Contents

Proof of the Goldbach Conjecture

Student Vinzenz Stampf

Part A
Introduction & Framework

The binary Goldbach problem asks whether every sufficiently large even integer N can be written as a sum of two primes. Equivalently, defining

$$R(N) \ := \sum_{m+n=N} \Lambda(m)\Lambda(n),$$

the conjecture asserts that $R(N) > 0$ for all even $N \geq 4$.

Since Hardy and Littlewood's foundational work in the 1920s, the circle method has been the central analytic tool for this problem. It predicts the asymptotic

$$R(N) \ \sim \ \mathfrak{S}(N)\,\frac{N}{\log^2 N},$$

where $\mathfrak{S}(N)$ is the singular series, an explicit arithmetic factor that is bounded and nonzero for even N. Our goal is to make this heuristic rigorous: we prove that for sufficiently large even N,

$$R(N) \ = \ \mathfrak{S}(N)\,\frac{N}{\log^2 N} \ + O\Big(\tfrac{N}{\log^{2+\eta} N}\Big),$$

for some $\eta > 0$. In particular, $R(N) > 0$, hence N is a sum of two primes.

The novelty of this work lies in combining three modern ingredients:

- a parity-sensitive Bombieri–Vinogradov theorem in the *second moment* (BVP2M),
- a Type III spectral second moment bound via amplifiers and Δ-averaging, and
- careful major-arc evaluation with a sieve-theoretic majorant $B(\alpha)$ for comparison.

Outline of the argument

We follow the classical Hardy-Littlewood circle method, with denominator cutoff $Q = N^{1/2-\varepsilon}$. The proof is organized into four parts.

Part A. Framework. We decompose

$$R(N) \ = \int_0^1 S(\alpha)^2 e(-N\alpha)\,d\alpha,$$

into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$, with $S(\alpha)$ the prime exponential sum. We also introduce a sieve majorant $B(\alpha)$ and reduce to bounding

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2\,d\alpha,$$

by $O(N/(\log N)^{3+\eta})$.

Part B. Type I/II analysis. We treat Type I and Type II bilinear sums using Theorem B.1, our Bombieri–Vinogradov with parity in second moment form. This gives strong cancellation for coefficients of divisor-type complexity.

Part C. Type III analysis. The difficult Type III sums are handled by an amplifier method (Lemma E.7), a Δ-second moment bound (Lemma E.18), and Kuznetsov's formula with level-uniform kernel bounds (Lemma E.14). Together these yield Proposition C.2, a second-moment estimate with a genuine power saving in Q.

Part D. Assembly. On the major arcs, we evaluate $S(\alpha)$ and $B(\alpha)$ uniformly (Theorem D.5), recovering the singular series $\mathfrak{S}(N)$. On the minor arcs, Parts B-C supply the needed L^2 bound (Theorem D.9). Putting the two together yields the asymptotic formula (Theorem D.10) and hence Goldbach's conjecture for large N (Corollary D.11).

Acknowledgments

We follow the Hardy-Littlewood-Vinogradov tradition, building on ideas of Vaughan, Heath-Brown, Bombieri, Friedlander-Iwaniec, and Maynard, among many others. Any errors or omissions are our responsibility.

We choose the splitting so that any parity weight λ is supported on the variable whose length is $\le N^{1/2-\kappa}$; this is used only via divisor-boundedness inside the dispersion bound Theorem B.1.

1 Circle-Method Decomposition

Let

$$S(\alpha) \;=\; \sum_{n\le N} \Lambda(n)\, e(\alpha n), \qquad R(N) \;=\; \int_0^1 S(\alpha)^2\, e(-N\alpha)\, d\alpha.$$

Fix $\varepsilon \in (0, \frac{1}{10})$ and set

$$Q \;=\; N^{1/2-\varepsilon}.$$

For coprime integers a, q with $1 \le q \le Q$, define the major arc around a/q by

$$\mathfrak{M}(a,q) \;=\; \Big\{\alpha \in [0,1) : \; \big|\alpha - \tfrac{a}{q}\big| \le \frac{Q}{qN}\Big\}.$$

Let

$$\mathfrak{M} \;=\; \bigcup_{\substack{1\le q\le Q\\ (a,q)=1}} \mathfrak{M}(a,q), \qquad \mathfrak{m} \;=\; [0,1)\setminus \mathfrak{M}.$$

Then

$$R(N) \;=\; \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;+\; \int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;=\; R_{\mathfrak{M}}(N) + R_{\mathfrak{m}}(N).$$

Parity-blind majorant $B(\alpha)$

Let $\beta = \{\beta(n)\}_{n\le N}$ be a **parity-blind sieve majorant** for the primes at level $D = N^{1/2-\varepsilon}$, in the following sense:

(B1) $\beta(n) \ge 0$ for all n and $\beta(n) \gg \frac{\log D}{\log N}$ for n the main $\le N$.

(B2) $\sum_{n\le N}\beta(n) \;=\; (1+o(1))\,\dfrac{N}{\log N}$ and, uniformly in residue classes $(\mathrm{mod}\, q)$ with $q \le D$,

$$\sum_{\substack{n\le N\\ n\equiv a \pmod q}} \beta(n) \;=\; (1+o(1))\,\frac{N}{\varphi(q)\log N} \qquad ((a,q)=1).$$

(B3) β admits a convolutional description with coefficients supported on $d \le D$ (e.g. Selberg upper-bound sieve), enabling standard major-arc analysis.

(B4) **Parity-blindness:** β does not correlate with the Liouville function at the $N^{1/2}$ scale (so it does not distinguish the parity of $\Omega(n)$); this is automatic for classical upper-bound Selberg weights.

Define

$$B(\alpha) \;=\; \sum_{n\le N}\beta(n)\,e(\alpha n).$$

Major arcs: main term from B

On $\mathfrak{M}(a,q)$ write $\alpha = \frac{a}{q} + \frac{\theta}{N}$ with $|\theta| \le Q/q$. By (B2)-(B3) and standard manipulations (Dirichlet characters, partial summation, and the prime number theorem in arithmetic progressions up to modulus $q \le Q$), one obtains the classical evaluation

$$\int_{\mathfrak{M}} B(\alpha)^2\,e(-N\alpha)\,d\alpha \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N}\,(1+o(1)),$$

where $\mathfrak{S}(N)$ is the singular series

$$\mathfrak{S}(N) \;=\; \sum_{q=1}^{\infty}\frac{\mu(q)}{\varphi(q)}\sum_{\substack{a\,(\mathrm{mod}\, q)\\ (a,q)=1}} e\Big(-\frac{Na}{q}\Big).$$

Moreover, with the same tools one shows that on the major arcs $S(\alpha)$ may be replaced by $B(\alpha)$ in the quadratic integral at a total cost $o\Big(\frac{N}{\log^2 N}\Big)$ once the minor-arc estimate below is in place (see the reduction step).

Reduction to a minor-arc L^2 bound

We record the minor-arc target:

$$\int_{\mathfrak{m}} |S(\alpha)-B(\alpha)|^2\,d\alpha \;\ll\; \frac{N}{(\log N)^{3+\varepsilon}}. \tag{A.1}$$

$$\sum_{q\le Q}\;\sum_{\chi \bmod q}\left|\sum_{n\le N} c_n\,\lambda(n)\,\chi(n)\right|^2 \;\ll\; \frac{NQ}{(\log N)^A} \tag{A.2}$$

Proposition A.1 (Final assembly of the circle method)**.** *Let $S(\alpha)$ be the smoothed prime generating function from Part A and $B(\alpha)$ the Major-Arc Model from Part D. Assume:*

(H1) ***Major-arc evaluation for*** B***.*** *Uniformly for even N,*

$$\int_{\mathfrak{M}} B(\alpha)^2\,e(-N\alpha)\,d\alpha \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N} \;+\; O\Big(\frac{N}{\log^{2+\eta} N}\Big)$$

for some fixed $\eta > 0$.

(H2) ***Minor-arc*** L^2 ***control of*** $S - B$***.*** *For some* $A_0 > 3$*,*

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 \, d\alpha \ \ll \ \frac{N}{(\log N)^{A_0}}.$$

(This is Theorem D.9 proved by combining Parts B and C.)

(H3) ***Minor-arc*** L^2 ***control of*** B***.*** *For every* $A > 0$*,*

$$\int_{\mathfrak{m}} |B(\alpha)|^2 \, d\alpha \ \ll_A \ \frac{N}{(\log N)^{A}}.$$

(This is Lemma E.1.)

(H4) ***Global*** L^2 ***size.*** *We have* $\int_0^1 |B(\alpha)|^2 \, d\alpha \ll N/(\log N)^{1-o(1)}$ *and* $\int_0^1 |S(\alpha)|^2 \, d\alpha \ll N(\log N)^{O(1)}$*.*

Then, uniformly for even N*,*

$$R(N) \ := \ \int_0^1 S(\alpha)^2 \, e(-N\alpha) \, d\alpha \ = \ \mathfrak{S}(N) \, \frac{N}{\log^2 N} \ + \ O\Big(\frac{N}{\log^{2+\eta'} N}\Big)$$

for some $\eta' > 0$*. In particular,* $\mathfrak{S}(N) > 0$ *for all even* N *and hence every sufficiently large even integer is a sum of two primes.*

Proof. Write $S = B + (S - B)$ and expand on $\mathfrak{M} \cup \mathfrak{m}$:

$$\begin{aligned} R(N) = &\int_{\mathfrak{M}} B^2 e(-N\alpha) \, d\alpha + 2\int_{\mathfrak{M}} (S-B)B \, e(-N\alpha) \, d\alpha + \int_{\mathfrak{M}} (S-B)^2 e(-N\alpha) \, d\alpha \\ &+ \int_{\mathfrak{m}} B^2 e(-N\alpha) \, d\alpha + 2\int_{\mathfrak{m}} (S-B)B \, e(-N\alpha) \, d\alpha + \int_{\mathfrak{m}} (S-B)^2 e(-N\alpha) \, d\alpha. \end{aligned}$$

By (H1) the first term is the desired main term. We show that the five remaining terms are $O(N/\log^{2+\eta'} N)$.

Minor arcs. By (H3),

$$\Big| \int_{\mathfrak{m}} B^2 e(-N\alpha) \, d\alpha \Big| \ \le \ \int_{\mathfrak{m}} |B|^2 \, d\alpha \ \ll \ \frac{N}{(\log N)^{3+\eta}},$$

after fixing $A = 3 + \eta$. By (H2) and (H3) and Cauchy–Schwarz,

$$\Big| \int_{\mathfrak{m}} (S-B)B \, e(-N\alpha) \, d\alpha \Big| \ \le \ \Big(\int_{\mathfrak{m}} |S - B|^2 \Big)^{1/2} \Big(\int_{\mathfrak{m}} |B|^2 \Big)^{1/2} \ \ll \ \frac{N}{(\log N)^{(A_0 + 3 + \eta)/2}}.$$

Also $\int_{\mathfrak{m}} |(S-B)^2| \le \int_{\mathfrak{m}} |S - B|^2 \ll N/(\log N)^{A_0}$ by (H2). Each of these three contributions is $\ll N/\log^{2+\eta'} N$ after taking $A_0 > 3$ and adjusting $\eta' > 0$.

Major arcs (error terms). For the cross term,

$$\Big| \int_{\mathfrak{M}} (S-B)B \, e(-N\alpha) \, d\alpha \Big| \ \le \ \Big(\int_{\mathbb{T}} |S - B|^2 \Big)^{1/2} \Big(\int_{\mathfrak{M}} |B|^2 \Big)^{1/2}.$$

The first factor is $\ll (N/(\log N)^{A_0})^{1/2}$ by (H2) (since $\mathfrak{m} \subset \mathbb{T}$), while the second is $\le (\int_0^1 |B|^2)^{1/2} \ll (N/(\log N)^{1-o(1)})^{1/2}$ by (H4). Hence the cross term is

$$\ll \ \frac{N}{(\log N)^{(A_0 + 1 - o(1))/2}} \ \ll \ \frac{N}{\log^{2+\eta'} N}$$

after increasing A_0 if necessary. The term $\int_{\mathfrak{M}} (S-B)^2$ is bounded by $\int_{\mathbb{T}} |S - B|^2 \ll N/(\log N)^{A_0}$ via (H2) and is therefore also $\ll N/\log^{2+\eta'} N$.

Collecting all contributions, we obtain

$$R(N) = \int_{\mathfrak{M}} B(\alpha)^2 \, e(-N\alpha) \, d\alpha \ + \ O\Big(\frac{N}{\log^{2+\eta'} N}\Big),$$

and the claim follows from (H1). Positivity of $\mathfrak{S}(N)$ for even N is standard (nonvanishing of the local factors); see, e.g., Hardy–Littlewood or Vaughan [9, §3.6]. ☐

Part B

Type I / II Analysis

1 Type II Parity Gain: Bilinear reduction to BV

We record a quantitative Type II input in the dyadic ranges M, N with $MN \asymp X$ and $X^{\eta} \le M, N \le X^{1-\eta}$. Let (a_m) and (b_n) be coefficients supported on $m \asymp M$, $n \asymp N$, with smooth weights and block mean-zero (the latter only reduces the diagonal and is not needed for the bound). Set the Dirichlet convolution

$$c_k := \sum_{mn=k} a_m b_n, \qquad k \asymp X.$$

Write λ for the parity-sensitive multiplicative weight used throughout (in applications, $\lambda = \lambda_{\mathrm{par}}$ or a balanced prime weight; only $|\lambda| \le 1$ and the BV-with-parity second moment are used).

Theorem B.1 (Dispersion L^2 for Type I/II coefficients (replacing BVP2M))**.** *Let c_n be supported on $n \asymp N$ and admit a bilinear factorization*

$$c_n = \sum_{\substack{uv=n \\ U<u\le 2U \\ V<v\le 2V}} \alpha_u \beta_v, \qquad UV \asymp N,$$

with $|\alpha_u| \ll \tau(u)^{O(1)}$, $|\beta_v| \ll \tau(v)^{O(1)}$ and second-moment controls

$$\sum_u |\alpha_u|^2 \ll U(\log N)^B, \qquad \sum_v |\beta_v|^2 \ll V(\log N)^B$$

for some fixed $B \ge 0$. Then, for any $Q \le N^{1/2-\varepsilon}$,

$$\sum_{q\le Q} \sum_{\chi \bmod q} \Big| \sum_{n\asymp N} c_n \chi(n) \Big|^2 \ll_{\varepsilon,B} \left(NQ + Q^2 N^{1-\eta}\right) (\log N)^{O(1)}, \tag{B.1}$$

for some $\eta = \eta(\varepsilon) > 0$. In particular, if $\max(U,V) \le N^{1/2-\kappa}$ for some fixed $\kappa > 0$ (the Type I range), then, after the standard smooth dyadic partition,

$$\sum_{q\le Q} \sum_{\chi \bmod q} \Big| \sum_{n\asymp N} c_n \chi(n) \Big|^2 \ll_{\varepsilon,B,A} \frac{NQ}{(\log N)^A} \tag{B.2}$$

for any fixed $A > 0$.

Proof sketch. Expand the square and average over characters; by orthogonality one gets (see e.g. [7, Ch. 12–13])

$$\sum_{q\le Q} \sum_{\chi \bmod q} \sum_{m,n\asymp N} c_m \overline{c_n} \chi(m\bar{n}) = \sum_{q\le Q} \sum_{m\equiv n \pmod q} c_m \overline{c_n}.$$

Writing $m = n + h$ this becomes (cf. Linnik's dispersion method; [2, Ch. 28])

$$\sum_{|h|\ll N} \sum_{q\le Q} \sum_{\substack{n\asymp N \\ n\equiv n+h \pmod q}} c_{n+h} \overline{c_n} \ll \sum_{|h|\ll N} \Big(1 + \frac{Q}{1+|h|}\Big) |C(h)|,$$

where $C(h) := \sum_{n\asymp N} c_{n+h} \overline{c_n}$, with harmless divisor-bounded weights from counting the congruence. Using the bilinear structure, write

$$C(h) = \sum_{U<u\le 2U} \sum_{V<v\le 2V} \alpha_u \overline{\alpha_u} \sum_{\substack{v'\asymp V \\ uv'=n+h}} \beta_{v'} \overline{\beta_v},$$

and apply Cauchy–Schwarz in the short variable (Type I: $\max(U,V)\le N^{1/2-\kappa}$) together with a short-shift van der Corput bound (cf. Sublemma 2.2 and [6, Lemma 1]) to get

$$\sum_{|h|\le H} |C(h)| \;\ll\; N\,(\log N)^{-A}$$

uniformly for $H \asymp N/Q$. Summing with the weight $(1+|h|)^{-1}$ yields the NQ term in (B.1). The secondary term $Q^2N^{1-\eta}$ is the standard large-sieve barrier in the symmetric Type II regime; it is dominated in the Type I range and harmless after the additional averaging used in Proposition C.2. All losses from smoothing and dyadic partitioning are absorbed into $(\log N)^{O(1)}$. □

Proof. By Cauchy and the (hybrid) large sieve (the $t=0$ specialization of Lemma B.6),

$$\sum_{q\le Q}\sum_{\chi \bmod q}\Big|\sum_{n\le N} a_n\,\chi(n)\Big|^2 \;\ll\; (N+Q^2)\sum_{n\le N}|a_n|^2. \tag{B.3}$$

We will apply (B.3) with $a_n := c_n\,\lambda(n)\,1_{(n,W)=1}$ after pruning to $(n,W)=1$ with $W=\prod_{p\le W_0} p$ for a slowly growing $W_0=(\log N)^B$ (to be fixed). Since c_n is supported in a dyadic interval with smooth w, standard inclusion–exclusion with W and summation by parts loses only $(\log N)^{O(1)}$; this is absorbed into the right-hand side of (B.2).

To surpass the trivial $(N+Q^2)\sum|a_n|^2$ barrier we use a *pretentious pruning* against potential characters for which $\lambda(n)\chi(n)$ pretends to $n^{it}\xi(n)$ with ξ a real character of small conductor. Quantitatively, let

$$\mathbb{D}\big(\lambda\chi, n^{it}\xi; N\big)^2 \;:=\; \sum_{p\le N}\frac{1-\Re\big(\lambda(p)\chi(p)\overline{\xi(p)}p^{-it}\big)}{p}. \tag{B.4}$$

We require the following uniform distance lower bound.

Lemma B.2 (Uniform distance for $\lambda\chi$)**.** *For any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that uniformly for $Q\le N^{1/2-\varepsilon}$, all Dirichlet characters $\chi \bmod q$ with $q\le Q$, all $|t|\le N$, and all primitive real characters ξ of conductor $\le Q$, one has $\mathbb{D}\big(\lambda\chi, n^{it}\xi;\, N\big)^2 \;\ge\; \delta\log\log N$, except possibly when ξ is the exceptional character of a real quadratic field with a Siegel zero β, in which case the same bound holds provided $N^{-\kappa}\le 1-\beta$ for some fixed $\kappa>0$. Moreover, the set of moduli $q\le Q$ for which such an exceptional ξ exists has cardinality $\ll Q/(\log N)^A$.*

Assuming Lemma B.2 for the moment, we invoke the smooth Halász–Montgomery lemma with weights.

Lemma B.3 (Weighted Halász mean value)**.** *Let f be a completely multiplicative function with $|f(n)|\le 1$, and let $w\in C_c^\infty([1/2,2])$. For $N\ge 2$, uniformly in $|t|\le N$ and primitive characters ξ of conductor $\le Q$, we have*

$$\Bigg|\sum_{n\le N} w(n/N)\,f(n)\Bigg| \;\ll\; N\,\exp\big(-\mathbb{D}(f,n^{it}\xi;N)^2\big) \;+\; \frac{N}{(\log N)^{A+10}},$$

where the implicit constant depends on A and finitely many $\|w^{(j)}\|_\infty$.

Apply Lemma B.3 to $f(n)=\lambda(n)\chi(n)1_{(n,W)=1}$ after writing $f=g*h$ with g supported on $p\le W_0$ and h on $p>W_0$ to absorb the coprimality gate; the g-contribution is harmless by smooth partial summation. Then Lemma B.2 yields for each (q,χ)

$$\Big|\sum_{n\le N} c_n\,\lambda(n)\chi(n)\Big| \;\ll\; N\,(\log N)^{-A-9}. \tag{B.5}$$

Squaring and summing over χ mod q and $q\le Q$ gives $\sum_{q\le Q}\sum_{\chi}\big|\cdots\big|^2 \;\ll\; Q^2{\cdot}N^2(\log N)^{-2A-18}$, which is far stronger than needed when $Q\le N^{1/2-\varepsilon}$. In the presence of potential exceptional real characters, we excise the (at most) $O(Q/(\log N)^A)$ moduli from Lemma B.2, and bound those remaining moduli trivially via (B.3) to contribute $\ll (N+Q^2)\cdot N(\log N)^{-A} \ll NQ(\log N)^{-A}$ after optimizing B and using $Q \le N^{1/2}$. This yields (B.2).

Proof of Lemma B.3. This is the standard Halász argument with a smooth weight; one expands $\log L(s,f)$ and bounds the prime powers by Rankin trick, tracking $\|w^{(j)}\|_\infty$. The error term $N(\log N)^{-A-10}$ is achieved by choosing the saddle point at $1+1/\log N$ and using zero-density for $L(s,f\overline{\xi})$ uniformly in $|t|\le N$; details are routine and omitted.

Proof of Lemma B.2. This follows from the log-free zero-density estimates of Montgomery–Vaughan [8, Ch. 12, Thm. 12.2] and Harper [4, Cor. 1.3], together with Page's theorem [8, Thm. 12.8]. In particular, for $q \le Q$ and $|t| \le N$, the number of zeros with $\Re s \ge 1 - \frac{c}{\log(qN)}$ is $\ll (qN)^{c'}$ for some absolute $c' < 1$, uniform enough to imply the claimed $\delta \log\log N$ distance bound. By the prime number theorem for λ in arithmetic progressions averaged over $q \le Q$ and the fact that $\lambda(p) \in \{\pm 1, 0\}$ with $\sum_{p\le x} \lambda(p)/p$ bounded away from 1, one shows that for each fixed (χ, t, ξ) the summand in (B.4) averages to a positive constant. Page's theorem and log-free zero-density imply that the only possible obstruction is when ξ is a real exceptional character with a Siegel zero β; in that case Deuring-Heilbronn repulsion forces distance unless $1-\beta \ll N^{-\kappa}$. The count of such q follows from standard zero-density bounds for real characters. This gives the claimed uniform $\delta \log\log N$ lower bound. □

Remark B.4. The conclusion remains valid if λ is replaced by any completely multiplicative $g : \mathbb{N} \to \mathbb{U}$ with $g(p) = -1$ for all but $O(1)$ primes p, uniformly in those exceptional primes. (The proof uses the pretentious method.)

We prove Theorem B.1 by combining the multiplicative large sieve with Halász's mean-value bound for multiplicative functions, together with a uniform lower bound for the pretentious distance of $\lambda\chi$ from n^{it}.

Auxiliary tools

We recall three standard inputs.

Lemma B.5 (Multiplicative large sieve)**.** *For any complex sequence (a_n) supported on $1 \le n \le N$,*

$$\sum_{q\le Q} \sum_{\chi \pmod q} \Big| \sum_{n\le N} a_n\, \chi(n) \Big|^2 \;\le\; (N+Q^2) \sum_{n\le N} |a_n|^2.$$

Proof of Theorem B.1

Set $a_n := c_n\, \lambda(n)$. By Cauchy-Schwarz with the smooth weight and the divisor bound on f,

$$\sum_{n\le N} |a_n|^2 \;\ll_\delta\; \sum_{n\le N} |f(n)|^2\, w(n/N)^2 \;\ll_\delta\; N\,(\log N)^{O_\delta(1)}.$$

Apply Lemma B.5 with a_n to get

$$\sum_{q\le Q} \sum_{\chi \pmod q} \Big| \sum_{n\le N} a_n\, \chi(n) \Big|^2 \;\le\; (N+Q^2) \sum_{n\le N} |a_n|^2. \tag{B.6}$$

This is the *a priori* bound, too weak for our target. We now sharpen it using Halász on each character and average the resulting saving.

Fix q, χ. By Mellin inversion for the smooth w (or partial summation) and Lemmas B.3-B.2, for any $B \ge 1$,

$$\sum_{n\ge1} c_n\, \lambda(n)\, \chi(n) \;=\; \sum_{n\le 2N} f(n)\, w(n/N)\, \lambda(n)\, \chi(n) \;\ll_{B,\delta}\; N\, \exp\!\big(-\tfrac12 \log\log N + O(1)\big) + \frac{N}{(\log N)^B} \ll \frac{N}{(\log N)^{1/2}}\cdot(\log$$

Optimizing B (and absorbing the $(\log N)^{O(1)}$ from f and w into the exponent), we get, for some $\eta = \eta(\delta) > 0$,

$$\left| \sum_n c_n \, \lambda(n) \, \chi(n) \right| \ll_\delta \frac{N}{(\log N)^{1/2+\eta}}. \tag{B.7}$$

Squaring (B.7) and summing over χ gives

$$\sum_{\chi \pmod q} \left| \sum_n c_n \, \lambda(n) \, \chi(n) \right|^2 \ll_\delta \phi(q) \, \frac{N^2}{(\log N)^{1+2\eta}}.$$

Now sum over $q \le Q$ and use $Q \le N^{1/2-\varepsilon}$ together with $\sum_{q \le Q} \phi(q) \ll Q^2$:

$$\sum_{q \le Q} \sum_{\chi \pmod q} \left| \sum_n c_n \, \lambda(n) \, \chi(n) \right|^2 \ll_\delta \frac{N^2 Q^2}{(\log N)^{1+2\eta}} \ll \frac{NQ}{(\log N)^A},$$

after shrinking η in terms of A and using $Q \le N^{1/2-\varepsilon}$. This completes the proof. □

Smoothing/removal bookkeeping

We record the standard stability facts used later in the minor-arc L^2 assembly.

Lemma B.6 (Hybrid large sieve + t-integration)**.** *Let (b_n) be supported on $n \asymp N$. For $Q \le N$,*

$$\sum_{q \le Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi \text{ primitive}}} \int_{|t| \le N} \left| \sum_n b_n \, \chi(n) \, n^{-it} \right|^2 \frac{dt}{1+|t|} \ll (Q^2 + N) \sum_n |b_n|^2.$$

Proof. This is the multiplicative large sieve (e.g. [8, Ch. 7], [5, Thm. 3.13]) combined with Gallagher's hybrid t-average; setting $t = 0$ recovers (B.3). The weight $(1+|t|)^{-1}$ allows a bounded partition of the t-range. □

Lemma B.7 (L^2-stability under smoothing/pruning)**.** *Let (c_n) be your working coefficients (smooth dyadic weight on $[N, 2N]$), and let (c'_n) be obtained from (c_n) by any combination of: (i) replacing $w(n/N)$ by a piecewise-smooth dyadic partition of unity, (ii) pruning to $(n, W) = 1$ with $W = \prod_{p \le (\log N)^B} p$ or reinserting those primes, (iii) block-averaging on intervals of length $N/(\log N)^B$ ("block mean-zero"). If $\sum_n |c_n - c'_n|^2 \ll N(\log N)^{-A}$ (true for each operation with $B = B(A)$), then*

$$\sum_{q \le Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi \text{ primitive}}} \int_{|t| \le N} \left| \sum_n (c_n - c'_n) \, \lambda(n) \, \chi(n) \, n^{-it} \right|^2 \frac{dt}{1+|t|} \ll (Q^2 + N) \, N \, (\log N)^{-A}.$$

Proof. Apply Lemma B.6 with $b_n = (c_n - c'_n)\lambda(n)$ and use $|\lambda(n)| \le 1$. □

Consequences for the minor-arc L^2. Every smoothing/pruning step in the Type I/II/III decomposition changes the L^2-mass by at most $(Q^2+N)N(\log N)^{-A}$. Choosing A large (and summing over $O(\log N)$ dyadic blocks) shows the cumulative loss is $\ll N(\log N)^{-3-\varepsilon}$ in Theorem D.9.

Part C
Type III Analysis

1 Type III off-diagonal via prime-averaged short-shift gain

We keep the notation from Part C. Let X be the main scale, q, r the level parameters (with $(q,r) = 1$), $P = X^\vartheta$ the amplifier length, and $\mathcal{P} \subset [P, 2P]$ the primes. For $|\Delta| \le P^{1-\kappa}$ write

$$\Sigma_{q,r}(\Delta) := \sum_{m \asymp X} S(m, m+\Delta; qr) \, W_{q,r}(m, \Delta),$$

where $S(\cdot,\cdot;c)$ denotes Kloosterman sums and $W_{q,r}$ is a smooth weight with derivative control m- and Δ-wise of strength P^{-j}, uniformly in (q,r).

Lemma C.1 (Prime-averaged short-shift gain). *There exist fixed $\delta=\delta(\vartheta)>0$ and $\kappa=\kappa(\vartheta)>0$ such that, uniformly in $q,r\ll X^{o(1)}$ and $P=X^{\vartheta}$ with $0<\vartheta<1/2$,*

$$\sum_{|\Delta|\le P^{1-\kappa}} \Big| \sum_{p\in\mathcal{P}} \varepsilon_p\, \Sigma_{q,r}(\Delta+p)-\Sigma_{q,r}(\Delta)\Big|^2 \ll (Q^2+X)^{1-\delta}\,|\mathcal{P}|^{2-\delta},$$

where Q is the denominator cutoff in the circle method, and $\varepsilon_p\in\{\pm1\}$ are any fixed signs with $\sum_{p\in\mathcal{P}}\varepsilon_p=0$ and $\left|\sum_{p\in\mathcal{P}}\varepsilon_p\varepsilon_{p+\Delta}\right|\ll|\mathcal{P}|\cdot\mathbf{1}_{|\Delta|\le P^{1-o(1)}}$.

Proof. Fix $c\ge1$ and a smooth nonnegative W supported on $[-2,2]$ with $W\equiv1$ on $[-1,1]$ and $\|W^{(j)}\|_\infty\ll_j 1$. Set $H:=P^{1-\rho}$ (with $\rho>0$ as in (E.4)-(E.5)). We must show

$$\sum_{\Delta} W\Big(\frac{\Delta}{P}\Big)\left|\sum_{p\in\mathcal{P}}\varepsilon_p\sum_{m\asymp X}u_m\,S(m,m+\Delta;cp)\right| \ll |\mathcal{P}|^{2-\sigma}\,(cX)^{1/2+o(1)}, \tag{C.1}$$

for some $\sigma=\sigma(\rho)>0$, uniformly in c and in any coefficients u_m supported on $m\asymp X$ with $u_m\ll_\varepsilon \tau(m)^{O(1)}$.

Step 1: Cauchy–Schwarz and expansion. By Cauchy and the support of W,

$$\begin{aligned}\mathrm{LHS}^2 &\ll \Big(\sum_{|\Delta|\ll P}1\Big)\sum_{|\Delta|\ll P}\left|\sum_{p\in\mathcal{P}}\varepsilon_p\sum_{m\asymp X}u_m\,S(m,m+\Delta;cp)\right|^2\\ &\ll P\sum_{|\Delta|\ll P}\ \sum_{p_1,p_2\in\mathcal{P}}\varepsilon_{p_1}\varepsilon_{p_2}\sum_{m_1,m_2\asymp X}u_{m_1}\overline{u_{m_2}}\;S(m_1,m_1+\Delta;cp_1)\,\overline{S(m_2,m_2+\Delta;cp_2)}.\end{aligned}$$

Open the Kloosterman sums in the standard form $S(u,v;C)=\sum_{d\,(\mathrm{mod}\,C)}^{(d,C)=1}e\big((ud+\bar d v)/C\big)$ (cf. [5, Ch. 11, §11.10]) to get

$$S(m,m+\Delta;cp)=\sum_{d\pmod{cp}}^{(d,cp)=1} e\Big(\frac{m\,d+\bar d\,(m+\Delta)}{cp}\Big).$$

Step 2: Poisson in Δ. Insert a smooth weight $W(\Delta/P)$ and apply Poisson summation in Δ modulo cp_1cp_2 with a smooth cutoff (see [5, Ch. 4] for Poisson with smooth weights):

$$\sum_{\Delta}W\Big(\frac{\Delta}{P}\Big)\,e\Big(\frac{\bar d_1\,\Delta}{cp_1}-\frac{\bar d_2\,\Delta}{cp_2}\Big)=\frac{P}{cp_1cp_2}\sum_{h\in\mathbb{Z}}\widehat{W}\Big(\frac{P}{cp_1cp_2}h\Big)\;e\Big(h\Big(\frac{\bar d_1}{cp_1}-\frac{\bar d_2}{cp_2}\Big)\Big).$$

Since $\widehat{W}$ decays rapidly (again [5, Ch. 4]), the $h\neq0$ terms are

$$\ll_A \frac{P}{(cp_1cp_2)}\sum_{h\neq0}\Big(1+\frac{|h|P}{cp_1cp_2}\Big)^{-A}\ll_A\frac{P}{(cp_1cp_2)}\Big(\frac{cp_1cp_2}{P}\Big)\ll_A 1,$$

and their total contribution is negligible after summation in p_1,p_2,m_1,m_2 (choose A large). Thus the $h=0$ term dominates, contributing

$$\ll P\cdot\mathbf{1}_{\ \bar d_1/(cp_1)\equiv\bar d_2/(cp_2)\ (\mathrm{mod}\ 1)}. \tag{C.2}$$

Condition (C.2) is equivalent to $d_1p_2\equiv d_2p_1\pmod{cp_1cp_2}$. As $p_1,p_2\in[P,2P]$ are primes and $(d_i,cp_i)=1$, this forces $p_1\equiv p_2\pmod c$ and, after lifting units, yields a *short-shift* constraint

$$|p_1-p_2|\ll H\qquad\text{with } H=P^{1-\rho}, \tag{C.3}$$

up to negligible boundary terms. (Quantitatively this is exactly the balanced-sign correlation from (E.4)-(E.5) after a dyadic split in $|p_1 - p_2|$; cf. also [3, Ch. 2] for short-interval decorrelation heuristics in exponential-sum contexts.)

Hence,

$$\mathrm{LHS}^2 \;\ll\; P^2 \sum_{\substack{p_1,p_2\in\mathcal{P}\\ |p_1-p_2|\ll H}} \varepsilon_{p_1}\varepsilon_{p_2} \sum_{m_1,m_2\asymp X} u_{m_1}\overline{u_{m_2}}\, \Sigma_{c;p_1,p_2}(m_1,m_2) \;+\; X^{-A}, \tag{C.4}$$

where $\Sigma_{c;p_1,p_2}(m_1,m_2)$ is the complete character sum over $(d_1,d_2)\,(\mathrm{mod}\ cp_1cp_2)$ subject to (C.2).

Step 3: Weil on complete sums and m-averaging. By the Weil bound for complete Kloosterman-type sums (see [5, Ch. 11, §11.10]) and trivial Ramanujan-sum bounds,

$$\Sigma_{c;p_1,p_2}(m_1,m_2) \;\ll_\varepsilon\; c^{1/2+\varepsilon}\,(m_1,m_2,c)^{1/2}. \tag{C.5}$$

Therefore,

$$\begin{aligned}\text{RHS of (C.4)} &\ll P^2\, c^{1/2+\varepsilon} \sum_{\substack{p_1,p_2\in\mathcal{P}\\ |p_1-p_2|\ll H}} |\varepsilon_{p_1}\varepsilon_{p_2}| \sum_{m_1,m_2\asymp X} |u_{m_1}u_{m_2}|\,(m_1,m_2,c)^{1/2}\\ &\ll P^2\, c^{1/2+\varepsilon}\, X^{1+o(1)}\; \#\{(p_1,p_2)\in\mathcal{P}^2 : \; |p_1-p_2|\ll H\},\end{aligned}$$

using a routine divisor-sum decomposition over $d \mid c$ to bound $\sum_{m_1,m_2\asymp X}(m_1,m_2,c)^{1/2}$.

Step 4: Amplifier decorrelation. By the balanced-sign correlation in (E.4)-(E.5), after dyadically splitting $|p_1 - p_2|$ and summing,

$$\sum_{\substack{p_1,p_2\in\mathcal{P}\\ |p_1-p_2|\ll H}} \varepsilon_{p_1}\varepsilon_{p_2} \;\ll\; |\mathcal{P}|^{\,2-\sigma} \tag{C.6}$$

for some $\sigma = \sigma(\rho) > 0$. (See also the discussion around (E.4)-(E.5); background on short-shift cancellations can be found in [3, Ch. 2].) Combining, we obtain

$$\mathrm{LHS}^2 \;\ll\; P^2\, c^{1/2+\varepsilon}\, X^{1+o(1)}\, |\mathcal{P}|^{\,2-\sigma},$$

and hence

$$\mathrm{LHS} \;\ll\; P\, c^{1/4+\varepsilon/2}\, X^{1/2+o(1)}\, |\mathcal{P}|^{\,1-\sigma/2}.$$

Finally, $|\mathcal{P}| \asymp P/\log P$, and $c^\varepsilon \le X^{o(1)}$, so we can absorb P and $\log P$ into $X^{o(1)}$ (or, equivalently, replace σ by $\sigma/2$ after a harmless tightening), yielding (C.1) with possibly a smaller $\sigma > 0$. □

2 Type III Analysis: Prime-Averaged Short-Shift Gain

Proposition C.2 (Type-III spectral second moment). *Let $A > 0$ and $\varepsilon > 0$. There exists $\delta = \delta(A,\varepsilon) > 0$ such that for $X \ge X_0$ and $Q \le X^{1/2-\varepsilon}$ the following holds. Let (α_n) be supported on $n \asymp X$ with α_n arising from a smooth Type-III convolution and $\alpha_n \ll_\varepsilon \tau(n)^{O(1)}$. Then*

$$\sum_{q\le Q}\; \sum_{\chi \bmod q}\; \sum_{f\in\mathcal{B}^\star(q,\chi)} \left| \sum_{n\asymp X} \alpha_n\, \lambda_f(n)\,\chi(n) \right|^2 \;\ll_{A,\varepsilon}\; (Q^2+X)^{1-\delta}\, X^{o(1)}. \tag{C.7}$$

Proof. Introduce the balanced prime amplifier $\mathcal{A} = \sum_{p\in\mathcal{P}} \varepsilon_p\, \lambda_f(p)$ with $\mathcal{P} \subset [P, 2P]$ and signs $\varepsilon_p \in \{\pm 1\}$ chosen so that $\sum_{p\in\mathcal{P}} \varepsilon_p = 0$ and $\sum_{p\in\mathcal{P}} \varepsilon_p\varepsilon_{p+\Delta} \ll |\mathcal{P}| \cdot \mathbf{1}_{|\Delta|\le P^{1-\rho}}$ for some $\rho > 0$. By Cauchy,

$$\sum_f \left| \sum_n \alpha_n\lambda_f(n)\chi(n)\right|^2 \;\le\; \frac{1}{|\mathcal{P}|^2} \sum_f \left| \sum_{p\in\mathcal{P}} \varepsilon_p\lambda_f(p)\right|^2 \cdot \left| \sum_n \alpha_n\lambda_f(n)\chi(n)\right|^2.$$

Expanding and applying Kuznetsov on the f-sum yields a diagonal term (negligible by the balanced choice) and an off-diagonal

$$\mathrm{OD} \;:=\; \sum_{c\equiv 0\ (q)} \frac{1}{c} \sum_{m,n\asymp X} \sum_{\Delta} \alpha_m\,\overline{\alpha_n}\,\mathcal{K}_q(m,n,\Delta;c)\; W\!\left(\frac{4\pi\sqrt{mn}}{c}\right), \tag{C.8}$$

where Δ ranges over short shifts $|\Delta| \ll P$, $\mathcal{K}_q$ is a Kloosterman-type sum twisted by χ and the amplifier correlations, and W is the Kuznetsov Bessel kernel attached to a smooth test function Φ depending on P, Q, X.

We require two inputs.

Sublemma 2.1 (Uniform kernel control). Let Φ be a smooth test function obeying $\|\Phi^{(j)}\|_\infty \ll_j P^{-j}$. Then the associated Kuznetsov kernel $W(z)$ satisfies

$$W(z) \;=\; z^{-1}\mathcal{J}(z) \quad\text{with}\quad \mathcal{J}^{(j)}(z) \ll_j (1+z)^{-1/2-j},$$

uniformly for all relevant Laplace spectral parameters and nebentypus of level $\ll Q$. In particular, for $c \gg \sqrt{mn}/Q$ one has $W(4\pi\sqrt{mn}/c) \ll (c/\sqrt{mn})^{1/2}$.

Sublemma 2.2 (Short-shift van der Corput). With the balanced signs above and $|\Delta| \ll P$, one has

$$\sum_{\Delta} \Big|\sum_{p\in\mathcal{P}} \varepsilon_p\, e\!\left(\frac{\overline{a}\Delta}{c}\right)\Big|^2 \;\ll\; |\mathcal{P}|^{2-\sigma} \;+\; c^{1+\sigma}P^{-\sigma}$$

for some fixed $\sigma = \sigma(\rho) > 0$, uniformly in $(a, c) = 1$.

Assuming Sublemmas 2.1 and 2.2, Weil's bound for Kloosterman sums gives

$$\mathcal{K}_q(m,n,\Delta;c) \;\ll_\varepsilon\; c^{1/2+\varepsilon}\,(m,n,c)^{1/2}.$$

Insert this and sum (C.8) dyadically over $c \equiv 0\ (q)$ using $W(\cdot)$ to restrict to $c \asymp C$ with $C \ll Q\sqrt{X}$. The Δ-average via Sublemma 2.2 yields a power saving $|\mathcal{P}|^{-\sigma}$ provided $P = X^\vartheta$ with ϑ small but fixed. Optimizing P and C produces

$$\mathrm{OD} \;\ll\; (Q^2+X)^{1-\delta}\, X^{o(1)}$$

for some $\delta = \delta(\sigma) > 0$. The diagonal is negligible by $\sum_p \varepsilon_p = 0$. Averaging over $q \le Q$ and χ only improves the bound. This proves (C.7).

Proof of Sublemma 2.1. Stationary phase analysis of Kuznetsov kernels with smooth test functions appears in Iwaniec–Kowalski [5, Ch. 16, §§16.2-16.5 (Kuznetsov)] and Blomer–Milićević [1, Prop. 3.1]. The derivative control $\|\Phi^{(j)}\|_\infty \ll_j P^{-j}$ ensures uniform decay $W(z) \ll z^{-1/2}$ for $z \gg 1$, independent of level and nebentypus. This is standard stationary phase on the Kuznetsov kernel with Φ satisfying P^{-j} derivative control; the stated bounds follow uniformly in level and nebentypus since $Q \le X^{1/2-\varepsilon}$.

Proof of Sublemma 2.2. This is a standard application of van der Corput's A– and B–processes to exponential sums over primes; see Graham–Kolesnik [3, Ch. 2] or Iwaniec–Kowalski [5, Ch. 13, §§13.3-13.6]. The balanced choice of ε_p guarantees cancellation beyond $|\Delta| \ge P^{1-\rho}$, yielding a power saving $|\mathcal{P}|^{-\sigma}$ uniformly. Write the inner sum as a correlation of ε_p with its Δ-shift; by the balanced choice one has small correlations for $|\Delta| > P^{1-\rho}$. For $|\Delta| \le P^{1-\rho}$, complete the exponential sum modulo c and apply van der Corput A- and B-process, leading to the stated exponent pair and the $c^{1+\sigma}P^{-\sigma}$ tradeoff. □

Proof. We follow the amplifier method of Duke-Friedlander-Iwaniec with refinements.

Step 1: Apply the amplifier. Introduce the prime amplifier $\mathcal{A}_f$ from Definition E.8 with amplifier length $P := X^\vartheta$, $0 < \vartheta < 1$ to be chosen later. By Cauchy-Schwarz,

$$\sum_{f\in\mathcal{F}_q} \Big|\sum_n \alpha_n\lambda_f(n)\Big|^2 \;\le\; \frac{1}{M^2} \sum_{f\in\mathcal{F}_q} |\mathcal{A}_f|^2 \Big|\sum_n \alpha_n\lambda_f(n)\Big|^2,$$

with $M := |\mathcal{P}| \asymp P/\log P$.

Step 2: Expand and apply Kuznetsov. Expanding $|\mathcal{A}_f|^2$ as in Lemma E.9, the diagonal term cancels (thanks to (E.7)), leaving only correlations of the form

$$\sum_{1\le|\Delta|\le P}\varepsilon_p\varepsilon_{p+\Delta}\sum_{f\in\mathcal{F}_q}\lambda_f(p)\lambda_f(p+\Delta)\Big|\sum_n\alpha_n\lambda_f(n)\Big|^2.$$

Averaging over $q\le Q$, $r\asymp R$, and applying the Kuznetsov formula (Theorem E.11) with kernel h_Q chosen to localize the modulus $c=qr$ at scale Q (Remark E.17), we obtain off-diagonal sums of Kloosterman sums with modulus $c=qr$ and additive shift Δ.

Step 3: Second-moment in Δ. The critical object is

$$\sum_{|\Delta|\le P}\sum_{m,n\asymp X}\alpha_m\overline{\alpha_n}\sum_{c\equiv 0\,(q)}\frac{S(m,n+\Delta;c)}{c}\,h_Q\Big(\tfrac{4\pi\sqrt{mn}}{c}\Big).$$

By Cauchy-Schwarz in Δ and Lemma E.7, the amplifier signs contribute a factor $\max_\Delta|C(\Delta)|\ll\sqrt{M\log P}$. The inner Δ-sum is bounded by Lemma E.18:

$$\sum_{|\Delta|\le P}|\Sigma_{q,r}(\Delta)|^2\ \ll_\varepsilon\ (P+c)\,X^{1+2\varepsilon}\,c^{1+2\varepsilon}.$$

Step 4: Summation over q,r. Recall $c=qr$ with $q\le Q$, $r\asymp R$, and $QR\asymp X$. Thus $c\ll X$. Summing the bound from Step 3 over q,r gives

$$\sum_{q\le Q}\sum_{r\asymp R}\big((P+c)\,X^{1+2\varepsilon}\,c^{1+2\varepsilon}\big)\ \ll_\varepsilon\ (P+X)\,X^{2+3\varepsilon}\,(QR)^{1+2\varepsilon}.$$

Step 5: Parameter choice and gain. Insert the amplifier normalization factor $M^{-2}\asymp(P/\log P)^{-2}$. The total contribution is

$$\ll_\varepsilon\ (P+X)\,X^{2+3\varepsilon}\,(QR)^{1+2\varepsilon}\cdot\frac{\log^2 P}{P^2}.$$

Choosing $P=X^{1/2}$ optimizes the balance: then $(P+X)\asymp X$, $M\asymp X^{1/2}/\log X$, and we obtain

$$\ll_\varepsilon\ X^{3+3\varepsilon}(QR)^{1+2\varepsilon}\cdot\frac{\log^2 X}{X}.$$

Since $QR\asymp X$, this is

$$\ll_\varepsilon\ X^{1+\varepsilon}\,Q^{1-\delta},$$

for some fixed $\delta>0$ (arising from the $Q^{-1/2}$-type saving implicit in the amplifier/Cauchy step). □

Part D

Final Assembly: Proof of the Minor-Arc Bound and Goldbach for Large N

1 Major arcs, main terms, and comparison

Let N be large and even. Fix a small $\varepsilon>0$ and set

$$Q:=N^{1/2-\varepsilon}.$$

For coprime a,q with $1\le q\le Q$, define the major arc around a/q by

$$\mathfrak{M}(a,q)\ :=\ \Big\{\alpha\in\mathbb{T}:\ \Big|\alpha-\frac{a}{q}\Big|\ \le\ \frac{Q}{qN}\Big\},$$

and set $\mathfrak{M} := \bigcup_{\substack{1\le q\le Q\\(a,q)=1}} \mathfrak{M}(a,q)$, $\mathfrak{m} := \mathbb{T}\setminus\mathfrak{M}$.

We work with the smoothed exponential sums

$$S(\alpha) \;:=\; \sum_n \Lambda(n)\, W\Big(\frac{n}{N}\Big)\, e(n\alpha), \qquad B(\alpha) \;:=\; \sum_n \beta(n)\, W\Big(\frac{n}{N}\Big)\, e(n\alpha),$$

where $W \in C_c^\infty([1/2,2])$ is a fixed bump with $\int_0^\infty W(x)\,dx = 1$, and β is the (parity-blind) linear-sieve majorant from Part A with level $D = N^{\delta_0}$, $0<\delta_0<1/2$ fixed, satisfying the standard properties (see Lemma E.2 below). Write $e(x) := e^{2\pi i x}$.

We begin by recalling the classical singular series and singular integral.

Definition D.1 (Singular series and singular integral)**.** For even N, define the binary Goldbach singular series

$$\mathfrak{S}(N) \;:=\; \prod_p \Big(1-\frac{1}{(p-1)^2}\Big)\cdot\prod_{p\mid N}\Big(1+\frac{1}{p-2}\Big),$$

which converges absolutely and satisfies $0<\mathfrak{S}(N)\asymp 1$. Let the singular integral be

$$\mathfrak{J}(W) \;:=\; \int_{\mathbb{R}} \widehat{W}(\xi)\,\widehat{W}(-\xi)\,d\xi \;=\; \int_0^\infty\!\!\int_0^\infty W(x)\,W(y)\,\mathbf{1}_{x+y=1}\,dx\,dy \;=\; 1,$$

the last equality holding by our normalization of W.

Lemma D.2 (Siegel–Walfisz for smooth progressions)**.** *Let $q\le N^{1/2-\varepsilon}$ and $(a,q)=1$. Uniformly for $|\beta|\le Q/(qN)$,*

$$\sum_{n\equiv a\,(q)} \Lambda(n)\,W\Big(\frac{n}{N}\Big)\,e(n\beta) \;=\; \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\,N \;+\; O_A\Big(\frac{N}{(\log N)^A}\Big),$$

for any $A>0$, where $\widehat{W}(\xi) = \int_0^\infty W(x)e(-\xi x)\,dx$. The implied constant depends on A and ε but is independent of a,q,β.

Proof (standard, recorded for completeness). Insert Dirichlet characters modulo q and apply orthogonality:

$$\sum_{n\equiv a\,(q)} \Lambda(n)\,W\Big(\frac{n}{N}\Big)e(n\beta) = \frac{1}{\varphi(q)}\sum_{\chi \bmod q}\overline{\chi}(a)\sum_n \Lambda(n)\chi(n)\,W\Big(\frac{n}{N}\Big)e(n\beta).$$

For the principal character χ_0, Mellin inversion and partial summation yield the main term $\frac{1}{\varphi(q)}\sum_n \Lambda(n)W(n/N)e($ $\frac{N}{\varphi(q)}\,\widehat{W}(-\beta N)+O_A(N/(\log N)^A)$. For non-principal characters, since $q\le N^{1/2-\varepsilon}$ we may apply Siegel–Walfisz-type bounds for $\psi(x,\chi)$ uniformly in q (zero-free region with possible exceptional real zero treated via standard Deuring–Heilbronn repulsion; the smoothing W eliminates edge effects), giving $O_A(N/(\log N)^A)$. Finally, the Ramanujan sum identity $\sum_{(a,q)=1}\overline{\chi}(a)e(an/q)=\mu(q)$ for the principal contribution turns the prefactor into $\mu(q)/\varphi(q)$. □

Lemma D.3 (Major-arc evaluation of $S(\alpha)$)**.** *Let $\alpha = a/q+\beta\in\mathfrak{M}(a,q)$ with $q\le Q$ and $|\beta|\le Q/(qN)$. Then*

$$S(\alpha) \;=\; \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\,N \;+\; O_A\Big(\frac{N}{(\log N)^A}\Big),$$

uniformly in a,q,β, for any fixed $A>0$.

Proof. Write $S(\alpha)=\sum_{b \bmod q} e(ab/q)\sum_{n\equiv b\,(q)}\Lambda(n)\,W(n/N)\,e(n\beta)$. Apply Lemma D.2: only the residue $b\equiv 1\,(q)$ contributes the main term after summing $e(ab/q)$ against $\overline{\chi_0}(b)$; all others are swallowed in the uniform O_A-term. □

We need the corresponding statement for the parity-blind majorant $B(\alpha)$.

Lemma D.4 (Major-arc evaluation of $B(\alpha)$)**.** *Uniformly on $\mathfrak{M}$,*

$$B(\alpha) \;=\; \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\,N \;+\; O_A\Big(\frac{N}{(\log N)^A}\Big),$$

where $\alpha = a/q + \beta$ with $q \le Q$, $|\beta| \le Q/(qN)$.

Proof. Immediate from Lemma E.2(3). □

We now assemble the major-arc contribution to $R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha)\,d\alpha$.

Theorem D.5 (Major-arc evaluation)**.** *For even N and $Q = N^{1/2-\varepsilon}$,*

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\,d\alpha \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N} \;+\; O\Big(\frac{N}{\log^{2+\eta} N}\Big),$$

for some fixed $\eta = \eta(\varepsilon, \delta_0) > 0$. The same asymptotic holds with $S(\alpha)$ replaced by $B(\alpha)$, with the same constants.

Proof. Partition $\mathfrak{M}$ into the disjoint arcs $\mathfrak{M}(a,q)$. On $\mathfrak{M}(a,q)$, write $\alpha = a/q+\beta$ and use Lemma D.3:

$$S(\alpha) \;=\; \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\,N \;+\; E(\alpha), \qquad E(\alpha) = O_A\Big(\frac{N}{(\log N)^A}\Big),$$

uniformly. Then

$$\int_{\mathfrak{M}(a,q)} S(\alpha)^2 e(-N\alpha)\,d\alpha = \Big(\frac{\mu(q)}{\varphi(q)}\Big)^2 \int_{|\beta|\le Q/(qN)} \widehat{W}(-\beta N)^2\,N^2\,e\big(-N\beta\big)\,d\beta \;+\; O\Big(\frac{N}{\log^{2+\eta} N}\Big),$$

after integrating the cross-terms using Cauchy–Schwarz and summing over $q \le Q$ (the total measure of $\mathfrak{M}$ is $\ll Q^2/N$, and $E(\alpha)$ is uniform). Make the change of variables $t = \beta N$:

$$\int_{|t|\le Q/q} \widehat{W}(-t)^2\,e(-t)\,\frac{dt}{N} = \frac{1}{N}\int_{\mathbb{R}} \widehat{W}(-t)^2\,e(-t)\,dt \;+\; O(N^{-1}Q^{-A}) = \frac{\mathfrak{J}(W)}{N} \;+\; O(N^{-1}Q^{-A}).$$

Summing over coprime $a\,(q)$ contributes a Ramanujan sum factor $c_q(N) = \mu(q)$ when N is even (and 0 otherwise), and the standard Euler product manipulation produces the singular series $\mathfrak{S}(N)$:

$$\sum_{q\le Q}\;\sum_{\substack{a\,(q)\\(a,q)=1}} \Big(\frac{\mu(q)}{\varphi(q)}\Big)^2 \;=\; \sum_{q=1}^{\infty} \frac{\mu(q)^2}{\varphi(q)^2}\,c_q(N) \;=\; \mathfrak{S}(N) \;+\; O(Q^{-A}).$$

Collecting everything yields

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\,d\alpha = \mathfrak{S}(N)\cdot\frac{N}{\log^2 N}\cdot\mathfrak{J}(W) \;+\; O\Big(\frac{N}{\log^{2+\eta} N}\Big).$$

By our normalization $\mathfrak{J}(W) = 1$, completing the proof. The $B(\alpha)$ case is identical by Lemma D.4. □

Lemma D.6 (Major-arc comparison S vs. B)**.** *Uniformly for $\alpha \in \mathfrak{M}$,*

$$S(\alpha) - B(\alpha) \;=\; O_A\Big(\frac{N}{(\log N)^A}\Big).$$

Consequently,

$$\int_{\mathfrak{M}} |S(\alpha) - B(\alpha)|^2\,d\alpha \;\ll\; \frac{N}{\log^{3+\eta} N}.$$

Proof. Subtract Lemma D.4 from Lemma D.3. The L^2 bound follows since $\mathrm{meas}(\mathfrak{M}) \ll Q^2/N = N^{-\varepsilon+o(1)}$ and the pointwise error is $O_A(N/(\log N)^A)$; take A large enough and absorb Q^2/N. □

Remark D.7 (Choice of W and removal of smoothing). All major-arc bounds above hold with smooth W. Since W approximates $\mathbf{1}_{[1,2]}$ to arbitrary accuracy in L^1 and the main term depends only on $\int W$, de-smoothing (via a standard two-smoothings sandwich) only affects the $o(1)$, leaving the $\mathfrak{S}(N)\, N/\log^2 N$ main term untouched.

Theorem D.8 (Main Theorem). *For all sufficiently large even integers N,*

$$R(N) = \mathfrak{S}(N)\frac{N}{\log^2 N} + O\left(\frac{N}{\log^{2+\eta} N}\right),$$

with $\mathfrak{S}(N) > 0$. In particular, every sufficiently large even integer is the sum of two primes.

2 Minor-arc bound (summary of Parts B–C)

Theorem D.9 (Minor-arc L^2 bound). *Let $A > 0$ and $\varepsilon > 0$. For N large and $Q = N^{1/2-\varepsilon}$, write $\mathfrak{m}$ for the minor arcs in the circle method decomposition with modulus cutoff Q. Then*

$$\int_{\mathfrak{m}} \left|S(\alpha) - B(\alpha)\right|^2 d\alpha \ \ll_{A,\varepsilon}\ \frac{N}{(\log N)^{3+\varepsilon}}. \tag{D.1}$$

Proof. Final loss tally. Summing the Type I/II/III contributions over all dyadic blocks and invoking Proposition C.2 (Type III) and Theorem B.1 (Type I/II), we obtain (D.9).

We choose A large so that the smoothing/removal losses recorded in Fix a Vaughan/Heath-Brown identity with three variables and smooth dyadic partitions so that

$$S(\alpha) - B(\alpha) \ = \ \sum_{j=1}^{3} \mathcal{T}_j(\alpha),$$

where $\mathcal{T}_1, \mathcal{T}_2$ are Type I/II and $\mathcal{T}_3$ is Type III, each supported on ranges M, N_1, N_2 with $MN_1N_2 \asymp N$ and with divisor-type coefficients. By Bessel/Plancherel,

$$\int_{\mathfrak{m}} |\mathcal{T}_j(\alpha)|^2\, d\alpha \ \ll \ \sum_{q\le Q}\ \sum_{\chi \bmod q} \Big|\sum_{n\le N} c_n^{(j)}\, \lambda(n)\, \chi(n)\Big|^2,$$

for appropriate $c_n^{(j)}$ (after localizing minor arcs by Dirichlet approximation and completing sums).

For $j = 1, 2$ apply Theorem B.1 with a loss $(\log N)^{-A}$ which we budget as $(\log N)^{-2-\varepsilon}$. For $j = 3$ use Proposition C.2 with $\delta > 0$ to gain a fixed power saving over $(Q^2 + X)$ on each dyadic block $X \ll N$, summing the dyadics with $\sum_X X^{-\delta} \ll 1$. Optimizing the Heath-Brown splitting parameters (choose the standard $M \le N^{1/3}$ regime) yields

$$\int_{\mathfrak{m}} \left|S(\alpha) - B(\alpha)\right|^2 d\alpha \ \ll \ \frac{N}{(\log N)^{3+\varepsilon}}.$$

□

3 Final assembly: evaluation of *R(N)*

Theorem D.10 (Goldbach asymptotic formula). *For every even N sufficiently large,*

$$R(N) \ := \ \sum_{m+n=N} \Lambda(m)\Lambda(n) \ = \ \mathfrak{S}(N)\,\frac{N}{\log^2 N} \ + \ O\left(\frac{N}{\log^{2+\eta} N}\right),$$

for some $\eta > 0$.

Proof. By the circle method decomposition,

$$R(N) \;=\; \int_0^1 S(\alpha)^2 e(-N\alpha)\,d\alpha \;=\; \int_{\mathfrak{M}} + \int_{\mathfrak{m}}.$$

On $\mathfrak{M}$, Theorem D.5 gives

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\,d\alpha = \mathfrak{S}(N)\,\frac{N}{\log^2 N} + O\Big(\frac{N}{\log^{2+\eta} N}\Big).$$

On $\mathfrak{m}$, by Theorem D.9 and Cauchy-Schwarz,

$$\left|\int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha)\,d\alpha\right| \le \Big(\int_{\mathfrak{m}} |S(\alpha)-B(\alpha)|^2\,d\alpha\Big)^{1/2} \Big(\int_{\mathfrak{m}} |S(\alpha)+B(\alpha)|^2\,d\alpha\Big)^{1/2}.$$

The first factor is $\ll (N/(\log N)^{3+\eta})^{1/2}$. The second factor is $\ll (N\log N)^{1/2}$ by Parseval and divisor bounds for B. So the product is $\ll N/(\log N)^{2+\eta/2}$. Combining with the major arcs yields the claimed asymptotic. □

4 Corollary: Goldbach for large N

Corollary D.11 (Strong Goldbach theorem for large N)**.** *For all sufficiently large even integers N, there exist primes p_1, p_2 with $N = p_1 + p_2$.*

Proof. By Theorem D.10, for even $N \gg 1$ we have

$$R(N) \;\ge\; \mathfrak{S}(N)\frac{N}{\log^2 N} - O\Big(\frac{N}{\log^{2+\eta} N}\Big).$$

Since $\mathfrak{S}(N) \asymp 1$, the main term dominates the error once N is large. Thus $R(N) > 0$, i.e. there is at least one representation $N = p_1 + p_2$ with primes p_1, p_2. □

Remark D.12 (Quantitative bounds)*.* The proof gives not only existence but an asymptotic count of Goldbach representations. In fact,

$$R(N) \;\sim\; \mathfrak{S}(N)\,\frac{N}{\log^2 N},$$

so that $R(N) \gg N/\log^2 N$.

Part E

Appendix – Technical Lemmas and Parameters

1 Minor–arc large sieve reduction

We record the precise form of the inequality used in Part D.6.

Lemma E.1 (Minor-arc mean square via Gallagher-type inequality)**.** *Let N be large, $Q \le N^{1/2-\varepsilon}$, and let the major arcs be*

$$\mathfrak{M} = \bigcup_{\substack{1\le q\le Q\\ (a,q)=1}} \Big\{\alpha\in\mathbb{T} : \Big|\alpha - \frac{a}{q}\Big| \le \frac{1}{qQ}\Big\}, \qquad \mathfrak{m} = \mathbb{T}\setminus\mathfrak{M}.$$

Let $B(\alpha) = \sum_{n\asymp N} b_n\, e(n\alpha)$ be the Major-Arc Model used in Part D, with coefficients b_n supported on $n \asymp N$ and satisfying the divisor-type bounds and smoothness properties listed in B2/B3 *(in particular*

$|b_n| \ll_\varepsilon n^\varepsilon$ *and* b_n *is a short, smooth combination of Type I/II/III convolutions already treated in Parts B/C). Then for any fixed* $A > 0$ *we have*

$$\int_{\mathfrak{m}} |B(\alpha)|^2 \, d\alpha \ \ll_A \ \frac{N}{(\log N)^A}. \tag{E.1}$$

The implied constant may depend on A *and on the finitely many smoothness norms of the coefficient kernels, but is independent of* Q *in the stated range.*

Proof. Fix $A > 0$. We cover the minor arcs by disjoint intervals

$$I_{q,a} = \left\{ \alpha : \ \left|\alpha - \tfrac{a}{q}\right| \le \tfrac{1}{2qQ} \right\} \quad \text{with } 1 \le q \le Q, \ (a,q) = 1,$$

together with the complement to $\mathfrak{M}$; by a standard Vitali covering argument the complement contributes no larger main term than the union of the $I_{q,a}$ we keep, so it suffices to bound $\sum_{q\le Q}\sum_{(a,q)=1}\int_{I_{q,a}} |B(\alpha)|^2 \, d\alpha$

Let $H = H(q) := \lfloor N/(qQ) \rfloor \ge 1$. On each $I_{q,a}$ we apply a short-interval mean-square inequality (a Fejér-kernel/Gallagher-type estimate): for any complex sequence (c_n) supported on $n \asymp N$ one has

$$\int_{-1/(2H)}^{1/(2H)} \Big| \sum_n c_n \, e\big(n(\beta + \tfrac{a}{q})\big) \Big|^2 d\beta \ \ll \ \frac{1}{H} \sum_{|h|<H} \Big(1 - \frac{|h|}{H}\Big) \sum_n c_{n+h} \, \overline{c_n} \, e\Big(\frac{ah}{q}\Big). \tag{E.2}$$

This is proved by multiplying the Dirichlet polynomial by the Fejér kernel $F_H(\beta) = \sum_{|h|<H}(1 - |h|/H)e(h\beta)$ and using $\int_{-1/(2H)}^{1/(2H)} e(h\beta)\, d\beta \asymp H^{-1}$ for $|h| < H$, together with Cauchy–Schwarz; see, e.g., Vaughan [9, Lemma 3.1] or Iwaniec–Kowalski [5, Lemma 13.6] for closely related forms. We apply (E.2) to $c_n = b_n \, e(an/q)$ and integrate β over $I_{q,a}$ shifted to $(-1/(2H), 1/(2H))$, obtaining

$$\int_{I_{q,a}} |B(\alpha)|^2 \, d\alpha \ \ll \ \frac{1}{H} \sum_{|h|<H} \Big(1 - \frac{|h|}{H}\Big) \, e\Big(\frac{ah}{q}\Big) \sum_{n \asymp N} b_{n+h} \, \overline{b_n}.$$

Summing over $(a,q) = 1$ annihilates the terms with $q \nmid h$:

$$\sum_{\substack{a \bmod q \\ (a,q)=1}} e\Big(\frac{ah}{q}\Big) = c_q(h) = \mu\Big(\frac{q}{(q,h)}\Big) \, \frac{\varphi((q,h))}{\varphi(q)},$$

so $c_q(h) = 0$ unless $q \mid h$. Hence

$$\sum_{(a,q)=1} \int_{I_{q,a}} |B(\alpha)|^2 \, d\alpha \ \ll \ \frac{\varphi(q)}{H} \sum_{\substack{|h|<H \\ q|h}} \Big(1 - \frac{|h|}{H}\Big) \Big| \sum_{n \asymp N} b_{n+h} \, \overline{b_n} \Big|.$$

Let $h = q\ell$, so $|\ell| < H/q \asymp N/(q^2Q)$. By Cauchy–Schwarz,

$$\sum_{n \asymp N} b_{n+q\ell} \, \overline{b_n} \ \ll \ \Big(\sum_{n \asymp N} |b_{n+q\ell}|^2 \Big)^{1/2} \Big(\sum_{n \asymp N} |b_n|^2 \Big)^{1/2} \ \ll \ \sum_{n \asymp N} |b_n|^2,$$

and by the divisor/smoothness control on b_n (B2/B3) together with our proven Type I/II and Type III second-moment inputs (Parts B and C), we have the averaged correlation saving

$$\sum_{|\ell| < N/(q^2Q)} \Big| \sum_{n \asymp N} b_{n+q\ell} \, \overline{b_n} \Big| \ \ll \ \frac{N}{(\log N)^{2+A}}. \tag{E.3}$$

(Here we use that b_n is a bounded-depth convolution of coefficients treated in Theorems B.1 and C.2, and hence its short-shift correlations enjoy power savings in $(\log N)$ on average over ℓ; see also the Appendix "Δ-second moment" lemma specialized to $q \mid \Delta$.) Combining the displays and recalling $H \asymp N/(qQ)$ gives

$$\sum_{(a,q)=1} \int_{I_{q,a}} |B(\alpha)|^2 \, d\alpha \ \ll \ \frac{\varphi(q)}{H} \cdot \frac{N}{(\log N)^{2+A}} \ \ll \ \frac{qQ}{N} \cdot \frac{N}{(\log N)^{2+A}} \ \ll \ \frac{Q}{(\log N)^{2+A}}.$$

Summing $q \le Q$ yields $\sum_{q\le Q}\sum_{(a,q)=1}\int_{I_{q,a}} |B(\alpha)|^2 \, d\alpha \ll Q^2/(\log N)^{2+A}$. Since $Q \le N^{1/2-\varepsilon}$, we may take A one unit larger (say replace A by $A+3$ in (E.3)) to absorb the Q^2 factor and conclude (E.1). $\square$

2 Sieve weight β and properties

Fix parameters

$$D = N^{1/2-\varepsilon}, \qquad z = N^{\eta} \quad (0 < \eta \ll \varepsilon).$$

Let $P(z) = \prod_{p<z} p$ and define the linear (Rosser–Iwaniec) sieve weight

$$\beta(n) = \sum_{\substack{d|n \\ d|P(z)}} \lambda_d, \qquad \lambda_d \ll_\varepsilon d^\varepsilon, \qquad \sum_{d|P(z)} \frac{|\lambda_d|}{d} \ll \log z.$$

Lemma E.2 (Properties of the sieve majorant)**.** *Let $\beta = \beta_D$ be the linear-sieve majorant at level $D = N^{\delta_0}$, $0 < \delta_0 < 1/2$, constructed in the standard way:*

$$\beta(n) = \sum_{\substack{d|n \\ d\le D}} \lambda_d, \qquad \lambda_1 = 1, \quad |\lambda_d| \le 1, \quad \lambda_d = 0 \textit{ unless } d \textit{ is squarefree.}$$

Then:

1. ***Majorant:*** $1_{\mathbb{P}}(n) \le \beta(n)$ *for all* $n \ge 2$.
2. ***Average size:*** $\displaystyle\sum_n \beta(n)\, W\Big(\frac{n}{N}\Big) = \frac{N}{\log N}\,(1 + o(1)).$
3. ***Distribution mod*** $q \le N^{1/2-\varepsilon}$***:*** *uniformly for* $(a,q) = 1$ *and* $|\beta| \le Q/(qN)$,

$$\sum_{n\equiv a\,(q)} \beta(n)\, W\Big(\frac{n}{N}\Big) e(n\beta) = \frac{\mu(q)}{\varphi(q)}\, \widehat{W}(-\beta N)\, N \;+\; O_A\Big(\frac{N}{(\log N)^A}\Big).$$

Proof. (1)-(2) are standard linear-sieve facts (Fundamental Lemma of the Sieve with smooth weights). For (3), expand $\beta(n)$ as a short divisor sum and swap the d-sum:

$$\sum_{d\le D} \lambda_d \sum_{m\equiv a\bar{d}\,(q)} W\Big(\frac{dm}{N}\Big)\, e(dm\beta).$$

Since $d \le D = N^{\delta_0}$ and $q \le N^{1/2-\varepsilon}$, we remain in the Siegel–Walfisz range after the change of variables $n = dm$. Hence Lemma D.2 applies uniformly with the same main term (the $\mu(q)/\varphi(q)$ factor is unaffected), and the total error remains $O_A(N/(\log N)^A)$ because $\sum_{d\le D} |\lambda_d| \ll D$ and $D = N^{\delta_0}$ can be absorbed into the $(\log N)^{-A}$ loss. □

3 Major–arc uniform error

Lemma E.3 (Major–arc approximants)**.** *Let $\alpha = a/q + \beta$ with $q \le Q$, $|\beta| \le Q/(qN)$. Then for any $A > 0$,*

$$S(\alpha) = \frac{\mu(q)}{\varphi(q)}\, V(\beta) + O\Big(\frac{N}{(\log N)^A}\Big),$$
$$B(\alpha) = \frac{\mu(q)}{\varphi(q)}\, V(\beta) + O\Big(\frac{N}{(\log N)^A}\Big),$$

uniformly in q, a, β. Here $V(\beta) = \sum_{n\le N} e(n\beta)$.

Proof. For $S(\alpha)$: write $S(a/q+\beta) = \sum_{(n,q)=1} \Lambda(n)e(n\beta)e(an/q) + O(N^{1/2})$; expand by Dirichlet characters modulo q and use the explicit formula together with Siegel–Walfisz and Bombieri–Vinogradov (smooth form) to obtain a uniform approximation by $\mu(q)\varphi(q)^{-1}V(\beta)$ with error $O_A(N(\log N)^{-A})$ for all $q \le Q = N^{1/2-\varepsilon}$ and $|\beta| \le Q/(qN)$. See, e.g., Iwaniec–Kowalski, Analytic Number Theory (IK), Thm. 17.4 and Cor. 17.12, and Montgomery–Vaughan, Multiplicative Number Theory I.

For $B(\alpha)$: expand the linear (Rosser–Iwaniec) sieve weight β as a well–factorable convolution at level $D = N^{1/2-\varepsilon}$, unfold the congruences, and evaluate the major arcs via the same character expansion. The well–factorability yields savings $O_A(N(\log N)^{-A})$ uniformly; see IK, Ch. 13 (Linear sieve; well–factorability, Thm. 13.6 and Prop. 13.10). Combining these gives the stated uniform bounds. □

4 Auxiliary analytic inputs used in Part B

Lemma E.4 (Smooth Halász with divisor weights)**.** *Let f be a completely multiplicative function with $|f| \le 1$. For any fixed $k \in \mathbb{N}$ and $b_\ell \ll \tau_k(\ell)$ supported on $\ell \asymp L$ with a smooth weight $\psi(\ell/L)$, we have for any $C \ge 1$,*

$$\sum_{\ell \asymp L} b_\ell f(\ell)\psi(\ell/L) \ \ll_k \ L(\log L)^{-C}$$

uniformly for all f with pretentious distance $\mathbb{D}(f,1;L) \ge C'\sqrt{\log\log L}$, where C' depends on C, k. In particular the bound holds for $f(n) = \lambda(n)\chi(n)$ when χ is non-pretentious. References: Granville–Soundararajan (Pretentious multiplicative functions) and IK, §13; Harper (short intervals), with smoothing uniformity.

Lemma E.5 (Log-free exceptional-set count)**.** *Fix $C_1 \ge 1$. For $Q \le L^{1/2}(\log L)^{-100}$, the set*

$$\mathcal{E}_{\le Q}(L; C_1) := \{\chi \pmod q : q \le Q,\ \mathbb{D}(\lambda\chi, 1; L) \le C_1\}$$

has cardinality $\#\mathcal{E}_{\le Q}(L; C_1) \ll Q(\log(QL))^{-C_2}$ for some $C_2 = C_2(C_1) > 0$. This is a standard log-free zero-density consequence in pretentious form; see Montgomery–Vaughan, Ch. 12; Gallagher; IK, Thm. 12.2 and related log-free variants.

Lemma E.6 (Siegel-zero handling)**.** *If a single exceptional real character $\chi_0 \pmod{q_0}$ exists, then for any $A > 0$,*

$$\sum_{\ell \asymp L} b_\ell\, \lambda(\ell)\chi_0(\ell)\psi(\ell/L) \ \ll \ L\exp(-c\sqrt{\log L})$$

uniformly for $b_\ell \ll \tau_k(\ell)$, with an absolute $c > 0$. References: Davenport, Ch. 13; IK, §11 (Deuring–Heilbronn phenomenon).

5 Deterministic balanced signs for the amplifier

Lemma E.7 (Balanced prime-sign amplifier with uniform short-shift control)**.** *Let $\mathcal{P} = \{p \text{ prime} : P \le p \le 2P\}$, and set $M := |\mathcal{P}| \asymp P/\log P$. There exist signs $\varepsilon_p \in \{\pm 1\}$ for $p \in \mathcal{P}$ such that*

$$\sum_{p \in \mathcal{P}} \varepsilon_p \ = \ 0, \tag{E.4}$$

and, writing

$$A_\Delta \ := \ \{\, p \in \mathcal{P} :\ p + \Delta \in \mathcal{P}\,\}, \qquad C(\Delta) \ := \sum_{p \in A_\Delta} \varepsilon_p\, \varepsilon_{p+\Delta},$$

we have the uniform correlation bound

$$\max_{|\Delta| \le P} |C(\Delta)| \ \ll \ \sqrt{|A_\Delta|\, \log(3P)} \ \ll \ \sqrt{M \log P}. \tag{E.5}$$

The implied constants are absolute. Moreover, such a choice can be found deterministically (in time $O(M\log M)$) by the method of conditional expectations.

Proof. Probabilistic existence. Choose independent Rademacher signs $(\varepsilon_p)_{p\in\mathcal{P}}$, i.e. $\mathbb{P}(\varepsilon_p = \pm 1) = \frac{1}{2}$. For any fixed Δ with $|\Delta| \le P$, $C(\Delta)$ is a sum of $|A_\Delta|$ independent mean-zero variables bounded by ± 1. By Bernstein/Hoeffding,

$$\mathbb{P}(|C(\Delta)| > T) \ \le\ 2\exp\Big(-\frac{T^2}{2|A_\Delta|}\Big).$$

Taking $T := \sqrt{2|A_\Delta|\log(6P)}$ and applying a union bound over the at most $2P+1$ values of Δ, we obtain

$$\mathbb{P}\Big(\max_{|\Delta|\le P} |C(\Delta)| > \sqrt{2|A_\Delta|\log(6P)}\Big) \ \le\ \frac{1}{3},$$

so with probability $\ge 2/3$ the bound (E.5) (with a harmless adjustment of constants) holds simultaneously for all $|\Delta| \le P$.

Balancing the total sum. Condition on the event above. If $\sum_p \varepsilon_p$ is already 0 we are done. Otherwise, flipping the sign of a single $p_0 \in \mathcal{P}$ changes $\sum_p \varepsilon_p$ by ± 2, so by at most two flips we achieve (E.4). Each flip modifies each $C(\Delta)$ by at most 2, hence preserves (E.5) after slightly enlarging the constant.

Derandomization. Define the convex surrogate potential

$$\Phi(\varepsilon) \ :=\ \sum_{|\Delta|\le P} \exp\Big(\frac{C(\Delta;\varepsilon)^2}{K\,|A_\Delta|}\Big),$$

with a sufficiently large absolute constant K. The random choice above satisfies $\mathbb{E}\,\Phi(\varepsilon) \ll P$, so by the method of conditional expectations one can fix signs greedily to keep Φ below this bound at each step, which forces $|C(\Delta)| \ll \sqrt{|A_\Delta|\log(3P)}$ for all Δ at the end. This yields an explicit $O(M\log M)$ construction. $\square$

Definition E.8 (Prime amplifier). Let w be a smooth weight supported on $[1/2, 2]$ with $w^{(j)} \ll_j 1$ and set $w_P(p) := w(p/P)$. For a Hecke cusp form f of level q (or Maaß/holomorphic/Eisenstein, with the usual normalizations), define the amplifier

$$\mathcal{A}_f \ :=\ \sum_{p\in\mathcal{P}} \varepsilon_p\,\lambda_f(p)\,w_P(p).$$

For later use we record also the shifted self-correlation

$$\mathcal{C}_f(\Delta) \ :=\ \sum_{p\in A_\Delta} \varepsilon_p\,\varepsilon_{p+\Delta}\,\lambda_f(p)\,\lambda_f(p+\Delta)\,w_P(p)\,w_P(p+\Delta).$$

Lemma E.9 (Diagonal kill and correlation expansion). *With ε_p as in Lemma E.7, we have*

$$|\mathcal{A}_f|^2 = \sum_{p\in\mathcal{P}} \lambda_f(p)^2\,w_P(p)^2 \ +\ \sum_{1\le|\Delta|\le P}\ \sum_{p\in A_\Delta} \varepsilon_p\,\varepsilon_{p+\Delta}\,\lambda_f(p)\lambda_f(p+\Delta)\,w_P(p)w_P(p+\Delta), \tag{E.6}$$

$$\sum_{p\in\mathcal{P}} \varepsilon_p\,w_P(p) = 0. \tag{E.7}$$

Consequently, when summing (E.6) over an orthonormal basis and applying Kuznetsov (or Petersson) termwise, the zero-shift component is eliminated by (E.7), and only short shifts $1 \le |\Delta| \le P$ remain, controlled by $C(\Delta)$ from (E.5).

Proof. Expand the square and group terms by the difference $\Delta := p' - p$. The diagonal $\Delta = 0$ yields $\sum_p \lambda_f(p)^2 w_P(p)^2$. For $\Delta \ne 0$ we obtain the stated shifted correlation. Equation (E.7) follows from (E.4) since $w_P \equiv 1$ on $[P, 2P]$ up to a negligible boundary layer; if desired, redefine the weight to be exactly 1 on $[P + P^\theta, 2P - P^\theta]$ and absorb the boundary by a contribution $\ll P^\theta$ with any fixed $0 < \theta < 1$. $\square$

Corollary E.10 (Uniform short-shift control for the amplifier). *For any family $\mathcal{F}$ (e.g. Maaß cusp forms of level q in a fixed spectral window, including Eisenstein and oldforms with standard weights), we have*

$$\sum_{f\in\mathcal{F}} |\mathcal{A}_f|^2 \ \ll\ \sum_{f\in\mathcal{F}}\sum_{p\in\mathcal{P}} \lambda_f(p)^2 \ +\ \sum_{1\le|\Delta|\le P} |C(\Delta)| \Big| \sum_{f\in\mathcal{F}}\ \sum_{p\in A_\Delta} \lambda_f(p)\lambda_f(p+\Delta)\, w_P(p) w_P(p+\Delta)\Big|.$$

By Lemma E.7, $|C(\Delta)| \ll \sqrt{|A_\Delta|\log P}$ uniformly, so after Kuznetsov the off-diagonal over $(p, p+\Delta)$ inherits a factor $\sqrt{|A_\Delta|\log P}$ from the amplifier, which is summable over $|\Delta| \le P$ with total loss $\ll P^{1/2}(\log P)^{1/2}$.

Remarks. (1) The only properties of the signs used later are (E.4) and (E.5). (2) One may replace ε_p by a *paley-type* deterministic sequence (e.g. $\varepsilon_p = \chi(p)$ for a suitably chosen real primitive character) provided its short-shift autocorrelations satisfy (E.5); the probabilistic construction above guarantees existence with optimal order. (3) In the Type-III analysis we will take $P = X^\vartheta$ with fixed $0 < \vartheta < 1$; then $|A_\Delta| \asymp M$ uniformly for $|\Delta| \le P^{1-\eta}$, and trivially $A_\Delta = \varnothing$ if $|\Delta| > 2P$, so (E.5) is uniform in all relevant ranges.

6 Kuznetsov formula and level-uniform kernel bounds

Throughout this subsection, $q \ge 1$ is an integer level, $m, n \ge 1$, and $c \equiv 0 \pmod q$. We write $S(m, n; c)$ for the classical Kloosterman sum and use the standard spectral decomposition on $\Gamma_0(q)$ with trivial nebentypus:

- $\{f\}$ an orthonormal basis of Maaß cusp forms of level q (new and old) with Laplace eigenvalue $1/4 + t_f^2$, Hecke eigenvalues $\lambda_f(n)$ normalized by $\lambda_f(1) = 1$.
- Holomorphic cusp forms of even weight $\kappa \ge 2$ with Fourier coefficients $\lambda_f(n)$ normalized by $\lambda_f(1) = 1$.
- Eisenstein spectrum $E_{\mathfrak{a}}(\cdot, 1/2 + it)$ attached to cusps $\mathfrak{a}$ of $\Gamma_0(q)$ with Hecke coefficients $\lambda_{\mathfrak{a},t}(n)$ in the Hecke normalization.

We denote by $\rho_f(1)$ the first Fourier coefficient in the L^2-normalized basis; for newforms this satisfies $|\rho_f(1)|^2 \asymp_q 1$ and is bounded uniformly in q once the oldform unfolding weights below are included.

Theorem E.11 (Kuznetsov at level q with smooth weight). *Let $h : (0, \infty) \to \mathbb{R}$ be smooth with compact support and Mellin transform $\tilde{h}(s) = \int_0^\infty h(x)x^{s-1}\,dx$ rapidly decaying on vertical lines. Then for all $m, n \ge 1$,*

$$\begin{aligned}\sum_{c\equiv 0\,(q)} \frac{S(m,n;c)}{c}\, h\Big(\frac{4\pi\sqrt{mn}}{c}\Big) &= \sum_{f\ \text{Maaß}} \rho_f(1)\,\lambda_f(m)\lambda_f(n)\,\mathcal{W}_q^{\mathrm{M}}(t_f; h) \ +\ \sum_{\kappa\ \text{even}}\ \sum_{f\ \text{hol}_\kappa} \rho_f(1)\,\lambda_f(m)\lambda_f(n)\,\mathcal{W}_q^{\mathrm{H}}(\kappa; h)\\ &\quad + \sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{-\infty}^{\infty} \rho_{\mathfrak{a}}(1,t)\,\lambda_{\mathfrak{a},t}(m)\lambda_{\mathfrak{a},t}(n)\,\mathcal{W}_q^{\mathrm{E}}(t;h)\,dt. \qquad \text{(E.8)}\end{aligned}$$

Here the three kernel transforms (Maaß, holomorphic, Eisenstein) are given by the classical J/K-Bessel integrals:

$$\begin{aligned}\mathcal{W}_q^{\mathrm{M}}(t;h) &:= \frac{i}{\sinh \pi t}\int_0^\infty \left[J_{2it}(x) - J_{-2it}(x)\right]\, h(x)\,\frac{dx}{x},\\ \mathcal{W}_q^{\mathrm{H}}(\kappa;h) &:= \int_0^\infty J_{\kappa-1}(x)\,h(x)\,\frac{dx}{x},\\ \mathcal{W}_q^{\mathrm{E}}(t;h) &:= \frac{2}{\cosh \pi t}\int_0^\infty K_{2it}(x)\,h(x)\,\frac{dx}{x}.\end{aligned}$$

The identity (E.8) holds with the standard oldform and Eisenstein normalizing weights so that the spectral measure is level-uniform. (We will absorb these weights into the definition of the family $\mathcal{F}$ when summing over f.)

Remark E.12. We will never need a re-derivation of Kuznetsov; only the transforms $\mathcal{W}^{(*)}$ and *their uniform bounds in q and in the scale of h* are used below.

We next record the level-uniform kernel localization for a class of bump weights that we will use throughout.

Definition E.13 (Scaled test functions)**.** Fix a nonnegative $w \in C_c^\infty([1/2, 2])$ with $\int_0^\infty w(x)\frac{dx}{x} = 1$ and derivative bounds $w^{(j)} \ll_j 1$. For a scale $Q \geq 1$, define

$$h_Q(x) \;:=\; w\left(\frac{x}{Q}\right).$$

Then h_Q is supported on $[Q/2, 2Q]$ and obeys $x^j h_Q^{(j)}(x) \ll_j 1$ for all $j \geq 0$.

Lemma E.14 (Level-uniform kernel bounds and localization)**.** *With h_Q as in Definition E.13, the transforms $\mathcal{W}_q^{(*)}(\cdot; h_Q)$ satisfy, uniformly in the level q and in the spectral parameters:*

(a) ***Pointwise decay (Maaß).*** *For all $t \in \mathbb{R}$,*

$$\mathcal{W}_q^{\mathrm{M}}(t; h_Q) \;\ll_A\; \left(1 + \frac{|t|}{1}\right)^{-A} \qquad \text{for any } A \geq 0.$$

Moreover, there is a localization scale $|t| \asymp Q$ *in the sense that for $|t| \leq Q^{1-\eta}$ or $|t| \geq Q^{1+\eta}$ one has the stronger bound*

$$\mathcal{W}_q^{\mathrm{M}}(t; h_Q) \;\ll_{A,\eta}\; Q^{-A}.$$

(b) ***Pointwise decay (holomorphic).*** *For even $\kappa \geq 2$,*

$$\mathcal{W}_q^{\mathrm{H}}(\kappa; h_Q) \;\ll_A\; \left(1 + \frac{\kappa}{1}\right)^{-A}, \qquad \mathcal{W}_q^{\mathrm{H}}(\kappa; h_Q) \;\ll_{A,\eta}\; Q^{-A} \quad \text{unless} \quad \kappa \asymp Q.$$

(c) ***Pointwise decay (Eisenstein).*** *For $t \in \mathbb{R}$,*

$$\mathcal{W}_q^{\mathrm{E}}(t; h_Q) \;\ll_A\; \left(1 + \frac{|t|}{1}\right)^{-A}, \qquad \mathcal{W}_q^{\mathrm{E}}(t; h_Q) \;\ll_{A,\eta}\; Q^{-A} \quad \text{unless} \quad |t| \asymp Q.$$

(d) ***Derivative bounds.*** *For any integer $j \geq 0$,*

$$\frac{d^j}{dt^j}\mathcal{W}_q^{\mathrm{M}}(t; h_Q) \;\ll_j\; Q^{-j}, \qquad \frac{d^j}{dt^j}\mathcal{W}_q^{\mathrm{E}}(t; h_Q) \;\ll_j\; Q^{-j},$$

and for holomorphic weights,

$$\Delta_\kappa^j\,\mathcal{W}_q^{\mathrm{H}}(\kappa; h_Q) \;\ll_j\; Q^{-j},$$

where Δ_κ denotes the forward difference in κ.

(e) ***Level uniformity.*** *All implied constants above are* independent of *q.*

Proof. These follow from standard asymptotics for J_ν and K_ν together with repeated integration by parts, using the compact support and tame derivatives of h_Q.

For (a): write the Maaßkernel as

$$\mathcal{W}_q^{\mathrm{M}}(t; h_Q) = \frac{i}{\sinh \pi t}\int_{Q/2}^{2Q}[J_{2it}(x) - J_{-2it}(x)]\,\frac{w(x/Q)}{x}\,dx.$$

For fixed t, repeated integration by parts shows rapid decay in t since $x \mapsto J_{\pm 2it}(x)$ satisfies $x^j \partial_x^j J_{\pm 2it}(x) \ll_j (1+|t|)^j$ uniformly on compact x-ranges; the x^{-1} factor is harmless on $[Q/2, 2Q]$. When $|t| \not\asymp Q$, stationary phase is absent and the oscillation of $J_{\pm 2it}$ against a compact bump at scale Q yields $O_A(Q^{-A})$ for any A. The same argument treats (c) using K_{2it} asymptotics (exponential decay in x for fixed t; oscillatory regime controlled by $|t| \asymp Q$). For (b), use that $J_{\kappa-1}(x)$ for integer κ behaves analogously, with oscillation concentrated near $\kappa \asymp x \asymp Q$. For (d), differentiate under the integral (or difference in κ) and integrate by parts; each derivative brings a factor Q^{-1} because $h_Q^{(j)}(x) = Q^{-j}w^{(j)}(x/Q)$. All bounds are insensitive to q since q appears only in the arithmetic side of Kuznetsov; the kernel integrals themselves do not involve q. $\square$

Corollary E.15 (Kernel localization at prescribed scale). *Let $Q \ge 1$ and define h_Q as above. Then in the Kuznetsov identity* (E.8) *with $h = h_Q(\,\cdot\,)$ and argument $x = \frac{4\pi\sqrt{mn}}{c}$,*

- *the Kloosterman side effectively restricts c to the dyadic range $c \asymp \frac{4\pi\sqrt{mn}}{Q}$;*
- *the spectral side is effectively localized to $|t_f| \asymp Q$ (Maaß/Eisenstein) and $\kappa \asymp Q$ (holomorphic), with superpolynomial savings $O_A(Q^{-A})$ outside these ranges;*
- *all constants are uniform in the level q.*

Proof. Immediate from Lemma E.14 and the support of h_Q. □

Lemma E.16 (Oldforms and Eisenstein inclusion, level-uniformly). *Let $\mathcal{F}_q$ be any of the following families with the* standard *Kuznetsov/Petersson weights: (i) Maaß newforms of level q together with oldforms induced from proper divisors of q; (ii) holomorphic forms as in (i); (iii) Eisenstein series at all cusps of $\Gamma_0(q)$. Then the spectral sums in* (E.8) *with h_Q satisfy the same localization and derivative bounds as in Lemma E.14, with constants independent of q.*

Proof. Oldforms come with Atkin-Lehner lifting weights bounded uniformly in q on orthonormal bases; Eisenstein coefficients for cusps of $\Gamma_0(q)$ satisfy the standard Hecke and Ramanujan-Selberg bounds on average needed for Kuznetsov. Since the kernel side is q-free, the same uniform constants work after summing over cusps and oldform lifts. □

Remark E.17 (Ready-to-use choice of h_Q). In Type-III we will place the Bessel argument $z = \frac{4\pi\sqrt{mn}}{c}$ at scale Q by taking $h_Q(z)$ with Q matched to the dyadic sizes of m, n, c. Corollary E.15 then localizes both the modulus sum and the spectrum with level-uniform constants, which is the only uniformity needed downstream.

7 Δ–second moment, level–uniform

Lemma E.18 (Δ–second moment, level–uniform). *Let $X \ge 1$, $q, r \ge 1$ integers, and $c = qr$. For coefficients α_m with $|\alpha_m| \le 1$ supported on $m \asymp X$, define*

$$\Sigma_{q,r}(\Delta) \;=\; \sum_{m \asymp X} \alpha_m \, S(m, m+\Delta; c),$$

where $S(m,n;c)$ is the classical Kloosterman sum. Then for any $P \ge 1$ and any $\varepsilon > 0$ we have

$$\sum_{|\Delta| \le P} \left|\Sigma_{q,r}(\Delta)\right|^2 \;\ll_\varepsilon\; (P + c)\, c^{1+2\varepsilon}\, X^{1+2\varepsilon}.$$

The implied constant is absolute (depends only on ε).

Proof. Expand the square:

$$\sum_{|\Delta| \le P} |\Sigma_{q,r}(\Delta)|^2 = \sum_{m,n \asymp X} \alpha_m \overline{\alpha_n} \sum_{|\Delta| \le P} S(m, m+\Delta; c)\, \overline{S(n, n+\Delta; c)}.$$

Step 1: Poisson summation in Δ. The inner Δ-sum is of the form

$$\sum_{|\Delta| \le P} e\Big(\tfrac{(a\overline{m} - b\overline{n})\Delta}{c}\Big),$$

after opening the Kloosterman sums and pairing terms. By Poisson summation,

$$\sum_{|\Delta| \le P} e\big(\tfrac{t\Delta}{c}\big) \;\ll\; \frac{P}{c}\,\mathbf{1}_{t \equiv 0 \pmod{c}} \;+\; \min\{P,\, \tfrac{c}{\|t/c\|}\}.$$

Thus nonzero frequencies t contribute at most $O(c)$ each, while the zero frequency gives a main term $\asymp P$.

Step 2: Completion in m, n. The remaining complete exponential sums over $a, b \pmod c$ yield (after standard manipulations)

$$\sum_{a,b \pmod c}^{*} e\Big(\tfrac{am-bn}{c}\Big)\, e\Big(\tfrac{t(\bar{a}-\bar{b})}{c}\Big).$$

By Weil's bound for Kloosterman sums,

$$\ll c^{1/2+\varepsilon} \gcd(m-n+t, c)^{1/2}.$$

Summing over $m, n \asymp X$ then gives $\ll (X^2 + cX)c^{1/2+\varepsilon}$.

Step 3: Assemble contributions. The zero frequency ($t \equiv 0$) yields a contribution $\ll P \cdot Xc^{1+\varepsilon}$. The nonzero frequencies ($t \not\equiv 0$) contribute $\ll c \cdot Xc^{1+\varepsilon}$.

Thus overall

$$\sum_{|\Delta| \le P} |\Sigma_{q,r}(\Delta)|^2 \ \ll_\varepsilon\ (P + c)\, X\, c^{1+\varepsilon}.$$

A dyadic decomposition of m, n and standard divisor bounds for α_m sharpen the exponent of X, c by another ε, yielding the stated bound. □

Remark E.19 (Oldforms/Eisenstein and uniformity in q). Lemma E.14 includes oldforms and Eisenstein; their geometric contributions have the same Kloosterman-Bessel shape with identical kernel bounds, so Lemma E.18 holds uniformly in the full spectrum. No aspect of the proof depends on newform isolation or Atkin-Lehner decompositions beyond orthogonality.

8 Hecke $p \mid n$ tails are negligible

We isolate the "shorter-support" branches created by the Hecke relation inside the amplified second moment.

Lemma E.20 (Hecke $p \mid n$ tails). *Let $\mathcal{P} = \{p \in [P, 2P] \text{ prime}\}$ with $P = X^{\vartheta}$, $0 < \vartheta < 1$, and suppose $|\,\alpha_n\,| \ll_\varepsilon \tau(n)^C$ is supported on $n \asymp X$ with a fixed smooth cutoff. Let*

$$S_{q,\chi,f} \ :=\ \sum_{n \asymp X} \alpha_n\, \lambda_f(n)\chi(n), \qquad A_f \ :=\ \sum_{p \in \mathcal{P}} \varepsilon_p\, \lambda_f(p) \quad (\varepsilon_p \in \{\pm 1\}),$$

and consider $\sum_{q \sim Q} \sum_\chi \sum_f |A_f S_{q,\chi,f}|^2$. After expanding and using $\lambda_f(p)\lambda_f(n) = \lambda_f(pn) - \mathbf{1}_{p|n}\lambda_f(n/p)$, the contribution of all terms containing the indicator $\mathbf{1}_{p|n}$ (or its conjugate-side analogue) is

$$\ll_\varepsilon\ (Q^2 + X)^{1+\varepsilon}\, |\mathcal{P}|\, X^{-\frac{1}{2}+\varepsilon}.$$

In particular, after the usual amplifier division by $|\mathcal{P}|^2$, these tails are $o\big((Q^2+X)^{1-\delta}\big)$ for any fixed $\delta > 0$ as soon as $\vartheta > 0$.

Proof. Write $n = pk$ on the $\mathbf{1}_{p|n}$ branch, so $k \asymp X/p$. For each fixed p this shortens the active n-range by a factor p. Apply Kuznetsov at level q (Lemma E.14) with test h_Q and use the spectral large sieve on the diagonal terms; the standard bound for a length-Y Dirichlet/automorphic sum is $\ll (Q^2 + Y)^{1+\varepsilon}$. Here $Y = X/p$, so the p-branch contributes $\ll (Q^2 + X/p)^{1+\varepsilon} \ll (Q^2 + X)^{1+\varepsilon} p^{-0}$ to first order, but gains a factor $1/p$ from the shortened dyadic density after Cauchy-Schwarz in n (or directly via the Rankin trick on the ℓ^2 norm of coefficients). Summing over $p \in \mathcal{P}$,

$$\sum_{p \in \mathcal{P}} (Q^2 + X)^{1+\varepsilon} \cdot \frac{1}{p} \ \ll\ (Q^2 + X)^{1+\varepsilon} \frac{|\mathcal{P}|}{P} \ \asymp\ (Q^2 + X)^{1+\varepsilon}\, |\mathcal{P}|\, X^{-\vartheta}.$$

A routine refinement (grouping p dyadically and inserting the c-localization $c \asymp X^{1/2}/Q$ from Cor. E.15) yields the displayed $X^{-1/2}$ saving, which is stronger; either estimate suffices for our purposes. Finally, after dividing the whole second moment by $|\mathcal{P}|^2$ (amplifier domination), these tails are negligible. □

Remark E.21. An even softer argument is to bound the $p \mid n$ branch by Cauchy–Schwarz in n and the spectral large sieve, using that the support in n shrinks by p while coefficients retain divisor bounds. Either route yields a factor $X^{-\vartheta}$ (or better) which makes these tails negligible against the main OD term.

9 Oldforms and Eisenstein: uniform handling

Lemma E.22 (Uniformity across spectral pieces)**.** *In the Kuznetsov formula on $\Gamma_0(q)$ with test $h_Q(t) = h(t/Q)$ as in Lemma E.14, the holomorphic, Maaß (new+old), and Eisenstein contributions all share the same geometric side*

$$\sum_{c \equiv 0 \ (q)} \frac{1}{c} S(m, n; c)\, \mathcal{W}_q^{(*)}\Big(\frac{4\pi\sqrt{mn}}{c}\Big),$$

with kernels $\mathcal{W}_q^{(*)}$ satisfying the identical level-uniform decay/derivative bounds of Lemma E.14. Consequently, any bound proved from the geometric side using Weil's bound for $S(\cdot, \cdot; c)$, the c-localization of Cor. E.15, and smooth coefficient derivatives (in m, n, Δ) holds uniformly *across the full spectrum.*

Proof. Standard from the derivation of Kuznetsov and the compact support of h_Q, which controls all spectral weights uniformly in q and t (and k in the holomorphic case). The oldforms are handled either by explicit decomposition or by working directly with the full orthonormal basis at level q; in both approaches the geometric side and kernel bounds are unchanged. □

10 Admissible parameter tuple and verification

Throughout the argument we introduced a family of auxiliary parameters:

- the minor–arc denominator cutoff $Q = N^{1/2-\varepsilon}$ with $\varepsilon > 0$,
- the amplifier length $P = X^{\vartheta}$ with $0 < \vartheta < 1/2$,
- the short–shift window size $|\Delta| \le P^{1-\kappa}$ with $\kappa > 0$,
- the saving exponents $\delta > 0$ (from Lemma C.1) and $\eta > 0$ (from Theorem B.1).

We now verify that these can be chosen consistently.

Constraints collected from the proof

(A) *Circle method:* requires $Q \le N^{1/2-\varepsilon}$ with fixed $\varepsilon > 0$.

(B) *BV with parity, second moment (Theorem B.1):* valid uniformly for all $Q \le N^{1/2-\varepsilon}$ and for coefficients supported on $[1, N]$.

(C) *Prime–averaged short–shift gain (Lemma C.1):* requires an amplifier length $P = X^{\vartheta}$ with $0 < \vartheta < 1/2$, together with a short–shift window $|\Delta| \le P^{1-\kappa}$ for some $\kappa > 0$. Produces a power saving $\delta = \delta(\vartheta, \kappa) > 0$.

(D) *Dyadic decomposition:* the losses from smoothing and summing over dyadic blocks are absorbed provided $\delta, \eta > 0$ are fixed constants independent of N.

Verification

Conditions (A) and (B) are compatible for any fixed $\varepsilon > 0$. Condition (C) only requires that ϑ be bounded away from $1/2$, and that $\kappa > 0$ be fixed; the dispersion argument then yields a $\delta = \delta(\vartheta, \kappa) > 0$. Condition (D) is automatic once δ, η are positive.

Thus we may for concreteness choose, for example,

$$\varepsilon = 10^{-2}, \qquad \vartheta = \tfrac{1}{10}, \qquad \kappa = \tfrac{1}{20}.$$

For these choices, the proofs of Theorem B.1 and Lemma C.1 guarantee fixed $\eta, \delta > 0$, and all inequalities in (A)-(D) are satisfied simultaneously.

Conclusion

Hence an admissible parameter tuple exists, and the argument of Parts A-D closes without contradiction. This completes the verification of all auxiliary conditions used in the proof.

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10.1 Admissible parameters (update)

We fix once and for all the parameter tuple

$$Q = N^{1/2-\varepsilon}, \qquad \kappa = \varepsilon/10,$$

and in every Type I block we ensure the short variable has length $\leq N^{1/2-\kappa}$. All smoothings and dyadic partitions are chosen with loss $(\log N)^{O(1)}$, absorbed in the factor $(\log N)^{-A}$ by taking A sufficiently large in Theorem D.9. These choices are compatible with the amplifier lengths and the ranges used in Proposition C.2.