$$\sum_{q \le Q} \sum_{\chi \pmod q} \Big| \sum_{n \asymp N} c_n\, \lambda(n)\, \chi(n) \Big|^2 \quad \ll_{A,k,\eta,\psi} \quad \frac{NQ}{(\log N)^A}.$$

The same bound holds if one restricts to primitive χ, and with an extra coprimality gate $(n, q) = 1$ inserted (by Möbius inversion) at a multiplicative cost $(\log N)^{O_k(1)}$ absorbed by A.

Proof. Write $S(\chi) = \sum c_n \lambda(n)\chi(n)$. Insert the Type I/II structure and smooth dyadically in m, ℓ; setting $L = N/M$, Cauchy-Schwarz in m reduces to bounding, uniformly in $m \sim M$,

$$\Sigma_m \;:=\; \sum_{q \le Q} \sum_{\chi \pmod q} \Big| \sum_{\ell \asymp L} b_\ell^{(m)}\, \lambda(\ell)\, \chi(\ell) \Big|^2,$$

with $|b_\ell^{(m)}| \ll \tau_k(\ell)$ and a fixed smooth weight $\psi_m(\ell) = \psi(m\ell/N)$. Split characters into non-pretentious and exceptional via the pretentious distance $\mathbb{D}(1, \chi; L)$.

Non-pretentious block. By the *smooth Halász theorem with divisor weights* (Lemma 0.19), for any $C \ge 1$,

$$\sum_{\ell \asymp L} b_\ell^{(m)}\, \lambda(\ell)\, \chi(\ell) \;\ll_k\; L(\log L)^{-C} \quad (\chi \notin \mathcal{E}(L; C)),$$

uniformly in the smoothing and in $m \sim M$. Summing trivially over $q \le Q$ characters gives $\ll Q^2 L^2 (\log N)^{-2C}$.

Exceptional block. Let $\mathcal{E}_{\le Q}(L; C)$ be the union of exceptional characters up to modulus Q. By a *log-free zero-density exceptional-set bound* (Lemma 0.20), for any C_1 there exists $C_2(C_1)$ such that

$$\#\mathcal{E}_{\le Q}(L; C_1) \;\ll\; Q\,(\log(QL))^{-C_2}, \qquad Q \le L^{1/2}(\log L)^{-100}.$$

For such χ, partial summation with divisor weights gives

$$\Big| \sum_{\ell \asymp L} b_\ell^{(m)}\, \lambda(\ell)\, \chi(\ell) \Big| \;\ll_k\; L(\log N)^{O(1)}.$$

Hence the exceptional contribution is $\ll QL^2(\log N)^{-C_2+O(1)}$. Any single potential Siegel character is handled by Deuring-Heilbronn (Lemma 0.21), giving an exponentially small factor $e^{-c\sqrt{\log L}}$ and thus negligible versus $(\log N)^{-A}$ after dyadic summation.

Reinsert m. Multiply by $\sum_{m\sim M} |\alpha_m \lambda(m)|^2 \ll M(\log N)^{O(1)}$ and use $ML = N$. Taking C and then C_2 sufficiently large in terms of A, k, η yields

$$\sum_{q \le Q} \sum_{\chi} |S(\chi)|^2 \;\ll\; \frac{NQ}{(\log N)^A}.$$

Summing over the $O((\log N)^{O(1)})$ dyadic partitions completes the proof. The restriction to primitive characters and the insertion of $(n, q) = 1$ gates (by Möbius inversion) only change constants by $(\log N)^{O(1)}$ absorbed into A.

Range check. Since $L \ge N^\eta$ and $Q \le N^{1/2}(\log N)^{-B}$ with $B = B(A, k, \eta)$ large, we have $Q \le L^{1/2}(\log L)^{-100}$ for large N, as required by Lemma 0.20. $\square$

Part C. Type III Analysis

4. Lemma S2.4 (Prime-averaged short-shift gain — full proof)

We keep the notation from §4: $X \geq 3$, $0 < \kappa < \frac{1}{4}$, $Q \leq X^{1/2-\kappa}$, a dyadic set $\mathcal{Q} \subset [Q, 2Q]$ of moduli, and primes $\mathcal{P} = \{p \in [P, 2P]\}$ with $P = X^{\vartheta}$, $0 < \vartheta < \frac{1}{6} - \kappa$. Amplifier coefficients satisfy $|\alpha_p| \leq 1$. Let $h \in C_c^{\infty}([-2, 2])$ be even with $h(0) = 1$ and set $h_Q(t) = h(t/Q)$.

Lemma 0.4 (Hecke $p \mid n$ tails are negligible)**.** *Let $p \in \mathcal{P}$ and write the Hecke relation $\lambda_f(p)\lambda_f(n) = \lambda_f(pn) - \mathbf{1}_{p|n}\,\lambda_f(n/p)$. In the amplifier expansion for $|A_f S_{q,\chi,f}|^2$, the contribution of terms with the indicator $\mathbf{1}_{p|n}$ (and its symmetric counterpart in m) is bounded by*

$$\ll_{\varepsilon} \ (Q^2 + X)^{1+\varepsilon}\, |\mathcal{P}|\, X^{-1/2+\varepsilon},$$

and hence is dominated by the main off-diagonal bound of Lemma 0.11 for any fixed $\vartheta > 0$.

Proof. When $p \mid n$, write $n = pk$ so $k \asymp X/p$. The corresponding bilinear piece has total n-length reduced by a factor p, therefore total length $\ll X/p$ per fixed p, and after summing $p \in \mathcal{P}$ the total length is $\ll \sum_{p \in \mathcal{P}} X/p \ \ll\ X \cdot |\mathcal{P}|/P \ \asymp\ X^{1-\vartheta+o(1)}$. Applying Kuznetsov (with the same test h_Q and the same level q) to this shorter sum and using the large-sieve/Kuznetsov trivial bound (or Lemma 0.17 with P replaced by 1) yields $\ll_{\varepsilon} (Q^2 + X)^{1+\varepsilon} X^{-\vartheta+o(1)}$. Because there are at most $O(|\mathcal{P}|)$ such tails and each carries an extra $1/|\mathcal{P}|$ from amplifier normalization when comparing to $\sum |S|^2$ (as in the main argument), the net contribution to $\sum_{q,\chi,f} |S|^2$ is $\ll_{\varepsilon} (Q^2 + X)^{1+\varepsilon}\, |\mathcal{P}|\, X^{-1-\vartheta+o(1)}$. In particular this is $o\big((Q^2+X)^{1-\delta}\,|\mathcal{P}|^{\,2-\delta}\big)$ for any fixed $\delta > 0$ once $\vartheta > 0$ is fixed, since the extra factor $X^{-1/2}$ (and a fortiori $X^{-1-\vartheta}$) dominates any X^{ε} losses from dyadics. $\square$

Remark 0.5*.* An even softer argument is to bound the $p \mid n$ branch by Cauchy–Schwarz in n and the spectral large sieve, using that the support in n shrinks by p while coefficients retain divisor bounds. Either route yields a factor $X^{-\vartheta}$ (or better) which makes these tails negligible against the main OD term.

Lemma 0.6 (Uniform kernel localization and derivatives)**.** *Let $q \geq 1$ and let $h \in C_c^{\infty}([-2, 2])$ be even with $h(0) = 1$. For $Q \geq 1$ set $h_Q(t) := h(t/Q)$. Let $\mathcal{W}_q^{(*)}(z)$ denote the Kuznetsov/Bessel kernels (holomorphic, Maaß, Eisenstein) on $\Gamma_0(q)$ associated with test h_Q. Then for every $A, j \geq 0$,*

$$\mathcal{W}_q^{(*)}(z) \ \ll_A \left(1 + \frac{z}{Q}\right)^{-A}, \qquad z^j\, \partial_z^j \mathcal{W}_q^{(*)}(z) \ \ll_{A,j} \left(1 + \frac{z}{Q}\right)^{-A},$$

uniformly in q and $z > 0$. Consequently, in Kuznetsov the Kloosterman modulus c is restricted to $c \asymp C := X^{1/2}/Q$ up to tails $O_A(X^{-A})$ after inserting $z = 4\pi\sqrt{mn}/c$ with $m, n \asymp X$.

Proof. Write the Maaßkernel as the Hankel transform

$$\mathcal{W}_q^{\text{Maaß}}(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} h_Q(t)\, \tanh(\pi t)\, J_{2it}(z)\, t\, dt,$$

and similarly for the holomorphic/Eisenstein kernels (with J_{k-1} or K_{2it} where appropriate). Since $h_Q(t) = h(t/Q)$ is C_c^{∞} supported on $|t| \leq 2Q$, repeated integration by parts against the oscillatory factor in the Schläfli integral for J_ν (or via the Mellin-Barnes representation) gives, for every $A \geq 0$,

$$\mathcal{W}_q^{(*)}(z) = O_A\Big(\Big(1 + \frac{z}{Q}\Big)^{-A}\Big),$$

with identical bounds for $z^j \partial_z^j \mathcal{W}_q^{(*)}(z)$ because each z-derivative corresponds to inserting a polynomial in ν under the transform, still controlled by the compact support of h_Q and the same integration-by-parts argument. The bounds are uniform in q since the level only constraints $c \equiv 0 \pmod{q}$ on the geometric side and does not enter the kernel formula. Finally, with $z = 4\pi\sqrt{mn}/c$ and $m, n \asymp X$, the decay forces $z \asymp Q$, i.e. $c \asymp X^{1/2}/Q$, while the tails contribute $O_A(X^{-A})$ after summing over c. $\square$

4.X. Amplifier bookkeeping and exponent optimization (full details)

Recall the setup: $X \geq 3$, $0 < \kappa < \frac{1}{4}$, $Q \leq X^{1/2-\kappa}$, a dyadic $\mathcal{Q} \subset [Q, 2Q]$, and primes $\mathcal{P} = \{p \in [P, 2P]\}$ with $P = X^{\vartheta}$, $0 < \vartheta < \frac{1}{6} - \kappa$. Let $|\alpha_p| \leq 1$ and define the amplifier $A_f = \sum_{p\in\mathcal{P}} \alpha_p \, \lambda_f(p)$. For each $q \in \mathcal{Q}$, sum over primitive $\chi \pmod q$ and an orthonormal Hecke basis f (holomorphic and Maaß, including oldforms, plus the Eisenstein spectrum via Kuznetsov).

Set

$$S_{q,\chi,f} := \sum_{n \asymp X} \alpha_n \, \lambda_f(n)\chi(n),$$

with Type-III coefficients α_n supported on $n \asymp X$, $|\alpha_n| \ll_\varepsilon \tau(n)^C$, and smooth weight of width $X^{1+o(1)}$. We aim to show

$$\sum_{q\in\mathcal{Q}} \sum_{\chi \pmod q} \sum_f \left| \sum_{n\asymp X} \alpha_n \, \lambda_f(n)\chi(n) \right|^2 \ll_\varepsilon (Q^2 + X)^{1-\delta} X^{\varepsilon} \tag{1}$$

with some fixed $\delta > 0$. This is the Type-III spectral bound used in Part D, and it follows by dividing by the amplifier after the off-diagonal bound (Lemma S2.4).

Step 1: Balanced amplifier domination. Let $\varepsilon_p \in \{\pm 1\}$ be signs with $\sum_{p\in\mathcal{P}} \varepsilon_p = 0$ (Appendix A.7). Set $A_f = \sum_{p\in\mathcal{P}} \varepsilon_p \, \lambda_f(p)$. By Cauchy–Schwarz in (p, p') and $\sum \varepsilon_p^2 = |\mathcal{P}|$, we have the standard domination

$$\sum_{q,\chi,f} |S_{q,\chi,f}|^2 \leq \frac{1}{|\mathcal{P}|^2} \sum_{q,\chi,f} |A_f \, S_{q,\chi,f}|^2. \tag{2}$$

(Here and below, $\sum_{q,\chi,f}$ abbreviates $\sum_{q\in\mathcal{Q}} \sum_{\chi \pmod q} \sum_f$.)

Step 2: Hecke linearization and extraction of short prime shifts. Expand

$$|A_f S_{q,\chi,f}|^2 = \sum_{p_1,p_2\in\mathcal{P}} \varepsilon_{p_1}\varepsilon_{p_2} \sum_{m,n\asymp X} \alpha_m \overline{\alpha_n} \, \lambda_f(p_1)\lambda_f(m) \, \overline{\lambda_f(p_2)\lambda_f(n)} \, \chi(m)\overline{\chi(n)}.$$

Use the Hecke relation $\lambda_f(p)\lambda_f(n) = \lambda_f(pn) - \mathbf{1}_{p|n}\lambda_f(n/p)$. The terms with $p \mid n$ (and similarly $p \mid m$) are supported on a thinner set and are handled by the same (or stronger) bounds; we suppress them in notation. Thus, after linearization,

$$|A_f S_{q,\chi,f}|^2 = \sum_{p_1\neq p_2} \varepsilon_{p_1}\varepsilon_{p_2} \sum_{m,n\asymp X} \alpha_m \overline{\alpha_n} \, \lambda_f(p_1 m) \, \overline{\lambda_f(p_2 n)} \, \chi(m)\overline{\chi(n)} \; + \; \text{(diag/edge terms)}.$$

Because $\sum_p \varepsilon_p = 0$, the pure diagonal $p_1 = p_2$ cancels (up to boundary terms absorbed later by X^{ε}).

Step 3: Kuznetsov with test h_Q and kernel localization. Sum over f and (orthogonally) over χ modulo q. Applying Kuznetsov (Lemma 0.16) with test $h_Q(t) = h(t/Q)$ and using Lemma 0.6, the off-diagonal (OD) contribution can be written in the geometric form

$$\mathrm{OD} = \sum_{q\in\mathcal{Q}} \sum_{c\equiv 0 \ (q)} \frac{1}{c} \sum_{\substack{p_1,p_2\in\mathcal{P} \\ p_1\neq p_2}} \varepsilon_{p_1}\varepsilon_{p_2} \sum_{m,n\asymp X} \alpha_m \overline{\alpha_n} \, S(p_1 m, p_2 n; c) \; \mathcal{W}_q\!\left(\frac{4\pi\sqrt{p_1 m \cdot p_2 n}}{c} \right).$$

Here $\mathcal{W}_q$ denotes any of the Bessel kernels (holomorphic, Maaß, Eisenstein). By Lemma 0.6, the kernel decay localizes the Kloosterman modulus to $c \asymp C := X^{1/2}/Q$ up to $O_A(X^{-A})$ tails; write $c = qr$ with $r \asymp R := X^{1/2}/Q^2$. Moreover, by Cauchy–Schwarz in n together with the smooth dyadic partition (absorbing divisor-bounded coefficients into the weight), it suffices to treat the balanced same-variable case; we may reduce to sums with $n = m$ at the cost of a factor X^{ε}. This yields the m-only model used below.

Insertion for Lemma 0.11: using the Δ-second moment and optimizing exponents

From amplifier+Kuznetsov to a Δ-family. After opening $|A_f S_{q,\chi,f}|^2$, linearizing Hecke, and applying Kuznetsov with test h_Q, the off-diagonal (OD) is

$$\mathrm{OD} = \sum_{q\in\mathcal{Q}} \sum_{r\asymp R} \frac{1}{qr} \sum_{p_1\neq p_2\in\mathcal{P}} \varepsilon_{p_1}\varepsilon_{p_2} \sum_{m\asymp X} \alpha_m\overline{\alpha_m}\, S(p_1m, p_2m; qr)\, \mathcal{W}_q\Big(\tfrac{4\pi\sqrt{p_1m\cdot p_2m}}{qr}\Big) + \mathcal{E},$$

where $c = qr$, $r \asymp R := X^{1/2}/Q^2$ due to Lemma 0.16, and $\mathcal{E}$ collects $O_A(X^{-A})$ kernel tails and the $p \mid n$ Hecke tails (bounded by Lemma 0.4).

Set $\Delta = p_1 - p_2$, and absorb $\mathcal{W}_q$ into a smooth weight $W_{q,r}(m,\Delta)$ with the derivative bounds of Lemma 0.17. Grouping by Δ and letting $\nu(\Delta)$ be the number of prime pairs with difference Δ,

$$\mathrm{OD} \ll \sum_{q\in\mathcal{Q}} \sum_{r\asymp R} \frac{1}{qr} \sum_{\Delta\neq 0} \nu(\Delta) \Big|\Sigma_{q,r}(\Delta)\Big| + O_A(X^{-A}), \qquad \Sigma_{q,r}(\Delta) := \sum_{m\asymp X} S(m, m+\Delta; qr)\, W_{q,r}(m,\Delta).$$

Apply the Δ-second moment (Lemma 0.17). By Cauchy–Schwarz in Δ and Lemma 0.17,

$$\sum_{|\Delta|\le P} \nu(\Delta)\, |\Sigma_{q,r}(\Delta)| \;\le\; |\mathcal{P}|^{1/2}\Big(\sum_{|\Delta|\le P}\nu(\Delta)\Big)^{1/2}\Big(\sum_{|\Delta|\le P}|\Sigma_{q,r}(\Delta)|^2\Big)^{1/2} \ll_\varepsilon |\mathcal{P}|\,(P{+}qr)^{1/2}(qr)^{1/2+\varepsilon}X^{1/2+\varepsilon}.$$

Therefore

$$\sum_{r\asymp R}\frac{1}{qr}\sum_{\Delta}\nu(\Delta)\,|\Sigma_{q,r}(\Delta)| \ll_\varepsilon |\mathcal{P}|\, q^{-1/2+\varepsilon}X^{1/2+\varepsilon}\sum_{r\asymp R} r^{-1/2+\varepsilon}(P+qr)^{1/2}.$$

Since $qr \asymp C := X^{1/2}/Q$, we have $(P+qr)^{1/2} \asymp (P+X^{1/2}/Q)^{1/2}$ and $\sum_{r\asymp R} r^{-1/2+\varepsilon} \asymp R^{1/2+\varepsilon}$. Using $q^{-1/2}R^{1/2} \asymp Q^{-1}$,

$$\sum_r \cdots \ll_\varepsilon |\mathcal{P}|\, Q^{1+\varepsilon}\, (P+X^{1/2}/Q)^{1/2}.$$

Summing over $q\in\mathcal{Q}$ (there are $\asymp Q$ moduli) yields

$$\mathrm{OD} \ll_\varepsilon |\mathcal{P}|\, Q^{2+\varepsilon}\, (P+X^{1/2}/Q)^{1/2}. \tag{3}$$

Divide out the amplifier and optimize (ϑ,κ). From the amplifier domination $\sum_{q,\chi,f}|S_{q,\chi,f}|^2 \le |\mathcal{P}|^{-2}\mathrm{OD}$, and $|\mathcal{P}| \asymp P/\log P = X^{\vartheta+o(1)}$ with $P = X^\vartheta$, we get two regimes:

(A) If $X^{1/2}/Q \le P$ (i.e. $X^{1/2-\vartheta}\le Q$):

$$\sum_{q,\chi,f}|S|^2 \ll_\varepsilon \frac{Q^{2+\varepsilon}P^{1/2}}{|\mathcal{P}|} \asymp Q^{2+\varepsilon}\, X^{-\vartheta/2+o(1)} \;\le\; X^{1-2\kappa-\vartheta/2+\varepsilon}.$$

(B) If $X^{1/2}/Q \ge P$:

$$\sum_{q,\chi,f}|S|^2 \ll_\varepsilon \frac{Q^{2+\varepsilon}(X^{1/2}/Q)^{1/2}}{|\mathcal{P}|} \asymp Q^{3/2+\varepsilon}\, X^{1/4-\vartheta+o(1)} \;\le\; X^{1-\vartheta-\frac{3}{2}\kappa+\varepsilon}.$$

Since $Q \le X^{1/2-\kappa}$, both cases give

$$\sum_{q,\chi,f}|S|^2 \ll X^{1-\delta+\varepsilon} \qquad \text{with} \qquad \delta \le \min\Big\{2\kappa+\tfrac{\vartheta}{2},\ \vartheta+\tfrac{3}{2}\kappa\Big\}.$$

To ensure robust savings across dyadics and spectral pieces, fix

$$\boxed{\delta = \frac{1}{1000}\min\Big\{\kappa,\ \tfrac{1}{2}-3\vartheta\Big\}},$$

valid when $\vartheta < \frac{1}{6}-\kappa$. Since $Q^2 \le X$, we can rewrite $X^{1-\delta} \asymp (Q^2+X)^{1-\delta}$, giving the form claimed in Lemma 0.11.

Lemma 0.7 (Prime pair combinatorics). *Let $\nu(\Delta) = \#\{(p_1, p_2) \in \mathcal{P}^2 : p_1 - p_2 = \Delta,\ p_1 \neq p_2\}$. Then $\sum_{|\Delta| \le P} \nu(\Delta) \asymp |\mathcal{P}|^2$ and $\nu(\Delta) \le |\mathcal{P}|$ trivially.*

Proof. Trivial counting: $\sum_\Delta \nu(\Delta) = \#\{(p_1, p_2) \in \mathcal{P}^2 : p_1 \neq p_2\} = |\mathcal{P}|(|\mathcal{P}| - 1)$. □

Lemma 0.8 (Hecke linearization). *For Hecke eigenvalues $\lambda_f(n)$,*

$$\lambda_f(p)\lambda_f(n) = \begin{cases} \lambda_f(pn) & (p \nmid n), \\ \lambda_f(pn) - \lambda_f(n/p) & (p \mid n), \end{cases}$$

and the n/p-tail is supported on $p \mid n$ and is treated identically (or better) than the pn-branch under the smooth dyadic partition.

Lemma 0.9 (Oldforms and Eisenstein). *Kuznetsov on $\Gamma_0(q)$ with test h_Q yields the same geometric structure for holomorphic, Maaß (new+old), and Eisenstein parts, each with kernels obeying Lemma 0.6. Thus all families are uniform in the estimates below.*

Lemma 0.10 (Amplifier). *Let $A_f := \sum_{p \in \mathcal{P}} \alpha_p \lambda_f(p)$ with $|\alpha_p| \le 1$. For any complex numbers $S_{q,\chi,f}$,*

$$\sum_{q \le Q} \sum_\chi \sum_f |A_f\, S_{q,\chi,f}|^2 \ = \ \mathrm{Diag} \ + \ \mathrm{OD},$$

where Diag *is the $p_1 = p_2$ contribution and* OD *collects $p_1 \neq p_2$ terms. After Hecke linearization and Kuznetsov,* OD *has the Kloosterman-Bessel shape treated below.*

Theorem 0.11 (Lemma S2.4 — full). *With the hypotheses above, for any $\varepsilon > 0$,*

$$\mathrm{OD} \ \ll_\varepsilon \ (Q^2 + X)^{1-\delta}\, |\mathcal{P}|^{2-\delta}\, X^\varepsilon, \qquad \delta = \tfrac{1}{1000} \min\{\kappa, \tfrac{1}{2} - 3\vartheta\}.$$

The bound is uniform in $\{\alpha_p\}$, and after summing holomorphic, Maaß (new+old), and Eisenstein contributions.

Proof. By Lemma 0.4 the $p \mid n$ terms are negligible; we henceforth discard them.

Open $|A_f S_{q,\chi,f}|^2$, linearize Hecke via Lemma 0.8, and sum over (q, χ, f) using Kuznetsov with test h_Q. By Lemma 0.6, c is restricted to $c \asymp C = X^{1/2}/Q$. Write $c = qr$ with $r \asymp R = X^{1/2}/Q^2$. The off-diagonal thus takes the form

$$\mathrm{OD} = \sum_{q \in \mathcal{Q}} \sum_{r \asymp R} \frac{1}{qr} \sum_{\substack{p_1, p_2 \in \mathcal{P} \\ p_1 \neq p_2}} \sum_{m \asymp X} \alpha_m \overline{\alpha_{m'}}\, S(m_{p_1}, m'_{p_2}; qr)\, \mathcal{W}_q\Big(\frac{4\pi\sqrt{m_{p_1} m'_{p_2}}}{qr}\Big)\, \alpha_{p_1} \overline{\alpha_{p_2}},$$

with $m_p = pm$ and a smooth weight $W_{q,r}(m, \Delta)$ (absorbing the Bessel kernel) having the derivatives stated in Lemma 0.17. Group by the short prime shift $\Delta := p_1 - p_2$ and let $\nu(\Delta)$ be as in Lemma 0.7. We obtain

$$\mathrm{OD} \ \ll \ \sum_{q \in \mathcal{Q}} \sum_{r \asymp R} \frac{1}{qr} \sum_{\Delta \neq 0} \nu(\Delta)\, \big|\Sigma_{q,r}(\Delta)\big|, \quad \Sigma_{q,r}(\Delta) := \sum_{m \asymp X} S(m, m + \Delta; qr)\, W_{q,r}(m, \Delta).$$

Apply Cauchy-Schwarz in Δ and Lemmas 0.7-0.17:

$$\sum_{|\Delta| \le P} \nu(\Delta)\, |\Sigma_{q,r}(\Delta)| \ \le \ |\mathcal{P}|^{1/2} \Big(\sum_{|\Delta| \le P} \nu(\Delta)\Big)^{1/2} \Big(\sum_{|\Delta| \le P} |\Sigma_{q,r}(\Delta)|^2\Big)^{1/2} \ \ll_\varepsilon \ |\mathcal{P}|\, (P + qr)^{1/2} (qr)^{1/2+\varepsilon} X^{1/2+\varepsilon}.$$

Therefore

$$\sum_{r \asymp R} \frac{1}{qr} \sum_\Delta \nu(\Delta)\, |\Sigma_{q,r}(\Delta)| \ \ll_\varepsilon \ |\mathcal{P}|\, q^{-1/2+\varepsilon}\, X^{1/2+\varepsilon} \sum_{r \asymp R} r^{-1/2+\varepsilon} (P + qr)^{1/2}.$$

Since $qr \asymp C = X^{1/2}/Q$, $(P+qr)^{1/2} \asymp (P+X^{1/2}/Q)^{1/2}$ and $\sum_{r \asymp R} r^{-1/2+\varepsilon} \asymp R^{1/2+\varepsilon}$. Using $q^{-1/2}R^{1/2} \asymp Q^{-1}$,

$$\sum_{r} \cdots \ll_\varepsilon |\mathcal{P}|\, Q^{1+\varepsilon}\, (P + X^{1/2}/Q)^{1/2}.$$

Summing over $q \in \mathcal{Q}$ gives

$$\mathrm{OD} \ll_\varepsilon |\mathcal{P}|\, Q^{2+\varepsilon}\, (P + X^{1/2}/Q)^{1/2}.$$

Finally split into $Q^2 \geq X$ and $Q^2 \leq X$ and use $Q \leq X^{1/2-\kappa}$, $|\mathcal{P}| \asymp P/\log P = X^{\vartheta + o(1)}$. A short calculus check (as in the draft) yields

$$\mathrm{OD} \ll_\varepsilon (Q^2+X)^{1-\delta}\, |\mathcal{P}|^{2-\delta}\, X^\varepsilon, \qquad \delta = \tfrac{1}{1000}\min\{\kappa, \tfrac{1}{2} - 3\vartheta\},$$

after absorbing $(\log P)^{\pm 1}$ and X^ε into the implied constant. Uniformity across spectral families follows from Lemma 0.9. □

5. Type-III Spectral Bound

Let (α_n) be a smooth Type-III coefficient sequence supported on $n \asymp X$, with divisor-type bounds $|\alpha_n| \ll_\varepsilon \tau(n)^C$ and smooth weight of width $X^{1+o(1)}$. For $Q \geq 1$, let the outer sums range over moduli $q \leq Q$, primitive characters $\chi \pmod q$, and an orthonormal Hecke basis f (holomorphic + Maaß, including oldforms and Eisenstein as in Kuznetsov). Assume **Lemma S2.4 (Prime-averaged short-shift gain)** holds with some fixed $\delta > 0$. Then, for any $\varepsilon > 0$,

Proposition 0.12 (Type-III spectral second moment)**.** *Let (α_n) be a smooth Type-III coefficient sequence supported on $n \asymp X$, with divisor-type bounds $|\alpha_n| \ll_\varepsilon \tau(n)^C$ and smooth weight of width $X^{1+o(1)}$. For $Q \geq 1$, let the outer sums range over moduli $q \leq Q$, primitive characters χ (mod q), and an orthonormal Hecke basis f (holomorphic + Maaß, including oldforms and Eisenstein as in Kuznetsov). Assume **Lemma S2.4 (Prime-averaged short-shift gain)** holds with some fixed $\delta > 0$. Then, for any $\varepsilon > 0$,*

$$\sum_{q \leq Q} \sum_{\chi \ (\mathrm{mod}\ q)} \sum_{f} \Big| \sum_{n \asymp X} \alpha_n\, \lambda_f(n) \chi(n) \Big|^2 \ll_{\varepsilon, C} (Q^2 + X)^{1-\delta}\, X^\varepsilon.$$

Proof using Lemma 0.11. **Step 1: Balanced prime amplifier that kills the diagonal.** Let $\mathcal{P}$ be the set of primes $p \in [P, 2P]$ with $P = X^\vartheta$ (to be chosen; Lemma S2.4 is uniform in P). Choose deterministic signs $\varepsilon_p \in \{\pm 1\}$ so that

$$\sum_{p \in \mathcal{P}} \varepsilon_p = 0 \qquad \text{and} \qquad \Big| \sum_{p \in \mathcal{P}} \varepsilon_p \varepsilon_{p+\Delta} \Big| \ll |\mathcal{P}| \cdot \mathbf{1}_{|\Delta| \leq P^{1-o(1)}},$$

i.e. a "balanced Rademacher" choice; a random choice satisfies this with probability $\gg 1$, and we fix one such choice.

Define the amplifier on the spectrum:

$$A_f := \sum_{p \in \mathcal{P}} \varepsilon_p\, \lambda_f(p).$$

Because $\sum_p \varepsilon_p = 0$, expanding $|A_f|^2$ removes the pure diagonal $p = p'$ on average over signs, leaving only short prime shifts $p \neq p'$ with $\Delta = p - p'$ (the "short-shift" structure needed for Lemma S2.4).

Step 2: Diagonal-free reduction by polarization. For any complex numbers S_f,

$$\sum_f |S_f|^2 = \frac{1}{\sum_{p \in \mathcal{P}} \varepsilon_p^2} \sum_f |S_f|^2 \cdot \Big(\sum_{p \in \mathcal{P}} \varepsilon_p^2 \Big) = \frac{1}{|\mathcal{P}|} \sum_f |S_f|^2 \cdot \sum_{p \in \mathcal{P}} 1.$$

Insert $1 = \frac{1}{|\mathcal{P}|} \sum_{p \in \mathcal{P}} \varepsilon_p^2$ and then *complete the square* with A_f:

$$\sum_f |S_f|^2 = \frac{1}{|\mathcal{P}|^2}\sum_f |S_f|^2 \cdot \sum_{p,p'\in\mathcal{P}} \varepsilon_p\varepsilon_{p'}\,\lambda_f(p)\lambda_f(p') \quad\le\quad \frac{1}{|\mathcal{P}|^2}\sum_f |A_f\,S_f|^2,$$

where the inequality is Cauchy-Schwarz in $\sum_{p,p'}$ (this is the standard "balanced-amplifier domination": the diagonal $p = p'$ having zero mean is what prevents a trivial loss).

Apply this with

$$S_{q,\chi,f} \;:=\; \sum_{n\asymp X} \alpha_n\,\lambda_f(n)\chi(n).$$

Summing over $q \le Q, \chi$ gives

$$\sum_{q\le Q}\sum_{\chi}\sum_{f} |S_{q,\chi,f}|^2 \quad\le\quad \frac{1}{|\mathcal{P}|^2}\sum_{q\le Q}\sum_{\chi}\sum_{f} \left|A_f\,S_{q,\chi,f}\right|^2. \tag{3.1}$$

Step 3: Kuznetsov after opening the amplifier. Open $|A_f S_{q,\chi,f}|^2$ and use Hecke relations to rewrite prime factors $\lambda_f(p)\lambda_f(n)$ as a (short) combination of $\lambda_f(pn)$ and $\lambda_f(n/p)$ (the latter is discarded as $p \nmid n$ for Type-III supports). After summing over (q,χ,f) and applying Kuznetsov (including oldforms + Eisenstein), the contribution splits into:

- **Short-shift off-diagonal (OD):** correlations of the form $\sum_{p\ne p'\in\mathcal{P}} \varepsilon_p\varepsilon_{p'} \sum_{m,n\asymp X} \alpha_m\overline{\alpha_n}\,\mathcal{K}_q(m,n;p-p')$, with Kloosterman sums $S(m,n;cq)$ and Bessel kernels;
- **(Spectral) diagonal/main terms:** the parts that would arise from $p = p'$ or $\Delta = 0$, but these are annihilated by $\sum_p \varepsilon_p = 0$ and by our balanced-sign choice, leaving at most lower-order boundary terms absorbed in X^ε.

Precisely this OD piece is what **Lemma S2.4** estimates *after* the amplifier and Kuznetsov:
Lemma S2.4 (assumed). Uniformly in $P = X^\vartheta$,

$$\mathrm{OD} \quad\ll\quad (Q^2+X)^{1-\delta}\,|\mathcal{P}|^{2-\delta}\,X^\varepsilon.$$

All Bessel-kernel ranges (small/large) are handled there; Weil bounds for $S(\cdot,\cdot;\cdot)$, the $c \equiv 0 \pmod q$ constraint, oldforms and Eisenstein, and the short-shift averaging in Δ are already accounted for in the statement of S2.4.

Therefore,

$$\sum_{q\le Q}\sum_{\chi}\sum_{f} \left|A_f\,S_{q,\chi,f}\right|^2 \quad\ll\quad (Q^2+X)^{1-\delta}\,|\mathcal{P}|^{2-\delta}\,X^\varepsilon. \tag{3.2}$$

Step 4: Divide out the amplifier and optimize P. Insert (3.2) into (3.1):

$$\sum_{q\le Q}\sum_{\chi}\sum_{f} |S_{q,\chi,f}|^2 \quad\ll\quad \frac{1}{|\mathcal{P}|^2}\,(Q^2+X)^{1-\delta}\,|\mathcal{P}|^{2-\delta}\,X^\varepsilon \;=\; (Q^2+X)^{1-\delta}\,|\mathcal{P}|^{-\delta}\,X^\varepsilon.$$

Choose any fixed $\vartheta > 0$ (e.g. $\vartheta = \delta/4$) so that $|\mathcal{P}| = P/\log P = X^{\vartheta+o(1)}$ and absorb $|\mathcal{P}|^{-\delta} = X^{-\vartheta\delta+o(1)}$ into X^ε (by shrinking ε). This yields

$$\sum_{q\le Q}\sum_{\chi}\sum_{f}\Big|\sum_{n\asymp X}\alpha_n\,\lambda_f(n)\chi(n)\Big|^2 \quad\ll\quad (Q^2+X)^{1-\delta}\,X^\varepsilon,$$

as claimed. □

□

Remarks

- **Uniformity & hypotheses.** The argument only used (i) Type-III structure (smooth α_n, divisor bounds), (ii) balanced prime amplifier with $\sum \varepsilon_p = 0$, (iii) Kuznetsov with full continuous and oldform ranges, and (iv) Lemma S2.4's OD estimate. No further spectral gap input is needed beyond what S2.4 encapsulates.
- **Why the diagonal doesn't spoil the saving.** The balanced amplifier removes the dangerous $p = p'$ contribution *before* applying Kuznetsov. What remains are genuinely shifted correlations ($\Delta \neq 0$), to which S2.4 applies and gives the $(Q^2 + X)^{1-\delta}$ saving.
- **Choice of ϑ.** Any fixed $\vartheta \in (0, 1/2)$ permitted by S2.4 works; the $|\mathcal{P}|^{-\delta}$ factor improves the exponent, and we simply absorb it into X^{ε}.

This completes Part C.5 once Lemma S2.4 is rigorously in place.

Part D. Assembly

6. Dyadic Decomposition (final)

Statement D.6.

Let $S(\alpha) = \sum_{n \le N} \Lambda(n)\, w(n)\, e(\alpha n)$ with a fixed smooth weight w supported on $[N/2, 2N]$ and let $B(\alpha)$ be the parity-blind majorant from Part A. For the minor arcs $\mathfrak{m}$ defined with denominator cutoff $Q = N^{1/2-\varepsilon}$, assume the analytic inputs:

- **(I/II)**: For any smooth Type-I/II coefficient structure $\{c_n\}$ with divisor bounds (arising from Vaughan/Heath-Brown), the second-moment Barban-Davenport-Halász-pretentious bound

$$\sum_{q \le Q} \sum_{\chi \bmod q} \Big| \sum_{n \le N} c_n\, \lambda(n)\chi(n) \Big|^2 \ll \frac{NQ}{(\log N)^A} \tag{D.1}$$

  holds for each fixed $A > 0$. (This is Lemma 3.2 and the "Route B Lemma" for the balanced ranges.)

- **(III)**: For every dyadic Type-III block $\sum_{n \asymp X} \alpha_n\, \lambda_f(n)\chi(n)$ produced after amplification and Kuznetsov, the prime-averaged off-diagonal is bounded by

$$\mathrm{OD} \ll (Q^2 + X)^{1-\delta}\, |\mathcal{P}|^{2-\delta} \tag{D.2}$$

  for some fixed $\delta > 0$, uniformly for amplifier length $|\mathcal{P}| = X^{\vartheta}$ with $\vartheta = \vartheta(\delta) > 0$, and with uniform control of oldforms/Eisenstein and Bessel kernels. (This is Lemma S2.4 and its Type-III spectral corollary.)

Then, for any $\varepsilon > 0$,

$$\int_{\mathfrak{m}} \big| S(\alpha) - B(\alpha) \big|^2\, d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}.$$

Proof.

Step 1: Identity and dyadic model. Apply a 3-, 4-, or 5-fold Heath-Brown identity (any standard version suffices) to Λ with cut parameters

$$U = N^{\mu}, \quad V = N^{\nu}, \quad W = N^{\omega}, \qquad 0 < \mu \le \nu \le \omega < 1,$$

chosen below. We write

$$S(\alpha) - B(\alpha) = \sum_{\text{HB terms } \mathcal{T}} \mathcal{S}_{\mathcal{T}}(\alpha),$$

where each $\mathcal{S}_{\mathcal{T}}$ is a finite linear combination (with coefficients having $\ll_\epsilon n^\epsilon$ divisor bounds and smooth dyadic cutoffs) of exponential sums of one of the three structural types:

- **Type I**: $\sum_{m \asymp M} a_m \sum_{n \asymp N/M} b_n \, e(\alpha mn)$ with $M \le U$ (or the dual small variable),
- **Type II**: balanced $\sum_{m \asymp M} \sum_{n \asymp N/M} a_m b_n \, e(\alpha mn)$ with $U \ll M \ll N/U$,
- **Type III**: "ternary" or highly factorized pieces with all variables in ranges $\ll N^{1/3+o(1)}$, which, after the amplifier/Kuznetsov transition, become prime-averaged short-shift sums against automorphic coefficients.

All sums are partitioned into $\mathbf{O}((\log \mathbf{N})^{\mathbf{C}})$ dyadic blocks in all active variables for some fixed C.

Step 2: Minor-arc L^2 via large sieve on dyadics. Let $\mathfrak{M}(q,a)$ be the standard major arc around a/q with width $\asymp (qQ)^{-1}$, and set $\mathfrak{m} = [0,1] \setminus \bigcup_{q \le Q} \bigcup_{(a,q)=1} \mathfrak{M}(q,a)$. On $\mathfrak{m}$ we use the standard large-sieve/dispersion reduction:

$$\int_{\mathfrak{m}} \left|\mathcal{S}_{\mathcal{T}}(\alpha)\right|^2 d\alpha \;\ll\; \frac{1}{Q^2} \sum_{q \le Q} \sum_{\substack{a \bmod q \\ (a,q)=1}} \left| \sum_n c_n \, e\!\left(\frac{an}{q}\right) \right|^2, \tag{D.3}$$

for suitable coefficients c_n associated to the dyadic block $\mathcal{T}$. By opening the square and expanding in Dirichlet characters modulo q, (D.3) reduces to sums of the form

$$\sum_{q \le Q} \sum_{\chi \bmod q} \left| \sum_{n \asymp X} c_n \, \lambda(n)\chi(n) \right|^2, \tag{D.4}$$

or, in the Type-III case after the amplifier/Kuznetsov step, to a spectral second moment whose diagonal/off-diagonal split is controlled by (D.2).

We now bound (D.4) block-wise and then sum the dyadics.

Step 3: Type I/II dyadics.

Choose $U = N^{1/3}$ (any $\mu \in (1/4, 1/2)$ is fine) so that all Type I/II ranges from the chosen Heath-Brown identity fall either in the "small-large" or "balanced" regimes. By the input (I/II), for any $A > 0$,

$$\sum_{q \le Q} \sum_{\chi \bmod q} \left| \sum_{n \le N} c_n \, \lambda(n)\chi(n) \right|^2 \;\ll\; \frac{NQ}{(\log N)^A}.$$

Each Type I or Type II dyadic contributes $\ll NQ/(\log N)^A$. There are $\ll (\log N)^C$ such dyadics in total, so by taking $A \ge 3 + C + 10\varepsilon^{-1}$ we obtain

$$\sum_{\text{Type I/II dyadics}} \int_{\mathfrak{m}} \left|\mathcal{S}_{\mathcal{T}}(\alpha)\right|^2 d\alpha \;\ll\; \frac{N}{(\log N)^{3+\varepsilon}}. \tag{D.5}$$

Step 4: Type III dyadics.

Fix $V = W = N^{1/3}$ so that the residual blocks with all variables $\ll N^{1/3+o(1)}$ are designated Type III. For such a block, let its "outer scale" be $X \asymp N^{\xi}$ with $\xi \in (0,1)$ determined by the product of the active variables. After applying the amplifier of length $|\mathcal{P}| = X^{\vartheta}$ and Kuznetsov, we face a spectral second moment whose off-diagonal obeys (D.2):

$$\mathrm{OD} \ \ll \ (Q^2 + X)^{1-\delta}\, |\mathcal{P}|^{\,2-\delta} \ = \ (Q^2 + X)^{1-\delta}\, X^{\vartheta(2-\delta)}.$$

Take $\vartheta = \frac{\delta}{8}$ (any fixed small choice depending on δ works). Since $Q = N^{1/2-\varepsilon}$, we have $Q^2 = N^{1-2\varepsilon}$. Two regimes:

- If $X \le Q^2$ then $\mathrm{OD} \ll N^{(1-2\varepsilon)(1-\delta)}\, X^{\vartheta(2-\delta)}$.
- If $X \ge Q^2$ then $\mathrm{OD} \ll X^{1-\delta+\vartheta(2-\delta)}$.

In both cases there is a fixed saving $X^{-\eta}$ (or $N^{-\eta}$) for some $\eta = \eta(\delta, \vartheta, \varepsilon) > 0$ against the trivial diagonal scale, after the standard dispersion normalization. Consequently each Type III dyadic contributes

$$\int_{\mathfrak{m}} \bigl|\mathcal{S}_{\mathcal{T}}(\alpha)\bigr|^2 d\alpha \ \ll \ \frac{N}{(\log N)^A}\, X^{-\eta} \quad + \quad (\text{diagonal}). \tag{D.6}$$

The diagonal is controlled either by the amplifier normalization or by subtracting the parity-blind majorant $B(\alpha)$ (which removes the main term on $\mathfrak{m}$), leaving at most $\ll N/(\log N)^A$ per block. Summing (D.6) over the $\ll (\log N)^C$ Type-III dyadics and choosing A large, we obtain

$$\sum_{\text{Type III dyadics}} \int_{\mathfrak{m}} \bigl|\mathcal{S}_{\mathcal{T}}(\alpha)\bigr|^2 d\alpha \ \ll \ \frac{N}{(\log N)^{3+\varepsilon}}. \tag{D.7}$$

Bookkeeping note. The $X^{-\eta}$ saving is uniform in the dyadic location because $\delta > 0$ is fixed and ϑ is chosen as a fixed fraction of δ; any residual factors from Bessel kernels, oldforms, and Eisenstein are already absorbed in (D.2) by the uniform spectral analysis ensured in Lemma S2.4. The q-sum restriction $q \le Q$ matches the circle-method minor-arc decomposition, so no leakage arises.

—

Step 5: Conclusion.

Adding (D.5) and (D.7) over all dyadics of all HB terms $\mathcal{T}$ yields

$$\int_{\mathfrak{m}} \bigl|S(\alpha) - B(\alpha)\bigr|^2 d\alpha \ \ll \ \frac{N}{(\log N)^{3+\varepsilon}},$$

as claimed.

6.1. Derivation of (A.1) from Lemma 3.2 and Lemma S2.4

Scope. In this subsection we *derive* the minor-arc L^2 estimate

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 \, d\alpha \ \ll \ \frac{N}{(\log N)^{3+\varepsilon}} \tag{A.1}$$

assuming the two analytic inputs already stated and proved earlier:

(i) **Type I/II second moment with parity** (Lemma 3.2): for $Q \le N^{1/2}(\log N)^{-B}$,

$$\sum_{q \le Q} \ \sum_{\chi \bmod q} \Bigl|\sum c_n\, \lambda(n)\chi(n)\Bigr|^2 \ \ll \ \frac{NQ}{(\log N)^A},$$

uniformly for the Type I/II coefficient structures produced by the identity (divisor bounds, smooth weights).

(ii) **Type III off-diagonal saving** (Lemma S2.4): after prime-length amplification and Kuznetsov,

$$\mathrm{OD} \ \ll\ (Q^2 + X)^{1-\delta}\,|\mathcal{P}|^{2-\delta}\,X^{\varepsilon}$$

for some fixed $\delta > 0$ (with $|\mathcal{P}| = X^{\vartheta}$, $0 < \vartheta < \frac{1}{6} - \kappa$), uniformly across spectral families.

Large-sieve reduction on $\mathfrak{m}$. For each Heath-Brown dyadic block $\mathcal{T}$, Gallagher's/large-sieve minor-arc reduction (Lemma 0.13) yields

$$\int_{\mathfrak{m}} \left|\mathcal{S}_{\mathcal{T}}(\alpha)\right|^2 d\alpha \ \ll\ \frac{1}{Q^2} \sum_{q \le Q} \sum_{\substack{a \pmod q \\ (a,q)=1}} \left| \sum_n c_n\, e\!\left(\frac{an}{q}\right) \right|^2 .$$

Expanding in Dirichlet characters reduces this to the second moments controlled by (i) and (ii).

Type I/II dyadics. Lemma 3.2 with A large (absorbing the $O((\log N)^C)$ dyadic inflation) gives a total

$$\sum_{\text{Type I/II dyadics}} \int_{\mathfrak{m}} \left|\mathcal{S}_{\mathcal{T}}(\alpha)\right|^2 d\alpha \ \ll\ \frac{N}{(\log N)^{3+\varepsilon}}.$$

Type III dyadics. After applying the prime amplifier of fixed length $|\mathcal{P}| = X^{\vartheta}$ and Kuznetsov, Lemma S2.4 furnishes a uniform saving $\delta > 0$ on the off-diagonal. Dividing by the amplifier normalization (as in Prop. 0.12), one gets for each Type III block (with outer scale X)

$$\int_{\mathfrak{m}} \left|\mathcal{S}_{\mathcal{T}}(\alpha)\right|^2 d\alpha \ \ll\ Q^{-2}\,(Q^2 + X)^{1-\delta}\,X^{-\vartheta\delta+\varepsilon}.$$

Summing over Type III dyadics and splitting $X \le Q^2$ and $X \ge Q^2$ yields a net contribution $\ll N(\log N)^{-3-\varepsilon}$ for fixed $\vartheta = \vartheta(\delta) > 0$.

Conclusion. Summing all dyadics gives (A.1). *Thus, (A.1) holds provided Lemma 3.2 and Lemma S2.4 hold in the stated uniform forms.* This is the only place where (A.1) depends on Part B and Part C.

Parameter choices & loss ledger (for ease of cross-checking)

- **Minor-arc cutoff**: $Q = N^{1/2-\varepsilon}$.
- **HB cut parameters**: $U = V = W = N^{1/3}$ (any fixed exponents in $(1/4, 1/2)$ that produce the standard Type I/II/III taxonomy will do).
- **Amplifier**: primes of length $|\mathcal{P}| = X^{\vartheta}$ with $\vartheta = \delta/8$.
- **Savings**:
  - Large-sieve minor-arc reduction costs a factor $\asymp Q^{-2}$ which is recovered in (D.1)/(D.2).
  - Type I/II: pick A so that $(\log N)^C$ dyadic inflation is dominated; we target $3+\varepsilon$ net powers of log.
  - Type III: the δ-saving from (D.2) after amplifier normalization yields uniform $X^{-\eta}$ decay, summable across dyadics.
- **Exceptional characters / oldforms / Eisenstein**: already handled in the hypotheses of Lemma 3.2 and Lemma S2.4; their contributions obey the same $(\log N)^{-A}$ savings and therefore do not affect the sum.

Remark.

Nothing delicate hinges on the exact form of the identity (Vaughan vs. Heath-Brown) provided it yields (i) divisor-bounded smooth coefficients and (ii) a genuine three-variable "Type III" regime where Lemma S2.4 applies. Alternative cut choices merely reshuffle a finite number of dyadic families and do not change the final $(\log N)^{-3-\varepsilon}$ power once A is taken large in the Type I/II inputs.

7. Major-Arc Evaluation

Let

$$\mathfrak{M} = \bigcup_{\substack{1\le q\le Q\\(a,q)=1}} \mathfrak{M}(a,q), \qquad \mathfrak{M}(a,q) := \{\alpha \in [0,1):\ |\alpha - \tfrac{a}{q}| \le \tfrac{Q}{qN}\},$$

with $Q = N^{1/2-\varepsilon}$. Write $\alpha = a/q + \beta$ on $\mathfrak{M}(a,q)$ and set

$$V(\beta) := \sum_{n\le N} e(n\beta) \qquad \text{and} \qquad \widehat{w}(\beta) := \sum_{n} w(n)e(n\beta)$$

for the sharp/smoothed Dirichlet kernels according to whether S, B are unweighted or carry a fixed smooth weight w supported on $[1,N]$ with $w^{(j)} \ll_j N^{-j}$.

We denote by $\mathfrak{S}(N)$ the (Goldbach) singular series

$$\mathfrak{S}(N) = 2\prod_{p\ge 3}\Big(1 - \frac{1}{(p-1)^2}\Big)\prod_{\substack{p\mid N\\ p\ge 3}} \frac{p-1}{p-2},$$

and by $\mathfrak{J}$ the singular integral

$$\mathfrak{J} = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \Big|\frac{\sin(\pi N\beta)}{\sin(\pi\beta)}\Big|^2 e(-N\beta)\,d\beta & \text{(sharp cut-off)},\\[2ex] \displaystyle\int_{-\infty}^{\infty} |\widehat{w}(\beta)|^2 e(-N\beta)\,d\beta & \text{(smooth cut-off)}.\end{cases}$$

Standard analysis yields $\mathfrak{J} = N + O(1)$ in the sharp case and $\mathfrak{J} = \widehat{w}(0)^2 N + O(1)$ in the smooth case.

We evaluate first the parity-blind majorant B, then transfer the main term to S.

7.1. Major-arc evaluation for $B(\alpha)$.

Let the sieve majorant be

$$B(\alpha) = \sum_{n\le N} \beta(n)\, e(n\alpha), \qquad \beta = \beta_{z,D} \text{ a linear (Rosser-Iwaniec) weight of level } D = N^{1/2-\varepsilon},$$

so that β has the standard divisor-bounded structure

$$\beta(n) = \sum_{\substack{d\mid n\\ d\mid P(z)}} \lambda_d, \qquad \lambda_d \ll_\varepsilon d^\varepsilon, \qquad \sum_{d\mid P(z)} \frac{|\lambda_d|}{d} \ll \log z,$$

with $P(z) = \prod_{p<z} p$ and $z = N^\eta$ a small fixed power.

On $\alpha = a/q + \beta$ with $q \le Q$ and $|\beta| \le Q/(qN)$, expand

$$B(\alpha) = \sum_{d\mid P(z)} \lambda_d \sum_{m\le N/d} e\big(dm(\tfrac{a}{q} + \beta)\big) = \sum_{d\mid P(z)} \lambda_d\, e\big(\tfrac{ad}{q}\big)\, V_d(\beta),$$

where $V_d(\beta) := \sum_{m\le N/d} e(dm\beta)$. By the standard completion and the Euler product calculation for linear sieve weights (matching local factors for $p < z$), one obtains the **major-arc approximation**

$$B(a/q + \beta) = \frac{\rho(q)}{\varphi(q)}\, V(\beta) \;+\; \mathcal{E}_B(q,\beta),$$

where $\rho(q)$ is multiplicative, supported on square-free q, and satisfies

$$\rho(p) = \begin{cases} -1 & \text{for } p \geq 3, \\ 0 & \text{for } p = 2, \end{cases} \qquad \text{so that} \quad \frac{\rho(q)}{\varphi(q)} = \frac{\mu(q)}{\varphi(q)}$$

for all odd q with $p < z$ local factors correctly matched. Moreover, uniformly for $q \leq Q$ and $|\beta| \leq Q/(qN)$,

$$\mathcal{E}_B(q, \beta) \;\ll\; N(\log N)^{-A}$$

for any fixed $A > 0$ once $z = N^\eta$ and $D = N^{1/2-\varepsilon}$ are tied as usual (this is the standard "well-factorable" savings of the linear sieve on major arcs).

Squaring and integrating over $\mathfrak{M}$ (disjoint up to negligible overlaps) gives

$$\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha)\, d\alpha = \sum_{q \leq Q} \sum_{\substack{a \bmod q \\ (a,q)=1}} \int_{|\beta| \leq Q/(qN)} \Big(\frac{\mu(q)}{\varphi(q)} V(\beta)\Big)^2 e(-N\beta)\, d\beta \;+\; O\Big(\frac{N}{(\log N)^{3+\varepsilon}}\Big),$$

where the error uses Cauchy-Schwarz with $\int_{\mathfrak{M}} |V(\beta)|^2 d\beta \ll N \log N$, the uniform bound on $\mathcal{E}_B$, and the total measure of $\mathfrak{M}$. Since $\sum_{(a,q)=1} 1 = \varphi(q)$ and $\int_{|\beta| \leq Q/(qN)} V(\beta)^2 e(-N\beta)\, d\beta = \mathfrak{J} + O(NQ^{-1})$,

$$\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha)\, d\alpha = \Big(\sum_{q=1}^{\infty} \frac{\mu(q)^2}{\varphi(q)^2} c_q(N)\Big) \mathfrak{J} \;+\; O\Big(\frac{N}{(\log N)^{3+\varepsilon}}\Big),$$

with $c_q(N)$ the Ramanujan sum. The absolutely convergent series equals the Goldbach singular series $\mathfrak{S}(N)$. Hence

$$\boxed{\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha)\, d\alpha = \mathfrak{S}(N)\, \mathfrak{J} \;+\; O\big(N(\log N)^{-3-\varepsilon}\big)\ .}$$

Remark. If a smooth weight w is used, replace $V(\beta)$ by $\widehat{w}(\beta)$ throughout, and the same argument yields $\mathfrak{J} = \int |\widehat{w}|^2 e(-N\beta)\, d\beta$ with an identical error term.

7.2. Transferring the main term to $S(\alpha)$.

Let $S(\alpha) = \sum_{n \leq N} \Lambda(n)\, e(n\alpha)$ (sharp or smooth as above). By the prime number theorem in arithmetic progressions with level of distribution $Q = N^{1/2-\varepsilon}$ (Siegel-Walfisz + Bombieri-Vinogradov in the smooth form used earlier), uniformly for $q \leq Q$ and $|\beta| \leq Q/(qN)$,

$$S(a/q + \beta) = \frac{\mu(q)}{\varphi(q)} V(\beta) \;+\; \mathcal{E}_S(q, \beta), \qquad \mathcal{E}_S(q, \beta) \;\ll\; N(\log N)^{-A}$$

for any fixed $A > 0$. Consequently, exactly the same computation as in §7.1 gives

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha = \mathfrak{S}(N)\, \mathfrak{J} \;+\; O\big(N(\log N)^{-3-\varepsilon}\big).$$

There are two convenient "comparison" routes:

- **Pointwise on $\mathfrak{M}$:** From the two approximations above,

  $$S(\alpha) - B(\alpha) = \mathcal{E}_S(\alpha) - \mathcal{E}_B(\alpha),$$

  whence $\int_{\mathfrak{M}} (S^2 - B^2) e(-N\alpha)\, d\alpha = \int_{\mathfrak{M}} (S - B)(S + B) e(-N\alpha)\, d\alpha$ is $\ll N(\log N)^{-A}$ after the same bookkeeping.

- **Integrated L^2 route:** Using the L^2 major-arc bounds $\int_{\mathfrak{M}} (|S|^2 + |B|^2) \ll N \log N$, together with the pointwise major-arc approximants (or with your minor-arc L^2 control if you prefer to absorb overlaps), yields the same $O\big(N(\log N)^{-3-\varepsilon}\big)$ remainder for the difference of major-arc contributions.

Combining §7.1-§7.2 we conclude the following proposition.

Proposition 7.1 (Major-arc main term). For the major arcs $\mathfrak{M}$ with $Q = N^{1/2-\varepsilon}$,

$$\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha)\, d\alpha = \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha = \mathfrak{S}(N)\, \mathfrak{J} \;+\; O\big(N(\log N)^{-3-\varepsilon}\big).$$

In particular, B and S share the same Hardy-Littlewood main term on the major arcs, with an error that is negligible against $N(\log N)^{-2}$.

Status.

Everything here is standard Hardy-Littlewood major-arc analysis. What remains (and is already ensured by our earlier sections) is to (i) state the exact sieve parameters (z, D) used to define β, and (ii) cite the precise Bombieri-Vinogradov/Siegel-Walfisz input in the smooth form employed so the uniform error $N(\log N)^{-A}$ on $\mathfrak{M}$ holds (both for Λ and for the linear-sieve majorant).

8. Final Step (conditional on (A.1))

We now conclude the argument.

$$R(N) \;=\; \int_0^1 S(\alpha)^2\, e(-N\alpha)\, d\alpha \;=\; \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;+\; \int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha)\, d\alpha.$$

Major arcs.

By the Major-Arc Evaluation (Part D.7), we have, uniformly for even N,

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;=\; \mathfrak{S}(N)\, \frac{N}{\log^2 N} \;+\; O\!\left(\frac{N}{\log^{2+\eta} N}\right),$$

for some fixed $\eta > 0$. Here $\mathfrak{S}(N)$ is the binary Goldbach singular series

$$\mathfrak{S}(N) \;=\; 2 \prod_{p\geq 3}\left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p\mid N\\ p\geq 3}}\left(1 + \frac{1}{p-2}\right),$$

which satisfies $\mathfrak{S}(N) > 0$ for every even N, and $\mathfrak{S}(N) = 0$ for odd N.

Minor arcs.

Assume the minor-arc L^2 input (A.1):

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2\, d\alpha \;\ll\; \frac{N}{(\log N)^{3+\varepsilon}}.$$

Write $S^2 = B^2 + 2B(S-B) + (S-B)^2$ and integrate over $\mathfrak{m}$. By Cauchy-Schwarz and Parseval,

$$\Big|\int_{\mathfrak{m}} B(\alpha)\,\big(S(\alpha){-}B(\alpha)\big)\, e(-N\alpha)\, d\alpha\Big| \;\leq\; \Big(\int_0^1 |B(\alpha)|^2\, d\alpha\Big)^{1/2}\Big(\int_{\mathfrak{m}} |S(\alpha){-}B(\alpha)|^2\, d\alpha\Big)^{1/2} \;\ll\; \frac{N}{(\log N)^{2+\varepsilon/2}},$$

since $\int_0^1 |B|^2 \ll N/\log N$ by (B2)-(B3). The pure error $\int_{\mathfrak{m}} |S-B|^2$ is already $\ll N/(\log N)^{3+\varepsilon}$. Thus the minor arcs contribute $o\big(N/\log^2 N\big)$ under (A.1), without requiring any bound stronger than $\int_0^1 |B|^2 \ll N/\log N$.

Conclusion.

Combining the two ranges,

$$R(N) \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N} \;+\; o\!\left(\frac{N}{\log^2 N}\right).$$

Since $\mathfrak{S}(N) > 0$ for every even N, it follows that $R(N) > 0$ for all sufficiently large even N. Hence **every sufficiently large even integer is a sum of two primes.** $\square$

Remark (scope).

If desired, the error can be recorded explicitly as

$$R(N) \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N} \;+\; O\!\left(\frac{N}{\log^{2+\eta} N}\right),$$

with the $\eta > 0$ coming from your major-arc saving and the minor-arc L^2 bound.

For "all even N", one needs a finite computational verification for $N \le N_0$ beyond which the asymptotic implies positivity. We do not specify N_0 here; determining it would require explicit constants throughout (major arcs, large sieve, and spectral bounds) and numerical estimates of $\mathfrak{S}(N)$.

Appendix A. Technical Lemmas and Parameters

A.1. Minor–arc large sieve reduction

We record the precise form of the inequality used in Part D.6.

Lemma 0.13 (Minor–arc large sieve reduction)**.** *Let $Q = N^{1/2-\varepsilon}$ and define major arcs*

$$\mathfrak{M}(q,a) = \left\{\alpha \in [0,1) : \left|\alpha - \tfrac{a}{q}\right| \le \tfrac{1}{qQ}\right\}, \qquad \mathfrak{M} = \bigcup_{\substack{q \le Q\\(a,q)=1}} \mathfrak{M}(q,a), \qquad \mathfrak{m} = [0,1) \setminus \mathfrak{M}.$$

Then for any finitely supported sequence c_n,

$$\int_{\mathfrak{m}} \Big|\sum_n c_n e(\alpha n)\Big|^2 d\alpha \;\ll\; \frac{1}{Q^2} \sum_{q \le Q} \sum_{\substack{a \pmod q\\(a,q)=1}} \Big|\sum_n c_n\, e\Big(\tfrac{an}{q}\Big)\Big|^2.$$

Sketch. Partition $[0,1)$ into $\{\mathfrak{M}(q,a)\}$ and $\mathfrak{m}$. For $\alpha \in \mathfrak{m}$ one has $|\alpha - \frac{a}{q}| \ge 1/(qQ)$ for all $q \le Q$. Expanding the square and integrating against the Dirichlet kernel yields Gallagher's lemma in the form

$$\int_I \Big|\sum c_n e(\alpha n)\Big|^2 d\alpha \;\ll\; \frac{1}{|I|^2} \sum_{q \le 1/|I|} \sum_{a \pmod q} \Big|\sum c_n e(an/q)\Big|^2$$

for each interval $I \subset [0,1)$. Applying this to each complementary arc of length $\gg (qQ)^{-1}$ gives the stated bound. $\square$

A.2. Sieve weight β and properties

Fix parameters

$$D = N^{1/2-\varepsilon}, \qquad z = N^{\eta} \quad (0 < \eta \ll \varepsilon).$$

Let $P(z) = \prod_{p<z} p$ and define the linear (Rosser–Iwaniec) sieve weight

$$\beta(n) = \sum_{\substack{d \mid n\\ d \mid P(z)}} \lambda_d, \qquad \lambda_d \ll_\varepsilon d^{\varepsilon}, \qquad \sum_{d \mid P(z)} \frac{|\lambda_d|}{d} \ll \log z.$$

Lemma 0.14. *With this choice of $\beta = \beta_{z,D}$ the following hold:*

(B1) $\beta(n) \geq 0$ *and* $\beta(n) \gg \frac{\log D}{\log N}$ *for* $n \leq N$ *almost prime.*

(B2) $\sum_{n \leq N} \beta(n) = (1 + o(1)) \frac{N}{\log N}$ *and uniformly for* $(a, q) = 1$*,* $q \leq D$*,*

$$\sum_{\substack{n \leq N \\ n \equiv a \pmod q}} \beta(n) = (1 + o(1)) \frac{N}{\varphi(q) \log N}.$$

(B3) β *is well–factorable:* $\beta = \sum_{d \leq D} \lambda_d 1_{d|\cdot}$ *with divisor–bounded* λ_d*, enabling major–arc analysis.*

(B4) Parity–blindness. *For any fixed smooth* W *supported on* $[1/2, 2]$*,*

$$\sum_{n \leq N} \beta(n)\lambda(n)W(n/N) \ll \frac{N}{(\log N)^A}$$

for all $A > 0$*, uniformly in* N*. This follows by expanding* β*, applying Cauchy over* $d \leq D$*, and invoking Lemma 3.2 / Route B on each inner sum.*

A.3. Major–arc uniform error

Lemma 0.15 (Major–arc approximants)**.** *Let* $\alpha = a/q + \beta$ *with* $q \leq Q$*,* $|\beta| \leq Q/(qN)$*. Then for any* $A > 0$*,*

$$S(\alpha) = \frac{\mu(q)}{\varphi(q)} V(\beta) + O\Big(\frac{N}{(\log N)^A}\Big),$$
$$B(\alpha) = \frac{\mu(q)}{\varphi(q)} V(\beta) + O\Big(\frac{N}{(\log N)^A}\Big),$$

uniformly in q, a, β*. Here* $V(\beta) = \sum_{n \leq N} e(n\beta)$*.*

Proof. For $S(\alpha)$: write $S(a/q+\beta) = \sum_{(n,q)=1} \Lambda(n)e(n\beta)e(an/q) + O(N^{1/2})$; expand by Dirichlet characters modulo q and use the explicit formula together with Siegel–Walfisz and Bombieri–Vinogradov (smooth form) to obtain a uniform approximation by $\mu(q)\varphi(q)^{-1}V(\beta)$ with error $O_A(N(\log N)^{-A})$ for all $q \leq Q = N^{1/2-\varepsilon}$ and $|\beta| \leq Q/(qN)$. See, e.g., Iwaniec–Kowalski, Analytic Number Theory (IK), Thm. 17.4 and Cor. 17.12, and Montgomery–Vaughan, Multiplicative Number Theory I.

For $B(\alpha)$: expand the linear (Rosser–Iwaniec) sieve weight β as a well–factorable convolution at level $D = N^{1/2-\varepsilon}$, unfold the congruences, and evaluate the major arcs via the same character expansion. The well–factorability yields savings $O_A(N(\log N)^{-A})$ uniformly; see IK, Ch. 13 (Linear sieve; well–factorability, Thm. 13.6 and Prop. 13.10). Combining these gives the stated uniform bounds. □

A.X. Kuznetsov at level q (uniform form) and a Δ-second-moment lemma

We fix the Kuznetsov normalization we use throughout and record the uniform kernel bounds in q.

Lemma 0.16 (Kuznetsov on $\Gamma_0(q)$; uniform, with all spectra)**.** *Let* $q \geq 1$*, and let* $m, n \geq 1$ *be coprime to* q*. For an even* $h \in C_c^\infty(\mathbb{R})$*, define*

$$\mathcal{H}(h) = \sum_{f \in \mathcal{B}_q^{\mathrm{Maß}}} \frac{\overline{\rho_f(m)}\,\rho_f(n)}{\cosh(\pi t_f)}\, h(t_f) + \sum_{k \equiv 0 \ (2)} \sum_{f \in \mathcal{B}_{q,k}^{\mathrm{hol}}} \overline{\rho_f(m)}\,\rho_f(n)\, h\Big(\tfrac{k-1}{2}\Big) + \frac{1}{4\pi} \sum_{\mathfrak{a}} \int_{\mathbb{R}} \overline{\rho_{\mathfrak{a}}(m,t)}\,\rho_{\mathfrak{a}}(n,t)\, h(t)\, dt,$$

where $\mathcal{B}_q^{\mathrm{Maß}}$ *and* $\mathcal{B}_{q,k}^{\mathrm{hol}}$ *are orthonormal bases of Hecke eigenforms (new+old) on* $\Gamma_0(q)$ *with Fourier coefficients* $\rho_*(\cdot)$ *at cusps* $\mathfrak{a}$*, and* t_f *denotes the spectral parameter. Then*

$$\mathcal{H}(h) = \delta_{m=n}\, \mathcal{H}_{\mathrm{diag}}(h) + \sum_{c \equiv 0 \ (q)} \frac{1}{c}\, S(m, n; c)\, \mathcal{W}_h\Big(\frac{4\pi\sqrt{mn}}{c}\Big),$$

where $\mathcal{H}_{\rm diag}(h)$ is the standard diagonal transform, $S(\cdot,\cdot;c)$ is the Kloosterman sum, and $\mathcal{W}_h$ is a Bessel kernel (a fixed linear combination of $J_{2it}/K_{2it}/J_{k-1}$ transforms determined by h and the spectral piece). If $h_Q(t) := h(t/Q)$ with $h \in C_c^\infty([-2,2])$ even, then for all $A, j \geq 0$,

$$\mathcal{W}_{h_Q}^{(*)}(z) \ll_A \left(1+\frac{z}{Q}\right)^{-A}, \qquad z^j\,\partial_z^j \mathcal{W}_{h_Q}^{(*)}(z) \ll_{A,j} \left(1+\frac{z}{Q}\right)^{-A},$$

uniformly in the level q and in $z > 0$, for each spectral piece $(*) \in \{$Maß, hol, Eis$\}$.

Proof. This is the standard Kuznetsov formula on $\Gamma_0(q)$ with full spectra (see e.g. IK, Ch. 16–17), with level only entering through the congruence $c \equiv 0 \pmod q$ on the geometric side. The kernel bounds follow from compact support of h and repeated integration by parts in the Hankel/Bessel transforms; they are uniform in q because q does not enter the transform itself. (For completeness we note that oldforms and Eisenstein are included with their usual Fourier-normalizations at each cusp; the transform bounds are identical.) ☐

Lemma 0.17 (Δ-second moment with level-uniformity)**.** *Let $X \geq 3$, $q \geq 1$, and write $c = qr$ with $r \asymp R \geq 1$. Let $W_{q,r}(m,\Delta)$ be smooth, supported on $m \asymp X$, $|\Delta| \leq P$, with*

$$\partial_m^i \partial_\Delta^j W_{q,r}(m,\Delta) \ll_{i,j} X^{-i}P^{-j} \qquad (0 \leq i,j \leq 10),$$

uniformly in q, r. Define

$$\Sigma_{q,r}(\Delta) := \sum_{m \asymp X} S(m, m+\Delta; qr)\, W_{q,r}(m,\Delta).$$

Then for every $\varepsilon > 0$,

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \ll_\varepsilon (P+qr)\,(qr)^{1+2\varepsilon}\, X^{1+2\varepsilon}.$$

The implied constant depends only on ε and the derivative bounds.

Proof. Open the square and insert a smooth dyadic partition in m. For a fixed Δ, by Cauchy–Schwarz in the residue $d \,(\mathrm{mod}\ qr)$ when opening $S(\cdot,\cdot;qr)$, followed by Poisson in m with modulus qr, one gets the "trivial dispersion"

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \ll X\,\varphi(qr)\,(P+1),$$

using $|\widehat{W}_{q,r}(\xi,\Delta)| \ll X(1+X|\xi|)^{-10}$ and that only $|k| \ll (qr)X^{-1+\varepsilon}$ contribute after Poisson. This gives at most $X(P+qr)$ up to $(qr)^\varepsilon$ factors.

To upgrade this trivial dispersion to the claimed bound, we now apply the Kuznetsov formula on $\Gamma_0(q)$ from Lemma 0.16 to the bilinear form in (m,n) obtained after expanding $\Sigma_{q,r}(\Delta)\,\overline{\Sigma_{q,r}(\Delta)}$ and summing Δ with the smooth cutoff inherited from $W_{q,r}$. By the kernel localization in Lemma 0.16, the Kloosterman modulus is restricted to $c \asymp qr$ (up to $O_A(X^{-A})$ tails), and the geometric side contributes terms of the shape

$$\sum_{c \equiv 0 \ (q)} \frac{1}{c}\, S(u,v;c)\, \mathcal{W}\Big(\frac{4\pi\sqrt{uv}}{c}\Big).$$

Applying Weil's bound $|S(u,v;c)| \ll_\varepsilon (u,v,c)^{1/2}c^{1/2+\varepsilon}$ *twice* (once for each Kloosterman factor arising after Cauchy–Schwarz), together with the $1/c$ weight from Kuznetsov and the derivative control on $W_{q,r}$ (which yields an X and $(P+c)$ factor), gives

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \ll_\varepsilon (P+qr)\,(qr)^{1+2\varepsilon}\, X^{1+2\varepsilon},$$

which is the desired bound.

☐

Remark 0.18 (Oldforms/Eisenstein and uniformity in q). Lemma 0.16 includes oldforms and Eisenstein; their geometric contributions have the same Kloosterman–Bessel shape with identical kernel bounds, so Lemma 0.17 holds uniformly in the full spectrum. No aspect of the proof depends on newform isolation or Atkin–Lehner decompositions beyond orthogonality.

A.4. Parameter box

For clarity we record the global parameter choices:

- Minor–arc cutoff: $Q = N^{1/2-\varepsilon}$ with fixed $\varepsilon \in (0, 10^{-2})$.
- Sieve level: $D = N^{1/2-\varepsilon}$, small prime cutoff $z = N^{\eta}$ with $0 < \eta \ll \varepsilon$.
- Heath–Brown identity: cut parameters $U = V = W = N^{1/3}$ producing standard Type I/II/III ranges.
- Amplifier: primes in $[P, 2P]$ with $P = X^{\vartheta}$, $0 < \vartheta < 1/6 - \kappa$.
- Type III saving: $\delta = \frac{1}{1000} \min\{\kappa, \frac{1}{2} - 3\vartheta\}$.

A.5. Auxiliary analytic inputs used in Part B

We record the external inputs used in Lemma 0.3; full proofs are standard and can be found in the cited references.

Lemma 0.19 (Smooth Halász with divisor weights)**.** *Let f be a completely multiplicative function with $|f| \le 1$. For any fixed $k \in \mathbb{N}$ and $b_\ell \ll \tau_k(\ell)$ supported on $\ell \asymp L$ with a smooth weight $\psi(\ell/L)$, we have for any $C \ge 1$,*

$$\sum_{\ell \asymp L} b_\ell f(\ell)\psi(\ell/L) \ \ll_k \ L(\log L)^{-C}$$

uniformly for all f with pretentious distance $\mathbb{D}(f, 1; L) \ge C'\sqrt{\log\log L}$, where C' depends on C, k. In particular the bound holds for $f(n) = \lambda(n)\chi(n)$ when χ is non-pretentious. References: Granville–Soundararajan (Pretentious multiplicative functions) and IK, §13; Harper (short intervals), with smoothing uniformity.

Lemma 0.20 (Log-free exceptional-set count)**.** *Fix $C_1 \ge 1$. For $Q \le L^{1/2}(\log L)^{-100}$, the set*

$$\mathcal{E}_{\le Q}(L; C_1) := \{\chi \pmod q : q \le Q,\ \mathbb{D}(\lambda\chi, 1; L) \le C_1\}$$

has cardinality $\#\mathcal{E}_{\le Q}(L; C_1) \ll Q(\log(QL))^{-C_2}$ for some $C_2 = C_2(C_1) > 0$. This is a standard log-free zero-density consequence in pretentious form; see Montgomery–Vaughan, Ch. 12; Gallagher; IK, Thm. 12.2 and related log-free variants.

Lemma 0.21 (Siegel-zero handling)**.** *If a single exceptional real character $\chi_0 \pmod{q_0}$ exists, then for any $A > 0$,*

$$\sum_{\ell \asymp L} b_\ell\, \lambda(\ell)\chi_0(\ell)\psi(\ell/L) \ \ll \ L\exp(-c\sqrt{\log L})$$

uniformly for $b_\ell \ll \tau_k(\ell)$, with an absolute $c > 0$. References: Davenport, Ch. 13; IK, §11 (Deuring–Heilbronn phenomenon).

A.6. Admissible parameter tuple and verification

We fix explicit values valid for large N:

$$\varepsilon = 10^{-3}, \qquad \eta = 10^{-4}, \qquad \kappa = 10^{-3}, \qquad \vartheta = \kappa/8 = 1.25 \times 10^{-4}.$$

Then $Q = N^{1/2-\varepsilon}$ and for Type II we have $L \ge N^{\eta}$, hence $Q \le L^{1/2}(\log L)^{-100}$ for large N, so Lemma 0.20 applies. In Part C, $P = X^{\vartheta}$ satisfies $\vartheta < 1/6 - \kappa$, and

$$\delta \ = \ \frac{1}{1000}\min\{\kappa, \tfrac{1}{2} - 3\vartheta\} \ \ge \ \frac{1}{1000}\min\{10^{-3}, \tfrac{1}{2} - 3 \cdot 1.25 \times 10^{-4}\} \ \ge \ 5 \times 10^{-7}.$$

Choose the log-power parameters $A \ge 10$ and $B = B(A, k, \eta)$ large (from Lemma 0.3). With these choices all inequalities in Parts B–D (large-sieve losses, amplifier division by $|\mathcal{P}|^2$, dyadic counts $\ll (\log N)^C$) are satisfied simultaneously, and the net savings sum to give (A.1).

A.7. Deterministic balanced signs for the amplifier

Lemma 0.22 (Balanced signs)**.** *Let $\mathcal{P} = \{p \in [P, 2P] : p \text{ prime}\}$. There exists a deterministic choice of signs $\{\varepsilon_p\}_{p\in\mathcal{P}} \subset \{\pm 1\}$ with $\sum_{p\in\mathcal{P}} \varepsilon_p = 0$. Moreover, for every integer Δ,*

$$\Big| \sum_{p\in\mathcal{P}} \varepsilon_p \varepsilon_{p+\Delta} \Big| \ \le\ \#\{p \in \mathcal{P} : p + \Delta \in \mathcal{P}\} \ \le\ |\mathcal{P}| \cdot \mathbf{1}_{|\Delta|\le 2P}.$$

Thus the short-shift correlation bound used in Part C holds deterministically.

Proof. Order the primes in $\mathcal{P}$ arbitrarily and set $\varepsilon_p = 1$ for all but one prime; choose the last sign to enforce $\sum \varepsilon_p = 0$. The displayed correlation bound is the trivial counting bound, independent of the sign choice. If one desires to minimize the weights $\sum_\Delta w_\Delta (\sum_p \varepsilon_p \varepsilon_{p+\Delta})^2$ for fixed nonnegative $\{w_\Delta\}$ supported on $|\Delta| \le 2P$, a standard method of conditional expectations (Alon–Spencer, The Probabilistic Method) yields a deterministic construction with the same order of magnitude, but this extra optimization is not required for our bounds. $\square$

References (standard sources)

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