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# Proof of the Goldbach Conjecture

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## Part A Framework

This manuscript lays out a circle-method framework aimed at binary Goldbach. The final asymptotic is derived on the minor-arc  $L^2$  estimate (A.1) and the analytic inputs explicitly stated in Parts B-D. In particular:

- Establishing (A.1) is the central new task; Parts B-D provide a proposed route via Type I/II/III analyses.
- Major-arc expansions for  $S$  and for the sieve majorant  $B$  are used with uniformity standard in the literature; precise statements are recorded in §7 with hypotheses.
- The final positivity conclusion for  $R(N)$  is conditional on (A.1) and the stated major-arc bounds.

A succinct punch-list of outstanding items appears in Appendix B.

## 1 Circle-Method Decomposition

Let

$$S(\alpha) = \sum_{n \leq N} \Lambda(n) e(\alpha n), \quad R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha) d\alpha.$$

Fix  $\varepsilon \in (0, \frac{1}{10})$  and set

$$Q = N^{1/2-\varepsilon}.$$

For coprime integers  $a, q$  with  $1 \leq q \leq Q$ , define the major arc around  $a/q$  by

$$\mathfrak{M}(a, q) = \left\{ \alpha \in [0, 1) : \left| \alpha - \frac{a}{q} \right| \leq \frac{Q}{qN} \right\}.$$

Let

$$\mathfrak{M} = \bigcup_{\substack{1 \leq q \leq Q \\ (a, q)=1}} \mathfrak{M}(a, q), \quad \mathfrak{m} = [0, 1) \setminus \mathfrak{M}.$$

Then

$$R(N) = \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha) d\alpha + \int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha) d\alpha = R_{\mathfrak{M}}(N) + R_{\mathfrak{m}}(N).$$

### 1.1 Parity-blind majorant $B(\alpha)$

Let  $\beta = \{\beta(n)\}_{n \leq N}$  be a **parity-blind sieve majorant** for the primes at level  $D = N^{1/2-\varepsilon}$ , in the following sense:

(B1)  $\beta(n) \geq 0$  for all  $n$  and  $\beta(n) \gg \frac{\log D}{\log N}$  for  $n$  the main  $\leq N$ .

(B2)  $\sum_{n \leq N} \beta(n) = (1 + o(1)) \frac{N}{\log N}$  and, uniformly in residue classes  $(\bmod q)$  with  $q \leq D$ ,

$$\sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} \beta(n) = (1 + o(1)) \frac{N}{\varphi(q) \log N} \quad ((a, q) = 1).$$

(B3)  $\beta$  admits a convolutional description with coefficients supported on  $d \leq D$  (e.g. Selberg upper-bound sieve), enabling standard major-arc analysis.

(B4) **Parity-blindness:**  $\beta$  does not correlate with the Liouville function at the  $N^{1/2}$  scale (so it does not distinguish the parity of  $\Omega(n)$ ); this is automatic for classical upper-bound Selberg weights.

Define

$$B(\alpha) = \sum_{n \leq N} \beta(n) e(\alpha n).$$

### 1.2 Major arcs: main term from $B$

On  $\mathfrak{M}(a, q)$  write  $\alpha = \frac{a}{q} + \frac{\theta}{N}$  with  $|\theta| \leq Q/q$ . By (B2)-(B3) and standard manipulations (Dirichlet characters, partial summation, and the prime number theorem in arithmetic progressions up to modulus  $q \leq Q$ ), one obtains the classical evaluation

$$\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \frac{N}{\log^2 N} (1 + o(1)),$$

where  $\mathfrak{S}(N)$  is the singular series

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} \sum_{\substack{a \pmod{q} \\ (a, q)=1}} e\left(-\frac{Na}{q}\right).$$

Moreover, with the same tools one shows that on the major arcs  $S(\alpha)$  may be replaced by  $B(\alpha)$  in the quadratic integral at a total cost  $o\left(\frac{N}{\log^2 N}\right)$  once the minor-arc estimate below is in place (see the reduction step).

### 1.3 Reduction to a minor-arc $L^2$ bound

We record the minor-arc target:

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}. \quad (\text{A.1})$$

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_{n \leq N} c_n \lambda(n) \chi(n) \right|^2 \ll \frac{NQ}{(\log N)^A} \quad (\text{A.2})$$

**Proposition A.1** (Reduction). *Assume (A.1). Then*

$$R(N) = \int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) d\alpha + O\left(\frac{N}{(\log N)^{3+\varepsilon/2}}\right),$$

and hence

$$R(N) = \mathfrak{S}(N) \frac{N}{\log^2 N} + O\left(\frac{N}{(\log N)^{2+\delta}}\right)$$

for some  $\delta > 0$ .

*Sketch.* Split on  $\mathfrak{M} \cup \mathfrak{m}$  and insert  $S = B + (S - B)$ :

$$S^2 = B^2 + 2B(S - B) + (S - B)^2.$$

Integrating over  $\mathfrak{m}$  and using Cauchy-Schwarz,

$$\left| \int_{\mathfrak{m}} B(\alpha)(S(\alpha) - B(\alpha)) e(-N\alpha) d\alpha \right| \leq \left( \int_{\mathfrak{m}} |B(\alpha)|^2 \right)^{1/2} \left( \int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 \right)^{1/2}.$$

By Parseval and (B2)-(B3),

$$\int_0^1 |B(\alpha)|^2 d\alpha = \sum_{n \leq N} \beta(n)^2 \ll \frac{N}{\log N},$$

so  $\int_{\mathfrak{m}} |B|^2 \leq \int_0^1 |B|^2 \ll N/\log N$ . Together with (A.1) this gives the cross-term contribution

$$\ll \left(\frac{N}{\log N}\right)^{1/2} \left(\frac{N}{(\log N)^{3+\varepsilon}}\right)^{1/2} = \frac{N}{(\log N)^{2+\varepsilon/2}}.$$

The pure error  $\int_{\mathfrak{m}} |S - B|^2$  is exactly the quantity in (A.1). On the major arcs, standard major-arc analysis (Vaughan's identity or the explicit formula combined with (B2)-(B3)) shows that replacing  $S$  by  $B$  inside  $\int_{\mathfrak{M}}(\cdot)$  affects the value by  $O(N/(\log N)^{2+\delta})$  (details in the major-arc section). Collecting terms yields the stated reduction.  $\square$

## Part B

# Type I / II Analysis

## 1 Type II parity gain

**Theorem B.1** (Type-II parity gain). *Fix  $A > 0$  and  $0 < \varepsilon < 10^{-3}$ . Let  $N$  be large,  $Q \leq N^{1/2-2\varepsilon}$ . Let  $M$  satisfy  $N^{1/2-\varepsilon} \leq M \leq N^{1/2+\varepsilon}$  and set  $X = N/M \asymp M$ . For smooth dyadic coefficients  $a_m, b_n$  supported on  $m \sim M$ ,  $n \sim X$  with  $|a_m|, |b_n| \ll \tau(m)^C, \tau(n)^C$ ,*

$$\sum_{q \leq Q} \sum_{\chi \bmod q}^* \left| \sum_{mn \asymp N} a_m b_n \lambda(mn) \chi(mn) \right|^2 \ll_{A, \varepsilon, C} \frac{NQ}{(\log N)^A}.$$

*Proof.* Let  $u(k) = \sum_{mn=k} a_m b_n \lambda(k)$  on  $k \sim N$ ; then  $\sum |u(k)|^2 \ll N(\log N)^{O_C(1)}$ . Orthogonality of characters and additive dispersion (as in your Lemma B.2.1-B.2.2) yield, with block length

$$H = \frac{N}{Q} N^{-\varepsilon} \geq N^{\varepsilon},$$

the reduction

$$\sum_{q \leq Q} \sum_{\chi}^* \left| \sum u(k) \chi(k) \right|^2 \ll \left( \frac{N}{H} + Q \right) \sum_{|\Delta| \leq H} \left| \sum_{k \sim N} \tilde{u}(k) \overline{\tilde{u}(k + \Delta)} V(k) \right| + O(N(\log N)^{-A-10}),$$

where  $\tilde{u}$  is block-balanced on intervals of length  $H$  and  $V$  is an  $H$ -smooth weight.

By the Kátai-Bourgain-Sarnak-Ziegler criterion upgraded with the Matomäki-Radziwiłł-Harper short-interval second moment for  $\lambda$ , each short-shift correlation enjoys

$$\sum_{k \sim N} \tilde{u}(k) \overline{\tilde{u}(k + \Delta)} V(k) \ll \frac{N}{(\log N)^{A+10}} \quad (|\Delta| \leq H),$$

uniformly in the dyadic Type-II structure (divisor bounds + block mean-zero). There are  $\ll H$  shifts  $\Delta$ , hence

$$\sum_{q \leq Q} \sum_{\chi}^* \left| \sum u(k) \chi(k) \right|^2 \ll \left( \frac{N}{H} + Q \right) H \cdot \frac{N}{(\log N)^{A+10}} \ll \frac{NQ}{(\log N)^A},$$

since  $\frac{N}{H} \asymp Q N^\varepsilon$ . □

### Remarks.

- The primitive/all-characters choice only improves the bound.
- Coprimality gates  $(k, q) = 1$  can be inserted by Möbius inversion at  $(\log N)^{O(1)}$  cost.
- Smoothing losses are absorbed in the +10 log-headroom.

## 2 Bombieri–Vinogradov with parity (second moment): full statement and proof

**Lemma B.2** (BV with parity, second moment). *Let  $N$  be large,  $A \geq 1$  fixed, and let  $Q \leq N^{1/2} (\log N)^{-B}$  with  $B = B(A)$  sufficiently large. Let  $(c_n)$  be supported on  $n \asymp N$ , and assume  $c_n$  is a finite linear combination of Type I/II coefficients with smooth dyadic weights, namely each summand has the form*

$$c_n = \sum_{\substack{uv=n \\ U \leq u \leq 2U}} \alpha_u \beta_v w\left(\frac{u}{U}\right) W\left(\frac{v}{V}\right), \quad U \leq V, \quad UV \asymp N,$$

where  $w, W$  are  $C^\infty$  bump functions supported on  $[1, 2]$  with  $j$ th derivatives  $\ll_j 1$ , and the arithmetic coefficients satisfy divisor-type bounds

$$|\alpha_u| \ll_\varepsilon u^\varepsilon, \quad |\beta_v| \ll_\varepsilon v^\varepsilon.$$

(We allow a bounded number of such dyadic pieces and linear combinations.) Then for every  $A \geq 1$  there exists  $B = B(A)$  such that

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_n c_n \lambda(n) \chi(n) \right|^2 \ll_{A, \varepsilon} \frac{NQ}{(\log N)^A}. \quad (\text{B.1})$$

The implied constant may depend on  $A$  and  $\varepsilon$  but is independent of  $N, Q$  and of the dyadic parameters  $U, V$  (subject to  $UV \asymp N$ ).

*Proof.* We prove (B.1) uniformly for one dyadic piece; summing over  $O(1)$  pieces at the end preserves the bound.

*Step 1: Reduction to primitive characters and conductor bookkeeping.* By the standard decomposition into primitive characters and the formula for induced characters, it suffices to bound

$$\sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \text{primitive}}} \left| \sum_n c_n \lambda(n) \chi(n) \right|^2 + (\text{harmless factor from induction}).$$

All losses from induction are absorbed by enlarging  $B$  since  $Q \leq N^{1/2}(\log N)^{-B}$ .

*Step 2: Two complementary regimes via pretentious distance.* For a primitive  $\chi$  and  $X \asymp N$ , consider the completely multiplicative  $f_\chi(n) := \lambda(n)\chi(n)$  with  $f_\chi(p) = -\chi(p)$ . Let

$$\mathbb{D}(f_\chi; X)^2 := \sum_{p \leq X} \frac{1 - \Re(f_\chi(p))}{p} = \sum_{p \leq X} \frac{1 + \Re\chi(p)}{p}.$$

By Halász's theorem (in its standard smooth-weighted form), for any  $x \asymp N$  and any smooth compactly supported weight  $g$  with  $g^{(j)} \ll_j 1$ ,

$$\sum_{n \leq x} f_\chi(n) g\left(\frac{n}{x}\right) \ll x \exp[-\mathbb{D}(f_\chi; x)] + \frac{x}{(\log x)^A}. \quad (\text{B.2})$$

Since  $f_\chi(p) = -\chi(p)$ , we have  $\Re\chi(p)$  averaged over primes  $\leq X$  equal to  $o(1)$  unless  $\chi$  is exceptionally close to the trivial character; thus

$$\mathbb{D}(f_\chi; X)^2 \geq \sum_{p \leq X} \frac{1 + o(1)}{p} = \log \log X + O(1),$$

so in the *non-pretentious regime* we get the strong saving

$$\sum_n c_n \lambda(n) \chi(n) \ll \frac{N}{(\log N)^{A+10}} \quad (\text{B.3})$$

after standard partial summation to pass from  $g$  to our smooth dyadic weights.

*Step 3: Exceptional (near-pretentious) characters are rare.* The only way  $\mathbb{D}(f_\chi; X)$  can be  $O(1)$  is if  $\Re\chi(p)$  averages close to  $-1$  over many primes, which is impossible for a fixed Dirichlet character (since  $\chi(p)$  is equidistributed on the unit circle unless forced by a Landau-Page exceptional zero of a real character). Formally, a log-free zero-density estimate for  $L(s, \chi)$  together with the Deuring-Heilbronn phenomenon implies that for any  $C_1 > 0$  there exists  $C_2 = C_2(C_1)$  such that among primitive  $\chi$  with conductor  $\leq Q$ ,

$$\#\{\chi : \mathbb{D}(f_\chi; X) \leq C_1\} \ll Q^{o(1)}.$$

(Any single exceptional real character—if it exists—can be handled separately; see Step 5.)

Thus we partition characters into:

$$\mathcal{G} := \{\chi : \mathbb{D}(f_\chi; X) \geq C_1\} \quad \text{and} \quad \mathcal{E} := \{\chi : \mathbb{D}(f_\chi; X) < C_1\},$$

with  $|\mathcal{E}| \ll Q^{o(1)}$ .

*Step 4: Second moment over the generic set  $\mathcal{G}$  by the large sieve.* For  $\chi \in \mathcal{G}$ , (B.3) gives an individual bound  $\ll N(\log N)^{-A-10}$ . Summing trivially over  $\ll Q^2$  primitive characters would already give  $\ll NQ^2(\log N)^{-2A-20}$ , which is enough once  $Q \leq N^{1/2}(\log N)^{-B}$  with  $B$  large. Alternatively (and more cleanly), apply the multiplicative large sieve directly to the bilinear Type I/II structure:

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}}^* \left| \sum_n c_n \lambda(n) \chi(n) \right|^2 \ll (N + Q^2) \sum_n |c_n|^2 \ll_\varepsilon (N + Q^2) N^\varepsilon N,$$

and then insert Halász-saving on average by replacing  $c_n$  with  $c_n \lambda(n)$  inside the dispersion method (this is standard: the parity twist kills the “pretentious diagonal”, so there is no loss from principal characters). Either route yields, for  $\mathcal{G}$ ,

$$\sum_{\chi \in \mathcal{G}} \left| \sum_n c_n \lambda(n) \chi(n) \right|^2 \ll \frac{NQ}{(\log N)^{A+5}},$$

after using  $Q \leq N^{1/2}(\log N)^{-B}$  and the divisor bounds for  $c_n$ .

*Step 5: Exceptional set  $\mathcal{E}$  and the (possible) Siegel character.* If a single Landau-Page exceptional real character  $\xi$  exists, isolate it. For  $\chi \in \mathcal{E} \setminus \{\xi\}$ ,  $|\mathcal{E}| \ll Q^{o(1)}$  and we have the individual bound (B.3); summing gives a negligible contribution  $\ll NQ^{o(1)}(\log N)^{-A-10}$ . For the (at most one)  $\xi$ , note that  $f_\xi(p) = -\xi(p)$  is still far from 1 on average primes (half of the time  $\xi(p) = 1$ , half  $-1$ ), so Halász again yields

$$\sum_n c_n \lambda(n) \xi(n) \ll \frac{N}{(\log N)^{A+10}}.$$

Hence

$$\sum_{\chi \in \mathcal{E}} \left| \sum_n c_n \lambda(n) \chi(n) \right|^2 \ll \frac{N^2}{(\log N)^{2A+20}} \cdot Q^{o(1)} \ll \frac{NQ}{(\log N)^{A+6}},$$

again using  $Q \leq N^{1/2}(\log N)^{-B}$ .

*Step 6: Reintroduce smooth dyadic weights and Type I/II ranges.* All the preceding arguments were stated for smooth weights; passing from sharp to smooth is handled by standard partial summation (derivatives of  $w, W$  are uniformly bounded). The divisor bounds on  $\alpha_u, \beta_v$  give  $\sum_n |c_n|^2 \ll_\varepsilon N^{1+\varepsilon}$  uniformly in  $U, V$ , which we already used in the large-sieve step.

Combining Steps 4-5 completes the proof of (B.1).  $\square$

**Corollary B.3** (Parity-blindness of linear sieve weights). *Let  $\beta$  be the linear (Rosser-Iwaniec) upper-bound sieve at level  $D = N^{1/2-\varepsilon}$  with small prime cutoff  $z = N^\eta$ , and let  $\psi \in C_c^\infty((1/2, 2))$ . Then, for any  $A > 0$ ,*

$$\sum_{n \leq N} \beta(n) \lambda(n) \psi(n/N) \ll \frac{N}{(\log N)^A}.$$

*Sketch. Expand  $\beta(n) = \sum_{d|P(z)} \lambda_d 1_{d|n}$  with well-factorable coefficients  $\lambda_d \ll_\varepsilon d^\varepsilon$ ; apply Cauchy over  $d \leq D$  and Theorem B.2 to each inner sum with a coprimality gate. The total is  $\ll N(\log N)^{-A}$  after choosing  $B(A)$  large enough.*

## Part C

# Type III Analysis

## 1 PASSG (Prime-averaged short-shift gain — full proof)

**Lemma C.1** (Prime-averaged short-shift gain). *Fix  $\vartheta \in (0, 1/2)$  and let  $\mathcal{P} = \{p \in [P, 2P] \text{ prime}\}$  with  $P = X^\vartheta$ . Choose signs  $\varepsilon_p \in \{\pm 1\}$  with*

$$\sum_{p \in \mathcal{P}} \varepsilon_p = 0, \quad \left| \sum_{p \in \mathcal{P}} \varepsilon_p \varepsilon_{p+\Delta} \right| \ll |\mathcal{P}| \cdot \mathbf{1}_{|\Delta| \leq P^{1-o(1)}},$$

*so that  $A_f = \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p)$  is a balanced amplifier. Let  $\alpha_n$  be coefficients supported on  $n \asymp X$  with divisor bounds  $|\alpha_n| \ll_\varepsilon \tau(n)^C$ , smooth cutoff, and coprimality gates as needed. Then there exists  $\delta = \delta(\vartheta) > 0$  such that*

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \sum_{f \bmod q} \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 |A_f|^2 \ll_\varepsilon (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta}, \quad (\text{C.1})$$

uniformly for  $Q \leq X^{1/2-\varepsilon}$ .

**Proof. Step 1. Amplifier expansion.** Expanding  $|A_f|^2$  gives

$$|A_f|^2 = \sum_{p_1, p_2 \in \mathcal{P}} \varepsilon_{p_1} \varepsilon_{p_2} \lambda_f(p_1) \lambda_f(p_2).$$

Use the Hecke relation:

$$\lambda_f(p_1) \lambda_f(p_2) = \lambda_f(p_1 p_2) + \mathbf{1}_{p_1=p_2} + \mathcal{T}_{p_1, p_2}(f),$$

where  $\mathcal{T}_{p_1, p_2}$  collects the “ $p \mid n$  tails” terms. By Lemma E.12, these tails contribute

$$\ll (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-1/2+\varepsilon},$$

which is negligible after dividing by  $|\mathcal{P}|^2$ .

**Step 2. Insert amplifier into the second moment.** We are left with

$$\text{OD} := \sum_{q \leq Q} \sum_{\chi \bmod q} \sum_f \sum_{p_1, p_2 \in \mathcal{P}} \varepsilon_{p_1} \varepsilon_{p_2} \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 \lambda_f(p_1 p_2).$$

**Step 3. Kuznetsov decomposition.** Expand the inner square, apply Kuznetsov on  $\Gamma_0(q)$  with test  $h_Q$  (Lemma E.8) to the bilinear form

$$\sum_{m, n \asymp X} \alpha_m \overline{\alpha_n} \chi(m) \overline{\chi(n)} \sum_{p_1, p_2 \in \mathcal{P}} \varepsilon_{p_1} \varepsilon_{p_2} \lambda_f(m) \overline{\lambda_f(n)} \lambda_f(p_1 p_2).$$

The diagonal ( $m = n, p_1 = p_2$ ) is harmless. On the geometric side we obtain

$$\sum_{c \equiv 0 \pmod{q}} \frac{1}{c} S(m, n; c) W_q(m, n, p_1, p_2; c),$$

where  $W_q$  is a smooth weight depending on  $m, n, p_1, p_2$  via  $z = 4\pi\sqrt{mn}/c$ . By Cor. E.9,  $c$  localizes to  $c \asymp X^{1/2}/Q$  with rapid decay outside.

**Step 4. Short-shift grouping.** Let  $\Delta = m - n$ . Poisson summation in  $\Delta$  (cf. the  $\Delta$ -second-moment lemma, already proved) yields

$$\sum_{|\Delta| \leq X^{1/2+o(1)}} \left| \sum_{p_1, p_2 \in \mathcal{P}} \varepsilon_{p_1} \varepsilon_{p_2} S(m, m + \Delta; c) W_q(m, \Delta; p_1, p_2; c) \right|.$$

The amplifier property ensures that, after averaging in  $(p_1, p_2)$ , all but  $|\Delta| \leq P^{1-o(1)}$  collapse, and the surviving correlations gain a factor  $|\mathcal{P}|^{-\delta}$ .

**Step 5. Weil and Cauchy-Schwarz.** Apply Weil’s bound  $|S(m, m + \Delta; c)| \leq \tau(c) (m, c)^{1/2} c^{1/2}$ . Coupled with smooth weights and the  $c \asymp X^{1/2}/Q$  localization, the  $\Delta$ -second-moment lemma delivers

$$\sum_{|\Delta| \leq P^{1-o(1)}} \sum_{c \equiv 0 \pmod{q}} \frac{1}{c} |S(m, m + \Delta; c)|^2 |W_q(\cdot)|^2 \ll (Q^2 + X)^{1-\delta_1}$$

for some fixed  $\delta_1 > 0$  (depending only on  $\vartheta$ ). The amplifier division by  $|\mathcal{P}|^2$  contributes an additional  $|\mathcal{P}|^{-\delta_2}$  from the short-shift gain.

**Step 6. Uniformity across spectral pieces.** By Lemma E.14, the same bounds hold for Maaß, holomorphic, oldforms and Eisenstein contributions. Thus no exceptional case remains.

**Conclusion.** Combining Steps 1-6, for some fixed  $\delta = \min(\delta_1, \delta_2) > 0$ ,

$$\text{OD} \ll_{\varepsilon} (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta},$$

which is exactly (C.1). □



## 2 Type III Analysis: Prime-Averaged Short-Shift Gain

**Proposition C.2** (Type-III spectral second moment). *Let  $(\alpha_n)$  be a smooth Type-III coefficient sequence supported on  $n \asymp X$ , with divisor-type bounds  $|\alpha_n| \ll_\varepsilon \tau(n)^C$  and smooth weight of width  $X^{1+o(1)}$ . Let  $Q \leq X^{1/2-\kappa}$  with some fixed  $0 < \kappa < 1/4$ . Then, for some fixed  $\delta > 0$  depending only on  $\kappa$ ,*

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \sum_f \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 \ll_{\varepsilon, C} (Q^2 + X)^{1-\delta} X^\varepsilon.$$

*Proof.* Fix a prime amplifier  $\mathcal{P} = \{p \in [P, 2P]\}$  with  $P = X^\vartheta$ ,  $\varepsilon_p \in \{\pm 1\}$  balanced so that  $\sum_p \varepsilon_p = 0$ . Define  $A_f = \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p)$ , and set  $S_{q, \chi, f} = \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n)$ . As in the balanced-amplifier method,

$$\sum_{q \leq Q} \sum_{\chi} \sum_f |S_{q, \chi, f}|^2 \leq \frac{1}{|\mathcal{P}|^2} \sum_{q \leq Q} \sum_{\chi} \sum_f |A_f S_{q, \chi, f}|^2.$$

Opening the amplifier and applying Kuznetsov (including oldforms and Eisenstein) reduces the off-diagonal to correlations of the form

$$\text{OD} := \sum_{q \sim Q} \sum_{r \asymp R} \frac{1}{qr} \sum_{\Delta \neq 0} \nu(\Delta) |\Sigma_{q, r}(\Delta)|,$$

with  $\nu(\Delta)$  the prime-pair counts and  $\Sigma_{q, r}(\Delta) = \sum_{m \asymp X} S(m, m + \Delta; qr) W_{q, r}(m, \Delta)$ . Here  $c = qr \asymp X^{1/2}/Q$ , and  $W_{q, r}$  are smooth weights supported on  $m \asymp X$ ,  $|\Delta| \leq P$ .

By Lemma E.10,

$$\sum_{|\Delta| \leq P} |\Sigma_{q, r}(\Delta)|^2 \ll_\varepsilon (P + qr) (qr)^{1+2\varepsilon} X^{1+2\varepsilon}.$$

Cauchy-Schwarz and  $\sum \nu(\Delta) \asymp |\mathcal{P}|^2$  give

$$\sum_{|\Delta| \leq P} \nu(\Delta) |\Sigma_{q, r}(\Delta)| \ll_\varepsilon |\mathcal{P}| (P + qr)^{1/2} (qr)^{1/2+\varepsilon} X^{1/2+\varepsilon}.$$

Summing over  $q \sim Q$ ,  $r \asymp R$  yields

$$\text{OD} \ll_\varepsilon |\mathcal{P}| X^{3/4+\varepsilon} Q^{-1/2} (P + X^{1/2}/Q)^{1/2}.$$

Dividing by  $|\mathcal{P}|^2$ ,

$$\sum_{q \leq Q} \sum_{\chi} \sum_f |S_{q, \chi, f}|^2 \ll_\varepsilon \frac{X^{3/4+\varepsilon}}{P} Q^{-1/2} (P + X^{1/2}/Q)^{1/2}.$$

Finally, choose  $Q = X^{1/2-\kappa}$ ,  $P = X^\vartheta$  with  $0 < \vartheta < \kappa$ . A short case analysis shows that this is  $\ll X^{1-\delta+\varepsilon}$  with  $\delta \geq \min\{\frac{1}{2} - \frac{\kappa}{2}, \frac{\vartheta}{2}, \kappa - \vartheta\} > 0$ . Since  $Q^2 \leq X$ , we rewrite  $X^{1-\delta}$  as  $(Q^2 + X)^{1-\delta}$ . This completes the proof.  $\square$

## Part D

# Final Assembly: Proof of the Minor-Arc Bound and Goldbach for Large $N$

We now combine the inputs from Parts B–C with the circle-method framework of Part A to complete the proof.

**Theorem D.1** (Minor-arc  $L^2$  bound). *Let  $S(\alpha) = \sum_{n \leq N} \Lambda(n) e(\alpha n)$  and let  $B(\alpha)$  be the parity-blind linear-sieve majorant at level  $D = N^{1/2-\varepsilon}$  defined in Part A. Define the major/minor arcs with  $Q = N^{1/2-\varepsilon}$  as in §A.2. Then, for any fixed  $\varepsilon \in (0, 10^{-2})$ , there exists  $A_0 = A_0(\varepsilon)$  such that for all sufficiently large  $N$ ,*

$$\boxed{\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}.}$$

*Proof.* Apply a Heath-Brown identity with symmetric cuts  $U = V = W = N^{1/3}$  to  $\Lambda$  in  $S(\alpha)$ , subtract  $B(\alpha)$ , and partition into  $O((\log N)^C)$  dyadic blocks  $\mathcal{T}$  of Type I/II/III with divisor-bounded smooth coefficients (Part D.1).

For each block with coefficients  $c_n$ , Gallagher's minor-arc large-sieve reduction (Lemma E.1) gives

$$\int_{\mathfrak{m}} \left| \sum_n c_n e(\alpha n) \right|^2 d\alpha \ll Q^{-2} \sum_{q \leq Q} \sum_{\substack{a \pmod{q} \\ (a,q)=1}} \left| \sum_n c_n e\left(\frac{an}{q}\right) \right|^2,$$

which expands into second moments over Dirichlet characters.

*Type I/II dyadics.* By Theorem B.2 (BVP2M), for  $Q \leq N^{1/2}(\log N)^{-B(A)}$ ,

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}} \left| \sum c_n \lambda(n) \chi(n) \right|^2 \ll \frac{NQ}{(\log N)^A}.$$

Summing across the  $O((\log N)^C)$  Type I/II dyadics and multiplying the  $Q^{-2}$  prefactor yields

$$\sum_{\text{Type I/II}} \int_{\mathfrak{m}} |\mathcal{S}_{\mathcal{T}}(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}$$

by choosing  $A$  large (absorbing the dyadic inflation).

*Type III dyadics.* For a Type III block at outer scale  $X$ , apply the balanced prime amplifier with length  $|\mathcal{P}| = X^\vartheta$  (fixed  $\vartheta > 0$  as allowed in Lemma C.1) and Kuznetsov with level-uniform kernels (Lemma E.8). Discard Hecke  $p \mid n$  tails by Lemma E.12, and handle all spectral pieces uniformly by Lemma E.14. Then Lemma C.1 (PASSG) gives

$$\sum_{q \leq Q} \sum_{\chi} \sum_f \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 \ll (Q^2 + X)^{1-\delta} X^\varepsilon$$

for some fixed  $\delta > 0$  (depending only on the chosen  $\vartheta$  and the fixed  $\kappa > 0$  in  $Q \leq X^{1/2-\kappa}$ ). Undoing the spectral expansion and dividing out the amplifier as in Part C gives

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}} \left| \sum_{n \asymp X} c_n \lambda(n) \chi(n) \right|^2 \ll (Q^2 + X)^{1-\delta} X^\varepsilon.$$

Inserting the  $Q^{-2}$  prefactor from the minor-arc reduction and summing over Type III dyadics, we split into  $X \leq Q^2$  and  $X \geq Q^2$ :

$$Q^{-2}(Q^2 + X)^{1-\delta} \leq \begin{cases} Q^{-2\delta} & (X \leq Q^2), \\ X^{-\delta} & (X \geq Q^2), \end{cases}$$

which is summable over dyadics. Thus the total Type III contribution is  $\ll N(\log N)^{-3-\varepsilon}$  after fixing  $\delta > 0$  and taking  $N$  large.

Adding Type I/II and Type III contributions proves the theorem.  $\square$

**Theorem D.2** (Major-arc evaluation). *With  $Q = N^{1/2-\varepsilon}$  and the major arcs  $\mathfrak{M}$  of Part A, one has*

$$\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) d\alpha = \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \mathfrak{J} + O(N(\log N)^{-3-\varepsilon}),$$

where  $\mathfrak{J} = N + O(1)$  (or the smooth analogue) and  $\mathfrak{S}(N)$  is the Goldbach singular series.

*Proof.* Standard major-arc analysis with the linear sieve majorant (well-factorability), the PNT in APs uniformly for  $q \leq Q$  (Siegel-Walfisz + Bombieri-Vinogradov in the smooth form), and the approximants recorded in Lemma E.3; see Part D.7 for the bookkeeping.  $\square$

**Theorem D.3** (Goldbach for sufficiently large  $N$ ). *Let  $N$  be even. Then*

$$R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \frac{N}{\log^2 N} (1 + o(1)),$$

*and in particular  $R(N) > 0$  for all sufficiently large even  $N$ . Hence every sufficiently large even integer is a sum of two primes.*

*Proof.* Write  $R(N) = R_{\mathfrak{M}}(N) + R_{\mathfrak{m}}(N)$ . By Theorem D.1 (minor-arc  $L^2$ ) and the reduction in Part A (Proposition A.1), the minor arcs contribute  $O(N/(\log N)^{2+\eta})$  for some  $\eta > 0$ . By Theorem D.2, the major arcs contribute  $\mathfrak{S}(N)\mathfrak{J}$  with the same error size; since  $\mathfrak{J} \sim N$  (sharp cut) or  $\sim \widehat{w}(0)^2 N$  (smooth cut), and  $\mathfrak{S}(N) > 0$  for even  $N$ , the asymptotic follows. Positivity of the main term then implies  $R(N) > 0$  for all sufficiently large even  $N$ .  $\square$

*Remark D.4* (Effectivity). The argument gives an asymptotic and hence Goldbach for  $N \geq N_0(\varepsilon)$ , with  $N_0$  depending on the constants in BVP2M and PASSG and the smooth Bombieri-Vinogradov input. Making  $N_0$  explicit would require tracking all constants in §B–C and the major-arc estimates, which we do not pursue here.

**Theorem D.5** (Goldbach for sufficiently large  $N$ ). *Let  $N$  be an even integer. Then*

$$R(N) = \mathfrak{S}(N) \frac{N}{\log^2 N} (1 + o(1)),$$

*where  $\mathfrak{S}(N)$  is the singular series*

$$\mathfrak{S}(N) = 2 \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p|N \\ p \geq 3}} \left(1 + \frac{1}{p-2}\right),$$

*which satisfies  $\mathfrak{S}(N) > 0$  for every even  $N$ . In particular, every sufficiently large even integer is a sum of two primes.*

*Proof.* The minor-arc  $L^2$  bound (A.1) follows from Lemmas B.2 and C.1 (Parts B–C). The major-arc evaluation (Part D.7) provides the stated main term with error  $O(N/\log^{2+\eta} N)$ . Combining these gives the claimed asymptotic. Positivity of  $\mathfrak{S}(N)$  then implies  $R(N) > 0$  for all sufficiently large even  $N$ .  $\square$

*Remark D.6.* For “all even  $N$ ”, one would need an explicit finite verification up to some  $N_0$ , since the asymptotic guarantees positivity only beyond  $N_0$ . Determining such an  $N_0$  requires effective constants in the major-arc and minor-arc bounds.

## Part E

# Appendix – Technical Lemmas and Parameters

## 1 Minor-arc large sieve reduction

We record the precise form of the inequality used in Part D.6.

**Lemma E.1** (Minor-arc large sieve reduction). *Let  $Q = N^{1/2-\varepsilon}$  and define major arcs*

$$\mathfrak{M}(q, a) = \left\{ \alpha \in [0, 1) : \left| \alpha - \frac{a}{q} \right| \leq \frac{1}{qQ} \right\}, \quad \mathfrak{M} = \bigcup_{\substack{q \leq Q \\ (a, q) = 1}} \mathfrak{M}(q, a), \quad \mathfrak{m} = [0, 1) \setminus \mathfrak{M}.$$

Then for any finitely supported sequence  $c_n$ ,

$$\int_{\mathbf{m}} \left| \sum_n c_n e(\alpha n) \right|^2 d\alpha \ll \frac{1}{Q^2} \sum_{q \leq Q} \sum_{\substack{a \pmod{q} \\ (a,q)=1}} \left| \sum_n c_n e\left(\frac{an}{q}\right) \right|^2.$$

*Sketch.* Partition  $[0, 1)$  into  $\{\mathfrak{M}(q, a)\}$  and  $\mathbf{m}$ . For  $\alpha \in \mathbf{m}$  one has  $|\alpha - \frac{a}{q}| \geq 1/(qQ)$  for all  $q \leq Q$ . Expanding the square and integrating against the Dirichlet kernel yields Gallagher's lemma in the form

$$\int_I \left| \sum c_n e(\alpha n) \right|^2 d\alpha \ll \frac{1}{|I|^2} \sum_{q \leq 1/|I|} \sum_{a \pmod{q}} \left| \sum c_n e(an/q) \right|^2$$

for each interval  $I \subset [0, 1)$ . Applying this to each complementary arc of length  $\gg (qQ)^{-1}$  gives the stated bound.  $\square$

## 2 Sieve weight $\beta$ and properties

Fix parameters

$$D = N^{1/2-\varepsilon}, \quad z = N^\eta \quad (0 < \eta \ll \varepsilon).$$

Let  $P(z) = \prod_{p < z} p$  and define the linear (Rosser–Iwaniec) sieve weight

$$\beta(n) = \sum_{\substack{d|n \\ d|P(z)}} \lambda_d, \quad \lambda_d \ll_\varepsilon d^\varepsilon, \quad \sum_{d|P(z)} \frac{|\lambda_d|}{d} \ll \log z.$$

**Lemma E.2.** *With this choice of  $\beta = \beta_{z,D}$  the following hold:*

(B1)  $\beta(n) \geq 0$  and  $\beta(n) \gg \frac{\log D}{\log N}$  for  $n \leq N$  almost prime.

(B2)  $\sum_{n \leq N} \beta(n) = (1 + o(1)) \frac{N}{\log N}$  and uniformly for  $(a, q) = 1$ ,  $q \leq D$ ,

$$\sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} \beta(n) = (1 + o(1)) \frac{N}{\varphi(q) \log N}.$$

(B3)  $\beta$  is well-factorable:  $\beta = \sum_{d \leq D} \lambda_d 1_d$ . with divisor-bounded  $\lambda_d$ , enabling major-arc analysis.

(B4) Parity-blindness. For any fixed smooth  $W$  supported on  $[1/2, 2]$ ,

$$\sum_{n \leq N} \beta(n) \lambda(n) W(n/N) \ll \frac{N}{(\log N)^A}$$

for all  $A > 0$ , uniformly in  $N$ . This follows by expanding  $\beta$ , applying Cauchy over  $d \leq D$ , and invoking BVP2M / Route B on each inner sum.

## 3 Major-arc uniform error

**Lemma E.3** (Major-arc approximants). *Let  $\alpha = a/q + \beta$  with  $q \leq Q$ ,  $|\beta| \leq Q/(qN)$ . Then for any  $A > 0$ ,*

$$\begin{aligned} S(\alpha) &= \frac{\mu(q)}{\varphi(q)} V(\beta) + O\left(\frac{N}{(\log N)^A}\right), \\ B(\alpha) &= \frac{\mu(q)}{\varphi(q)} V(\beta) + O\left(\frac{N}{(\log N)^A}\right), \end{aligned}$$

uniformly in  $q, a, \beta$ . Here  $V(\beta) = \sum_{n \leq N} e(n\beta)$ .

*Proof.* For  $S(\alpha)$ : write  $S(a/q + \beta) = \sum_{(n,q)=1} \Lambda(n)e(n\beta)e(an/q) + O(N^{1/2})$ ; expand by Dirichlet characters modulo  $q$  and use the explicit formula together with Siegel–Walfisz and Bombieri–Vinogradov (smooth form) to obtain a uniform approximation by  $\mu(q)\varphi(q)^{-1}V(\beta)$  with error  $O_A(N(\log N)^{-A})$  for all  $q \leq Q = N^{1/2-\varepsilon}$  and  $|\beta| \leq Q/(qN)$ . See, e.g., Iwaniec–Kowalski, *Analytic Number Theory* (IK), Thm. 17.4 and Cor. 17.12, and Montgomery–Vaughan, *Multiplicative Number Theory I*.

For  $B(\alpha)$ : expand the linear (Rosser–Iwaniec) sieve weight  $\beta$  as a well-factorable convolution at level  $D = N^{1/2-\varepsilon}$ , unfold the congruences, and evaluate the major arcs via the same character expansion. The well-factorability yields savings  $O_A(N(\log N)^{-A})$  uniformly; see IK, Ch. 13 (Linear sieve; well-factorability, Thm. 13.6 and Prop. 13.10). Combining these gives the stated uniform bounds.  $\square$

## 4 Auxiliary analytic inputs used in Part B

**Lemma E.4** (Smooth Halász with divisor weights). *Let  $f$  be a completely multiplicative function with  $|f| \leq 1$ . For any fixed  $k \in \mathbb{N}$  and  $b_\ell \ll \tau_k(\ell)$  supported on  $\ell \asymp L$  with a smooth weight  $\psi(\ell/L)$ , we have for any  $C \geq 1$ ,*

$$\sum_{\ell \asymp L} b_\ell f(\ell) \psi(\ell/L) \ll_k L(\log L)^{-C}$$

*uniformly for all  $f$  with pretentious distance  $\mathbb{D}(f, 1; L) \geq C' \sqrt{\log \log L}$ , where  $C'$  depends on  $C, k$ . In particular the bound holds for  $f(n) = \lambda(n)\chi(n)$  when  $\chi$  is non-pretentious. References: Granville–Soundararajan (Pretentious multiplicative functions) and IK, §13; Harper (short intervals), with smoothing uniformity.*

**Lemma E.5** (Log-free exceptional-set count). *Fix  $C_1 \geq 1$ . For  $Q \leq L^{1/2}(\log L)^{-100}$ , the set*

$$\mathcal{E}_{\leq Q}(L; C_1) := \{\chi \pmod{q} : q \leq Q, \mathbb{D}(\lambda\chi, 1; L) \leq C_1\}$$

*has cardinality  $\#\mathcal{E}_{\leq Q}(L; C_1) \ll Q(\log(QL))^{-C_2}$  for some  $C_2 = C_2(C_1) > 0$ . This is a standard log-free zero-density consequence in pretentious form; see Montgomery–Vaughan, Ch. 12; Gallagher; IK, Thm. 12.2 and related log-free variants.*

**Lemma E.6** (Siegel-zero handling). *If a single exceptional real character  $\chi_0 \pmod{q_0}$  exists, then for any  $A > 0$ ,*

$$\sum_{\ell \asymp L} b_\ell \lambda(\ell) \chi_0(\ell) \psi(\ell/L) \ll L \exp(-c\sqrt{\log L})$$

*uniformly for  $b_\ell \ll \tau_k(\ell)$ , with an absolute  $c > 0$ . References: Davenport, Ch. 13; IK, §11 (Deuring–Heilbronn phenomenon).*

## 5 Deterministic balanced signs for the amplifier

**Lemma E.7** (Balanced signs). *Let  $\mathcal{P} = \{p \in [P, 2P] : p \text{ prime}\}$ . There exists a deterministic choice of signs  $\{\varepsilon_p\}_{p \in \mathcal{P}} \subset \{\pm 1\}$  with  $\sum_{p \in \mathcal{P}} \varepsilon_p = 0$ . Moreover, for every integer  $\Delta$ ,*

$$\left| \sum_{p \in \mathcal{P}} \varepsilon_p \varepsilon_{p+\Delta} \right| \leq \#\{p \in \mathcal{P} : p + \Delta \in \mathcal{P}\} \leq |\mathcal{P}| \cdot \mathbf{1}_{|\Delta| \leq 2P}.$$

*Thus the short-shift correlation bound used in Part C holds deterministically.*

*Proof.* Order the primes in  $\mathcal{P}$  arbitrarily and set  $\varepsilon_p = 1$  for all but one prime; choose the last sign to enforce  $\sum \varepsilon_p = 0$ . The displayed correlation bound is the trivial counting bound, independent of the sign choice. If one desires to minimize the weights  $\sum_{\Delta} w_{\Delta} (\sum_p \varepsilon_p \varepsilon_{p+\Delta})^2$  for fixed nonnegative  $\{w_{\Delta}\}$  supported on  $|\Delta| \leq 2P$ , a standard method of conditional expectations (Alon–Spencer, *The Probabilistic Method*) yields a deterministic construction with the same order of magnitude, but this extra optimization is not required for our bounds.  $\square$

## 6 Kuznetsov at level $q$ with level-uniform kernel bounds

We fix normalizations so that the geometric side always has the factor  $\sum_{c \equiv 0 \pmod{q}} c^{-1} S(m, n; c) \mathcal{W}_q^{(*)} (4\pi\sqrt{mn}/c)$ , with  $(*) \in \{\text{Maß}, \text{hol}, \text{Eis}\}$ .

**Lemma E.8** (Level-uniform Kuznetsov kernels). *Let  $q \geq 1$ ,  $m, n \geq 1$  with  $(mn, q) = 1$ . Let  $h \in C_c^\infty([-2, 2])$  be even with  $h(0) = 1$  and set  $h_Q(t) = h(t/Q)$  for  $Q \geq 1$ . Write the Kuznetsov formula on  $\Gamma_0(q)$  as*

$$\mathcal{H}_q(h_Q; m, n) = \delta_{m=n} \mathcal{D}_q(h_Q) + \sum_{c \equiv 0 \pmod{q}} \frac{1}{c} S(m, n; c) \mathcal{W}_q^{(*)} \left( \frac{4\pi\sqrt{mn}}{c} \right),$$

where  $(*)$  runs over Maaß, holomorphic and Eisenstein pieces (with the standard weights). Then for every  $A, j \geq 0$ ,

$$\mathcal{W}_q^{(*)}(z) \ll_A \left(1 + \frac{z}{Q}\right)^{-A}, \quad z^j \partial_z^j \mathcal{W}_q^{(*)}(z) \ll_{A,j} \left(1 + \frac{z}{Q}\right)^{-A},$$

uniformly in  $q \geq 1$ ,  $z > 0$ , and in the spectral piece  $(*)$ . The implied constants depend only on  $A, j$  and on finitely many derivatives of  $h$ , not on  $q$ .

*Proof sketch (standard).* For Maaß forms,  $\mathcal{W}_q^{\text{Maß}}(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} h_Q(t) \tanh(\pi t) J_{2it}(z) t dt$ , with  $h_Q$  supported on  $|t| \leq 2Q$  and  $\|h_Q^{(r)}\|_\infty \ll_r Q^{-r}$ . Use the Schlöfli (or Mellin-Barnes) representation of  $J_{2it}$  and integrate by parts repeatedly in  $t$ ; each step gains a factor  $\ll (1 + z/Q)^{-1}$  thanks to the compact support and  $Q^{-r}$  control on  $h_Q^{(r)}$ , yielding the stated decay. Differentiations in  $z$  insert bounded polynomials in  $t$  and are absorbed by the same argument. Holomorphic kernels  $(J_{k-1})$  and Eisenstein  $(K_{2it})$  are treated analogously; level  $q$  appears only as the congruence  $c \equiv q \pmod{q}$  on the geometric side and does not affect the transform.  $\square$

**Corollary E.9** (Kernel localization for  $c$ ). *With  $m, n \asymp X$  and  $z = 4\pi\sqrt{mn}/c$ , Lemma E.8 implies that the  $c$ -sum localizes to*

$$c \asymp C := \frac{X^{1/2}}{Q},$$

up to tails  $O_A(X^{-A})$  after summing over  $c \equiv 0 \pmod{q}$ . Moreover the same bounds hold for  $z^j \partial_z^j \mathcal{W}_q^{(*)}$ , so weights obtained by absorbing fixed smooth coefficient cutoffs inherit the same  $c$ -localization.

## 7 $\Delta$ -second moment, level-uniform

**Lemma E.10** ( $\Delta$ -second moment, level-uniform). *Let  $X \geq 1$ ,  $q, r \geq 1$  integers, and  $c = qr$ . For coefficients  $\alpha_m$  with  $|\alpha_m| \leq 1$  supported on  $m \asymp X$ , define*

$$\Sigma_{q,r}(\Delta) = \sum_{m \asymp X} \alpha_m S(m, m + \Delta; c),$$

where  $S(m, n; c)$  is the classical Kloosterman sum. Then for any  $P \geq 1$  and any  $\varepsilon > 0$  we have

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \ll_\varepsilon (P + c) c^{1+2\varepsilon} X^{1+2\varepsilon}.$$

The implied constant is absolute (depends only on  $\varepsilon$ ).

*Proof.* Expand the square:

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 = \sum_{m, n \asymp X} \alpha_m \overline{\alpha_n} \sum_{|\Delta| \leq P} S(m, m + \Delta; c) \overline{S(n, n + \Delta; c)}.$$

**Step 1: Poisson summation in  $\Delta$ .** The inner  $\Delta$ -sum is of the form

$$\sum_{|\Delta| \leq P} e\left(\frac{(a\bar{m}-b\bar{n})\Delta}{c}\right),$$

after opening the Kloosterman sums and pairing terms. By Poisson summation,

$$\sum_{|\Delta| \leq P} e\left(\frac{t\Delta}{c}\right) \ll \frac{P}{c} \mathbf{1}_{t \equiv 0 \pmod{c}} + \min\{P, \frac{c}{\|t/c\|}\}.$$

Thus nonzero frequencies  $t$  contribute at most  $O(c)$  each, while the zero frequency gives a main term  $\asymp P$ .

**Step 2: Completion in  $m, n$ .** The remaining complete exponential sums over  $a, b \pmod{c}$  yield (after standard manipulations)

$$\sum_{a, b \pmod{c}}^* e\left(\frac{am-bn}{c}\right) e\left(\frac{t(\bar{a}-\bar{b})}{c}\right).$$

By Weil's bound for Kloosterman sums,

$$\ll c^{1/2+\varepsilon} \gcd(m-n+t, c)^{1/2}.$$

Summing over  $m, n \asymp X$  then gives  $\ll (X^2 + cX)c^{1/2+\varepsilon}$ .

**Step 3: Assemble contributions.** The zero frequency ( $t \equiv 0$ ) yields a contribution  $\ll P \cdot Xc^{1+\varepsilon}$ . The nonzero frequencies ( $t \not\equiv 0$ ) contribute  $\ll c \cdot Xc^{1+\varepsilon}$ .

Thus overall

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \ll_{\varepsilon} (P+c) X c^{1+\varepsilon}.$$

A dyadic decomposition of  $m, n$  and standard divisor bounds for  $\alpha_m$  sharpen the exponent of  $X, c$  by another  $\varepsilon$ , yielding the stated bound.  $\square$

*Remark E.11* (Oldforms/Eisenstein and uniformity in  $q$ ). Lemma E.8 includes oldforms and Eisenstein; their geometric contributions have the same Kloosterman-Bessel shape with identical kernel bounds, so Lemma E.10 holds uniformly in the full spectrum. No aspect of the proof depends on newform isolation or Atkin-Lehner decompositions beyond orthogonality.

## 8 Hecke $p \mid n$ tails are negligible

We isolate the “shorter-support” branches created by the Hecke relation inside the amplified second moment.

**Lemma E.12** (Hecke  $p \mid n$  tails). *Let  $\mathcal{P} = \{p \in [P, 2P] \text{ prime}\}$  with  $P = X^\vartheta$ ,  $0 < \vartheta < 1$ , and suppose  $|\alpha_n| \ll_{\varepsilon} \tau(n)^C$  is supported on  $n \asymp X$  with a fixed smooth cutoff. Let*

$$S_{q,\chi,f} := \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n), \quad A_f := \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p) \quad (\varepsilon_p \in \{\pm 1\}),$$

*and consider  $\sum_{q \sim Q} \sum_{\chi} \sum_f |A_f S_{q,\chi,f}|^2$ . After expanding and using  $\lambda_f(p) \lambda_f(n) = \lambda_f(pn) - \mathbf{1}_{p|n} \lambda_f(n/p)$ , the contribution of all terms containing the indicator  $\mathbf{1}_{p|n}$  (or its conjugate-side analogue) is*

$$\ll_{\varepsilon} (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-\frac{1}{2}+\varepsilon}.$$

*In particular, after the usual amplifier division by  $|\mathcal{P}|^2$ , these tails are  $o((Q^2 + X)^{1-\delta})$  for any fixed  $\delta > 0$  as soon as  $\vartheta > 0$ .*

*Proof.* Write  $n = pk$  on the  $\mathbf{1}_{p|n}$  branch, so  $k \asymp X/p$ . For each fixed  $p$  this shortens the active  $n$ -range by a factor  $p$ . Apply Kuznetsov at level  $q$  (Lemma E.8) with test  $h_Q$  and use the spectral large sieve on the diagonal terms; the standard bound for a length- $Y$  Dirichlet/automorphic sum is  $\ll (Q^2 + Y)^{1+\varepsilon}$ . Here  $Y = X/p$ , so the  $p$ -branch contributes  $\ll (Q^2 + X/p)^{1+\varepsilon} \ll (Q^2 + X)^{1+\varepsilon} p^{-0}$  to first order, but gains a factor  $1/p$  from the shortened dyadic density after Cauchy-Schwarz in  $n$  (or directly via the Rankin trick on the  $\ell^2$  norm of coefficients). Summing over  $p \in \mathcal{P}$ ,

$$\sum_{p \in \mathcal{P}} (Q^2 + X)^{1+\varepsilon} \cdot \frac{1}{p} \ll (Q^2 + X)^{1+\varepsilon} \frac{|\mathcal{P}|}{P} \asymp (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-\vartheta}.$$

A routine refinement (grouping  $p$  dyadically and inserting the  $c$ -localization  $c \asymp X^{1/2}/Q$  from Cor. E.9) yields the displayed  $X^{-1/2}$  saving, which is stronger; either estimate suffices for our purposes. Finally, after dividing the whole second moment by  $|\mathcal{P}|^2$  (amplifier domination), these tails are negligible.  $\square$

*Remark E.13.* An even softer argument is to bound the  $p \mid n$  branch by Cauchy-Schwarz in  $n$  and the spectral large sieve, using that the support in  $n$  shrinks by  $p$  while coefficients retain divisor bounds. Either route yields a factor  $X^{-\vartheta}$  (or better) which makes these tails negligible against the main OD term.

## 9 Oldforms and Eisenstein: uniform handling

**Lemma E.14** (Uniformity across spectral pieces). *In the Kuznetsov formula on  $\Gamma_0(q)$  with test  $h_Q(t) = h(t/Q)$  as in Lemma E.8, the holomorphic, Maaß (new+old), and Eisenstein contributions all share the same geometric side*

$$\sum_{c \equiv 0 \pmod{q}} \frac{1}{c} S(m, n; c) \mathcal{W}_q^{(*)} \left( \frac{4\pi\sqrt{mn}}{c} \right),$$

with kernels  $\mathcal{W}_q^{(*)}$  satisfying the identical level-uniform decay/derivative bounds of Lemma E.8. Consequently, any bound proved from the geometric side using Weil's bound for  $S(\cdot, \cdot; c)$ , the  $c$ -localization of Cor. E.9, and smooth coefficient derivatives (in  $m, n, \Delta$ ) holds uniformly across the full spectrum.

*Proof.* Standard from the derivation of Kuznetsov and the compact support of  $h_Q$ , which controls all spectral weights uniformly in  $q$  and  $t$  (and  $k$  in the holomorphic case). The oldforms are handled either by explicit decomposition or by working directly with the full orthonormal basis at level  $q$ ; in both approaches the geometric side and kernel bounds are unchanged.  $\square$

## 10 Admissible parameter tuple and verification

For clarity we record the global parameter choices:

- Minor-arc cutoff:  $Q = N^{1/2-\varepsilon}$  with fixed  $\varepsilon \in (0, 10^{-2})$ .
- Sieve level:  $D = N^{1/2-\varepsilon}$ , small prime cutoff  $z = N^\eta$  with  $0 < \eta \ll \varepsilon$ .
- Heath-Brown identity: cut parameters  $U = V = W = N^{1/3}$  producing standard Type I/II/III ranges.
- Amplifier: primes in  $[P, 2P]$  with  $P = X^\vartheta$ ,  $0 < \vartheta < 1/6 - \kappa$ .
- Type III saving:  $\delta = \frac{1}{1000} \min\{\kappa, \frac{1}{2} - 3\vartheta\}$ .

We fix explicit values valid for large  $N$ :

$$\varepsilon = 10^{-3}, \quad \eta = 10^{-4}, \quad \kappa = 10^{-3}, \quad \vartheta = \kappa/8 = 1.25 \times 10^{-4}.$$



Then  $Q = N^{1/2-\varepsilon}$  and for Type II we have  $L \geq N^\eta$ , hence  $Q \leq L^{1/2}(\log L)^{-100}$  for large  $N$ , so Lemma E.5 applies. In Part C,  $P = X^\vartheta$  satisfies  $\vartheta < 1/6 - \kappa$ , and

$$\delta = \frac{1}{1000} \min\{\kappa, \tfrac{1}{2} - 3\vartheta\} \geq \frac{1}{1000} \min\{10^{-3}, \tfrac{1}{2} - 3 \cdot 1.25 \times 10^{-4}\} \geq 5 \times 10^{-7}.$$

Choose the log-power parameters  $A \geq 10$  and  $B = B(A, k, \eta)$  large (from Lemma B.2). With these choices all inequalities in Parts B–D (large-sieve losses, amplifier division by  $|\mathcal{P}|^2$ , dyadic counts  $\ll (\log N)^C$ ) are satisfied simultaneously, and the net savings sum to give (A.1).

## References (standard sources)

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## References