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Proof of the Goldbach Conjecture

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Part A Framework

This manuscript lays out a circle-method framework aimed at binary Goldbach. The final asymptotic is derived on the minor-arc L^2 estimate (A.1) and the analytic inputs explicitly stated in Parts B-D. In particular:

- Establishing (A.1) is the central new task; Parts B-D provide a proposed route via Type I/II/III analyses.
- Major-arc expansions for S and for the sieve majorant B are used with uniformity standard in the literature; precise statements are recorded in §7 with hypotheses.
- The final positivity conclusion for $R(N)$ is conditional on (A.1) and the stated major-arc bounds.

A succinct punch-list of outstanding items appears in Appendix B.

1 Circle-Method Decomposition

Let

$$S(\alpha) = \sum_{n \leq N} \Lambda(n) e(\alpha n), \quad R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha) d\alpha.$$

Fix $\varepsilon \in (0, \frac{1}{10})$ and set

$$Q = N^{1/2-\varepsilon}.$$

For coprime integers a, q with $1 \leq q \leq Q$, define the major arc around a/q by

$$\mathfrak{M}(a, q) = \left\{ \alpha \in [0, 1) : \left| \alpha - \frac{a}{q} \right| \leq \frac{Q}{qN} \right\}.$$

Let

$$\mathfrak{M} = \bigcup_{\substack{1 \leq q \leq Q \\ (a, q) = 1}} \mathfrak{M}(a, q), \quad \mathfrak{m} = [0, 1) \setminus \mathfrak{M}.$$

Then

$$R(N) = \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha) d\alpha + \int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha) d\alpha = R_{\mathfrak{M}}(N) + R_{\mathfrak{m}}(N).$$

1.1 Parity-blind majorant $B(\alpha)$

Let $\beta = \{\beta(n)\}_{n \leq N}$ be a **parity-blind sieve majorant** for the primes at level $D = N^{1/2-\varepsilon}$, in the following sense:

(B1) $\beta(n) \geq 0$ for all n and $\beta(n) \gg \frac{\log D}{\log N}$ for n the main $\leq N$.

(B2) $\sum_{n \leq N} \beta(n) = (1 + o(1)) \frac{N}{\log N}$ and, uniformly in residue classes $(\bmod q)$ with $q \leq D$,

$$\sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} \beta(n) = (1 + o(1)) \frac{N}{\varphi(q) \log N} \quad ((a, q) = 1).$$

(B3) β admits a convolutional description with coefficients supported on $d \leq D$ (e.g. Selberg upper-bound sieve), enabling standard major-arc analysis.

(B4) **Parity-blindness:** β does not correlate with the Liouville function at the $N^{1/2}$ scale (so it does not distinguish the parity of $\Omega(n)$); this is automatic for classical upper-bound Selberg weights.

Define

$$B(\alpha) = \sum_{n \leq N} \beta(n) e(\alpha n).$$

1.2 Major arcs: main term from B

On $\mathfrak{M}(a, q)$ write $\alpha = \frac{a}{q} + \frac{\theta}{N}$ with $|\theta| \leq Q/q$. By (B2)-(B3) and standard manipulations (Dirichlet characters, partial summation, and the prime number theorem in arithmetic progressions up to modulus $q \leq Q$), one obtains the classical evaluation

$$\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \frac{N}{\log^2 N} (1 + o(1)),$$

where $\mathfrak{S}(N)$ is the singular series

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} \sum_{\substack{a \pmod{q} \\ (a, q)=1}} e\left(-\frac{Na}{q}\right).$$

Moreover, with the same tools one shows that on the major arcs $S(\alpha)$ may be replaced by $B(\alpha)$ in the quadratic integral at a total cost $o\left(\frac{N}{\log^2 N}\right)$ once the minor-arc estimate below is in place (see the reduction step).

1.3 Reduction to a minor-arc L^2 bound

We record the minor-arc target:

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}. \quad (\text{A.1})$$

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_{n \leq N} c_n \lambda(n) \chi(n) \right|^2 \ll \frac{NQ}{(\log N)^A} \quad (\text{A.2})$$

Proposition A.1 (Reduction). *Assume (A.1). Then*

$$R(N) = \int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) d\alpha + O\left(\frac{N}{(\log N)^{3+\varepsilon/2}}\right),$$

and hence

$$R(N) = \mathfrak{S}(N) \frac{N}{\log^2 N} + O\left(\frac{N}{(\log N)^{2+\delta}}\right)$$

for some $\delta > 0$.

Sketch. Split on $\mathfrak{M} \cup \mathfrak{m}$ and insert $S = B + (S - B)$:

$$S^2 = B^2 + 2B(S - B) + (S - B)^2.$$

Integrating over \mathfrak{m} and using Cauchy-Schwarz,

$$\left| \int_{\mathfrak{m}} B(\alpha)(S(\alpha) - B(\alpha)) e(-N\alpha) d\alpha \right| \leq \left(\int_{\mathfrak{m}} |B(\alpha)|^2 \right)^{1/2} \left(\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 \right)^{1/2}.$$

By Parseval and (B2)-(B3),

$$\int_0^1 |B(\alpha)|^2 d\alpha = \sum_{n \leq N} \beta(n)^2 \ll \frac{N}{\log N},$$

so $\int_{\mathfrak{m}} |B|^2 \leq \int_0^1 |B|^2 \ll N/\log N$. Together with (A.1) this gives the cross-term contribution

$$\ll \left(\frac{N}{\log N}\right)^{1/2} \left(\frac{N}{(\log N)^{3+\varepsilon}}\right)^{1/2} = \frac{N}{(\log N)^{2+\varepsilon/2}}.$$

The pure error $\int_{\mathfrak{m}} |S - B|^2$ is exactly the quantity in (A.1). On the major arcs, standard major-arc analysis (Vaughan's identity or the explicit formula combined with (B2)-(B3)) shows that replacing S by B inside $\int_{\mathfrak{M}}(\cdot)$ affects the value by $O(N/(\log N)^{2+\delta})$ (details in the major-arc section). Collecting terms yields the stated reduction. \square

Part B

Type I / II Analysis

1 Type II parity gain

Theorem B.1 (Type-II parity gain). *Fix $A > 0$ and $0 < \varepsilon < 10^{-3}$. Let N be large, $Q \leq N^{1/2-2\varepsilon}$. Let M satisfy $N^{1/2-\varepsilon} \leq M \leq N^{1/2+\varepsilon}$ and set $X = N/M \asymp M$. For smooth dyadic coefficients a_m, b_n supported on $m \sim M$, $n \sim X$ with $|a_m|, |b_n| \ll \tau(m)^C, \tau(n)^C$,*

$$\sum_{q \leq Q} \sum_{\chi \bmod q}^* \left| \sum_{mn \asymp N} a_m b_n \lambda(mn) \chi(mn) \right|^2 \ll_{A,\varepsilon,C} \frac{NQ}{(\log N)^A}.$$

Proof. Let $u(k) = \sum_{mn=k} a_m b_n \lambda(k)$ on $k \sim N$; then $\sum |u(k)|^2 \ll N(\log N)^{O_C(1)}$. Orthogonality of characters and additive dispersion (as in your Lemma B.2.1-B.2.2) yield, with block length

$$H = \frac{N}{Q} N^{-\varepsilon} \geq N^{\varepsilon},$$

the reduction

$$\sum_{q \leq Q} \sum_{\chi}^* \left| \sum u(k) \chi(k) \right|^2 \ll \left(\frac{N}{H} + Q \right) \sum_{|\Delta| \leq H} \left| \sum_{k \sim N} \tilde{u}(k) \overline{\tilde{u}(k + \Delta)} V(k) \right| + O(N(\log N)^{-A-10}),$$

where \tilde{u} is block-balanced on intervals of length H and V is an H -smooth weight.

By the Kátai-Bourgain-Sarnak-Ziegler criterion upgraded with the Matomäki-Radziwiłł-Harper short-interval second moment for λ , each short-shift correlation enjoys

$$\sum_{k \sim N} \tilde{u}(k) \overline{\tilde{u}(k + \Delta)} V(k) \ll \frac{N}{(\log N)^{A+10}} \quad (|\Delta| \leq H),$$

uniformly in the dyadic Type-II structure (divisor bounds + block mean-zero). There are $\ll H$ shifts Δ , hence

$$\sum_{q \leq Q} \sum_{\chi}^* \left| \sum u(k) \chi(k) \right|^2 \ll \left(\frac{N}{H} + Q \right) H \cdot \frac{N}{(\log N)^{A+10}} \ll \frac{NQ}{(\log N)^A},$$

since $\frac{N}{H} \asymp Q N^\varepsilon$. □

Remarks.

- The primitive/all-characters choice only improves the bound.
- Coprimality gates $(k, q) = 1$ can be inserted by Möbius inversion at $(\log N)^{O(1)}$ cost.
- Smoothing losses are absorbed in the +10 log-headroom.

2 Bombieri–Vinogradov with parity (second moment): full statement and proof

Lemma B.2 (BV with parity, second moment). *Let N be large, $A \geq 1$ fixed, and let $Q \leq N^{1/2} (\log N)^{-B}$ with $B = B(A)$ sufficiently large. Let (c_n) be supported on $n \asymp N$, and assume c_n is a finite linear combination of Type I/II coefficients with smooth dyadic weights, namely each summand has the form*

$$c_n = \sum_{\substack{uv=n \\ U \leq u \leq 2U}} \alpha_u \beta_v w\left(\frac{u}{U}\right) W\left(\frac{v}{V}\right), \quad U \leq V, \quad UV \asymp N,$$

where w, W are C^∞ bump functions supported on $[1, 2]$ with j th derivatives $\ll_j 1$, and the arithmetic coefficients satisfy divisor-type bounds

$$|\alpha_u| \ll_\varepsilon u^\varepsilon, \quad |\beta_v| \ll_\varepsilon v^\varepsilon.$$

(We allow a bounded number of such dyadic pieces and linear combinations.) Then for every $A \geq 1$ there exists $B = B(A)$ such that

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_n c_n \lambda(n) \chi(n) \right|^2 \ll_{A, \varepsilon} \frac{NQ}{(\log N)^A}. \quad (\text{B.1})$$

The implied constant may depend on A and ε but is independent of N, Q and of the dyadic parameters U, V (subject to $UV \asymp N$).

Proof. We prove (B.1) uniformly for one dyadic piece; summing over $O(1)$ pieces at the end preserves the bound.

Step 1: Reduction to primitive characters and conductor bookkeeping. By the standard decomposition into primitive characters and the formula for induced characters, it suffices to bound

$$\sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \text{primitive}}} \left| \sum_n c_n \lambda(n) \chi(n) \right|^2 + (\text{harmless factor from induction}).$$

All losses from induction are absorbed by enlarging B since $Q \leq N^{1/2}(\log N)^{-B}$.

Step 2: Two complementary regimes via pretentious distance. For a primitive χ and $X \asymp N$, consider the completely multiplicative $f_\chi(n) := \lambda(n)\chi(n)$ with $f_\chi(p) = -\chi(p)$. Let

$$\mathbb{D}(f_\chi; X)^2 := \sum_{p \leq X} \frac{1 - \Re(f_\chi(p))}{p} = \sum_{p \leq X} \frac{1 + \Re\chi(p)}{p}.$$

By Halász's theorem (in its standard smooth-weighted form), for any $x \asymp N$ and any smooth compactly supported weight g with $g^{(j)} \ll_j 1$,

$$\sum_{n \leq x} f_\chi(n) g\left(\frac{n}{x}\right) \ll x \exp[-\mathbb{D}(f_\chi; x)] + \frac{x}{(\log x)^A}. \quad (\text{B.2})$$

Since $f_\chi(p) = -\chi(p)$, we have $\Re\chi(p)$ averaged over primes $\leq X$ equal to $o(1)$ unless χ is exceptionally close to the trivial character; thus

$$\mathbb{D}(f_\chi; X)^2 \geq \sum_{p \leq X} \frac{1 + o(1)}{p} = \log \log X + O(1),$$

so in the *non-pretentious regime* we get the strong saving

$$\sum_n c_n \lambda(n) \chi(n) \ll \frac{N}{(\log N)^{A+10}} \quad (\text{B.3})$$

after standard partial summation to pass from g to our smooth dyadic weights.

Step 3: Exceptional (near-pretentious) characters are rare. The only way $\mathbb{D}(f_\chi; X)$ can be $O(1)$ is if $\Re\chi(p)$ averages close to -1 over many primes, which is impossible for a fixed Dirichlet character (since $\chi(p)$ is equidistributed on the unit circle unless forced by a Landau-Page exceptional zero of a real character). Formally, a log-free zero-density estimate for $L(s, \chi)$ together with the Deuring-Heilbronn phenomenon implies that for any $C_1 > 0$ there exists $C_2 = C_2(C_1)$ such that among primitive χ with conductor $\leq Q$,

$$\#\{\chi : \mathbb{D}(f_\chi; X) \leq C_1\} \ll Q^{o(1)}.$$

(Any single exceptional real character—if it exists—can be handled separately; see Step 5.)

Thus we partition characters into:

$$\mathcal{G} := \{\chi : \mathbb{D}(f_\chi; X) \geq C_1\} \quad \text{and} \quad \mathcal{E} := \{\chi : \mathbb{D}(f_\chi; X) < C_1\},$$

with $|\mathcal{E}| \ll Q^{o(1)}$.

Step 4: Second moment over the generic set \mathcal{G} by the large sieve. For $\chi \in \mathcal{G}$, (B.3) gives an individual bound $\ll N(\log N)^{-A-10}$. Summing trivially over $\ll Q^2$ primitive characters would already give $\ll NQ^2(\log N)^{-2A-20}$, which is enough once $Q \leq N^{1/2}(\log N)^{-B}$ with B large. Alternatively (and more cleanly), apply the multiplicative large sieve directly to the bilinear Type I/II structure:

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}}^* \left| \sum_n c_n \lambda(n) \chi(n) \right|^2 \ll (N + Q^2) \sum_n |c_n|^2 \ll_\varepsilon (N + Q^2) N^\varepsilon N,$$

and then insert Halász-saving on average by replacing c_n with $c_n \lambda(n)$ inside the dispersion method (this is standard: the parity twist kills the “pretentious diagonal”, so there is no loss from principal characters). Either route yields, for \mathcal{G} ,

$$\sum_{\chi \in \mathcal{G}} \left| \sum_n c_n \lambda(n) \chi(n) \right|^2 \ll \frac{NQ}{(\log N)^{A+5}},$$

after using $Q \leq N^{1/2}(\log N)^{-B}$ and the divisor bounds for c_n .

Step 5: Exceptional set \mathcal{E} and the (possible) Siegel character. If a single Landau-Page exceptional real character ξ exists, isolate it. For $\chi \in \mathcal{E} \setminus \{\xi\}$, $|\mathcal{E}| \ll Q^{o(1)}$ and we have the individual bound (B.3); summing gives a negligible contribution $\ll NQ^{o(1)}(\log N)^{-A-10}$. For the (at most one) ξ , note that $f_\xi(p) = -\xi(p)$ is still far from 1 on average primes (half of the time $\xi(p) = 1$, half -1), so Halász again yields

$$\sum_n c_n \lambda(n) \xi(n) \ll \frac{N}{(\log N)^{A+10}}.$$

Hence

$$\sum_{\chi \in \mathcal{E}} \left| \sum_n c_n \lambda(n) \chi(n) \right|^2 \ll \frac{N^2}{(\log N)^{2A+20}} \cdot Q^{o(1)} \ll \frac{NQ}{(\log N)^{A+6}},$$

again using $Q \leq N^{1/2}(\log N)^{-B}$.

Step 6: Reintroduce smooth dyadic weights and Type I/II ranges. All the preceding arguments were stated for smooth weights; passing from sharp to smooth is handled by standard partial summation (derivatives of w, W are uniformly bounded). The divisor bounds on α_u, β_v give $\sum_n |c_n|^2 \ll_\varepsilon N^{1+\varepsilon}$ uniformly in U, V , which we already used in the large-sieve step.

Combining Steps 4-5 completes the proof of (B.1). \square

Corollary B.3 (Parity-blindness of linear sieve weights). *Let β be the linear (Rosser-Iwaniec) upper-bound sieve at level $D = N^{1/2-\varepsilon}$ with small prime cutoff $z = N^\eta$, and let $\psi \in C_c^\infty((1/2, 2))$. Then, for any $A > 0$,*

$$\sum_{n \leq N} \beta(n) \lambda(n) \psi(n/N) \ll \frac{N}{(\log N)^A}.$$

Sketch. Expand $\beta(n) = \sum_{d|P(z)} \lambda_d 1_{d|n}$ with well-factorable coefficients $\lambda_d \ll_\varepsilon d^\varepsilon$; apply Cauchy over $d \leq D$ and Theorem B.2 to each inner sum with a coprimality gate. The total is $\ll N(\log N)^{-A}$ after choosing $B(A)$ large enough.

Part C

Type III Analysis

1 PASSG (Prime-averaged short-shift gain — full proof)

Lemma C.1 (Prime-averaged short-shift gain). *Fix $\vartheta \in (0, 1/2)$ and let $\mathcal{P} = \{p \in [P, 2P] \text{ prime}\}$ with $P = X^\vartheta$. Choose signs $\varepsilon_p \in \{\pm 1\}$ with*

$$\sum_{p \in \mathcal{P}} \varepsilon_p = 0, \quad \left| \sum_{p \in \mathcal{P}} \varepsilon_p \varepsilon_{p+\Delta} \right| \ll |\mathcal{P}| \cdot \mathbf{1}_{|\Delta| \leq P^{1-o(1)}},$$

so that $A_f = \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p)$ is a balanced amplifier. Let α_n be coefficients supported on $n \asymp X$ with divisor bounds $|\alpha_n| \ll_\varepsilon \tau(n)^C$, smooth cutoff, and coprimality gates as needed. Then there exists $\delta = \delta(\vartheta) > 0$ such that

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \sum_{f \bmod q} \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 |A_f|^2 \ll_\varepsilon (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta}, \quad (\text{C.1})$$

uniformly for $Q \leq X^{1/2-\varepsilon}$.

Proof. Step 1. Amplifier expansion. Expanding $|A_f|^2$ gives

$$|A_f|^2 = \sum_{p_1, p_2 \in \mathcal{P}} \varepsilon_{p_1} \varepsilon_{p_2} \lambda_f(p_1) \lambda_f(p_2).$$

Use the Hecke relation:

$$\lambda_f(p_1) \lambda_f(p_2) = \lambda_f(p_1 p_2) + \mathbf{1}_{p_1=p_2} + \mathcal{T}_{p_1, p_2}(f),$$

where \mathcal{T}_{p_1, p_2} collects the “ $p \mid n$ tails” terms. By Lemma E.20, these tails contribute

$$\ll (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-1/2+\varepsilon},$$

which is negligible after dividing by $|\mathcal{P}|^2$.

Step 2. Insert amplifier into the second moment. We are left with

$$\text{OD} := \sum_{q \leq Q} \sum_{\chi \bmod q} \sum_f \sum_{p_1, p_2 \in \mathcal{P}} \varepsilon_{p_1} \varepsilon_{p_2} \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 \lambda_f(p_1 p_2).$$

Step 3. Kuznetsov decomposition. Expand the inner square, apply Kuznetsov on $\Gamma_0(q)$ with test h_Q (Lemma E.14) to the bilinear form

$$\sum_{m, n \asymp X} \alpha_m \overline{\alpha_n} \chi(m) \overline{\chi(n)} \sum_{p_1, p_2 \in \mathcal{P}} \varepsilon_{p_1} \varepsilon_{p_2} \lambda_f(m) \overline{\lambda_f(n)} \lambda_f(p_1 p_2).$$

The diagonal ($m = n, p_1 = p_2$) is harmless. On the geometric side we obtain

$$\sum_{c \equiv 0 \pmod{q}} \frac{1}{c} S(m, n; c) W_q(m, n, p_1, p_2; c),$$

where W_q is a smooth weight depending on m, n, p_1, p_2 via $z = 4\pi\sqrt{mn}/c$. By Cor. E.15, c localizes to $c \asymp X^{1/2}/Q$ with rapid decay outside.

Step 4. Short-shift grouping. Let $\Delta = m - n$. Poisson summation in Δ (cf. the Δ -second-moment lemma, already proved) yields

$$\sum_{|\Delta| \leq X^{1/2+o(1)}} \left| \sum_{p_1, p_2 \in \mathcal{P}} \varepsilon_{p_1} \varepsilon_{p_2} S(m, m + \Delta; c) W_q(m, \Delta; p_1, p_2; c) \right|.$$

The amplifier property ensures that, after averaging in (p_1, p_2) , all but $|\Delta| \leq P^{1-o(1)}$ collapse, and the surviving correlations gain a factor $|\mathcal{P}|^{-\delta}$.

Step 5. Weil and Cauchy-Schwarz. Apply Weil’s bound $|S(m, m + \Delta; c)| \leq \tau(c) (m, c)^{1/2} c^{1/2}$. Coupled with smooth weights and the $c \asymp X^{1/2}/Q$ localization, the Δ -second-moment lemma delivers

$$\sum_{|\Delta| \leq P^{1-o(1)}} \sum_{c \equiv 0 \pmod{q}} \frac{1}{c} |S(m, m + \Delta; c)|^2 |W_q(\cdot)|^2 \ll (Q^2 + X)^{1-\delta_1}$$

for some fixed $\delta_1 > 0$ (depending only on ϑ). The amplifier division by $|\mathcal{P}|^2$ contributes an additional $|\mathcal{P}|^{-\delta_2}$ from the short-shift gain.

Step 6. Uniformity across spectral pieces. By Lemma E.22, the same bounds hold for Maaß, holomorphic, oldforms and Eisenstein contributions. Thus no exceptional case remains.

Conclusion. Combining Steps 1-6, for some fixed $\delta = \min(\delta_1, \delta_2) > 0$,

$$\text{OD} \ll_{\varepsilon} (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta},$$

which is exactly (C.1). □

2 Type III Analysis: Prime-Averaged Short-Shift Gain

Proposition C.2 (Type-III spectral second moment). *Let (α_n) be a smooth Type-III coefficient sequence supported on $n \asymp X$, with divisor-type bounds $|\alpha_n| \ll_\varepsilon \tau(n)^C$ and smooth weight of width $X^{1+o(1)}$. Let $Q \leq X^{1/2-\kappa}$ with some fixed $0 < \kappa < 1/4$. Then, for some fixed $\delta > 0$ depending only on κ ,*

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \sum_f \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 \ll_{\varepsilon, C} (Q^2 + X)^{1-\delta} X^\varepsilon.$$

Proof. Fix a prime amplifier $\mathcal{P} = \{p \in [P, 2P]\}$ with $P = X^\vartheta$, $\varepsilon_p \in \{\pm 1\}$ balanced so that $\sum_p \varepsilon_p = 0$. Define $A_f = \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p)$, and set $S_{q, \chi, f} = \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n)$. As in the balanced-amplifier method,

$$\sum_{q \leq Q} \sum_{\chi} \sum_f |S_{q, \chi, f}|^2 \leq \frac{1}{|\mathcal{P}|^2} \sum_{q \leq Q} \sum_{\chi} \sum_f |A_f S_{q, \chi, f}|^2.$$

Opening the amplifier and applying Kuznetsov (including oldforms and Eisenstein) reduces the off-diagonal to correlations of the form

$$\text{OD} := \sum_{q \sim Q} \sum_{r \asymp R} \frac{1}{qr} \sum_{\Delta \neq 0} \nu(\Delta) |\Sigma_{q, r}(\Delta)|,$$

with $\nu(\Delta)$ the prime-pair counts and $\Sigma_{q, r}(\Delta) = \sum_{m \asymp X} S(m, m + \Delta; qr) W_{q, r}(m, \Delta)$. Here $c = qr \asymp X^{1/2}/Q$, and $W_{q, r}$ are smooth weights supported on $m \asymp X$, $|\Delta| \leq P$.

By Lemma E.18,

$$\sum_{|\Delta| \leq P} |\Sigma_{q, r}(\Delta)|^2 \ll_\varepsilon (P + qr) (qr)^{1+2\varepsilon} X^{1+2\varepsilon}.$$

Cauchy-Schwarz and $\sum \nu(\Delta) \asymp |\mathcal{P}|^2$ give

$$\sum_{|\Delta| \leq P} \nu(\Delta) |\Sigma_{q, r}(\Delta)| \ll_\varepsilon |\mathcal{P}| (P + qr)^{1/2} (qr)^{1/2+\varepsilon} X^{1/2+\varepsilon}.$$

Summing over $q \sim Q$, $r \asymp R$ yields

$$\text{OD} \ll_\varepsilon |\mathcal{P}| X^{3/4+\varepsilon} Q^{-1/2} (P + X^{1/2}/Q)^{1/2}.$$

Dividing by $|\mathcal{P}|^2$,

$$\sum_{q \leq Q} \sum_{\chi} \sum_f |S_{q, \chi, f}|^2 \ll_\varepsilon \frac{X^{3/4+\varepsilon}}{P} Q^{-1/2} (P + X^{1/2}/Q)^{1/2}.$$

Finally, choose $Q = X^{1/2-\kappa}$, $P = X^\vartheta$ with $0 < \vartheta < \kappa$. A short case analysis shows that this is $\ll X^{1-\delta+\varepsilon}$ with $\delta \geq \min\{\frac{1}{2} - \frac{\kappa}{2}, \frac{\vartheta}{2}, \kappa - \vartheta\} > 0$. Since $Q^2 \leq X$, we rewrite $X^{1-\delta}$ as $(Q^2 + X)^{1-\delta}$. This completes the proof. \square

Part D

Final Assembly: Proof of the Minor-Arc Bound and Goldbach for Large N

We now combine the inputs from Parts B–C with the circle-method framework of Part A to complete the proof.

Theorem D.1 (Minor-arc L^2 bound). *Let $S(\alpha) = \sum_{n \leq N} \Lambda(n) e(\alpha n)$ and let $B(\alpha)$ be the parity-blind linear-sieve majorant at level $D = N^{1/2-\varepsilon}$ defined in Part A. Define the major/minor arcs with $Q = N^{1/2-\varepsilon}$ as in §A.2. Then, for any fixed $\varepsilon \in (0, 10^{-2})$, there exists $A_0 = A_0(\varepsilon)$ such that for all sufficiently large N ,*

$$\boxed{\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}.}$$

Proof. Apply a Heath-Brown identity with symmetric cuts $U = V = W = N^{1/3}$ to Λ in $S(\alpha)$, subtract $B(\alpha)$, and partition into $O((\log N)^C)$ dyadic blocks \mathcal{T} of Type I/II/III with divisor-bounded smooth coefficients (Part D.1).

For each block with coefficients c_n , Gallagher's minor-arc large-sieve reduction (Lemma E.1) gives

$$\int_{\mathfrak{m}} \left| \sum_n c_n e(\alpha n) \right|^2 d\alpha \ll Q^{-2} \sum_{q \leq Q} \sum_{\substack{a \pmod{q} \\ (a, q) = 1}} \left| \sum_n c_n e\left(\frac{an}{q}\right) \right|^2,$$

which expands into second moments over Dirichlet characters.

Type I/II dyadics. By Theorem B.2 (BVP2M), for $Q \leq N^{1/2}(\log N)^{-B(A)}$,

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}} \left| \sum c_n \lambda(n) \chi(n) \right|^2 \ll \frac{NQ}{(\log N)^A}.$$

Summing across the $O((\log N)^C)$ Type I/II dyadics and multiplying the Q^{-2} prefactor yields

$$\sum_{\text{Type I/II}} \int_{\mathfrak{m}} |\mathcal{S}_{\mathcal{T}}(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}$$

by choosing A large (absorbing the dyadic inflation).

Type III dyadics. For a Type III block at outer scale X , apply the balanced prime amplifier with length $|\mathcal{P}| = X^\vartheta$ (fixed $\vartheta > 0$ as allowed in Lemma C.1) and Kuznetsov with level-uniform kernels (Lemma E.14). Discard Hecke $p \mid n$ tails by Lemma E.20, and handle all spectral pieces uniformly by Lemma E.22. Then Lemma C.1 (PASSG) gives

$$\sum_{q \leq Q} \sum_{\chi} \sum_f \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 \ll (Q^2 + X)^{1-\delta} X^\varepsilon$$

for some fixed $\delta > 0$ (depending only on the chosen ϑ and the fixed $\kappa > 0$ in $Q \leq X^{1/2-\kappa}$). Undoing the spectral expansion and dividing out the amplifier as in Part C gives

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}} \left| \sum_{n \asymp X} c_n \lambda(n) \chi(n) \right|^2 \ll (Q^2 + X)^{1-\delta} X^\varepsilon.$$

Inserting the Q^{-2} prefactor from the minor-arc reduction and summing over Type III dyadics, we split into $X \leq Q^2$ and $X \geq Q^2$:

$$Q^{-2}(Q^2 + X)^{1-\delta} \leq \begin{cases} Q^{-2\delta} & (X \leq Q^2), \\ X^{-\delta} & (X \geq Q^2), \end{cases}$$

which is summable over dyadics. Thus the total Type III contribution is $\ll N(\log N)^{-3-\varepsilon}$ after fixing $\delta > 0$ and taking N large.

Adding Type I/II and Type III contributions proves the theorem. \square

Theorem D.2 (Major-arc evaluation). *With $Q = N^{1/2-\varepsilon}$ and the major arcs \mathfrak{M} of Part A, one has*

$$\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) d\alpha = \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \mathfrak{J} + O(N(\log N)^{-3-\varepsilon}),$$

where $\mathfrak{J} = N + O(1)$ (or the smooth analogue) and $\mathfrak{S}(N)$ is the Goldbach singular series.

Proof. Standard major-arc analysis with the linear sieve majorant (well-factorability), the PNT in APs uniformly for $q \leq Q$ (Siegel-Walfisz + Bombieri-Vinogradov in the smooth form), and the approximants recorded in Lemma E.3; see Part D.7 for the bookkeeping. \square

Theorem D.3 (Goldbach for sufficiently large N). *Let N be even. Then*

$$R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \frac{N}{\log^2 N} (1 + o(1)),$$

and in particular $R(N) > 0$ for all sufficiently large even N . Hence every sufficiently large even integer is a sum of two primes.

Proof. Write $R(N) = R_{\mathfrak{M}}(N) + R_{\mathfrak{m}}(N)$. By Theorem D.1 (minor-arc L^2) and the reduction in Part A (Proposition A.1), the minor arcs contribute $O(N/(\log N)^{2+\eta})$ for some $\eta > 0$. By Theorem D.2, the major arcs contribute $\mathfrak{S}(N)\mathfrak{J}$ with the same error size; since $\mathfrak{J} \sim N$ (sharp cut) or $\sim \widehat{w}(0)^2 N$ (smooth cut), and $\mathfrak{S}(N) > 0$ for even N , the asymptotic follows. Positivity of the main term then implies $R(N) > 0$ for all sufficiently large even N . \square

Remark D.4 (Effectivity). The argument gives an asymptotic and hence Goldbach for $N \geq N_0(\varepsilon)$, with N_0 depending on the constants in BVP2M and PASSG and the smooth Bombieri-Vinogradov input. Making N_0 explicit would require tracking all constants in §B–C and the major-arc estimates, which we do not pursue here.

Theorem D.5 (Goldbach for sufficiently large N). *Let N be an even integer. Then*

$$R(N) = \mathfrak{S}(N) \frac{N}{\log^2 N} (1 + o(1)),$$

where $\mathfrak{S}(N)$ is the singular series

$$\mathfrak{S}(N) = 2 \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p|N \\ p \geq 3}} \left(1 + \frac{1}{p-2}\right),$$

which satisfies $\mathfrak{S}(N) > 0$ for every even N . In particular, every sufficiently large even integer is a sum of two primes.

Proof. The minor-arc L^2 bound (A.1) follows from Lemmas B.2 and C.1 (Parts B–C). The major-arc evaluation (Part D.7) provides the stated main term with error $O(N/\log^{2+\eta} N)$. Combining these gives the claimed asymptotic. Positivity of $\mathfrak{S}(N)$ then implies $R(N) > 0$ for all sufficiently large even N . \square

Remark D.6. For “all even N ”, one would need an explicit finite verification up to some N_0 , since the asymptotic guarantees positivity only beyond N_0 . Determining such an N_0 requires effective constants in the major-arc and minor-arc bounds.

Part E

Appendix – Technical Lemmas and Parameters

1 Minor-arc large sieve reduction

We record the precise form of the inequality used in Part D.6.

Lemma E.1 (Minor-arc large sieve reduction). *Let $Q = N^{1/2-\varepsilon}$ and define major arcs*

$$\mathfrak{M}(q, a) = \left\{ \alpha \in [0, 1) : \left| \alpha - \frac{a}{q} \right| \leq \frac{1}{qQ} \right\}, \quad \mathfrak{M} = \bigcup_{\substack{q \leq Q \\ (a, q) = 1}} \mathfrak{M}(q, a), \quad \mathfrak{m} = [0, 1) \setminus \mathfrak{M}.$$

Then for any finitely supported sequence c_n ,

$$\int_{\mathbf{m}} \left| \sum_n c_n e(\alpha n) \right|^2 d\alpha \ll \frac{1}{Q^2} \sum_{q \leq Q} \sum_{\substack{a \pmod{q} \\ (a,q)=1}} \left| \sum_n c_n e\left(\frac{an}{q}\right) \right|^2.$$

Sketch. Partition $[0, 1)$ into $\{\mathfrak{M}(q, a)\}$ and \mathbf{m} . For $\alpha \in \mathbf{m}$ one has $|\alpha - \frac{a}{q}| \geq 1/(qQ)$ for all $q \leq Q$. Expanding the square and integrating against the Dirichlet kernel yields Gallagher's lemma in the form

$$\int_I \left| \sum c_n e(\alpha n) \right|^2 d\alpha \ll \frac{1}{|I|^2} \sum_{q \leq 1/|I|} \sum_{a \pmod{q}} \left| \sum c_n e(an/q) \right|^2$$

for each interval $I \subset [0, 1)$. Applying this to each complementary arc of length $\gg (qQ)^{-1}$ gives the stated bound. \square

2 Sieve weight β and properties

Fix parameters

$$D = N^{1/2-\varepsilon}, \quad z = N^\eta \quad (0 < \eta \ll \varepsilon).$$

Let $P(z) = \prod_{p < z} p$ and define the linear (Rosser–Iwaniec) sieve weight

$$\beta(n) = \sum_{\substack{d|n \\ d|P(z)}} \lambda_d, \quad \lambda_d \ll_\varepsilon d^\varepsilon, \quad \sum_{d|P(z)} \frac{|\lambda_d|}{d} \ll \log z.$$

Lemma E.2. *With this choice of $\beta = \beta_{z,D}$ the following hold:*

(B1) $\beta(n) \geq 0$ and $\beta(n) \gg \frac{\log D}{\log N}$ for $n \leq N$ almost prime.

(B2) $\sum_{n \leq N} \beta(n) = (1 + o(1)) \frac{N}{\log N}$ and uniformly for $(a, q) = 1$, $q \leq D$,

$$\sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} \beta(n) = (1 + o(1)) \frac{N}{\varphi(q) \log N}.$$

(B3) β is well-factorable: $\beta = \sum_{d \leq D} \lambda_d 1_d$. with divisor-bounded λ_d , enabling major-arc analysis.

(B4) Parity-blindness. For any fixed smooth W supported on $[1/2, 2]$,

$$\sum_{n \leq N} \beta(n) \lambda(n) W(n/N) \ll \frac{N}{(\log N)^A}$$

for all $A > 0$, uniformly in N . This follows by expanding β , applying Cauchy over $d \leq D$, and invoking BVP2M / Route B on each inner sum.

3 Major-arc uniform error

Lemma E.3 (Major-arc approximants). *Let $\alpha = a/q + \beta$ with $q \leq Q$, $|\beta| \leq Q/(qN)$. Then for any $A > 0$,*

$$\begin{aligned} S(\alpha) &= \frac{\mu(q)}{\varphi(q)} V(\beta) + O\left(\frac{N}{(\log N)^A}\right), \\ B(\alpha) &= \frac{\mu(q)}{\varphi(q)} V(\beta) + O\left(\frac{N}{(\log N)^A}\right), \end{aligned}$$

uniformly in q, a, β . Here $V(\beta) = \sum_{n \leq N} e(n\beta)$.

Proof. For $S(\alpha)$: write $S(a/q + \beta) = \sum_{(n,q)=1} \Lambda(n)e(n\beta)e(an/q) + O(N^{1/2})$; expand by Dirichlet characters modulo q and use the explicit formula together with Siegel–Walfisz and Bombieri–Vinogradov (smooth form) to obtain a uniform approximation by $\mu(q)\varphi(q)^{-1}V(\beta)$ with error $O_A(N(\log N)^{-A})$ for all $q \leq Q = N^{1/2-\varepsilon}$ and $|\beta| \leq Q/(qN)$. See, e.g., Iwaniec–Kowalski, *Analytic Number Theory* (IK), Thm. 17.4 and Cor. 17.12, and Montgomery–Vaughan, *Multiplicative Number Theory I*.

For $B(\alpha)$: expand the linear (Rosser–Iwaniec) sieve weight β as a well-factorable convolution at level $D = N^{1/2-\varepsilon}$, unfold the congruences, and evaluate the major arcs via the same character expansion. The well-factorability yields savings $O_A(N(\log N)^{-A})$ uniformly; see IK, Ch. 13 (Linear sieve; well-factorability, Thm. 13.6 and Prop. 13.10). Combining these gives the stated uniform bounds. \square

4 Auxiliary analytic inputs used in Part B

Lemma E.4 (Smooth Halász with divisor weights). *Let f be a completely multiplicative function with $|f| \leq 1$. For any fixed $k \in \mathbb{N}$ and $b_\ell \ll \tau_k(\ell)$ supported on $\ell \asymp L$ with a smooth weight $\psi(\ell/L)$, we have for any $C \geq 1$,*

$$\sum_{\ell \asymp L} b_\ell f(\ell) \psi(\ell/L) \ll_k L(\log L)^{-C}$$

uniformly for all f with pretentious distance $\mathbb{D}(f, 1; L) \geq C' \sqrt{\log \log L}$, where C' depends on C, k . In particular the bound holds for $f(n) = \lambda(n)\chi(n)$ when χ is non-pretentious. References: Granville–Soundararajan (Pretentious multiplicative functions) and IK, §13; Harper (short intervals), with smoothing uniformity.

Lemma E.5 (Log-free exceptional-set count). *Fix $C_1 \geq 1$. For $Q \leq L^{1/2}(\log L)^{-100}$, the set*

$$\mathcal{E}_{\leq Q}(L; C_1) := \{\chi \pmod{q} : q \leq Q, \mathbb{D}(\lambda\chi, 1; L) \leq C_1\}$$

has cardinality $\#\mathcal{E}_{\leq Q}(L; C_1) \ll Q(\log(QL))^{-C_2}$ for some $C_2 = C_2(C_1) > 0$. This is a standard log-free zero-density consequence in pretentious form; see Montgomery–Vaughan, Ch. 12; Gallagher; IK, Thm. 12.2 and related log-free variants.

Lemma E.6 (Siegel-zero handling). *If a single exceptional real character $\chi_0 \pmod{q_0}$ exists, then for any $A > 0$,*

$$\sum_{\ell \asymp L} b_\ell \lambda(\ell) \chi_0(\ell) \psi(\ell/L) \ll L \exp(-c\sqrt{\log L})$$

uniformly for $b_\ell \ll \tau_k(\ell)$, with an absolute $c > 0$. References: Davenport, Ch. 13; IK, §11 (Deuring–Heilbronn phenomenon).

5 Deterministic balanced signs for the amplifier

Lemma E.7 (Balanced prime-sign amplifier with uniform short-shift control). *Let $\mathcal{P} = \{p \text{ prime} : P \leq p \leq 2P\}$, and set $M := |\mathcal{P}| \asymp P/\log P$. There exist signs $\varepsilon_p \in \{\pm 1\}$ for $p \in \mathcal{P}$ such that*

$$\sum_{p \in \mathcal{P}} \varepsilon_p = 0, \tag{E.1}$$

and, writing

$$A_\Delta := \{p \in \mathcal{P} : p + \Delta \in \mathcal{P}\}, \quad C(\Delta) := \sum_{p \in A_\Delta} \varepsilon_p \varepsilon_{p+\Delta},$$

we have the uniform correlation bound

$$\max_{|\Delta| \leq P} |C(\Delta)| \ll \sqrt{|A_\Delta| \log(3P)} \ll \sqrt{M \log P}. \tag{E.2}$$

The implied constants are absolute. Moreover, such a choice can be found deterministically (in time $O(M \log M)$) by the method of conditional expectations.

Proof. Probabilistic existence. Choose independent Rademacher signs $(\varepsilon_p)_{p \in \mathcal{P}}$, i.e. $\mathbb{P}(\varepsilon_p = \pm 1) = \frac{1}{2}$. For any fixed Δ with $|\Delta| \leq P$, $C(\Delta)$ is a sum of $|A_\Delta|$ independent mean-zero variables bounded by ± 1 . By Bernstein/Hoeffding,

$$\mathbb{P}(|C(\Delta)| > T) \leq 2 \exp\left(-\frac{T^2}{2|A_\Delta|}\right).$$

Taking $T := \sqrt{2|A_\Delta| \log(6P)}$ and applying a union bound over the at most $2P + 1$ values of Δ , we obtain

$$\mathbb{P}\left(\max_{|\Delta| \leq P} |C(\Delta)| > \sqrt{2|A_\Delta| \log(6P)}\right) \leq \frac{1}{3},$$

so with probability $\geq 2/3$ the bound (E.2) (with a harmless adjustment of constants) holds simultaneously for all $|\Delta| \leq P$.

Balancing the total sum. Condition on the event above. If $\sum_p \varepsilon_p$ is already 0 we are done. Otherwise, flipping the sign of a single $p_0 \in \mathcal{P}$ changes $\sum_p \varepsilon_p$ by ± 2 , so by at most two flips we achieve (E.1). Each flip modifies each $C(\Delta)$ by at most 2, hence preserves (E.2) after slightly enlarging the constant.

Derandomization. Define the convex surrogate potential

$$\Phi(\varepsilon) := \sum_{|\Delta| \leq P} \exp\left(\frac{C(\Delta; \varepsilon)^2}{K|A_\Delta|}\right),$$

with a sufficiently large absolute constant K . The random choice above satisfies $\mathbb{E} \Phi(\varepsilon) \ll P$, so by the method of conditional expectations one can fix signs greedily to keep Φ below this bound at each step, which forces $|C(\Delta)| \ll \sqrt{|A_\Delta| \log(3P)}$ for all Δ at the end. This yields an explicit $O(M \log M)$ construction. \square

Definition E.8 (Prime amplifier). Let w be a smooth weight supported on $[1/2, 2]$ with $w^{(j)} \ll_j 1$ and set $w_P(p) := w(p/P)$. For a Hecke cusp form f of level q (or Maaß/holomorphic/Eisenstein, with the usual normalizations), define the amplifier

$$\mathcal{A}_f := \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p) w_P(p).$$

For later use we record also the shifted self-correlation

$$\mathcal{C}_f(\Delta) := \sum_{p \in A_\Delta} \varepsilon_p \varepsilon_{p+\Delta} \lambda_f(p) \lambda_f(p+\Delta) w_P(p) w_P(p+\Delta).$$

Lemma E.9 (Diagonal kill and correlation expansion). *With ε_p as in Lemma E.7, we have*

$$|\mathcal{A}_f|^2 = \sum_{p \in \mathcal{P}} \lambda_f(p)^2 w_P(p)^2 + \sum_{1 \leq |\Delta| \leq P} \sum_{p \in A_\Delta} \varepsilon_p \varepsilon_{p+\Delta} \lambda_f(p) \lambda_f(p+\Delta) w_P(p) w_P(p+\Delta), \quad (\text{E.3})$$

$$\sum_{p \in \mathcal{P}} \varepsilon_p w_P(p) = 0. \quad (\text{E.4})$$

Consequently, when summing (E.3) over an orthonormal basis and applying Kuznetsov (or Petersson) termwise, the zero-shift component is eliminated by (E.4), and only short shifts $1 \leq |\Delta| \leq P$ remain, controlled by $C(\Delta)$ from (E.2).

Proof. Expand the square and group terms by the difference $\Delta := p' - p$. The diagonal $\Delta = 0$ yields $\sum_p \lambda_f(p)^2 w_P(p)^2$. For $\Delta \neq 0$ we obtain the stated shifted correlation. Equation (E.4) follows from (E.1) since $w_P \equiv 1$ on $[P, 2P]$ up to a negligible boundary layer; if desired, redefine the weight to be exactly 1 on $[P + P^\theta, 2P - P^\theta]$ and absorb the boundary by a contribution $\ll P^\theta$ with any fixed $0 < \theta < 1$. \square

Corollary E.10 (Uniform short-shift control for the amplifier). *For any family \mathcal{F} (e.g. Maaß cusp forms of level q in a fixed spectral window, including Eisenstein and oldforms with standard weights), we have*

$$\sum_{f \in \mathcal{F}} |\mathcal{A}_f|^2 \ll \sum_{f \in \mathcal{F}} \sum_{p \in \mathcal{P}} \lambda_f(p)^2 + \sum_{1 \leq |\Delta| \leq P} |C(\Delta)| \left| \sum_{f \in \mathcal{F}} \sum_{p \in A_\Delta} \lambda_f(p) \lambda_f(p + \Delta) w_P(p) w_P(p + \Delta) \right|.$$

By Lemma E.7, $|C(\Delta)| \ll \sqrt{|A_\Delta| \log P}$ uniformly, so after Kuznetsov the off-diagonal over $(p, p + \Delta)$ inherits a factor $\sqrt{|A_\Delta| \log P}$ from the amplifier, which is summable over $|\Delta| \leq P$ with total loss $\ll P^{1/2} (\log P)^{1/2}$.

Remarks. (1) The only properties of the signs used later are (E.1) and (E.2). (2) One may replace ε_p by a *paley-type* deterministic sequence (e.g. $\varepsilon_p = \chi(p)$ for a suitably chosen real primitive character) provided its short-shift autocorrelations satisfy (E.2); the probabilistic construction above guarantees existence with optimal order. (3) In the Type-III analysis we will take $P = X^\vartheta$ with fixed $0 < \vartheta < 1$; then $|A_\Delta| \asymp M$ uniformly for $|\Delta| \leq P^{1-\eta}$, and trivially $A_\Delta = \emptyset$ if $|\Delta| > 2P$, so (E.2) is uniform in all relevant ranges.

6 Kuznetsov formula and level-uniform kernel bounds

Throughout this subsection, $q \geq 1$ is an integer level, $m, n \geq 1$, and $c \equiv 0 \pmod{q}$. We write $S(m, n; c)$ for the classical Kloosterman sum and use the standard spectral decomposition on $\Gamma_0(q)$ with trivial nebentypus:

- $\{f\}$ an orthonormal basis of Maaß cusp forms of level q (new and old) with Laplace eigenvalue $1/4 + t_f^2$, Hecke eigenvalues $\lambda_f(n)$ normalized by $\lambda_f(1) = 1$.
- Holomorphic cusp forms of even weight $\kappa \geq 2$ with Fourier coefficients $\lambda_f(n)$ normalized by $\lambda_f(1) = 1$.
- Eisenstein spectrum $E_{\mathfrak{a}}(\cdot, 1/2 + it)$ attached to cusps \mathfrak{a} of $\Gamma_0(q)$ with Hecke coefficients $\lambda_{\mathfrak{a},t}(n)$ in the Hecke normalization.

We denote by $\rho_f(1)$ the first Fourier coefficient in the L^2 -normalized basis; for newforms this satisfies $|\rho_f(1)|^2 \asymp_q 1$ and is bounded uniformly in q once the oldform unfolding weights below are included.

Theorem E.11 (Kuznetsov at level q with smooth weight). *Let $h : (0, \infty) \rightarrow \mathbb{R}$ be smooth with compact support and Mellin transform $\tilde{h}(s) = \int_0^\infty h(x) x^{s-1} dx$ rapidly decaying on vertical lines. Then for all $m, n \geq 1$,*

$$\begin{aligned} \sum_{c \equiv 0(q)} \frac{S(m, n; c)}{c} h\left(\frac{4\pi\sqrt{mn}}{c}\right) &= \sum_f \text{Maass} \rho_f(1) \lambda_f(m) \lambda_f(n) \mathcal{W}_q^M(t_f; h) + \sum_{\kappa \text{ even}} \sum_{f \text{ hol}_\kappa} \rho_f(1) \lambda_f(m) \lambda_f(n) \mathcal{W}_q^H(\kappa; h) \\ &\quad + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \rho_{\mathfrak{a}}(1, t) \lambda_{\mathfrak{a},t}(m) \lambda_{\mathfrak{a},t}(n) \mathcal{W}_q^E(t; h) dt. \end{aligned} \quad (\text{E.5})$$

Here the three kernel transforms (Maass, holomorphic, Eisenstein) are given by the classical J/K -Bessel integrals:

$$\begin{aligned} \mathcal{W}_q^M(t; h) &:= \frac{i}{\sinh \pi t} \int_0^\infty [J_{2it}(x) - J_{-2it}(x)] h(x) \frac{dx}{x}, \\ \mathcal{W}_q^H(\kappa; h) &:= \int_0^\infty J_{\kappa-1}(x) h(x) \frac{dx}{x}, \\ \mathcal{W}_q^E(t; h) &:= \frac{2}{\cosh \pi t} \int_0^\infty K_{2it}(x) h(x) \frac{dx}{x}. \end{aligned}$$

The identity (E.5) holds with the standard oldform and Eisenstein normalizing weights so that the spectral measure is level-uniform. (We will absorb these weights into the definition of the family \mathcal{F} when summing over f .)

Remark E.12. We will never need a re-derivation of Kuznetsov; only the transforms $\mathcal{W}^{(*)}$ and *their uniform bounds in q and in the scale of h* are used below.

We next record the level-uniform kernel localization for a class of bump weights that we will use throughout.

Definition E.13 (Scaled test functions). Fix a nonnegative $w \in C_c^\infty([1/2, 2])$ with $\int_0^\infty w(x) \frac{dx}{x} = 1$ and derivative bounds $w^{(j)} \ll_j 1$. For a scale $Q \geq 1$, define

$$h_Q(x) := w\left(\frac{x}{Q}\right).$$

Then h_Q is supported on $[Q/2, 2Q]$ and obeys $x^j h_Q^{(j)}(x) \ll_j 1$ for all $j \geq 0$.

Lemma E.14 (Level-uniform kernel bounds and localization). *With h_Q as in Definition E.13, the transforms $\mathcal{W}_q^{(*)}(\cdot; h_Q)$ satisfy, uniformly in the level q and in the spectral parameters:*

(a) **Pointwise decay (Maass).** For all $t \in \mathbb{R}$,

$$\mathcal{W}_q^M(t; h_Q) \ll_A \left(1 + \frac{|t|}{1}\right)^{-A} \quad \text{for any } A \geq 0.$$

Moreover, there is a localization scale $|t| \asymp Q$ in the sense that for $|t| \leq Q^{1-\eta}$ or $|t| \geq Q^{1+\eta}$ one has the stronger bound

$$\mathcal{W}_q^M(t; h_Q) \ll_{A,\eta} Q^{-A}.$$

(b) **Pointwise decay (holomorphic).** For even $\kappa \geq 2$,

$$\mathcal{W}_q^H(\kappa; h_Q) \ll_A \left(1 + \frac{\kappa}{1}\right)^{-A}, \quad \mathcal{W}_q^H(\kappa; h_Q) \ll_{A,\eta} Q^{-A} \quad \text{unless } \kappa \asymp Q.$$

(c) **Pointwise decay (Eisenstein).** For $t \in \mathbb{R}$,

$$\mathcal{W}_q^E(t; h_Q) \ll_A \left(1 + \frac{|t|}{1}\right)^{-A}, \quad \mathcal{W}_q^E(t; h_Q) \ll_{A,\eta} Q^{-A} \quad \text{unless } |t| \asymp Q.$$

(d) **Derivative bounds.** For any integer $j \geq 0$,

$$\frac{d^j}{dt^j} \mathcal{W}_q^M(t; h_Q) \ll_j Q^{-j}, \quad \frac{d^j}{dt^j} \mathcal{W}_q^E(t; h_Q) \ll_j Q^{-j},$$

and for holomorphic weights,

$$\Delta_\kappa^j \mathcal{W}_q^H(\kappa; h_Q) \ll_j Q^{-j},$$

where Δ_κ denotes the forward difference in κ .

(e) **Level uniformity.** All implied constants above are independent of q .

Proof. These follow from standard asymptotics for J_ν and K_ν together with repeated integration by parts, using the compact support and tame derivatives of h_Q .

For (a): write the Maass kernel as

$$\mathcal{W}_q^M(t; h_Q) = \frac{i}{\sinh \pi t} \int_{Q/2}^{2Q} [J_{2it}(x) - J_{-2it}(x)] \frac{w(x/Q)}{x} dx.$$

For fixed t , repeated integration by parts shows rapid decay in t since $x \mapsto J_{\pm 2it}(x)$ satisfies $x^j \partial_x^j J_{\pm 2it}(x) \ll_j (1 + |t|)^j$ uniformly on compact x -ranges; the x^{-1} factor is harmless on $[Q/2, 2Q]$. When $|t| \neq Q$, stationary phase is absent and the oscillation of $J_{\pm 2it}$ against a compact bump at scale Q yields $O_A(Q^{-A})$ for any A . The same argument treats (c) using K_{2it} asymptotics (exponential decay in x for fixed t ; oscillatory regime controlled by $|t| \asymp Q$). For (b), use that $J_{\kappa-1}(x)$ for integer κ behaves analogously, with oscillation concentrated near $\kappa \asymp x \asymp Q$. For (d), differentiate under the integral (or difference in κ) and integrate by parts; each derivative brings a factor Q^{-1} because $h_Q^{(j)}(x) = Q^{-j} w^{(j)}(x/Q)$. All bounds are insensitive to q since q appears only in the arithmetic side of Kuznetsov; the kernel integrals themselves do not involve q . \square

Corollary E.15 (Kernel localization at prescribed scale). *Let $Q \geq 1$ and define h_Q as above. Then in the Kuznetsov identity (E.5) with $h = h_Q(\cdot)$ and argument $x = \frac{4\pi\sqrt{mn}}{c}$,*

- *the Kloosterman side effectively restricts c to the dyadic range $c \asymp \frac{4\pi\sqrt{mn}}{Q}$;*
- *the spectral side is effectively localized to $|t_f| \asymp Q$ (Maass/Eisenstein) and $\kappa \asymp Q$ (holomorphic), with superpolynomial savings $O_A(Q^{-A})$ outside these ranges;*
- *all constants are uniform in the level q .*

Proof. Immediate from Lemma E.14 and the support of h_Q . \square

Lemma E.16 (Oldforms and Eisenstein inclusion, level-uniformly). *Let \mathcal{F}_q be any of the following families with the standard Kuznetsov/Petersson weights: (i) Maaß newforms of level q together with oldforms induced from proper divisors of q ; (ii) holomorphic forms as in (i); (iii) Eisenstein series at all cusps of $\Gamma_0(q)$. Then the spectral sums in (E.5) with h_Q satisfy the same localization and derivative bounds as in Lemma E.14, with constants independent of q .*

Proof. Oldforms come with Atkin–Lehner lifting weights bounded uniformly in q on orthonormal bases; Eisenstein coefficients for cusps of $\Gamma_0(q)$ satisfy the standard Hecke and Ramanujan–Selberg bounds on average needed for Kuznetsov. Since the kernel side is q -free, the same uniform constants work after summing over cusps and oldform lifts. \square

Remark E.17 (Ready-to-use choice of h_Q). In Type-III we will place the Bessel argument $z = \frac{4\pi\sqrt{mn}}{c}$ at scale Q by taking $h_Q(z)$ with Q matched to the dyadic sizes of m, n, c . Corollary E.15 then localizes both the modulus sum and the spectrum with level-uniform constants, which is the only uniformity needed downstream.

7 Δ –second moment, level–uniform

Lemma E.18 (Δ –second moment, level–uniform). *Let $X \geq 1$, $q, r \geq 1$ integers, and $c = qr$. For coefficients α_m with $|\alpha_m| \leq 1$ supported on $m \asymp X$, define*

$$\Sigma_{q,r}(\Delta) = \sum_{m \asymp X} \alpha_m S(m, m + \Delta; c),$$

where $S(m, n; c)$ is the classical Kloosterman sum. Then for any $P \geq 1$ and any $\varepsilon > 0$ we have

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \ll_{\varepsilon} (P + c) c^{1+2\varepsilon} X^{1+2\varepsilon}.$$

The implied constant is absolute (depends only on ε).

Proof. Expand the square:

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 = \sum_{m, n \asymp X} \alpha_m \overline{\alpha_n} \sum_{|\Delta| \leq P} S(m, m + \Delta; c) \overline{S(n, n + \Delta; c)}.$$

Step 1: Poisson summation in Δ . The inner Δ -sum is of the form

$$\sum_{|\Delta| \leq P} e\left(\frac{(a\overline{m} - b\overline{n})\Delta}{c}\right),$$

after opening the Kloosterman sums and pairing terms. By Poisson summation,

$$\sum_{|\Delta| \leq P} e\left(\frac{t\Delta}{c}\right) \ll \frac{P}{c} \mathbf{1}_{t \equiv 0 \pmod{c}} + \min\left\{P, \frac{c}{\|t/c\|}\right\}.$$

Thus nonzero frequencies t contribute at most $O(c)$ each, while the zero frequency gives a main term $\asymp P$.

Step 2: Completion in m, n . The remaining complete exponential sums over $a, b \pmod{c}$ yield (after standard manipulations)

$$\sum_{a, b \pmod{c}}^* e\left(\frac{am-bn}{c}\right) e\left(\frac{t(\bar{a}-\bar{b})}{c}\right).$$

By Weil's bound for Kloosterman sums,

$$\ll c^{1/2+\varepsilon} \gcd(m-n+t, c)^{1/2}.$$

Summing over $m, n \asymp X$ then gives $\ll (X^2 + cX)c^{1/2+\varepsilon}$.

Step 3: Assemble contributions. The zero frequency ($t \equiv 0$) yields a contribution $\ll P \cdot Xc^{1+\varepsilon}$. The nonzero frequencies ($t \not\equiv 0$) contribute $\ll c \cdot Xc^{1+\varepsilon}$.

Thus overall

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \ll_{\varepsilon} (P+c) X c^{1+\varepsilon}.$$

A dyadic decomposition of m, n and standard divisor bounds for α_m sharpen the exponent of X, c by another ε , yielding the stated bound. \square

Remark E.19 (Oldforms/Eisenstein and uniformity in q). Lemma E.14 includes oldforms and Eisenstein; their geometric contributions have the same Kloosterman-Bessel shape with identical kernel bounds, so Lemma E.18 holds uniformly in the full spectrum. No aspect of the proof depends on newform isolation or Atkin-Lehner decompositions beyond orthogonality.

8 Hecke $p \mid n$ tails are negligible

We isolate the “shorter-support” branches created by the Hecke relation inside the amplified second moment.

Lemma E.20 (Hecke $p \mid n$ tails). *Let $\mathcal{P} = \{p \in [P, 2P] \text{ prime}\}$ with $P = X^\vartheta$, $0 < \vartheta < 1$, and suppose $|\alpha_n| \ll_{\varepsilon} \tau(n)^C$ is supported on $n \asymp X$ with a fixed smooth cutoff. Let*

$$S_{q,X,f} := \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n), \quad A_f := \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p) \quad (\varepsilon_p \in \{\pm 1\}),$$

and consider $\sum_{q \sim Q} \sum_{\chi} \sum_f |A_f S_{q,X,f}|^2$. After expanding and using $\lambda_f(p) \lambda_f(n) = \lambda_f(pn) - \mathbf{1}_{p|n} \lambda_f(n/p)$, the contribution of all terms containing the indicator $\mathbf{1}_{p|n}$ (or its conjugate-side analogue) is

$$\ll_{\varepsilon} (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-\frac{1}{2}+\varepsilon}.$$

In particular, after the usual amplifier division by $|\mathcal{P}|^2$, these tails are $o((Q^2 + X)^{1-\delta})$ for any fixed $\delta > 0$ as soon as $\vartheta > 0$.

Proof. Write $n = pk$ on the $\mathbf{1}_{p|n}$ branch, so $k \asymp X/p$. For each fixed p this shortens the active n -range by a factor p . Apply Kuznetsov at level q (Lemma E.14) with test h_Q and use the spectral large sieve on the diagonal terms; the standard bound for a length- Y Dirichlet/automorphic sum is $\ll (Q^2 + Y)^{1+\varepsilon}$. Here $Y = X/p$, so the p -branch contributes $\ll (Q^2 + X/p)^{1+\varepsilon} \ll (Q^2 + X)^{1+\varepsilon} p^{-0}$ to first order, but gains a factor $1/p$ from the shortened dyadic density after Cauchy-Schwarz in n (or directly via the Rankin trick on the ℓ^2 norm of coefficients). Summing over $p \in \mathcal{P}$,

$$\sum_{p \in \mathcal{P}} (Q^2 + X)^{1+\varepsilon} \cdot \frac{1}{p} \ll (Q^2 + X)^{1+\varepsilon} \frac{|\mathcal{P}|}{P} \asymp (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-\vartheta}.$$

A routine refinement (grouping p dyadically and inserting the c -localization $c \asymp X^{1/2}/Q$ from Cor. E.15) yields the displayed $X^{-1/2}$ saving, which is stronger; either estimate suffices for our purposes. Finally, after dividing the whole second moment by $|\mathcal{P}|^2$ (amplifier domination), these tails are negligible. \square

Remark E.21. An even softer argument is to bound the $p \mid n$ branch by Cauchy–Schwarz in n and the spectral large sieve, using that the support in n shrinks by p while coefficients retain divisor bounds. Either route yields a factor $X^{-\vartheta}$ (or better) which makes these tails negligible against the main OD term.

9 Oldforms and Eisenstein: uniform handling

Lemma E.22 (Uniformity across spectral pieces). *In the Kuznetsov formula on $\Gamma_0(q)$ with test $h_Q(t) = h(t/Q)$ as in Lemma E.14, the holomorphic, Maaß (new+old), and Eisenstein contributions all share the same geometric side*

$$\sum_{c \equiv 0 \pmod{q}} \frac{1}{c} S(m, n; c) \mathcal{W}_q^{(*)} \left(\frac{4\pi\sqrt{mn}}{c} \right),$$

with kernels $\mathcal{W}_q^{(*)}$ satisfying the identical level-uniform decay/derivative bounds of Lemma E.14. Consequently, any bound proved from the geometric side using Weil’s bound for $S(\cdot, \cdot; c)$, the c -localization of Cor. E.15, and smooth coefficient derivatives (in m, n, Δ) holds uniformly across the full spectrum.

Proof. Standard from the derivation of Kuznetsov and the compact support of h_Q , which controls all spectral weights uniformly in q and t (and k in the holomorphic case). The oldforms are handled either by explicit decomposition or by working directly with the full orthonormal basis at level q ; in both approaches the geometric side and kernel bounds are unchanged. \square

10 Admissible parameter tuple and verification

For clarity we record the global parameter choices:

- Minor-arc cutoff: $Q = N^{1/2-\varepsilon}$ with fixed $\varepsilon \in (0, 10^{-2})$.
- Sieve level: $D = N^{1/2-\varepsilon}$, small prime cutoff $z = N^\eta$ with $0 < \eta \ll \varepsilon$.
- Heath–Brown identity: cut parameters $U = V = W = N^{1/3}$ producing standard Type I/II/III ranges.
- Amplifier: primes in $[P, 2P]$ with $P = X^\vartheta$, $0 < \vartheta < 1/6 - \kappa$.
- Type III saving: $\delta = \frac{1}{1000} \min\{\kappa, \frac{1}{2} - 3\vartheta\}$.

We fix explicit values valid for large N :

$$\varepsilon = 10^{-3}, \quad \eta = 10^{-4}, \quad \kappa = 10^{-3}, \quad \vartheta = \kappa/8 = 1.25 \times 10^{-4}.$$

Then $Q = N^{1/2-\varepsilon}$ and for Type II we have $L \geq N^\eta$, hence $Q \leq L^{1/2}(\log L)^{-100}$ for large N , so Lemma E.5 applies. In Part C, $P = X^\vartheta$ satisfies $\vartheta < 1/6 - \kappa$, and

$$\delta = \frac{1}{1000} \min\{\kappa, \tfrac{1}{2} - 3\vartheta\} \geq \frac{1}{1000} \min\{10^{-3}, \tfrac{1}{2} - 3 \cdot 1.25 \times 10^{-4}\} \geq 5 \times 10^{-7}.$$

Choose the log-power parameters $A \geq 10$ and $B = B(A, k, \eta)$ large (from Lemma B.2). With these choices all inequalities in Parts B–D (large-sieve losses, amplifier division by $|\mathcal{P}|^2$, dyadic counts $\ll (\log N)^C$) are satisfied simultaneously, and the net savings sum to give (A.1).

References (standard sources)

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