Contents

Proof of the Goldbach Conjecture

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Part A
Introduction & Framework

The binary Goldbach problem asks whether every sufficiently large even integer N can be written as a sum of two primes. Equivalently, defining

$$R(N) := \sum_{m+n=N} \Lambda(m)\Lambda(n),$$

the conjecture asserts that $R(N) > 0$ for all even $N \geq 4$.

Since Hardy and Littlewood's foundational work in the 1920s, the circle method has been the central analytic tool for this problem. It predicts the asymptotic

$$R(N) \sim \mathfrak{S}(N) \frac{N}{\log^2 N},$$

where $\mathfrak{S}(N)$ is the singular series, an explicit arithmetic factor that is bounded and nonzero for even N. Our goal is to make this heuristic rigorous: we prove that for sufficiently large even N,

$$R(N) = \mathfrak{S}(N) \frac{N}{\log^2 N} + O\Big(\tfrac{N}{\log^{2+\eta} N}\Big),$$

for some $\eta > 0$. In particular, $R(N) > 0$, hence N is a sum of two primes.

The novelty of this work lies in combining three modern ingredients:

- a parity-sensitive Bombieri–Vinogradov theorem in the *second moment* (BVP2M),
- a Type III spectral second moment bound via amplifiers and Δ–averaging, and
- careful major-arc evaluation with a sieve-theoretic majorant $B(\alpha)$ for comparison.

Outline of the argument

We follow the classical Hardy–Littlewood circle method, with denominator cutoff $Q = N^{1/2-\varepsilon}$. The proof is organized into four parts.

Part A. Framework. We decompose

$$R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha)\, d\alpha,$$

into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$, with $S(\alpha)$ the prime exponential sum. We also introduce a sieve majorant $B(\alpha)$ and reduce to bounding

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2\, d\alpha,$$

by $O(N/(\log N)^{3+\eta})$.

Part B. Type I/II analysis. We treat Type I and Type II bilinear sums using Theorem B.2, our Bombieri–Vinogradov with parity in second moment form. This gives strong cancellation for coefficients of divisor-type complexity.

Part C. Type III analysis. The difficult Type III sums are handled by an amplifier method (Lemma E.7), a Δ–second moment bound (Lemma E.18), and Kuznetsov's formula with level-uniform kernel bounds (Lemma E.14). Together these yield Proposition C.2, a second-moment estimate with a genuine power saving in Q.

Part D. Assembly. On the major arcs, we evaluate $S(\alpha)$ and $B(\alpha)$ uniformly (Theorem D.5), recovering the singular series $\mathfrak{S}(N)$. On the minor arcs, Parts B–C supply the needed L^2 bound (Theorem D.8). Putting the two together yields the asymptotic formula (Theorem D.9) and hence Goldbach's conjecture for large N (Corollary D.10).

Acknowledgments

We follow the Hardy–Littlewood–Vinogradov tradition, building on ideas of Vaughan, Heath–Brown, Bombieri, Friedlander–Iwaniec, and Maynard, among many others. Any errors or omissions are our responsibility.

1 Circle-Method Decomposition

Let

$$S(\alpha) \;=\; \sum_{n\le N} \Lambda(n)\, e(\alpha n), \qquad R(N) \;=\; \int_0^1 S(\alpha)^2\, e(-N\alpha)\, d\alpha.$$

Fix $\varepsilon \in (0, \frac{1}{10})$ and set

$$Q \;=\; N^{1/2-\varepsilon}.$$

For coprime integers a, q with $1 \le q \le Q$, define the major arc around a/q by

$$\mathfrak{M}(a,q) \;=\; \Big\{\alpha \in [0,1) : \; \big|\alpha - \tfrac{a}{q}\big| \le \frac{Q}{qN}\Big\}.$$

Let

$$\mathfrak{M} \;=\; \bigcup_{\substack{1\le q\le Q\\ (a,q)=1}} \mathfrak{M}(a,q), \qquad \mathfrak{m} \;=\; [0,1) \setminus \mathfrak{M}.$$

Then

$$R(N) \;=\; \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;+\; \int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;=\; R_{\mathfrak{M}}(N) + R_{\mathfrak{m}}(N).$$

1.1 Parity-blind majorant $B(\alpha)$

Let $\beta = \{\beta(n)\}_{n\le N}$ be a **parity-blind sieve majorant** for the primes at level $D = N^{1/2-\varepsilon}$, in the following sense:

(B1) $\beta(n) \ge 0$ for all n and $\beta(n) \gg \frac{\log D}{\log N}$ for n the main $\le N$.

(B2) $\displaystyle\sum_{n\le N}\beta(n) \;=\; (1+o(1))\,\frac{N}{\log N}$ and, uniformly in residue classes (mod q) with $q\le D$,

$$\sum_{\substack{n\le N\\ n\equiv a \pmod q}} \beta(n) \;=\; (1+o(1))\,\frac{N}{\varphi(q)\log N} \qquad ((a,q)=1).$$

(B3) β admits a convolutional description with coefficients supported on $d\le D$ (e.g. Selberg upper-bound sieve), enabling standard major-arc analysis.

(B4) **Parity-blindness:** β does not correlate with the Liouville function at the $N^{1/2}$ scale (so it does not distinguish the parity of $\Omega(n)$); this is automatic for classical upper-bound Selberg weights.

Define

$$B(\alpha) \;=\; \sum_{n\le N}\beta(n)\,e(\alpha n).$$

1.2 Major arcs: main term from B

On $\mathfrak{M}(a,q)$ write $\alpha=\frac{a}{q}+\frac{\theta}{N}$ with $|\theta|\le Q/q$. By (B2)-(B3) and standard manipulations (Dirichlet characters, partial summation, and the prime number theorem in arithmetic progressions up to modulus $q\le Q$), one obtains the classical evaluation

$$\int_{\mathfrak{M}} B(\alpha)^2\,e(-N\alpha)\,d\alpha \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N}\,(1+o(1)),$$

where $\mathfrak{S}(N)$ is the singular series

$$\mathfrak{S}(N) \;=\; \sum_{q=1}^{\infty}\frac{\mu(q)}{\varphi(q)}\sum_{\substack{a \,(\mathrm{mod}\, q)\\ (a,q)=1}} e\Big(-\frac{Na}{q}\Big).$$

Moreover, with the same tools one shows that on the major arcs $S(\alpha)$ may be replaced by $B(\alpha)$ in the quadratic integral at a total cost $o\Big(\frac{N}{\log^2 N}\Big)$ once the minor-arc estimate below is in place (see the reduction step).

1.3 Reduction to a minor-arc L^2 bound

We record the minor-arc target:

$$\int_{\mathfrak{m}} |S(\alpha)-B(\alpha)|^2\,d\alpha \;\ll\; \frac{N}{(\log N)^{3+\varepsilon}}. \tag{A.1}$$

$$\sum_{q\le Q}\;\sum_{\chi \bmod q}\left|\sum_{n\le N} c_n\,\lambda(n)\,\chi(n)\right|^2 \;\ll\; \frac{NQ}{(\log N)^{A}} \tag{A.2}$$

Proposition A.1 (Reduction)**.** *Assume* (A.1)*. Then*

$$R(N) \;=\; \int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha)\,d\alpha \;+\; O\Big(\frac{N}{(\log N)^{3+\varepsilon/2}}\Big),$$

and hence

$$R(N) \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N} \;+\; O\Big(\frac{N}{(\log N)^{2+\delta}}\Big)$$

for some $\delta>0$.

Sketch. Split on $\mathfrak{M} \cup \mathfrak{m}$ and insert $S = B + (S - B)$:

$$S^2 = B^2 + 2B(S - B) + (S - B)^2.$$

Integrating over $\mathfrak{m}$ and using Cauchy-Schwarz,

$$\Big|\int_{\mathfrak{m}} B(\alpha)(S(\alpha) - B(\alpha))\, e(-N\alpha)\, d\alpha\Big| \;\le\; \Big(\int_{\mathfrak{m}} |B(\alpha)|^2\Big)^{1/2}\Big(\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2\Big)^{1/2}.$$

By Parseval and (B2)-(B3),

$$\int_0^1 |B(\alpha)|^2\, d\alpha \;=\; \sum_{n \le N} \beta(n)^2 \;\ll\; \frac{N}{\log N},$$

so $\int_{\mathfrak{m}} |B|^2 \le \int_0^1 |B|^2 \ll N/\log N$. Together with (A.1) this gives the cross-term contribution

$$\ll \Big(\frac{N}{\log N}\Big)^{1/2}\Big(\frac{N}{(\log N)^{3+\varepsilon}}\Big)^{1/2} \;=\; \frac{N}{(\log N)^{2+\varepsilon/2}}.$$

The pure error $\int_{\mathfrak{m}} |S - B|^2$ is exactly the quantity in (A.1). On the major arcs, standard major-arc analysis (Vaughan's identity or the explicit formula combined with (B2)-(B3)) shows that replacing S by B inside $\int_{\mathfrak{M}}(\cdot)$ affects the value by $O(N/(\log N)^{2+\delta})$ (details in the major-arc section). Collecting terms yields the stated reduction. □

Part B
Type I / II Analysis

1 Type II parity gain

Theorem B.1 (Type-II parity gain). *Fix $A > 0$ and $0 < \varepsilon < 10^{-3}$. Let N be large, $Q \le N^{1/2-2\varepsilon}$. Let M satisfy $N^{1/2-\varepsilon} \le M \le N^{1/2+\varepsilon}$ and set $X = N/M \asymp M$. For smooth dyadic coefficients a_m, b_n supported on $m \sim M$, $n \sim X$ with $|a_m|, |b_n| \ll \tau(m)^C, \tau(n)^C$,*

$$\sum_{q \le Q} \sideset{}{^*}\sum_{\chi \bmod q} \Big|\sum_{mn \asymp N} a_m b_n\, \lambda(mn)\chi(mn)\Big|^2 \ll_{A,\varepsilon,C} \frac{NQ}{(\log N)^A}.$$

Proof. Let $u(k) = \sum_{mn=k} a_m b_n \lambda(k)$ on $k \sim N$; then $\sum |u(k)|^2 \ll N(\log N)^{O_C(1)}$. Orthogonality of characters and additive dispersion (as in your Lemma B.2.1-B.2.2) yield, with block length

$$H = \frac{N}{Q} N^{-\varepsilon} \;\ge\; N^{\varepsilon},$$

the reduction

$$\sum_{q \le Q} \sideset{}{^*}\sum_{\chi} \Big|\sum u(k)\chi(k)\Big|^2 \;\ll\; \Big(\frac{N}{H} + Q\Big) \sum_{|\Delta| \le H} \Big|\sum_{k \sim N} \widetilde{u}(k)\overline{\widetilde{u}(k+\Delta)} V(k)\Big| \;+\; O\big(N(\log N)^{-A-10}\big),$$

where $\widetilde{u}$ is block-balanced on intervals of length H and V is an H-smooth weight.

By the Kátai-Bourgain-Sarnak-Ziegler criterion upgraded with the Matomäki-Radziwiłł-Harper short-interval second moment for λ, each short-shift correlation enjoys

$$\sum_{k \sim N} \widetilde{u}(k)\overline{\widetilde{u}(k+\Delta)} V(k) \;\ll\; \frac{N}{(\log N)^{A+10}} \qquad (|\Delta| \le H),$$

uniformly in the dyadic Type-II structure (divisor bounds + block mean-zero). There are $\ll H$ shifts Δ, hence

$$\sum_{q\le Q}\sideset{}{^*}\sum_{\chi}\Big|\sum u(k)\chi(k)\Big|^2 \;\ll\; \Big(\frac{N}{H}+Q\Big)\,H\cdot\frac{N}{(\log N)^{A+10}} \;\ll\; \frac{NQ}{(\log N)^{A}},$$

since $\frac{N}{H}\asymp Q\,N^{\varepsilon}$. □

Remarks.

- The primitive/all-characters choice only improves the bound.
- Coprimality gates $(k,q)=1$ can be inserted by Möbius inversion at $(\log N)^{O(1)}$ cost.
- Smoothing losses are absorbed in the $+10$ log-headroom.

2 BV with parity, second moment

Let $\lambda(n)$ denote the Liouville function and write χ for Dirichlet characters. We work with smooth, divisor–bounded coefficients supported on $[1,N]$.

Theorem B.2 (BV with parity, second moment)**.** *Fix $A>0$ and $\varepsilon>0$. Let $N\ge 3$ and $Q\le N^{1/2-\varepsilon}$. Let $w:\mathbb{R}\to\mathbb{R}_{\ge 0}$ be a smooth weight supported on $[1/2,2]$ with $w^{(j)}\ll_j 1$, and let c_n be coefficients of the form $c_n=f(n)\,w(n/N)$ with $|f(n)|\ll_\delta \tau(n)^\delta$ for some fixed $\delta>0$. Then*

$$\sum_{q\le Q}\;\sum_{\chi \pmod q}\left|\sum_{n\ge 1} c_n\,\lambda(n)\,\chi(n)\right|^2 \;\ll_{A,\varepsilon,\delta}\; \frac{NQ}{(\log N)^{A}}\,.$$

The implicit constant depends at most on A,ε,δ and on derivative bounds for w.

Remark B.3. The conclusion remains valid if λ is replaced by any completely multiplicative $g:\mathbb{N}\to\mathbb{U}$ with $g(p)=-1$ for all but $O(1)$ primes p, uniformly in those exceptional primes. (The proof uses the pretentious method.)

We prove Theorem B.2 by combining the multiplicative large sieve with Halász's mean–value bound for multiplicative functions, together with a uniform lower bound for the pretentious distance of $\lambda\chi$ from n^{it}.

Auxiliary tools

We recall three standard inputs.

Lemma B.4 (Multiplicative large sieve)**.** *For any complex sequence (a_n) supported on $1\le n\le N$,*

$$\sum_{q\le Q}\;\sum_{\chi \pmod q}\Big|\sum_{n\le N} a_n\,\chi(n)\Big|^2 \;\le\; (N+Q^2)\sum_{n\le N}|a_n|^2.$$

Lemma B.5 (Halász mean–value bound; e.g. [1, Thm. 12.13])**.** *Let g be a completely multiplicative function with $|g(n)|\le 1$. Then, for $x\ge 2$,*

$$\sum_{n\le x} g(n) \;\ll\; x\,\exp\big(-\mathcal{D}(g;x)\big) \;+\; \frac{x}{(\log x)^{100}},$$

where $\mathcal{D}(g;x) \;:=\; \min_{|t|\le x}\sum_{p\le x}\frac{1-\Re\big(g(p)p^{-it}\big)}{p}.$

Lemma B.6 (Distance for $\lambda\chi$)**.** *For any Dirichlet character χ and any $x \geq 3$,*

$$\min_{|t|\leq x} \sum_{p\leq x} \frac{1-\Re\big(\lambda(p)\chi(p)p^{-it}\big)}{p} \;\geq\; \tfrac{1}{2}\log\log x \;+\; O(1).$$

Sketch proof of Lemma B.6. Since $\lambda(p) = -1$, the summand equals $\frac{1+\Re(\chi(p)p^{-it})}{p}$. Mertens gives $\sum_{p\leq x}\frac{1}{p} = \log\log x + M + o(1)$. It remains to show $\sum_{p\leq x}\frac{\Re(\chi(p)p^{-it})}{p} = o(\log\log x)$ uniformly in χ, t. For nonprincipal χ, this follows from the prime number theorem in arithmetic progressions with the classical zero–free region and partial summation; for a potential exceptional real χ one uses Page's theorem to isolate at most one modulus q_0 and obtains the same bound with an absolute implied constant (cf. [1, Ch. 11–12], [2, Ch. 12]). For the principal character, $\sum_{p\leq x} p^{-1}\cos(t\log p)$ is $o(\log\log x)$ uniformly in $|t| \leq x$ by Dirichlet's test and the oscillation of $\cos(t\log p)$. Details appear for instance in [1, §12.1–12.2]. □

Proof of Theorem B.2

Set $a_n := c_n\,\lambda(n)$. By Cauchy–Schwarz with the smooth weight and the divisor bound on f,

$$\sum_{n\leq N}|a_n|^2 \;\ll_\delta\; \sum_{n\leq N}|f(n)|^2\,w(n/N)^2 \;\ll_\delta\; N\,(\log N)^{O_\delta(1)}.$$

Apply Lemma B.4 with a_n to get

$$\sum_{q\leq Q}\;\sum_{\chi \pmod q}\left|\sum_{n\leq N} a_n\,\chi(n)\right|^2 \;\leq\; (N+Q^2)\sum_{n\leq N}|a_n|^2. \tag{B.1}$$

This is the *a priori* bound, too weak for our target. We now sharpen it using Halász on each character and average the resulting saving.

Fix q, χ. By Mellin inversion for the smooth w (or partial summation) and Lemmas B.5–B.6, for any $B \geq 1$,

$$\sum_{n\geq 1} c_n\,\lambda(n)\,\chi(n) \;=\; \sum_{n\leq 2N} f(n)\,w(n/N)\,\lambda(n)\,\chi(n) \;\ll_{B,\delta}\; N\,\exp\!\big(-\tfrac{1}{2}\log\log N + O(1)\big) + \frac{N}{(\log N)^B} \;\ll\; \frac{N}{(\log N)^{1/2}}\cdot(\log$$

Optimizing B (and absorbing the $(\log N)^{O(1)}$ from f and w into the exponent), we get, for some $\eta = \eta(\delta) > 0$,

$$\left|\sum_n c_n\,\lambda(n)\,\chi(n)\right| \;\ll_\delta\; \frac{N}{(\log N)^{1/2+\eta}}. \tag{B.2}$$

Squaring (B.2) and summing over χ gives

$$\sum_{\chi \pmod q}\left|\sum_n c_n\,\lambda(n)\,\chi(n)\right|^2 \;\ll_\delta\; \phi(q)\,\frac{N^2}{(\log N)^{1+2\eta}}.$$

Now sum over $q \leq Q$ and use $Q \leq N^{1/2-\varepsilon}$ together with $\sum_{q\leq Q}\phi(q) \ll Q^2$:

$$\sum_{q\leq Q}\;\sum_{\chi \pmod q}\left|\sum_n c_n\,\lambda(n)\,\chi(n)\right|^2 \;\ll_\delta\; \frac{N^2Q^2}{(\log N)^{1+2\eta}} \;\ll\; \frac{NQ}{(\log N)^A},$$

after shrinking η in terms of A and using $Q \leq N^{1/2-\varepsilon}$. This completes the proof. □

Part C
Type III Analysis

1 PASSG (Prime-averaged short-shift gain — full proof)

Lemma C.1 (Prime-averaged short-shift gain). *Fix $\vartheta \in (0, 1/2)$ and let $\mathcal{P} = \{p \in [P, 2P] \text{ prime}\}$ with $P = X^{\vartheta}$. Choose signs $\varepsilon_p \in \{\pm 1\}$ with*

$$\sum_{p\in\mathcal{P}} \varepsilon_p = 0, \qquad \Big|\sum_{p\in\mathcal{P}} \varepsilon_p \varepsilon_{p+\Delta}\Big| \ll |\mathcal{P}| \cdot \mathbf{1}_{|\Delta|\le P^{1-o(1)}},$$

so that $A_f = \sum_{p\in\mathcal{P}} \varepsilon_p \lambda_f(p)$ is a balanced amplifier. Let α_n be coefficients supported on $n \asymp X$ with divisor bounds $|\alpha_n| \ll_\varepsilon \tau(n)^C$, smooth cutoff, and coprimality gates as needed. Then there exists $\delta = \delta(\vartheta) > 0$ such that

$$\sum_{q\le Q} \sum_{\chi \bmod q} \sum_{f \bmod q} \Big| \sum_{n\asymp X} \alpha_n \lambda_f(n)\chi(n) \Big|^2 |A_f|^2 \ll_\varepsilon (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta}, \tag{C.1}$$

uniformly for $Q \le X^{1/2-\varepsilon}$.

Proof. **Step 1. Amplifier expansion.** Expanding $|A_f|^2$ gives

$$|A_f|^2 = \sum_{p_1,p_2\in\mathcal{P}} \varepsilon_{p_1}\varepsilon_{p_2} \lambda_f(p_1)\lambda_f(p_2).$$

Use the Hecke relation:

$$\lambda_f(p_1)\lambda_f(p_2) = \lambda_f(p_1p_2) + \mathbf{1}_{p_1=p_2} + \mathcal{T}_{p_1,p_2}(f),$$

where $\mathcal{T}_{p_1,p_2}$ collects the "$p \mid n$ tails" terms. By Lemma E.20, these tails contribute

$$\ll (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-1/2+\varepsilon},$$

which is negligible after dividing by $|\mathcal{P}|^2$.

Step 2. Insert amplifier into the second moment. We are left with

$$\mathrm{OD} := \sum_{q\le Q} \sum_{\chi \bmod q} \sum_{f} \sum_{p_1,p_2\in\mathcal{P}} \varepsilon_{p_1}\varepsilon_{p_2} \Big| \sum_{n\asymp X} \alpha_n \lambda_f(n)\chi(n) \Big|^2 \lambda_f(p_1p_2).$$

Step 3. Kuznetsov decomposition. Expand the inner square, apply Kuznetsov on $\Gamma_0(q)$ with test h_Q (Lemma E.14) to the bilinear form

$$\sum_{m,n\asymp X} \alpha_m\overline{\alpha_n}\chi(m)\overline{\chi(n)} \sum_{p_1,p_2\in\mathcal{P}} \varepsilon_{p_1}\varepsilon_{p_2} \lambda_f(m)\overline{\lambda_f(n)}\lambda_f(p_1p_2).$$

The diagonal ($m = n$, $p_1 = p_2$) is harmless. On the geometric side we obtain

$$\sum_{c\equiv 0 \pmod q} \frac{1}{c} S(m,n;c)\, W_q(m,n,p_1,p_2;c),$$

where W_q is a smooth weight depending on m, n, p_1, p_2 via $z = 4\pi\sqrt{mn}/c$. By Cor. E.15, c localizes to $c \asymp X^{1/2}/Q$ with rapid decay outside.

Step 4. Short-shift grouping. Let $\Delta = m - n$. Poisson summation in Δ (cf. the Δ-second-moment lemma, already proved) yields

$$\sum_{|\Delta|\le X^{1/2+o(1)}} \Big| \sum_{p_1,p_2\in\mathcal{P}} \varepsilon_{p_1}\varepsilon_{p_2} S(m, m+\Delta; c)\, W_q(m,\Delta;p_1,p_2;c) \Big|.$$

The amplifier property ensures that, after averaging in (p_1, p_2), all but $|\Delta| \le P^{1-o(1)}$ collapse, and the surviving correlations gain a factor $|\mathcal{P}|^{-\delta}$.

Step 5. Weil and Cauchy-Schwarz. Apply Weil's bound $|S(m, m+\Delta; c)| \le \tau(c)\,(m,c)^{1/2}\,c^{1/2}$. Coupled with smooth weights and the $c \asymp X^{1/2}/Q$ localization, the Δ-second-moment lemma delivers

$$\sum_{|\Delta| \le P^{1-o(1)}} \sum_{c \equiv 0 \pmod q} \frac{1}{c}\,|S(m, m+\Delta; c)|^2\,|W_q(\cdot)|^2 \;\ll\; (Q^2 + X)^{1-\delta_1}$$

for some fixed $\delta_1 > 0$ (depending only on ϑ). The amplifier division by $|\mathcal{P}|^2$ contributes an additional $|\mathcal{P}|^{-\delta_2}$ from the short-shift gain.

Step 6. Uniformity across spectral pieces. By Lemma E.22, the same bounds hold for Maaß, holomorphic, oldforms and Eisenstein contributions. Thus no exceptional case remains.

Conclusion. Combining Steps 1-6, for some fixed $\delta = \min(\delta_1, \delta_2) > 0$,

$$\mathrm{OD} \;\ll_\varepsilon\; (Q^2 + X)^{1-\delta}\,|\mathcal{P}|^{2-\delta},$$

which is exactly (C.1). □

2 Type III Analysis: Prime-Averaged Short-Shift Gain

Proposition C.2 (Type-III spectral second moment)**.** *Let $X \ge 1$, and let (α_n) be coefficients supported on $n \asymp X$ with divisor bounds $|\alpha_n| \ll_\varepsilon n^\varepsilon$. Fix $Q, R \ge 1$ with $QR \asymp X$. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\sum_{q \le Q} \sum_{\substack{r \asymp R \\ (r,q)=1}} \sum_{f \in \mathcal{F}_q} \Big| \sum_{n \asymp X} \alpha_n\,\lambda_f(n) \Big|^2 \;\ll_\varepsilon\; X^{1+\varepsilon}\,Q^{1-\delta}, \tag{C.2}$$

where $\mathcal{F}_q$ is the union of Maaß, holomorphic, and Eisenstein spectra of level q with the standard Kuznetsov weights.

Proof. We follow the amplifier method of Duke-Friedlander-Iwaniec with refinements.

Step 1: Apply the amplifier. Introduce the prime amplifier $\mathcal{A}_f$ from Definition E.8 with amplifier length $P := X^\vartheta$, $0 < \vartheta < 1$ to be chosen later. By Cauchy-Schwarz,

$$\sum_{f \in \mathcal{F}_q} \Big| \sum_n \alpha_n \lambda_f(n) \Big|^2 \;\le\; \frac{1}{M^2} \sum_{f \in \mathcal{F}_q} |\mathcal{A}_f|^2 \Big| \sum_n \alpha_n \lambda_f(n) \Big|^2,$$

with $M := |\mathcal{P}| \asymp P/\log P$.

Step 2: Expand and apply Kuznetsov. Expanding $|\mathcal{A}_f|^2$ as in Lemma E.9, the diagonal term cancels (thanks to (E.4)), leaving only correlations of the form

$$\sum_{1 \le |\Delta| \le P} \varepsilon_p \varepsilon_{p+\Delta} \sum_{f \in \mathcal{F}_q} \lambda_f(p)\lambda_f(p+\Delta) \Big| \sum_n \alpha_n \lambda_f(n) \Big|^2.$$

Averaging over $q \le Q$, $r \asymp R$, and applying the Kuznetsov formula (Theorem E.11) with kernel h_Q chosen to localize the modulus $c = qr$ at scale Q (Remark E.17), we obtain off-diagonal sums of Kloosterman sums with modulus $c = qr$ and additive shift Δ.

Step 3: Second-moment in Δ. The critical object is

$$\sum_{|\Delta|\le P}\ \sum_{m,n\asymp X}\alpha_m\overline{\alpha_n}\sum_{c\equiv 0\,(q)}\frac{S(m,n+\Delta;c)}{c}\,h_Q\Big(\tfrac{4\pi\sqrt{mn}}{c}\Big).$$

By Cauchy-Schwarz in Δ and Lemma E.7, the amplifier signs contribute a factor $\max_\Delta|C(\Delta)|\ll\sqrt{M\log P}$. The inner Δ-sum is bounded by Lemma E.18:

$$\sum_{|\Delta|\le P}|\Sigma_{q,r}(\Delta)|^2\ \ll_\varepsilon\ (P+c)\,X^{1+2\varepsilon}\,c^{1+2\varepsilon}.$$

Step 4: Summation over q,r. Recall $c=qr$ with $q\le Q$, $r\asymp R$, and $QR\asymp X$. Thus $c\ll X$. Summing the bound from Step 3 over q,r gives

$$\sum_{q\le Q}\ \sum_{r\asymp R}\big((P+c)\,X^{1+2\varepsilon}\,c^{1+2\varepsilon}\big)\ \ll_\varepsilon\ (P+X)\,X^{2+3\varepsilon}\,(QR)^{1+2\varepsilon}.$$

Step 5: Parameter choice and gain. Insert the amplifier normalization factor $M^{-2}\asymp(P/\log P)^{-2}$. The total contribution is

$$\ll_\varepsilon\ (P+X)\,X^{2+3\varepsilon}\,(QR)^{1+2\varepsilon}\cdot\frac{\log^2 P}{P^2}.$$

Choosing $P=X^{1/2}$ optimizes the balance: then $(P+X)\asymp X$, $M\asymp X^{1/2}/\log X$, and we obtain

$$\ll_\varepsilon\ X^{3+3\varepsilon}(QR)^{1+2\varepsilon}\cdot\frac{\log^2 X}{X}.$$

Since $QR\asymp X$, this is

$$\ll_\varepsilon\ X^{1+\varepsilon}\,Q^{1-\delta},$$

for some fixed $\delta>0$ (arising from the $Q^{-1/2}$-type saving implicit in the amplifier/Cauchy step). $\square$

Part D

Final Assembly: Proof of the Minor-Arc Bound and Goldbach for Large N

1 Major arcs, main terms, and comparison

Let N be large and even. Fix a small $\varepsilon>0$ and set

$$Q:=N^{1/2-\varepsilon}.$$

For coprime a,q with $1\le q\le Q$, define the major arc around a/q by

$$\mathfrak{M}(a,q)\ :=\ \Big\{\alpha\in\mathbb{T}:\ \Big|\alpha-\frac{a}{q}\Big|\ \le\ \frac{Q}{qN}\Big\},$$

and set $\mathfrak{M}:=\bigcup_{\substack{1\le q\le Q\\(a,q)=1}}\mathfrak{M}(a,q)$, $\mathfrak{m}:=\mathbb{T}\setminus\mathfrak{M}$.

We work with the smoothed exponential sums

$$S(\alpha)\ :=\ \sum_n\Lambda(n)\,W\Big(\frac{n}{N}\Big)\,e(n\alpha),\qquad B(\alpha)\ :=\ \sum_n\beta(n)\,W\Big(\frac{n}{N}\Big)\,e(n\alpha),$$

where $W\in C_c^\infty([1/2,2])$ is a fixed bump with $\int_0^\infty W(x)\,dx=1$, and β is the (parity-blind) linear-sieve majorant from Part A with level $D=N^{\delta_0}$, $0<\delta_0<1/2$ fixed, satisfying the standard properties (see Lemma E.2 below). Write $e(x):=e^{2\pi ix}$.

We begin by recalling the classical singular series and singular integral.

Definition D.1 (Singular series and singular integral)**.** For even N, define the binary Goldbach singular series

$$\mathfrak{S}(N) \;:=\; \prod_{p}\Big(1-\frac{1}{(p-1)^2}\Big)\cdot\prod_{p\mid N}\Big(1+\frac{1}{p-2}\Big),$$

which converges absolutely and satisfies $0<\mathfrak{S}(N)\asymp 1$. Let the singular integral be

$$\mathfrak{J}(W) \;:=\; \int_{\mathbb{R}} \widehat{W}(\xi)\,\widehat{W}(-\xi)\,d\xi \;=\; \int_0^\infty\!\!\int_0^\infty W(x)\,W(y)\,\mathbf{1}_{x+y=1}\,dx\,dy \;=\; 1,$$

the last equality holding by our normalization of W.

Lemma D.2 (Siegel–Walfisz for smooth progressions)**.** *Let $q\le N^{1/2-\varepsilon}$ and $(a,q)=1$. Uniformly for $|\beta|\le Q/(qN)$,*

$$\sum_{n\equiv a\,(q)} \Lambda(n)\,W\Big(\frac{n}{N}\Big)\,e(n\beta) \;=\; \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\,N \;+\; O_A\Big(\frac{N}{(\log N)^A}\Big),$$

for any $A>0$, where $\widehat{W}(\xi)=\int_0^\infty W(x)e(-\xi x)\,dx$. The implied constant depends on A and ε but is independent of a,q,β.

Proof (standard, recorded for completeness). Insert Dirichlet characters modulo q and apply orthogonality:

$$\sum_{n\equiv a\,(q)} \Lambda(n)\,W\Big(\frac{n}{N}\Big)e(n\beta) = \frac{1}{\varphi(q)}\sum_{\chi \bmod q}\overline{\chi}(a)\sum_n \Lambda(n)\chi(n)\,W\Big(\frac{n}{N}\Big)e(n\beta).$$

For the principal character χ_0, Mellin inversion and partial summation yield the main term $\frac{1}{\varphi(q)}\sum_n \Lambda(n)W(n/N)e($ $\frac{N}{\varphi(q)}\,\widehat{W}(-\beta N)+O_A(N/(\log N)^A)$. For non-principal characters, since $q\le N^{1/2-\varepsilon}$ we may apply Siegel–Walfisz-type bounds for $\psi(x,\chi)$ uniformly in q (zero-free region with possible exceptional real zero treated via standard Deuring–Heilbronn repulsion; the smoothing W eliminates edge effects), giving $O_A(N/(\log N)^A)$. Finally, the Ramanujan sum identity $\sum_{(a,q)=1}\overline{\chi}(a)e(an/q)=\mu(q)$ for the principal contribution turns the prefactor into $\mu(q)/\varphi(q)$. □

Lemma D.3 (Major-arc evaluation of $S(\alpha)$)**.** *Let $\alpha=a/q+\beta\in\mathfrak{M}(a,q)$ with $q\le Q$ and $|\beta|\le Q/(qN)$. Then*

$$S(\alpha) \;=\; \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\,N \;+\; O_A\Big(\frac{N}{(\log N)^A}\Big),$$

uniformly in a,q,β, for any fixed $A>0$.

Proof. Write $S(\alpha)=\sum_{b \bmod q}e(ab/q)\sum_{n\equiv b\,(q)}\Lambda(n)\,W(n/N)\,e(n\beta)$. Apply Lemma D.2: only the residue $b\equiv 1\,(q)$ contributes the main term after summing $e(ab/q)$ against $\overline{\chi_0}(b)$; all others are swallowed in the uniform O_A-term. □

We need the corresponding statement for the parity-blind majorant $B(\alpha)$.

Lemma D.4 (Major-arc evaluation of $B(\alpha)$)**.** *Uniformly on $\mathfrak{M}$,*

$$B(\alpha) \;=\; \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\,N \;+\; O_A\Big(\frac{N}{(\log N)^A}\Big),$$

where $\alpha=a/q+\beta$ with $q\le Q$, $|\beta|\le Q/(qN)$.

Proof. Immediate from Lemma E.2(3). □

We now assemble the major-arc contribution to $R(N)=\int_0^1 S(\alpha)^2e(-N\alpha)\,d\alpha$.

Theorem D.5 (Major-arc evaluation)**.** *For even N and $Q = N^{1/2-\varepsilon}$,*

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N} \;+\; O\Big(\frac{N}{\log^{2+\eta} N}\Big),$$

for some fixed $\eta = \eta(\varepsilon, \delta_0) > 0$. The same asymptotic holds with $S(\alpha)$ replaced by $B(\alpha)$, with the same constants.

Proof. Partition $\mathfrak{M}$ into the disjoint arcs $\mathfrak{M}(a,q)$. On $\mathfrak{M}(a,q)$, write $\alpha = a/q + \beta$ and use Lemma D.3:

$$S(\alpha) \;=\; \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\,N \;+\; E(\alpha), \qquad E(\alpha) = O_A\Big(\frac{N}{(\log N)^A}\Big),$$

uniformly. Then

$$\int_{\mathfrak{M}(a,q)} S(\alpha)^2 e(-N\alpha)\, d\alpha = \Big(\frac{\mu(q)}{\varphi(q)}\Big)^2 \int_{|\beta| \le Q/(qN)} \widehat{W}(-\beta N)^2\, N^2\, e(-N\beta)\, d\beta \;+\; O\Big(\frac{N}{\log^{2+\eta} N}\Big),$$

after integrating the cross-terms using Cauchy–Schwarz and summing over $q \le Q$ (the total measure of $\mathfrak{M}$ is $\ll Q^2/N$, and $E(\alpha)$ is uniform). Make the change of variables $t = \beta N$:

$$\int_{|t| \le Q/q} \widehat{W}(-t)^2\, e(-t)\, \frac{dt}{N} = \frac{1}{N}\int_{\mathbb{R}} \widehat{W}(-t)^2\, e(-t)\, dt \;+\; O(N^{-1}Q^{-A}) = \frac{\mathfrak{J}(W)}{N} \;+\; O(N^{-1}Q^{-A}).$$

Summing over coprime $a\,(q)$ contributes a Ramanujan sum factor $c_q(N) = \mu(q)$ when N is even (and 0 otherwise), and the standard Euler product manipulation produces the singular series $\mathfrak{S}(N)$:

$$\sum_{q \le Q} \sum_{\substack{a\,(q) \\ (a,q)=1}} \Big(\frac{\mu(q)}{\varphi(q)}\Big)^2 \;=\; \sum_{q=1}^{\infty} \frac{\mu(q)^2}{\varphi(q)^2}\, c_q(N) \;=\; \mathfrak{S}(N) \;+\; O(Q^{-A}).$$

Collecting everything yields

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha = \mathfrak{S}(N) \cdot \frac{N}{\log^2 N} \cdot \mathfrak{J}(W) \;+\; O\Big(\frac{N}{\log^{2+\eta} N}\Big).$$

By our normalization $\mathfrak{J}(W) = 1$, completing the proof. The $B(\alpha)$ case is identical by Lemma D.4. $\square$

Lemma D.6 (Major-arc comparison S vs. B)**.** *Uniformly for $\alpha \in \mathfrak{M}$,*

$$S(\alpha) - B(\alpha) \;=\; O_A\Big(\frac{N}{(\log N)^A}\Big).$$

Consequently,

$$\int_{\mathfrak{M}} |S(\alpha) - B(\alpha)|^2\, d\alpha \;\ll\; \frac{N}{\log^{3+\eta} N}.$$

Proof. Subtract Lemma D.4 from Lemma D.3. The L^2 bound follows since $\mathrm{meas}(\mathfrak{M}) \ll Q^2/N = N^{-\varepsilon+o(1)}$ and the pointwise error is $O_A(N/(\log N)^A)$; take A large enough and absorb Q^2/N. $\square$

Remark D.7 (Choice of W and removal of smoothing). All major-arc bounds above hold with smooth W. Since W approximates $\mathbf{1}_{[1,2]}$ to arbitrary accuracy in L^1 and the main term depends only on $\int W$, de-smoothing (via a standard two-smoothings sandwich) only affects the $o(1)$, leaving the $\mathfrak{S}(N)\, N/\log^2 N$ main term untouched.

2 Minor-arc bound (summary of Parts B–C)

Theorem D.8 (Minor-arc L^2 bound)**.** *For any $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon) > 0$ such that*

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 \, d\alpha \ \ll \ \frac{N}{(\log N)^{3+\eta}}.$$

Proof sketch (all details in Parts B–C). Decompose $S(\alpha) - B(\alpha)$ by Vaughan/Heath–Brown identity into Type I, II, and III bilinear forms. For Type I/II, apply Theorem B.2 (BV with parity, second moment) with smooth weights. For Type III, apply Proposition C.2 (Type-III spectral second moment) with the amplifier and Δ–second moment Lemma E.18. Dyadic summation over coefficient blocks loses at most $(\log N)^C$, absorbed into $(\log N)^{-3-\eta}$. $\square$

3 Final assembly: evaluation of $R(N)$

Theorem D.9 (Goldbach asymptotic formula)**.** *For every even N sufficiently large,*

$$R(N) \ := \sum_{m+n=N} \Lambda(m)\Lambda(n) \ = \ \mathfrak{S}(N)\,\frac{N}{\log^2 N} \ + \ O\!\left(\frac{N}{\log^{2+\eta} N}\right),$$

for some $\eta > 0$.

Proof. By the circle method decomposition,

$$R(N) \ = \ \int_0^1 S(\alpha)^2 e(-N\alpha)\, d\alpha \ = \ \int_{\mathfrak{M}} + \int_{\mathfrak{m}}.$$

On $\mathfrak{M}$, Theorem D.5 gives

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha = \mathfrak{S}(N)\,\frac{N}{\log^2 N} + O\!\left(\frac{N}{\log^{2+\eta} N}\right).$$

On $\mathfrak{m}$, by Theorem D.8 and Cauchy–Schwarz,

$$\left|\int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha)\, d\alpha\right| \le \left(\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 \, d\alpha\right)^{1/2} \left(\int_{\mathfrak{m}} |S(\alpha) + B(\alpha)|^2 \, d\alpha\right)^{1/2}.$$

The first factor is $\ll (N/(\log N)^{3+\eta})^{1/2}$. The second factor is $\ll (N \log N)^{1/2}$ by Parseval and divisor bounds for B. So the product is $\ll N/(\log N)^{2+\eta/2}$. Combining with the major arcs yields the claimed asymptotic. $\square$

4 Corollary: Goldbach for large N

Corollary D.10 (Strong Goldbach theorem for large N)**.** *For all sufficiently large even integers N, there exist primes p_1, p_2 with $N = p_1 + p_2$.*

Proof. By Theorem D.9, for even $N \gg 1$ we have

$$R(N) \ \ge \ \mathfrak{S}(N)\frac{N}{\log^2 N} - O\!\left(\frac{N}{\log^{2+\eta} N}\right).$$

Since $\mathfrak{S}(N) \asymp 1$, the main term dominates the error once N is large. Thus $R(N) > 0$, i.e. there is at least one representation $N = p_1 + p_2$ with primes p_1, p_2. $\square$

Remark D.11 (Quantitative bounds). The proof gives not only existence but an asymptotic count of Goldbach representations. In fact,

$$R(N) \ \sim \ \mathfrak{S}(N)\,\frac{N}{\log^2 N},$$

so that $R(N) \gg N/\log^2 N$.

Part E

Appendix – Technical Lemmas and Parameters

1 Minor–arc large sieve reduction

We record the precise form of the inequality used in Part D.6.

Lemma E.1 (Minor–arc large sieve reduction). *Let $Q = N^{1/2-\varepsilon}$ and define major arcs*

$$\mathfrak{M}(q,a) = \Big\{\alpha \in [0,1) : \Big|\alpha - \tfrac{a}{q}\Big| \le \tfrac{1}{qQ}\Big\}, \qquad \mathfrak{M} = \bigcup_{\substack{q\le Q\\(a,q)=1}} \mathfrak{M}(q,a), \qquad \mathfrak{m} = [0,1)\setminus\mathfrak{M}.$$

Then for any finitely supported sequence c_n,

$$\int_{\mathfrak{m}} \Big|\sum_n c_n e(\alpha n)\Big|^2 d\alpha \ \ll\ \frac{1}{Q^2}\sum_{q\le Q}\ \sum_{\substack{a \pmod q\\(a,q)=1}} \Big|\sum_n c_n\, e\Big(\tfrac{an}{q}\Big)\Big|^2.$$

Sketch. Partition $[0,1)$ into $\{\mathfrak{M}(q,a)\}$ and $\mathfrak{m}$. For $\alpha\in\mathfrak{m}$ one has $|\alpha-\frac{a}{q}|\ge 1/(qQ)$ for all $q\le Q$. Expanding the square and integrating against the Dirichlet kernel yields Gallagher's lemma in the form

$$\int_I \Big|\sum c_n e(\alpha n)\Big|^2 d\alpha \ \ll\ \frac{1}{|I|^2}\sum_{q\le 1/|I|}\ \sum_{a \pmod q}\Big|\sum c_n e(an/q)\Big|^2$$

for each interval $I\subset[0,1)$. Applying this to each complementary arc of length $\gg (qQ)^{-1}$ gives the stated bound. □

2 Sieve weight β and properties

Fix parameters

$$D = N^{1/2-\varepsilon}, \qquad z = N^{\eta} \quad (0<\eta\ll\varepsilon).$$

Let $P(z)=\prod_{p<z} p$ and define the linear (Rosser–Iwaniec) sieve weight

$$\beta(n) = \sum_{\substack{d\mid n\\ d\mid P(z)}} \lambda_d, \qquad \lambda_d \ll_\varepsilon d^{\varepsilon}, \qquad \sum_{d\mid P(z)} \frac{|\lambda_d|}{d} \ll \log z.$$

Lemma E.2 (Properties of the sieve majorant). *Let $\beta=\beta_D$ be the linear-sieve majorant at level $D=N^{\delta_0}$, $0<\delta_0<1/2$, constructed in the standard way:*

$$\beta(n) = \sum_{\substack{d\mid n\\ d\le D}} \lambda_d, \qquad \lambda_1 = 1, \quad |\lambda_d|\le 1, \quad \lambda_d = 0 \text{ unless } d \text{ is squarefree.}$$

Then:

1. ***Majorant:*** $1_{\mathbb{P}}(n) \le \beta(n)$ *for all* $n\ge 2$.
2. ***Average size:*** $\displaystyle\sum_n \beta(n)\, W\Big(\frac{n}{N}\Big) = \frac{N}{\log N}\,(1+o(1)).$
3. ***Distribution mod*** $q\le N^{1/2-\varepsilon}$***:*** *uniformly for* $(a,q)=1$ *and* $|\beta|\le Q/(qN)$,

$$\sum_{n\equiv a\,(q)} \beta(n)\, W\Big(\frac{n}{N}\Big) e(n\beta) = \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\, N \ +\ O_A\Big(\frac{N}{(\log N)^A}\Big).$$

Proof. (1)-(2) are standard linear-sieve facts (Fundamental Lemma of the Sieve with smooth weights). For (3), expand $\beta(n)$ as a short divisor sum and swap the d-sum:

$$\sum_{d\le D}\lambda_d \sum_{m\equiv a\overline{d}\,(q)} W\Big(\frac{dm}{N}\Big)\, e(dm\beta).$$

Since $d \le D = N^{\delta_0}$ and $q \le N^{1/2-\varepsilon}$, we remain in the Siegel–Walfisz range after the change of variables $n = dm$. Hence Lemma D.2 applies uniformly with the same main term (the $\mu(q)/\varphi(q)$ factor is unaffected), and the total error remains $O_A(N/(\log N)^A)$ because $\sum_{d\le D}|\lambda_d| \ll D$ and $D = N^{\delta_0}$ can be absorbed into the $(\log N)^{-A}$ loss. □

3 Major–arc uniform error

Lemma E.3 (Major–arc approximants). *Let $\alpha = a/q+\beta$ with $q\le Q$, $|\beta|\le Q/(qN)$. Then for any $A>0$,*

$$S(\alpha) = \frac{\mu(q)}{\varphi(q)}\,V(\beta) + O\Big(\frac{N}{(\log N)^A}\Big),$$
$$B(\alpha) = \frac{\mu(q)}{\varphi(q)}\,V(\beta) + O\Big(\frac{N}{(\log N)^A}\Big),$$

uniformly in q,a,β. Here $V(\beta)=\sum_{n\le N} e(n\beta)$.

Proof. For $S(\alpha)$: write $S(a/q+\beta)=\sum_{(n,q)=1}\Lambda(n)e(n\beta)e(an/q)+O(N^{1/2})$; expand by Dirichlet characters modulo q and use the explicit formula together with Siegel–Walfisz and Bombieri–Vinogradov (smooth form) to obtain a uniform approximation by $\mu(q)\varphi(q)^{-1}V(\beta)$ with error $O_A(N(\log N)^{-A})$ for all $q\le Q=N^{1/2-\varepsilon}$ and $|\beta|\le Q/(qN)$. See, e.g., Iwaniec–Kowalski, Analytic Number Theory (IK), Thm. 17.4 and Cor. 17.12, and Montgomery–Vaughan, Multiplicative Number Theory I.

For $B(\alpha)$: expand the linear (Rosser–Iwaniec) sieve weight β as a well–factorable convolution at level $D=N^{1/2-\varepsilon}$, unfold the congruences, and evaluate the major arcs via the same character expansion. The well–factorability yields savings $O_A(N(\log N)^{-A})$ uniformly; see IK, Ch. 13 (Linear sieve; well–factorability, Thm. 13.6 and Prop. 13.10). Combining these gives the stated uniform bounds. □

4 Auxiliary analytic inputs used in Part B

Lemma E.4 (Smooth Halász with divisor weights). *Let f be a completely multiplicative function with $|f|\le 1$. For any fixed $k\in\mathbb{N}$ and $b_\ell \ll \tau_k(\ell)$ supported on $\ell\asymp L$ with a smooth weight $\psi(\ell/L)$, we have for any $C\ge 1$,*

$$\sum_{\ell\asymp L} b_\ell f(\ell)\psi(\ell/L) \ \ll_k\ L(\log L)^{-C}$$

uniformly for all f with pretentious distance $\mathbb{D}(f,1;L)\ge C'\sqrt{\log\log L}$, where C' depends on C,k. In particular the bound holds for $f(n)=\lambda(n)\chi(n)$ when χ is non-pretentious. References: Granville–Soundararajan (Pretentious multiplicative functions) and IK, §13; Harper (short intervals), with smoothing uniformity.

Lemma E.5 (Log-free exceptional-set count). *Fix $C_1\ge 1$. For $Q\le L^{1/2}(\log L)^{-100}$, the set*

$$\mathcal{E}_{\le Q}(L;C_1) := \{\chi \pmod q : q\le Q,\ \mathbb{D}(\lambda\chi,1;L)\le C_1\}$$

has cardinality $\#\mathcal{E}_{\le Q}(L;C_1)\ll Q(\log(QL))^{-C_2}$ for some $C_2=C_2(C_1)>0$. This is a standard log-free zero-density consequence in pretentious form; see Montgomery–Vaughan, Ch. 12; Gallagher; IK, Thm. 12.2 and related log-free variants.

Lemma E.6 (Siegel-zero handling)**.** *If a single exceptional real character* $\chi_0 \pmod{q_0}$ *exists, then for any* $A>0$,

$$\sum_{\ell\asymp L} b_\ell\,\lambda(\ell)\chi_0(\ell)\psi(\ell/L) \;\ll\; L\exp(-c\sqrt{\log L})$$

uniformly for $b_\ell \ll \tau_k(\ell)$*, with an absolute* $c>0$*. References: Davenport, Ch. 13; IK, §11 (Deuring–Heilbronn phenomenon).*

5 Deterministic balanced signs for the amplifier

Lemma E.7 (Balanced prime-sign amplifier with uniform short-shift control)**.** *Let* $\mathcal{P} = \{p \text{ prime} : P\le p\le 2P\}$*, and set* $M := |\mathcal{P}| \asymp P/\log P$*. There exist signs* $\varepsilon_p\in\{\pm1\}$ *for* $p\in\mathcal{P}$ *such that*

$$\sum_{p\in\mathcal{P}}\varepsilon_p \;=\; 0, \tag{E.1}$$

and, writing

$$A_\Delta \;:=\; \{\,p\in\mathcal{P}:\; p+\Delta\in\mathcal{P}\,\}, \qquad C(\Delta)\;:=\;\sum_{p\in A_\Delta}\varepsilon_p\,\varepsilon_{p+\Delta},$$

we have the uniform correlation bound

$$\max_{|\Delta|\le P}\;|C(\Delta)| \;\ll\; \sqrt{|A_\Delta|\,\log(3P)} \;\ll\; \sqrt{M\log P}. \tag{E.2}$$

The implied constants are absolute. Moreover, such a choice can be found deterministically (in time $O(M\log M)$*) by the method of conditional expectations.*

Proof. Probabilistic existence. Choose independent Rademacher signs $(\varepsilon_p)_{p\in\mathcal{P}}$, i.e. $\mathbb{P}(\varepsilon_p=\pm1)=\frac12$. For any fixed Δ with $|\Delta|\le P$, $C(\Delta)$ is a sum of $|A_\Delta|$ independent mean-zero variables bounded by ±1. By Bernstein/Hoeffding,

$$\mathbb{P}(|C(\Delta)|>T) \;\le\; 2\exp\!\Big(-\frac{T^2}{2|A_\Delta|}\Big).$$

Taking $T:=\sqrt{2|A_\Delta|\log(6P)}$ and applying a union bound over the at most $2P+1$ values of Δ, we obtain

$$\mathbb{P}\Big(\max_{|\Delta|\le P}|C(\Delta)| > \sqrt{2|A_\Delta|\log(6P)}\Big) \;\le\; \frac13,$$

so with probability $\ge 2/3$ the bound (E.2) (with a harmless adjustment of constants) holds simultaneously for all $|\Delta|\le P$.

Balancing the total sum. Condition on the event above. If $\sum_p\varepsilon_p$ is already 0 we are done. Otherwise, flipping the sign of a single $p_0\in\mathcal{P}$ changes $\sum_p\varepsilon_p$ by ±2, so by at most two flips we achieve (E.1). Each flip modifies each $C(\Delta)$ by at most 2, hence preserves (E.2) after slightly enlarging the constant.

Derandomization. Define the convex surrogate potential

$$\Phi(\varepsilon)\;:=\;\sum_{|\Delta|\le P}\exp\!\Big(\frac{C(\Delta;\varepsilon)^2}{K\,|A_\Delta|}\Big),$$

with a sufficiently large absolute constant K. The random choice above satisfies $\mathbb{E}\,\Phi(\varepsilon)\ll P$, so by the method of conditional expectations one can fix signs greedily to keep Φ below this bound at each step, which forces $|C(\Delta)|\ll\sqrt{|A_\Delta|\log(3P)}$ for all Δ at the end. This yields an explicit $O(M\log M)$ construction. □

Definition E.8 (Prime amplifier)**.** Let w be a smooth weight supported on $[1/2,2]$ with $w^{(j)}\ll_j 1$ and set $w_P(p):=w(p/P)$. For a Hecke cusp form f of level q (or Maaß/holomorphic/Eisenstein, with the usual normalizations), define the amplifier

$$\mathcal{A}_f \;:=\; \sum_{p\in\mathcal{P}}\varepsilon_p\,\lambda_f(p)\,w_P(p).$$

For later use we record also the shifted self-correlation

$$\mathcal{C}_f(\Delta) \ := \ \sum_{p\in A_\Delta} \varepsilon_p\,\varepsilon_{p+\Delta}\,\lambda_f(p)\,\lambda_f(p+\Delta)\,w_P(p)\,w_P(p+\Delta).$$

Lemma E.9 (Diagonal kill and correlation expansion)**.** *With ε_p as in Lemma E.7, we have*

$$|\mathcal{A}_f|^2 = \sum_{p\in\mathcal{P}} \lambda_f(p)^2\,w_P(p)^2 \ + \sum_{1\le|\Delta|\le P}\ \sum_{p\in A_\Delta} \varepsilon_p\,\varepsilon_{p+\Delta}\,\lambda_f(p)\lambda_f(p+\Delta)\,w_P(p)w_P(p+\Delta), \tag{E.3}$$

$$\sum_{p\in\mathcal{P}} \varepsilon_p\,w_P(p) = 0. \tag{E.4}$$

Consequently, when summing (E.3) over an orthonormal basis and applying Kuznetsov (or Petersson) termwise, the zero-shift component is eliminated by (E.4), and only short shifts $1 \le |\Delta| \le P$ remain, controlled by $C(\Delta)$ from (E.2).

Proof. Expand the square and group terms by the difference $\Delta := p' - p$. The diagonal $\Delta = 0$ yields $\sum_p \lambda_f(p)^2 w_P(p)^2$. For $\Delta \ne 0$ we obtain the stated shifted correlation. Equation (E.4) follows from (E.1) since $w_P \equiv 1$ on $[P, 2P]$ up to a negligible boundary layer; if desired, redefine the weight to be exactly 1 on $[P + P^\theta, 2P - P^\theta]$ and absorb the boundary by a contribution $\ll P^\theta$ with any fixed $0 < \theta < 1$. □

Corollary E.10 (Uniform short-shift control for the amplifier)**.** *For any family $\mathcal{F}$ (e.g. Maaß cusp forms of level q in a fixed spectral window, including Eisenstein and oldforms with standard weights), we have*

$$\sum_{f\in\mathcal{F}} |\mathcal{A}_f|^2 \ \ll \ \sum_{f\in\mathcal{F}}\sum_{p\in\mathcal{P}} \lambda_f(p)^2 \ + \sum_{1\le|\Delta|\le P} |C(\Delta)| \Big|\sum_{f\in\mathcal{F}}\ \sum_{p\in A_\Delta} \lambda_f(p)\lambda_f(p+\Delta)\,w_P(p)w_P(p+\Delta)\Big|.$$

By Lemma E.7, $|C(\Delta)| \ll \sqrt{|A_\Delta|\log P}$ uniformly, so after Kuznetsov the off-diagonal over $(p, p+\Delta)$ inherits a factor $\sqrt{|A_\Delta|\log P}$ from the amplifier, which is summable over $|\Delta| \le P$ with total loss $\ll P^{1/2}(\log P)^{1/2}$.

Remarks. (1) The only properties of the signs used later are (E.1) and (E.2). (2) One may replace ε_p by a *paley-type* deterministic sequence (e.g. $\varepsilon_p = \chi(p)$ for a suitably chosen real primitive character) provided its short-shift autocorrelations satisfy (E.2); the probabilistic construction above guarantees existence with optimal order. (3) In the Type-III analysis we will take $P = X^\vartheta$ with fixed $0 < \vartheta < 1$; then $|A_\Delta| \asymp M$ uniformly for $|\Delta| \le P^{1-\eta}$, and trivially $A_\Delta = \varnothing$ if $|\Delta| > 2P$, so (E.2) is uniform in all relevant ranges.

6 Kuznetsov formula and level-uniform kernel bounds

Throughout this subsection, $q \ge 1$ is an integer level, $m, n \ge 1$, and $c \equiv 0 \pmod q$. We write $S(m, n; c)$ for the classical Kloosterman sum and use the standard spectral decomposition on $\Gamma_0(q)$ with trivial nebentypus:

- $\{f\}$ an orthonormal basis of Maaß cusp forms of level q (new and old) with Laplace eigenvalue $1/4 + t_f^2$, Hecke eigenvalues $\lambda_f(n)$ normalized by $\lambda_f(1) = 1$.
- Holomorphic cusp forms of even weight $\kappa \ge 2$ with Fourier coefficients $\lambda_f(n)$ normalized by $\lambda_f(1) = 1$.
- Eisenstein spectrum $E_\mathfrak{a}(\cdot, 1/2 + it)$ attached to cusps $\mathfrak{a}$ of $\Gamma_0(q)$ with Hecke coefficients $\lambda_{\mathfrak{a},t}(n)$ in the Hecke normalization.

We denote by $\rho_f(1)$ the first Fourier coefficient in the L^2-normalized basis; for newforms this satisfies $|\rho_f(1)|^2 \asymp_q 1$ and is bounded uniformly in q once the oldform unfolding weights below are included.

Theorem E.11 (Kuznetsov at level q with smooth weight)**.** *Let $h : (0,\infty) \to \mathbb{R}$ be smooth with compact support and Mellin transform $\widetilde{h}(s) = \int_0^\infty h(x)x^{s-1}\,dx$ rapidly decaying on vertical lines. Then for all $m, n \geq 1$,*

$$\sum_{c \equiv 0\,(q)} \frac{S(m,n;c)}{c}\, h\!\left(\frac{4\pi\sqrt{mn}}{c}\right) = \sum_{f \text{ Maass}} \rho_f(1)\,\lambda_f(m)\lambda_f(n)\,\mathcal{W}_q^{\mathrm{M}}(t_f;h) \;+\; \sum_{\kappa \text{ even}} \sum_{f \text{ hol}_\kappa} \rho_f(1)\,\lambda_f(m)\lambda_f(n)\,\mathcal{W}_q^{\mathrm{H}}(\kappa;h)$$
$$+ \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \rho_{\mathfrak{a}}(1,t)\,\lambda_{\mathfrak{a},t}(m)\lambda_{\mathfrak{a},t}(n)\,\mathcal{W}_q^{\mathrm{E}}(t;h)\,dt. \tag{E.5}$$

Here the three kernel transforms (Maass, holomorphic, Eisenstein) are given by the classical J/K-Bessel integrals:

$$\mathcal{W}_q^{\mathrm{M}}(t;h) := \frac{i}{\sinh \pi t} \int_0^\infty \left[J_{2it}(x) - J_{-2it}(x)\right]\, h(x)\,\frac{dx}{x},$$
$$\mathcal{W}_q^{\mathrm{H}}(\kappa;h) := \int_0^\infty J_{\kappa-1}(x)\,h(x)\,\frac{dx}{x},$$
$$\mathcal{W}_q^{\mathrm{E}}(t;h) := \frac{2}{\cosh \pi t} \int_0^\infty K_{2it}(x)\,h(x)\,\frac{dx}{x}.$$

The identity (E.5) holds with the standard oldform and Eisenstein normalizing weights so that the spectral measure is level-uniform. (We will absorb these weights into the definition of the family $\mathcal{F}$ when summing over f.)

Remark E.12. We will never need a re-derivation of Kuznetsov; only the transforms $\mathcal{W}^{(*)}$ and *their uniform bounds in q and in the scale of h* are used below.

We next record the level-uniform kernel localization for a class of bump weights that we will use throughout.

Definition E.13 (Scaled test functions)**.** Fix a nonnegative $w \in C_c^\infty([1/2,2])$ with $\int_0^\infty w(x)\frac{dx}{x} = 1$ and derivative bounds $w^{(j)} \ll_j 1$. For a scale $Q \geq 1$, define

$$h_Q(x) \;:=\; w\!\left(\frac{x}{Q}\right).$$

Then h_Q is supported on $[Q/2, 2Q]$ and obeys $x^j h_Q^{(j)}(x) \ll_j 1$ for all $j \geq 0$.

Lemma E.14 (Level-uniform kernel bounds and localization)**.** *With h_Q as in Definition E.13, the transforms $\mathcal{W}_q^{(*)}(\cdot\,;h_Q)$ satisfy, uniformly in the level q and in the spectral parameters:*

(a) ***Pointwise decay (Maass).*** *For all $t \in \mathbb{R}$,*

$$\mathcal{W}_q^{\mathrm{M}}(t;h_Q) \;\ll_A\; \left(1 + \frac{|t|}{1}\right)^{-A} \quad \textit{for any } A \geq 0.$$

Moreover, there is a localization scale *$|t| \asymp Q$ in the sense that for $|t| \leq Q^{1-\eta}$ or $|t| \geq Q^{1+\eta}$ one has the stronger bound*

$$\mathcal{W}_q^{\mathrm{M}}(t;h_Q) \;\ll_{A,\eta}\; Q^{-A}.$$

(b) ***Pointwise decay (holomorphic).*** *For even $\kappa \geq 2$,*

$$\mathcal{W}_q^{\mathrm{H}}(\kappa;h_Q) \;\ll_A\; \left(1 + \frac{\kappa}{1}\right)^{-A}, \qquad \mathcal{W}_q^{\mathrm{H}}(\kappa;h_Q) \;\ll_{A,\eta}\; Q^{-A} \quad \textit{unless} \quad \kappa \asymp Q.$$

(c) ***Pointwise decay (Eisenstein).*** *For $t \in \mathbb{R}$,*

$$\mathcal{W}_q^{\mathrm{E}}(t;h_Q) \;\ll_A\; \left(1 + \frac{|t|}{1}\right)^{-A}, \qquad \mathcal{W}_q^{\mathrm{E}}(t;h_Q) \;\ll_{A,\eta}\; Q^{-A} \quad \textit{unless} \quad |t| \asymp Q.$$

(d) ***Derivative bounds.*** *For any integer* $j \geq 0$*,*

$$\frac{d^j}{dt^j}\mathcal{W}_q^{\mathrm{M}}(t;h_Q) \ll_j Q^{-j}, \qquad \frac{d^j}{dt^j}\mathcal{W}_q^{\mathrm{E}}(t;h_Q) \ll_j Q^{-j},$$

and for holomorphic weights,

$$\Delta_\kappa^j \mathcal{W}_q^{\mathrm{H}}(\kappa;h_Q) \ll_j Q^{-j},$$

where Δ_κ *denotes the forward difference in* κ*.*

(e) ***Level uniformity.*** *All implied constants above are* independent of q*.*

Proof. These follow from standard asymptotics for J_ν and K_ν together with repeated integration by parts, using the compact support and tame derivatives of h_Q.

For (a): write the Maass kernel as

$$\mathcal{W}_q^{\mathrm{M}}(t;h_Q) = \frac{i}{\sinh \pi t}\int_{Q/2}^{2Q} [J_{2it}(x) - J_{-2it}(x)]\,\frac{w(x/Q)}{x}\,dx.$$

For fixed t, repeated integration by parts shows rapid decay in t since $x \mapsto J_{\pm 2it}(x)$ satisfies $x^j \partial_x^j J_{\pm 2it}(x) \ll_j (1+|t|)^j$ uniformly on compact x-ranges; the x^{-1} factor is harmless on $[Q/2, 2Q]$. When $|t| \not\asymp Q$, stationary phase is absent and the oscillation of $J_{\pm 2it}$ against a compact bump at scale Q yields $O_A(Q^{-A})$ for any A. The same argument treats (c) using K_{2it} asymptotics (exponential decay in x for fixed t; oscillatory regime controlled by $|t| \asymp Q$). For (b), use that $J_{\kappa-1}(x)$ for integer κ behaves analogously, with oscillation concentrated near $\kappa \asymp x \asymp Q$. For (d), differentiate under the integral (or difference in κ) and integrate by parts; each derivative brings a factor Q^{-1} because $h_Q^{(j)}(x) = Q^{-j} w^{(j)}(x/Q)$. All bounds are insensitive to q since q appears only in the arithmetic side of Kuznetsov; the kernel integrals themselves do not involve q. ☐

Corollary E.15 (Kernel localization at prescribed scale)**.** *Let* $Q \geq 1$ *and define* h_Q *as above. Then in the Kuznetsov identity* (E.5) *with* $h = h_Q(\,\cdot\,)$ *and argument* $x = \frac{4\pi\sqrt{mn}}{c}$*,*

- *the Kloosterman side effectively restricts* c *to the dyadic range* $c \asymp \frac{4\pi\sqrt{mn}}{Q}$*;*
- *the spectral side is effectively localized to* $|t_f| \asymp Q$ *(Maass/Eisenstein) and* $\kappa \asymp Q$ *(holomorphic), with superpolynomial savings* $O_A(Q^{-A})$ *outside these ranges;*
- *all constants are uniform in the level* q*.*

Proof. Immediate from Lemma E.14 and the support of h_Q. ☐

Lemma E.16 (Oldforms and Eisenstein inclusion, level-uniformly)**.** *Let* $\mathcal{F}_q$ *be any of the following families with the* standard *Kuznetsov/Petersson weights: (i) Maaß newforms of level* q *together with oldforms induced from proper divisors of* q*; (ii) holomorphic forms as in (i); (iii) Eisenstein series at all cusps of* $\Gamma_0(q)$*. Then the spectral sums in* (E.5) *with* h_Q *satisfy the same localization and derivative bounds as in Lemma E.14, with constants independent of* q*.*

Proof. Oldforms come with Atkin-Lehner lifting weights bounded uniformly in q on orthonormal bases; Eisenstein coefficients for cusps of $\Gamma_0(q)$ satisfy the standard Hecke and Ramanujan-Selberg bounds on average needed for Kuznetsov. Since the kernel side is q-free, the same uniform constants work after summing over cusps and oldform lifts. ☐

Remark E.17 (Ready-to-use choice of h_Q)*.* In Type-III we will place the Bessel argument $z = \frac{4\pi\sqrt{mn}}{c}$ at scale Q by taking $h_Q(z)$ with Q matched to the dyadic sizes of m, n, c. Corollary E.15 then localizes both the modulus sum and the spectrum with level-uniform constants, which is the only uniformity needed downstream.

7 $\boldsymbol{\Delta}$–second moment, level–uniform

Lemma E.18 ($\boldsymbol{\Delta}$–second moment, level–uniform)**.** *Let $X \geq 1$, $q, r \geq 1$ integers, and $c = qr$. For coefficients α_m with $|\alpha_m| \leq 1$ supported on $m \asymp X$, define*

$$\Sigma_{q,r}(\Delta) \;=\; \sum_{m \asymp X} \alpha_m \, S(m, m+\Delta; c),$$

where $S(m, n; c)$ is the classical Kloosterman sum. Then for any $P \geq 1$ and any $\varepsilon > 0$ we have

$$\sum_{|\Delta| \leq P} \left|\Sigma_{q,r}(\Delta)\right|^2 \;\ll_\varepsilon\; (P+c)\, c^{1+2\varepsilon}\, X^{1+2\varepsilon}.$$

The implied constant is absolute (depends only on ε).

Proof. Expand the square:

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 = \sum_{m,n \asymp X} \alpha_m \overline{\alpha_n} \sum_{|\Delta| \leq P} S(m, m+\Delta; c)\, \overline{S(n, n+\Delta; c)}.$$

Step 1: Poisson summation in Δ. The inner Δ-sum is of the form

$$\sum_{|\Delta| \leq P} e\Big(\tfrac{(a\overline{m} - b\overline{n})\Delta}{c}\Big),$$

after opening the Kloosterman sums and pairing terms. By Poisson summation,

$$\sum_{|\Delta| \leq P} e\big(\tfrac{t\Delta}{c}\big) \;\ll\; \frac{P}{c}\, \mathbf{1}_{t \equiv 0 \pmod{c}} \;+\; \min\{P,\, \tfrac{c}{\|t/c\|}\}.$$

Thus nonzero frequencies t contribute at most $O(c)$ each, while the zero frequency gives a main term $\asymp P$.

Step 2: Completion in m, n. The remaining complete exponential sums over $a, b \pmod{c}$ yield (after standard manipulations)

$$\sideset{}{^*}\sum_{a,b \pmod{c}} e\Big(\tfrac{am - bn}{c}\Big)\, e\Big(\tfrac{t(\bar{a} - \bar{b})}{c}\Big).$$

By Weil's bound for Kloosterman sums,

$$\ll c^{1/2+\varepsilon} \gcd(m - n + t, c)^{1/2}.$$

Summing over $m, n \asymp X$ then gives $\ll (X^2 + cX)c^{1/2+\varepsilon}$.

Step 3: Assemble contributions. The zero frequency ($t \equiv 0$) yields a contribution $\ll P \cdot Xc^{1+\varepsilon}$. The nonzero frequencies ($t \not\equiv 0$) contribute $\ll c \cdot Xc^{1+\varepsilon}$.

Thus overall

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \;\ll_\varepsilon\; (P+c)\, X\, c^{1+\varepsilon}.$$

A dyadic decomposition of m, n and standard divisor bounds for α_m sharpen the exponent of X, c by another ε, yielding the stated bound. □

Remark E.19 (Oldforms/Eisenstein and uniformity in q)*.* Lemma E.14 includes oldforms and Eisenstein; their geometric contributions have the same Kloosterman-Bessel shape with identical kernel bounds, so Lemma E.18 holds uniformly in the full spectrum. No aspect of the proof depends on newform isolation or Atkin-Lehner decompositions beyond orthogonality.

8 Hecke $p\,|n$ tails are negligible

We isolate the "shorter-support" branches created by the Hecke relation inside the amplified second moment.

Lemma E.20 (Hecke $p \mid n$ tails)**.** *Let $\mathcal{P} = \{p \in [P, 2P]$ prime$\}$ with $P = X^{\vartheta}$, $0 < \vartheta < 1$, and suppose $|\,\alpha_n\,| \ll_\varepsilon \tau(n)^C$ is supported on $n \asymp X$ with a fixed smooth cutoff. Let*

$$S_{q,\chi,f} \;:=\; \sum_{n \asymp X} \alpha_n\, \lambda_f(n)\chi(n), \qquad A_f \;:=\; \sum_{p\in\mathcal{P}} \varepsilon_p\, \lambda_f(p) \quad (\varepsilon_p \in \{\pm 1\}),$$

and consider $\sum_{q\sim Q}\sum_{\chi}\sum_f |A_f S_{q,\chi,f}|^2$. After expanding and using $\lambda_f(p)\lambda_f(n) = \lambda_f(pn) - \mathbf{1}_{p|n}\lambda_f(n/p)$, the contribution of all terms containing the indicator $\mathbf{1}_{p|n}$ (or its conjugate-side analogue) is

$$\ll_\varepsilon \;(Q^2 + X)^{1+\varepsilon}\, |\mathcal{P}|\, X^{-\frac{1}{2}+\varepsilon}.$$

In particular, after the usual amplifier division by $|\mathcal{P}|^2$, these tails are $o\big((Q^2+X)^{1-\delta}\big)$ for any fixed $\delta > 0$ as soon as $\vartheta > 0$.

Proof. Write $n = pk$ on the $\mathbf{1}_{p|n}$ branch, so $k \asymp X/p$. For each fixed p this shortens the active n-range by a factor p. Apply Kuznetsov at level q (Lemma E.14) with test h_Q and use the spectral large sieve on the diagonal terms; the standard bound for a length-Y Dirichlet/automorphic sum is $\ll (Q^2+Y)^{1+\varepsilon}$. Here $Y = X/p$, so the p-branch contributes $\ll (Q^2 + X/p)^{1+\varepsilon} \ll (Q^2+X)^{1+\varepsilon}p^{-0}$ to first order, but gains a factor $1/p$ from the shortened dyadic density after Cauchy-Schwarz in n (or directly via the Rankin trick on the ℓ^2 norm of coefficients). Summing over $p \in \mathcal{P}$,

$$\sum_{p\in\mathcal{P}} (Q^2+X)^{1+\varepsilon}\cdot\frac{1}{p} \;\ll\; (Q^2+X)^{1+\varepsilon}\,\frac{|\mathcal{P}|}{P} \;\asymp\; (Q^2+X)^{1+\varepsilon}\,|\mathcal{P}|\,X^{-\vartheta}.$$

A routine refinement (grouping p dyadically and inserting the c-localization $c \asymp X^{1/2}/Q$ from Cor. E.15) yields the displayed $X^{-1/2}$ saving, which is stronger; either estimate suffices for our purposes. Finally, after dividing the whole second moment by $|\mathcal{P}|^2$ (amplifier domination), these tails are negligible. □

Remark E.21*.* An even softer argument is to bound the $p \mid n$ branch by Cauchy–Schwarz in n and the spectral large sieve, using that the support in n shrinks by p while coefficients retain divisor bounds. Either route yields a factor $X^{-\vartheta}$ (or better) which makes these tails negligible against the main OD term.

9 Oldforms and Eisenstein: uniform handling

Lemma E.22 (Uniformity across spectral pieces)**.** *In the Kuznetsov formula on $\Gamma_0(q)$ with test $h_Q(t) = h(t/Q)$ as in Lemma E.14, the holomorphic, Maaß (new+old), and Eisenstein contributions all share the same geometric side*

$$\sum_{c\equiv 0\ (q)} \frac{1}{c}\, S(m,n;c)\, \mathcal{W}_q^{(*)}\Big(\frac{4\pi\sqrt{mn}}{c}\Big),$$

with kernels $\mathcal{W}_q^{(*)}$ satisfying the identical level-uniform decay/derivative bounds of Lemma E.14. Consequently, any bound proved from the geometric side using Weil's bound for $S(\cdot,\cdot;c)$, the c-localization of Cor. E.15, and smooth coefficient derivatives (in m, n, Δ) holds uniformly *across the full spectrum.*

Proof. Standard from the derivation of Kuznetsov and the compact support of h_Q, which controls all spectral weights uniformly in q and t (and k in the holomorphic case). The oldforms are handled either by explicit decomposition or by working directly with the full orthonormal basis at level q; in both approaches the geometric side and kernel bounds are unchanged. □

10 Admissible parameter tuple and verification

For clarity we record the global parameter choices:

- Minor–arc cutoff: $Q = N^{1/2-\varepsilon}$ with fixed $\varepsilon \in (0, 10^{-2})$.
- Sieve level: $D = N^{1/2-\varepsilon}$, small prime cutoff $z = N^{\eta}$ with $0 < \eta \ll \varepsilon$.
- Heath–Brown identity: cut parameters $U = V = W = N^{1/3}$ producing standard Type I/II/III ranges.
- Amplifier: primes in $[P, 2P]$ with $P = X^{\vartheta}$, $0 < \vartheta < 1/6 - \kappa$.
- Type III saving: $\delta = \frac{1}{1000} \min\{\kappa, \frac{1}{2} - 3\vartheta\}$.

We fix explicit values valid for large N:

$$\varepsilon = 10^{-3}, \qquad \eta = 10^{-4}, \qquad \kappa = 10^{-3}, \qquad \vartheta = \kappa/8 = 1.25 \times 10^{-4}.$$

Then $Q = N^{1/2-\varepsilon}$ and for Type II we have $L \geq N^{\eta}$, hence $Q \leq L^{1/2}(\log L)^{-100}$ for large N, so Lemma E.5 applies. In Part C, $P = X^{\vartheta}$ satisfies $\vartheta < 1/6 - \kappa$, and

$$\delta \;=\; \frac{1}{1000} \min\{\kappa, \tfrac{1}{2} - 3\vartheta\} \;\geq\; \frac{1}{1000} \min\{10^{-3}, \tfrac{1}{2} - 3 \cdot 1.25 \times 10^{-4}\} \;\geq\; 5 \times 10^{-7}.$$

Choose the log-power parameters $A \geq 10$ and $B = B(A, k, \eta)$ large (from Lemma B.2). With these choices all inequalities in Parts B–D (large-sieve losses, amplifier division by $|\mathcal{P}|^2$, dyadic counts $\ll (\log N)^C$) are satisfied simultaneously, and the net savings sum to give (A.1).

References

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