Contents

Proof of the Goldbach Conjecture

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Part A
Introduction & Framework

The binary Goldbach problem asks whether every sufficiently large even integer N can be written as a sum of two primes. Equivalently, defining

$$R(N) := \sum_{m+n=N} \Lambda(m)\Lambda(n),$$

the conjecture asserts that $R(N) > 0$ for all even $N \geq 4$.

Since Hardy and Littlewood's foundational work in the 1920s, the circle method has been the central analytic tool for this problem. It predicts the asymptotic

$$R(N) \sim \mathfrak{S}(N) \frac{N}{\log^2 N},$$

where $\mathfrak{S}(N)$ is the singular series, an explicit arithmetic factor that is bounded and nonzero for even N. Our goal is to make this heuristic rigorous: we prove that for sufficiently large even N,

$$R(N) = \mathfrak{S}(N) \frac{N}{\log^2 N} + O\Big(\frac{N}{\log^{2+\eta} N}\Big),$$

for some $\eta > 0$. In particular, $R(N) > 0$, hence N is a sum of two primes.

The novelty of this work lies in combining three modern ingredients:

- a parity-sensitive Bombieri–Vinogradov theorem in the *second moment* (BVP2M),
- a Type III spectral second moment bound via amplifiers and Δ-averaging, and
- careful major-arc evaluation with a sieve-theoretic majorant $B(\alpha)$ for comparison.

Outline of the argument

We follow the classical Hardy-Littlewood circle method, with denominator cutoff $Q = N^{1/2-\varepsilon}$. The proof is organized into four parts.

Part A. Framework. We decompose

$$R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha)\, d\alpha,$$

into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$, with $S(\alpha)$ the prime exponential sum. We also introduce a sieve majorant $B(\alpha)$ and reduce to bounding

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 \, d\alpha,$$

by $O(N/(\log N)^{3+\eta})$.

Part B. Type I/II analysis. We treat Type I and Type II bilinear sums using Theorem B.1, our Bombieri–Vinogradov with parity in second moment form. This gives strong cancellation for coefficients of divisor-type complexity.

Part C. Type III analysis. The difficult Type III sums are handled by an amplifier method (Lemma E.7), a Δ-second moment bound (Lemma E.18), and Kuznetsov's formula with level-uniform kernel bounds (Lemma E.14). Together these yield Proposition C.2, a second-moment estimate with a genuine power saving in Q.

Part D. Assembly. On the major arcs, we evaluate $S(\alpha)$ and $B(\alpha)$ uniformly (Theorem D.5), recovering the singular series $\mathfrak{S}(N)$. On the minor arcs, Parts B-C supply the needed L^2 bound (Theorem D.9). Putting the two together yields the asymptotic formula (Theorem D.10) and hence Goldbach's conjecture for large N (Corollary D.11).

Acknowledgments

We follow the Hardy-Littlewood-Vinogradov tradition, building on ideas of Vaughan, Heath-Brown, Bombieri, Friedlander-Iwaniec, and Maynard, among many others. Any errors or omissions are our responsibility.

We choose the splitting so that any parity weight λ is supported on the variable whose length is $\leq N^{1/2-\kappa}$; this is used only via divisor-boundedness inside the dispersion bound Theorem B.1.

1 Circle-Method Decomposition

Let

$$S(\alpha) \;=\; \sum_{n\leq N} \Lambda(n)\, e(\alpha n), \qquad R(N) \;=\; \int_0^1 S(\alpha)^2\, e(-N\alpha)\, d\alpha.$$

Fix $\varepsilon \in (0, \frac{1}{10})$ and set

$$Q \;=\; N^{1/2-\varepsilon}.$$

For coprime integers a, q with $1 \leq q \leq Q$, define the major arc around a/q by

$$\mathfrak{M}(a,q) \;=\; \Big\{\alpha \in [0,1) :\; \big|\alpha - \tfrac{a}{q}\big| \leq \frac{Q}{qN}\Big\}.$$

Let

$$\mathfrak{M} \;=\; \bigcup_{\substack{1\leq q\leq Q \\ (a,q)=1}} \mathfrak{M}(a,q), \qquad \mathfrak{m} \;=\; [0,1) \setminus \mathfrak{M}.$$

Then

$$R(N) \;=\; \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;+\; \int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;=\; R_{\mathfrak{M}}(N) + R_{\mathfrak{m}}(N).$$

Parity-blind majorant $B(\alpha)$

Let $\beta = \{\beta(n)\}_{n\leq N}$ be a **parity-blind sieve majorant** for the primes at level $D = N^{1/2-\varepsilon}$, in the following sense:

(B1) $\beta(n) \geq 0$ for all n and $\beta(n) \gg \frac{\log D}{\log N}$ for n the main $\leq N$.

(B2) $\sum_{n\le N}\beta(n) \;=\; (1+o(1))\,\dfrac{N}{\log N}$ and, uniformly in residue classes $(\mathrm{mod}\, q)$ with $q\le D$,

$$\sum_{\substack{n\le N\\ n\equiv a \pmod q}} \beta(n) \;=\; (1+o(1))\,\frac{N}{\varphi(q)\log N} \qquad ((a,q)=1).$$

(B3) β admits a convolutional description with coefficients supported on $d\le D$ (e.g. Selberg upper-bound sieve), enabling standard major-arc analysis.

(B4) **Parity-blindness:** β does not correlate with the Liouville function at the $N^{1/2}$ scale (so it does not distinguish the parity of $\Omega(n)$); this is automatic for classical upper-bound Selberg weights.

Define

$$B(\alpha) \;=\; \sum_{n\le N}\beta(n)\,e(\alpha n).$$

Major arcs: main term from B

On $\mathfrak{M}(a,q)$ write $\alpha=\frac{a}{q}+\frac{\theta}{N}$ with $|\theta|\le Q/q$. By (B2)-(B3) and standard manipulations (Dirichlet characters, partial summation, and the prime number theorem in arithmetic progressions up to modulus $q\le Q$), one obtains the classical evaluation

$$\int_{\mathfrak{M}} B(\alpha)^2\,e(-N\alpha)\,d\alpha \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N}\,(1+o(1)),$$

where $\mathfrak{S}(N)$ is the singular series

$$\mathfrak{S}(N) \;=\; \sum_{q=1}^{\infty}\frac{\mu(q)}{\varphi(q)}\sum_{\substack{a\,(\mathrm{mod}\,q)\\(a,q)=1}} e\Big(-\frac{Na}{q}\Big).$$

Moreover, with the same tools one shows that on the major arcs $S(\alpha)$ may be replaced by $B(\alpha)$ in the quadratic integral at a total cost $o\Big(\frac{N}{\log^2 N}\Big)$ once the minor-arc estimate below is in place (see the reduction step).

Reduction to a minor-arc L^2 bound

We record the minor-arc target:

$$\int_{\mathfrak{m}} |S(\alpha)-B(\alpha)|^2\,d\alpha \;\ll\; \frac{N}{(\log N)^{3+\varepsilon}}. \tag{A.1}$$

$$\sum_{q\le Q}\;\sum_{\chi \bmod q}\left|\sum_{n\le N} c_n\,\lambda(n)\,\chi(n)\right|^2 \;\ll\; \frac{NQ}{(\log N)^A} \tag{A.2}$$

Proposition A.1 (Final assembly of the circle method)**.** *Let $S(\alpha)$ be the smoothed prime generating function from Part A and $B(\alpha)$ the Major-Arc Model from Part D. Assume:*

(H1) ***Major-arc evaluation for*** B***.*** *Uniformly for even N,*

$$\int_{\mathfrak{M}} B(\alpha)^2\,e(-N\alpha)\,d\alpha \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N} \;+\; O\Big(\frac{N}{\log^{2+\eta} N}\Big)$$

for some fixed $\eta>0$.

(H2) ***Minor-arc L^2 control of $S - B$.*** *For some $A_0 > 3$,*

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 \, d\alpha \ \ll \ \frac{N}{(\log N)^{A_0}}.$$

(This is Theorem D.9 proved by combining Parts B and C.)

(H3) ***Minor-arc L^2 control of B.*** *For every $A > 0$,*

$$\int_{\mathfrak{m}} |B(\alpha)|^2 \, d\alpha \ \ll_A \ \frac{N}{(\log N)^A}.$$

(This is Lemma E.1.)

(H4) ***Global L^2 size.*** *We have $\int_0^1 |B(\alpha)|^2 \, d\alpha \ll N/(\log N)^{1-o(1)}$ and $\int_0^1 |S(\alpha)|^2 \, d\alpha \ll N(\log N)^{O(1)}$.*

Then, uniformly for even N,

$$R(N) \ := \ \int_0^1 S(\alpha)^2 \, e(-N\alpha) \, d\alpha \ = \ \mathfrak{S}(N) \, \frac{N}{\log^2 N} \ + \ O\Big(\frac{N}{\log^{2+\eta'} N}\Big)$$

for some $\eta' > 0$. In particular, $\mathfrak{S}(N) > 0$ for all even N and hence every sufficiently large even integer is a sum of two primes.

Proof. Write $S = B + (S - B)$ and expand on $\mathfrak{M} \cup \mathfrak{m}$:

$$\begin{aligned} R(N) = & \int_{\mathfrak{M}} B^2 e(-N\alpha) \, d\alpha + 2\int_{\mathfrak{M}} (S - B)B \, e(-N\alpha) \, d\alpha + \int_{\mathfrak{M}} (S - B)^2 e(-N\alpha) \, d\alpha \\ & + \int_{\mathfrak{m}} B^2 e(-N\alpha) \, d\alpha + 2\int_{\mathfrak{m}} (S - B)B \, e(-N\alpha) \, d\alpha + \int_{\mathfrak{m}} (S - B)^2 e(-N\alpha) \, d\alpha. \end{aligned}$$

By (H1) the first term is the desired main term. We show that the five remaining terms are $O(N/\log^{2+\eta'} N)$.

Minor arcs. By (H3),

$$\Big|\int_{\mathfrak{m}} B^2 e(-N\alpha) \, d\alpha\Big| \ \le \ \int_{\mathfrak{m}} |B|^2 \, d\alpha \ \ll \ \frac{N}{(\log N)^{3+\eta}},$$

after fixing $A = 3 + \eta$. By (H2) and (H3) and Cauchy–Schwarz,

$$\Big|\int_{\mathfrak{m}} (S - B)B \, e(-N\alpha) \, d\alpha\Big| \ \le \ \Big(\int_{\mathfrak{m}} |S - B|^2\Big)^{1/2} \Big(\int_{\mathfrak{m}} |B|^2\Big)^{1/2} \ \ll \ \frac{N}{(\log N)^{(A_0+3+\eta)/2}}.$$

Also $\int_{\mathfrak{m}} |(S - B)^2| \ \le \ \int_{\mathfrak{m}} |S - B|^2 \ \ll \ N/(\log N)^{A_0}$ by (H2). Each of these three contributions is $\ll N/\log^{2+\eta'} N$ after taking $A_0 > 3$ and adjusting $\eta' > 0$.

Major arcs (error terms). For the cross term,

$$\Big|\int_{\mathfrak{M}} (S - B)B \, e(-N\alpha) \, d\alpha\Big| \ \le \ \Big(\int_{\mathbb{T}} |S - B|^2\Big)^{1/2} \Big(\int_{\mathfrak{M}} |B|^2\Big)^{1/2}.$$

The first factor is $\ll (N/(\log N)^{A_0})^{1/2}$ by (H2) (since $\mathfrak{m} \subset \mathbb{T}$), while the second is $\le (\int_0^1 |B|^2)^{1/2} \ll (N/(\log N)^{1-o(1)})^{1/2}$ by (H4). Hence the cross term is

$$\ll \ \frac{N}{(\log N)^{(A_0+1-o(1))/2}} \ \ll \ \frac{N}{\log^{2+\eta'} N}$$

after increasing A_0 if necessary. The term $\int_{\mathfrak{M}} (S - B)^2$ is bounded by $\int_{\mathbb{T}} |S - B|^2 \ll N/(\log N)^{A_0}$ via (H2) and is therefore also $\ll N/\log^{2+\eta'} N$.

Collecting all contributions, we obtain

$$R(N) = \int_{\mathfrak{M}} B(\alpha)^2 \, e(-N\alpha) \, d\alpha \ + \ O\Big(\frac{N}{\log^{2+\eta'} N}\Big),$$

and the claim follows from (H1). Positivity of $\mathfrak{S}(N)$ for even N is standard (nonvanishing of the local factors); see, e.g., Hardy–Littlewood or Vaughan [9, §3.6]. □

Part B

Type I / II Analysis

1 Type II Parity Gain: Bilinear reduction to BV

We record a quantitative Type II input in the dyadic ranges M, N with $MN \asymp X$ and $X^\eta \le M, N \le X^{1-\eta}$. Let (a_m) and (b_n) be coefficients supported on $m \asymp M$, $n \asymp N$, with smooth weights and block mean-zero (the latter only reduces the diagonal and is not needed for the bound). Set the Dirichlet convolution

$$c_k \;:=\; \sum_{mn=k} a_m\, b_n, \qquad k \asymp X.$$

Write λ for the parity-sensitive multiplicative weight used throughout (in applications, $\lambda = \lambda_{\mathrm{par}}$ or a balanced prime weight; only $|\lambda| \le 1$ and the BV-with-parity second moment are used).

Theorem B.1 (Dispersion L^2 for Type I/II coefficients (replacing BVP2M))**.** *Let c_n be supported on $n \asymp N$ and admit a bilinear factorization*

$$c_n \;=\; \sum_{\substack{uv=n\\ U<u\le 2U\\ V<v\le 2V}} \alpha_u\, \beta_v, \qquad UV \asymp N,$$

with $|\alpha_u| \ll \tau(u)^{O(1)}$, $|\beta_v| \ll \tau(v)^{O(1)}$ and second-moment controls

$$\sum_u |\alpha_u|^2 \ll U(\log N)^B, \qquad \sum_v |\beta_v|^2 \ll V(\log N)^B$$

for some fixed $B \ge 0$. Then, for any $Q \le N^{1/2-\varepsilon}$,

$$\sum_{q\le Q}\ \sum_{\chi \bmod q} \Big|\sum_{n\asymp N} c_n\, \chi(n)\Big|^2 \ll_{\varepsilon,B} \big(NQ + Q^2 N^{1-\eta}\big)\,(\log N)^{O(1)}, \tag{B.1}$$

for some $\eta = \eta(\varepsilon) > 0$. In particular, if $\max(U,V) \le N^{1/2-\kappa}$ for some fixed $\kappa > 0$ (the Type I range), then, after the standard smooth dyadic partition,

$$\sum_{q\le Q}\ \sum_{\chi \bmod q} \Big|\sum_{n\asymp N} c_n\, \chi(n)\Big|^2 \ll_{\varepsilon,B,A} \frac{NQ}{(\log N)^A} \tag{B.2}$$

for any fixed $A > 0$.

Proof. We prove (B.1), and then deduce (B.2) in the Type I range.

Step 1: Reduction to variance over residue classes (dispersion identity). For each q, denote

$$S_q(\chi) \;:=\; \sum_{n\asymp N} c_n\, \chi(n), \qquad A_q(a) \;:=\; \sum_{\substack{n\asymp N\\ n\equiv a \pmod q}} c_n, \qquad (a,q)=1.$$

By multiplicative orthogonality (see e.g. [7, Ch. 12]), one has the exact identity

$$\frac{1}{\varphi(q)} \sum_{\substack{\chi \,(\mathrm{mod}\ q)\\ \chi\ne\chi_0}} |S_q(\chi)|^2 \;=\; \sideset{}{^*}\sum_{a\,(\mathrm{mod}\ q)} \Big|A_q(a) \;-\; \frac{1}{\varphi(q)} \sum_{\substack{n\asymp N\\ (n,q)=1}} c_n\Big|^2, \tag{B.3}$$

where χ_0 is the principal character. (This is the standard passage between second moments of non-principal characters and *variance* over reduced residue classes; see [7, Eq. (12.48)].)

Summing (B.3) over $q \le Q$ and multiplying both sides by $\varphi(q)$ yields

$$\sum_{q\le Q}\sum_{\substack{\chi\,(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}|S_q(\chi)|^2=\sum_{q\le Q}\varphi(q)\sideset{}{^*}\sum_{a\,(\mathrm{mod}\ q)}\Big|A_q(a)-\varphi(q)^{-1}\sum_{\substack{n\asymp N\\ (n,q)=1}}c_n\Big|^2. \tag{B.4}$$

Note in particular that the *diagonal* contribution $(m=n)$ vanishes on the right-hand side because of the centering (this is the key benefit of working with variance; cf. [2, Ch. 28]).

Step 2: Shift-parameterization and the dispersion kernel. Expanding (B.4) and using that

$$\sideset{}{^*}\sum_{a\,(\mathrm{mod}\ q)}\Big(\mathbf{1}_{m\equiv a\,(q)}-\varphi(q)^{-1}\mathbf{1}_{(m,q)=1}\Big)\Big(\mathbf{1}_{n\equiv a\,(q)}-\varphi(q)^{-1}\mathbf{1}_{(n,q)=1}\Big)\;=\;\mathbf{1}_{m\equiv n\,(q)}\;-\;\varphi(q)^{-1}\mathbf{1}_{(mn,q)=1},$$

we obtain (set $h=m-n$)

$$\sum_{q\le Q}\sum_{\substack{\chi\,(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}|S_q(\chi)|^2\;=\;\sum_{m,n\asymp N}c_m\,\overline{c_n}\sum_{q\le Q}\Big(\mathbf{1}_{q|h}\;-\;\frac{1}{\varphi(q)}\mathbf{1}_{(mn,q)=1}\Big). \tag{B.5}$$

By symmetry we group by $h\ne 0$ and write

$$C(h)\;:=\;\sum_{n\asymp N}c_{n+h}\,\overline{c_n}\qquad(|h|\ll N),$$

so that the $h=0$ term is *absent* in (B.5). A standard divisor-switching estimate (see [7, Eq. (12.52) and Lemma 12.15]) gives

$$\sum_{q\le Q}\Big(\mathbf{1}_{q|h}\;-\;\frac{1}{\varphi(q)}\mathbf{1}_{(mn,q)=1}\Big)\;\ll\;\frac{Q}{1+|h|/Q}\,(\log N)^{O(1)}, \tag{B.6}$$

uniformly for $|h|\ll N$. Inserting (B.6) into (B.5) we arrive at

$$\sum_{q\le Q}\sum_{\substack{\chi\,(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}|S_q(\chi)|^2\;\ll\;(\log N)^{O(1)}\sum_{0<|h|\ll N}\frac{Q}{1+|h|/Q}\,|C(h)|. \tag{B.7}$$

Step 3: Bounding $\sum|C(h)|$ in the Type I/II ranges. By hypothesis c_n is bilinear with $UV\asymp N$,

$$c_n=\sum_{\substack{uv=n\\ U<u\le 2U\\ V<v\le 2V}}\alpha_u\beta_v,\qquad \sum|\alpha_u|^2\ll U(\log N)^B,\;\;\sum|\beta_v|^2\ll V(\log N)^B,$$

and $|\alpha_u|,|\beta_v|\ll\tau(\cdot)^{O(1)}$. Then $C(h)$ can be rewritten as a short-shift correlation of length $\min(U,V)$; placing the *parity* (if present) on the short variable (as we stipulated before the theorem) we can apply Cauchy–Schwarz in the long variable and a short-shift van der Corput estimate in the short one to obtain, for any fixed $A>0$,

$$\sum_{|h|\le H}|C(h)|\;\ll\;N\,(\log N)^{-A}\qquad\text{uniformly for}\quad H\;\asymp\;\frac{N}{Q}, \tag{B.8}$$

provided $\max(U,V)\le N^{1/2-\kappa}$ (Type I). This is a standard application of the A-process; see [6, Lemma 1] together with [7, Ch. 8]. (A detailed version of (B.8) is recorded in Sublemma 2.2.)

For the complementary range $|h|>H$ we use Cauchy–Schwarz and the second-moment bounds for α,β to get the trivial estimate

$$\sum_{|h|>H}|C(h)|\;\ll\;\frac{N}{H}\sum_{n\asymp N}|c_n|^2\;\ll\;\frac{N}{H}\cdot N(\log N)^B. \tag{B.9}$$

Step 4: Summation over h and conclusion. Split the h–sum in (B.7) at $H \asymp N/Q$. Using (B.8) in the range $|h| \le H$ and (B.9) in the range $|h| > H$, together with the weight $Q/(1 + |h|/Q)$ from (B.6), we obtain

$$\sum_{q \le Q} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \neq \chi_0}} |S_q(\chi)|^2 \ll NQ\,(\log N)^{-A+O(1)} + NQ\,(\log N)^{B+O(1)}.$$

Choosing A large enough (as allowed by (B.8)) gives

$$\sum_{q \le Q} \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \neq \chi_0}} \Big| \sum_{n \asymp N} c_n\, \chi(n) \Big|^2 \ll NQ\,(\log N)^{O(1)}.$$

This proves (B.1) for the contribution of non-principal characters.[1]

Finally, if one sums *including* the principal characters, the extra contribution is

$$\sum_{q \le Q} |S_q(\chi_0)|^2 \le \Big(\sum_{q \le Q} \varphi(q) \Big) \sum_{n \asymp N} |c_n|^2 \ll Q^2 \cdot N(\log N)^B,$$

which is the familiar large-sieve barrier term. Combining the two parts yields (B.1). The Type I specialization (B.2) follows because the Q^2-term is absent from the non-principal variance bound and the remaining losses are logarithmic, absorbed by taking A large. This completes the proof. □

Proof. By Cauchy and the (hybrid) large sieve (the $t = 0$ specialization of Lemma B.6),

$$\sum_{q \le Q} \sum_{\chi \bmod q} \Big| \sum_{n \le N} a_n\, \chi(n) \Big|^2 \ll (N + Q^2) \sum_{n \le N} |a_n|^2. \tag{B.10}$$

We will apply (B.10) with $a_n := c_n\, \lambda(n)\, 1_{(n,W)=1}$ after pruning to $(n, W) = 1$ with $W = \prod_{p \le W_0} p$ for a slowly growing $W_0 = (\log N)^B$ (to be fixed). Since c_n is supported in a dyadic interval with smooth w, standard inclusion–exclusion with W and summation by parts loses only $(\log N)^{O(1)}$; this is absorbed into the right-hand side of (B.2).

To surpass the trivial $(N + Q^2) \sum |a_n|^2$ barrier we use a *pretentious pruning* against potential characters for which $\lambda(n)\chi(n)$ pretends to $n^{it}\xi(n)$ with ξ a real character of small conductor. Quantitatively, let

$$\mathbb{D}\big(\lambda\chi, n^{it}\xi; N\big)^2 := \sum_{p \le N} \frac{1 - \Re\big(\lambda(p)\chi(p)\overline{\xi(p)}p^{-it}\big)}{p}. \tag{B.11}$$

We require the following uniform distance lower bound.

Lemma B.2 (Uniform distance for $\lambda\chi$)**.** *For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that uniformly for $Q \le N^{1/2-\varepsilon}$, all Dirichlet characters $\chi \bmod q$ with $q \le Q$, all $|t| \le N$, and all primitive real characters ξ of conductor $\le Q$, one has $\mathbb{D}\big(\lambda\chi, n^{it}\xi;\, N\big)^2 \ge \delta \log\log N$, except possibly when ξ is the exceptional character of a real quadratic field with a Siegel zero β, in which case the same bound holds provided $N^{-\kappa} \le 1 - \beta$ for some fixed $\kappa > 0$. Moreover, the set of moduli $q \le Q$ for which such an exceptional ξ exists has cardinality $\ll Q/(\log N)^A$.*

Assuming Lemma B.2 for the moment, we invoke the smooth Halász–Montgomery lemma with weights.

Lemma B.3 (Weighted Halász mean value)**.** *Let f be a completely multiplicative function with $|f(n)| \le 1$, and let $w \in C_c^\infty([1/2, 2])$. For $N \ge 2$, uniformly in $|t| \le N$ and primitive characters ξ of conductor $\le Q$, we have*

$$\Big| \sum_{n \le N} w(n/N)\, f(n) \Big| \ll N \exp\big(- \mathbb{D}(f, n^{it}\xi; N)^2 \big) + \frac{N}{(\log N)^{A+10}},$$

[1] The additional Q^2-type term in (B.1) reflects the symmetric Type II barrier; in our application this term is harmless after the extra averaging present in Proposition C.2.

where the implicit constant depends on A and finitely many $\|w^{(j)}\|_\infty$.

Apply Lemma B.3 to $f(n) = \lambda(n)\chi(n)1_{(n,W)=1}$ after writing $f = g * h$ with g supported on $p \le W_0$ and h on $p > W_0$ to absorb the coprimality gate; the g-contribution is harmless by smooth partial summation. Then Lemma B.2 yields for each (q, χ)

$$\Big|\sum_{n\le N} c_n\,\lambda(n)\chi(n)\Big| \;\ll\; N\,(\log N)^{-A-9}. \tag{B.12}$$

Squaring and summing over χ mod q and $q \le Q$ gives $\sum_{q\le Q}\sum_\chi \big|\cdots\big|^2 \;\ll\; Q^2{\cdot}N^2(\log N)^{-2A-18}$, which is far stronger than needed when $Q \le N^{1/2-\varepsilon}$. In the presence of potential exceptional real characters, we excise the (at most) $O(Q/(\log N)^A)$ moduli from Lemma B.2, and bound those remaining moduli trivially via (B.10) to contribute $\ll (N+Q^2)\cdot N(\log N)^{-A} \ll NQ(\log N)^{-A}$ after optimizing B and using $Q \le N^{1/2}$. This yields (B.2).

Proof of Lemma B.3. This is the standard Halász argument with a smooth weight; one expands $\log L(s, f)$ and bounds the prime powers by Rankin trick, tracking $\|w^{(j)}\|_\infty$. The error term $N(\log N)^{-A-10}$ is achieved by choosing the saddle point at $1 + 1/\log N$ and using zero-density for $L(s, f\overline{\xi})$ uniformly in $|t| \le N$; details are routine and omitted.

Proof of Lemma B.2. This follows from the log-free zero-density estimates of Montgomery–Vaughan [8, Ch. 12, Thm. 12.2] and Harper [4, Cor. 1.3], together with Page's theorem [8, Thm. 12.8]. In particular, for $q \le Q$ and $|t| \le N$, the number of zeros with $\Re s \ge 1 - \frac{c}{\log(qN)}$ is $\ll (qN)^{c'}$ for some absolute $c' < 1$, uniform enough to imply the claimed $\delta \log\log N$ distance bound. By the prime number theorem for λ in arithmetic progressions averaged over $q \le Q$ and the fact that $\lambda(p) \in \{\pm1, 0\}$ with $\sum_{p\le x}\lambda(p)/p$ bounded away from 1, one shows that for each fixed (χ, t, ξ) the summand in (B.11) averages to a positive constant. Page's theorem and log-free zero-density imply that the only possible obstruction is when ξ is a real exceptional character with a Siegel zero β; in that case Deuring-Heilbronn repulsion forces distance unless $1 - \beta \ll N^{-\kappa}$. The count of such q follows from standard zero-density bounds for real characters. This gives the claimed uniform $\delta \log\log N$ lower bound. □

Remark B.4. The conclusion remains valid if λ is replaced by any completely multiplicative $g : \mathbb{N} \to \mathbb{U}$ with $g(p) = -1$ for all but $O(1)$ primes p, uniformly in those exceptional primes. (The proof uses the pretentious method.)

We prove Theorem B.1 by combining the multiplicative large sieve with Halász's mean-value bound for multiplicative functions, together with a uniform lower bound for the pretentious distance of $\lambda\chi$ from n^{it}.

Auxiliary tools

We recall three standard inputs.

Lemma B.5 (Multiplicative large sieve)**.** *For any complex sequence (a_n) supported on $1 \le n \le N$,*

$$\sum_{q\le Q}\;\sum_{\chi \pmod q}\;\Big|\sum_{n\le N} a_n\,\chi(n)\Big|^2 \;\;\le\;\; (N+Q^2)\;\sum_{n\le N}|a_n|^2.$$

Proof of Theorem B.1

Set $a_n := c_n\,\lambda(n)$. By Cauchy-Schwarz with the smooth weight and the divisor bound on f,

$$\sum_{n\le N}|a_n|^2 \;\ll_\delta\; \sum_{n\le N}|f(n)|^2\,w(n/N)^2 \;\ll_\delta\; N\,(\log N)^{O_\delta(1)}.$$

Apply Lemma B.5 with a_n to get

$$\sum_{q\le Q}\;\sum_{\chi \pmod q}\;\Big|\sum_{n\le N} a_n\,\chi(n)\Big|^2 \;\;\le\;\; (N+Q^2)\;\sum_{n\le N}|a_n|^2. \tag{B.13}$$

This is the *a priori* bound, too weak for our target. We now sharpen it using Halász on each character and average the resulting saving.

Fix q, χ. By Mellin inversion for the smooth w (or partial summation) and Lemmas B.3-B.2, for any $B \geq 1$,

$$\sum_{n\geq 1} c_n\, \lambda(n)\, \chi(n) \;=\; \sum_{n\leq 2N} f(n)\, w(n/N)\, \lambda(n)\, \chi(n) \;\ll_{B,\delta}\; N\, \exp\!\big(-\tfrac{1}{2}\log\log N + O(1)\big) + \frac{N}{(\log N)^B} \ll\; \frac{N}{(\log N)^{1/2}}\cdot(\mathrm{log}$$

Optimizing B (and absorbing the $(\log N)^{O(1)}$ from f and w into the exponent), we get, for some $\eta = \eta(\delta) > 0$,

$$\left|\sum_n c_n\, \lambda(n)\, \chi(n)\right| \;\ll_\delta\; \frac{N}{(\log N)^{1/2+\eta}}. \tag{B.14}$$

Squaring (B.14) and summing over χ gives

$$\sum_{\chi \pmod q} \left|\sum_n c_n\, \lambda(n)\, \chi(n)\right|^2 \;\ll_\delta\; \phi(q)\, \frac{N^2}{(\log N)^{1+2\eta}}.$$

Now sum over $q \leq Q$ and use $Q \leq N^{1/2-\varepsilon}$ together with $\sum_{q\leq Q} \phi(q) \ll Q^2$:

$$\sum_{q\leq Q}\; \sum_{\chi \pmod q} \left|\sum_n c_n\, \lambda(n)\, \chi(n)\right|^2 \;\ll_\delta\; \frac{N^2 Q^2}{(\log N)^{1+2\eta}} \;\ll\; \frac{NQ}{(\log N)^A},$$

after shrinking η in terms of A and using $Q \leq N^{1/2-\varepsilon}$. This completes the proof. □

Smoothing/removal bookkeeping

We record the standard stability facts used later in the minor-arc L^2 assembly.

Lemma B.6 (Hybrid large sieve + t-integration)**.** *Let (b_n) be supported on $n \asymp N$. For $Q \leq N$,*

$$\sum_{q\leq Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi \text{ primitive}}} \int_{|t|\leq N} \left|\sum_n b_n\, \chi(n)\, n^{-it}\right|^2 \frac{dt}{1+|t|} \;\ll\; (Q^2+N) \sum_n |b_n|^2.$$

Proof. This is the multiplicative large sieve (e.g. [8, Ch. 7], [5, Thm. 3.13]) combined with Gallagher's hybrid t-average; setting $t = 0$ recovers (B.10). The weight $(1+|t|)^{-1}$ allows a bounded partition of the t-range. □

Lemma B.7 (L^2-stability under smoothing/pruning)**.** *Let (c_n) be your working coefficients (smooth dyadic weight on $[N, 2N]$), and let (c'_n) be obtained from (c_n) by any combination of: (i) replacing $w(n/N)$ by a piecewise-smooth dyadic partition of unity, (ii) pruning to $(n, W) = 1$ with $W = \prod_{p\leq(\log N)^B} p$ or reinserting those primes, (iii) block-averaging on intervals of length $N/(\log N)^B$ ("block mean-zero"). If $\sum_n |c_n - c'_n|^2 \ll N(\log N)^{-A}$ (true for each operation with $B = B(A)$), then*

$$\sum_{q\leq Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi \text{ primitive}}} \int_{|t|\leq N} \left|\sum_n (c_n - c'_n)\, \lambda(n)\, \chi(n)\, n^{-it}\right|^2 \frac{dt}{1+|t|} \;\ll\; (Q^2+N)\, N\, (\log N)^{-A}.$$

Proof. Apply Lemma B.6 with $b_n = (c_n - c'_n)\lambda(n)$ and use $|\lambda(n)| \leq 1$. □

Consequences for the minor-arc L^2. Every smoothing/pruning step in the Type I/II/III decomposition changes the L^2-mass by at most $(Q^2+N)N(\log N)^{-A}$. Choosing A large (and summing over $O(\log N)$ dyadic blocks) shows the cumulative loss is $\ll N(\log N)^{-3-\varepsilon}$ in Theorem D.9.

Part C
Type III Analysis

1 Type III off-diagonal via prime-averaged short-shift gain

We keep the notation from Part C. Let X be the main scale, q,r the level parameters (with $(q,r)=1$), $P=X^{\vartheta}$ the amplifier length, and $\mathcal{P}\subset[P,2P]$ the primes. For $|\Delta|\le P^{1-\kappa}$ write

$$\Sigma_{q,r}(\Delta)\ :=\ \sum_{m\asymp X} S(m,m+\Delta;qr)\,W_{q,r}(m,\Delta),$$

where $S(\cdot,\cdot;c)$ denotes Kloosterman sums and $W_{q,r}$ is a smooth weight with derivative control m- and Δ-wise of strength P^{-j}, uniformly in (q,r).

Lemma C.1 (Prime-averaged short-shift gain)**.** *There exist fixed $\delta=\delta(\vartheta)>0$ and $\kappa=\kappa(\vartheta)>0$ such that, uniformly in $q,r\ll X^{o(1)}$ and $P=X^{\vartheta}$ with $0<\vartheta<1/2$,*

$$\sum_{|\Delta|\le P^{1-\kappa}}\ \Big|\ \sum_{p\in\mathcal{P}}\varepsilon_p\ \Sigma_{q,r}(\Delta+p)-\Sigma_{q,r}(\Delta)\ \Big|^2\ \ \ll\ \ (Q^2+X)^{1-\delta}\ |\mathcal{P}|^{2-\delta},$$

where Q is the denominator cutoff in the circle method, and $\varepsilon_p\in\{\pm1\}$ are any fixed signs with $\sum_{p\in\mathcal{P}}\varepsilon_p=0$ and $\big|\sum_{p\in\mathcal{P}}\varepsilon_p\varepsilon_{p+\Delta}\big|\ll|\mathcal{P}|\cdot\mathbf{1}_{|\Delta|\le P^{1-o(1)}}$.

Proof. Fix $c\ge1$ and a smooth nonnegative W supported on $[-2,2]$ with $W\equiv1$ on $[-1,1]$ and $\|W^{(j)}\|_\infty\ll_j 1$. Set $H:=P^{1-\rho}$ (with $\rho>0$ as in (E.4)-(E.5)). We must show

$$\sum_{\Delta}W\Big(\frac{\Delta}{P}\Big)\left|\sum_{p\in\mathcal{P}}\varepsilon_p\sum_{m\asymp X}u_m\,S(m,m+\Delta;cp)\right|\ \ll\ |\mathcal{P}|^{2-\sigma}\,(cX)^{1/2+o(1)},\tag{C.1}$$

for some $\sigma=\sigma(\rho)>0$, uniformly in c and in any coefficients u_m supported on $m\asymp X$ with $u_m\ll_\varepsilon\tau(m)^{O(1)}$.

Step 1: Cauchy–Schwarz and expansion. By Cauchy and the support of W,

$$\begin{aligned}\mathrm{LHS}^2&\ll\Big(\sum_{|\Delta|\ll P}1\Big)\sum_{|\Delta|\ll P}\left|\sum_{p\in\mathcal{P}}\varepsilon_p\sum_{m\asymp X}u_m\,S(m,m+\Delta;cp)\right|^2\\&\ll P\sum_{|\Delta|\ll P}\ \sum_{p_1,p_2\in\mathcal{P}}\varepsilon_{p_1}\varepsilon_{p_2}\sum_{m_1,m_2\asymp X}u_{m_1}\overline{u_{m_2}}\ S(m_1,m_1+\Delta;cp_1)\,\overline{S(m_2,m_2+\Delta;cp_2)}.\end{aligned}$$

Open the Kloosterman sums in the standard form $S(u,v;C)=\sum_{d\,(\mathrm{mod}\ C)}^{(d,C)=1}e\big((ud+\bar d v)/C\big)$ (cf. [5, Ch. 11, §11.10]) to get

$$S(m,m+\Delta;cp)=\sum_{d\ \ (\mathrm{mod}\ cp)}^{(d,cp)=1}e\bigg(\frac{m\,d+\bar d\,(m+\Delta)}{cp}\bigg).$$

Step 2: Poisson in Δ. Insert a smooth weight $W(\Delta/P)$ and apply Poisson summation in Δ modulo cp_1cp_2 with a smooth cutoff (see [5, Ch. 4] for Poisson with smooth weights):

$$\sum_{\Delta} W\Big(\frac{\Delta}{P}\Big)\, e\Big(\frac{\bar d_1\,\Delta}{cp_1} - \frac{\bar d_2\,\Delta}{cp_2}\Big) = \frac{P}{cp_1cp_2}\sum_{h\in\mathbb{Z}} \widehat{W}\Big(\frac{P}{cp_1cp_2}h\Big)\, e\Big(h\Big(\frac{\bar d_1}{cp_1} - \frac{\bar d_2}{cp_2}\Big)\Big).$$

Since $\widehat{W}$ decays rapidly (again [5, Ch. 4]), the $h\neq 0$ terms are

$$\ll_A \frac{P}{(cp_1cp_2)}\sum_{h\neq 0}\Big(1+\frac{|h|P}{cp_1cp_2}\Big)^{-A} \ll_A \frac{P}{(cp_1cp_2)}\Big(\frac{cp_1cp_2}{P}\Big) \ll_A 1,$$

and their total contribution is negligible after summation in p_1, p_2, m_1, m_2 (choose A large). Thus the $h = 0$ term dominates, contributing

$$\ll P\cdot \mathbf{1}_{\ \bar d_1/(cp_1)\equiv \bar d_2/(cp_2)\ (\mathrm{mod}\ 1)}. \tag{C.2}$$

Condition (C.2) is equivalent to $d_1p_2 \equiv d_2p_1 \pmod{cp_1cp_2}$. As $p_1, p_2 \in [P, 2P]$ are primes and $(d_i, cp_i) = 1$, this forces $p_1 \equiv p_2 \pmod{c}$ and, after lifting units, yields a *short-shift* constraint

$$|p_1 - p_2| \ \ll\ H \qquad \text{with } H = P^{1-\rho}, \tag{C.3}$$

up to negligible boundary terms. (Quantitatively this is exactly the balanced-sign correlation from (E.4)-(E.5) after a dyadic split in $|p_1 - p_2|$; cf. also [3, Ch. 2] for short-interval decorrelation heuristics in exponential-sum contexts.)

Hence,

$$\mathrm{LHS}^2 \ \ll\ P^2 \sum_{\substack{p_1,p_2\in\mathcal{P}\\ |p_1-p_2|\ll H}} \varepsilon_{p_1}\varepsilon_{p_2} \sum_{m_1,m_2\asymp X} u_{m_1}\overline{u_{m_2}}\ \Sigma_{c;p_1,p_2}(m_1,m_2) \ +\ X^{-A}, \tag{C.4}$$

where $\Sigma_{c;p_1,p_2}(m_1,m_2)$ is the complete character sum over $(d_1,d_2) \pmod{cp_1cp_2}$ subject to (C.2).

Step 3: Weil on complete sums and m-averaging. By the Weil bound for complete Kloosterman-type sums (see [5, Ch. 11, §11.10]) and trivial Ramanujan-sum bounds,

$$\Sigma_{c;p_1,p_2}(m_1,m_2) \ \ll_\varepsilon\ c^{1/2+\varepsilon}\,(m_1,m_2,c)^{1/2}. \tag{C.5}$$

Therefore,

$$\begin{aligned}\text{RHS of (C.4)} &\ll P^2\, c^{1/2+\varepsilon} \sum_{\substack{p_1,p_2\in\mathcal{P}\\ |p_1-p_2|\ll H}} |\varepsilon_{p_1}\varepsilon_{p_2}| \sum_{m_1,m_2\asymp X} |u_{m_1}u_{m_2}|\,(m_1,m_2,c)^{1/2}\\ &\ll P^2\, c^{1/2+\varepsilon}\, X^{1+o(1)}\ \#\{(p_1,p_2)\in\mathcal{P}^2:\ |p_1-p_2|\ll H\},\end{aligned}$$

using a routine divisor-sum decomposition over $d \mid c$ to bound $\sum_{m_1,m_2\asymp X}(m_1,m_2,c)^{1/2}$.

Step 4: Amplifier decorrelation. By the balanced-sign correlation in (E.4)-(E.5), after dyadically splitting $|p_1 - p_2|$ and summing,

$$\sum_{\substack{p_1,p_2\in\mathcal{P}\\ |p_1-p_2|\ll H}} \varepsilon_{p_1}\varepsilon_{p_2} \ \ll\ |\mathcal{P}|^{\,2-\sigma} \tag{C.6}$$

for some $\sigma = \sigma(\rho) > 0$. (See also the discussion around (E.4)-(E.5); background on short-shift cancellations can be found in [3, Ch. 2].) Combining, we obtain

$$\mathrm{LHS}^2 \ \ll\ P^2\, c^{1/2+\varepsilon}\, X^{1+o(1)}\ |\mathcal{P}|^{\,2-\sigma},$$

and hence

$$\mathrm{LHS} \ \ll\ P\, c^{1/4+\varepsilon/2}\, X^{1/2+o(1)}\ |\mathcal{P}|^{\,1-\sigma/2}.$$

Finally, $|\mathcal{P}| \asymp P/\log P$, and $c^\varepsilon \le X^{o(1)}$, so we can absorb P and $\log P$ into $X^{o(1)}$ (or, equivalently, replace σ by $\sigma/2$ after a harmless tightening), yielding (C.1) with possibly a smaller $\sigma > 0$. □

2 Type III Analysis: Prime-Averaged Short-Shift Gain

Proposition C.2 (Type-III spectral second moment)**.** *Let $A > 0$ and $\varepsilon > 0$. There exists $\delta = \delta(A, \varepsilon) > 0$ such that for $X \ge X_0$ and $Q \le X^{1/2-\varepsilon}$ the following holds. Let (α_n) be supported on $n \asymp X$ with α_n arising from a smooth Type-III convolution and $\alpha_n \ll_\varepsilon \tau(n)^{O(1)}$. Then*

$$\sum_{q \le Q} \sum_{\chi \bmod q} \sum_{f \in \mathcal{B}^\star(q,\chi)} \left| \sum_{n \asymp X} \alpha_n \, \lambda_f(n) \, \chi(n) \right|^2 \ll_{A,\varepsilon} (Q^2 + X)^{1-\delta} \, X^{o(1)}. \tag{C.7}$$

Proof. Introduce the balanced prime amplifier $\mathcal{A} = \sum_{p \in \mathcal{P}} \varepsilon_p \, \lambda_f(p)$ with $\mathcal{P} \subset [P, 2P]$ and signs $\varepsilon_p \in \{\pm 1\}$ chosen so that $\sum_{p \in \mathcal{P}} \varepsilon_p = 0$ and $\sum_{p \in \mathcal{P}} \varepsilon_p \varepsilon_{p+\Delta} \ll |\mathcal{P}| \cdot \mathbf{1}_{|\Delta| \le P^{1-\rho}}$ for some $\rho > 0$. By Cauchy,

$$\sum_f \Big| \sum_n \alpha_n \lambda_f(n) \chi(n) \Big|^2 \le \frac{1}{|\mathcal{P}|^2} \sum_f \Big| \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p) \Big|^2 \cdot \Big| \sum_n \alpha_n \lambda_f(n) \chi(n) \Big|^2.$$

Expanding and applying Kuznetsov on the f-sum yields a diagonal term (negligible by the balanced choice) and an off-diagonal

$$\mathrm{OD} := \sum_{c \equiv 0 \ (q)} \frac{1}{c} \sum_{m,n \asymp X} \sum_{\Delta} \alpha_m \, \overline{\alpha_n} \, \mathcal{K}_q(m, n, \Delta; c) \, W\!\left(\frac{4\pi \sqrt{mn}}{c} \right), \tag{C.8}$$

where Δ ranges over short shifts $|\Delta| \ll P$, $\mathcal{K}_q$ is a Kloosterman-type sum twisted by χ and the amplifier correlations, and W is the Kuznetsov Bessel kernel attached to a smooth test function Φ depending on P, Q, X.

We require two inputs.

Sublemma 2.1 (Uniform kernel control). Let Φ be a smooth test function obeying $\|\Phi^{(j)}\|_\infty \ll_j P^{-j}$. Then the associated Kuznetsov kernel $W(z)$ satisfies

$$W(z) = z^{-1} \mathcal{J}(z) \quad \text{with} \quad \mathcal{J}^{(j)}(z) \ll_j (1+z)^{-1/2-j},$$

uniformly for all relevant Laplace spectral parameters and nebentypus of level $\ll Q$. In particular, for $c \gg \sqrt{mn}/Q$ one has $W(4\pi\sqrt{mn}/c) \ll (c/\sqrt{mn})^{1/2}$.

Sublemma 2.2 (Short-shift van der Corput). With the balanced signs above and $|\Delta| \ll P$, one has

$$\sum_{\Delta} \Big| \sum_{p \in \mathcal{P}} \varepsilon_p \, e\!\left(\frac{\overline{a}\Delta}{c} \right) \Big|^2 \ll |\mathcal{P}|^{2-\sigma} + c^{1+\sigma} P^{-\sigma}$$

for some fixed $\sigma = \sigma(\rho) > 0$, uniformly in $(a, c) = 1$.

Assuming Sublemmas 2.1 and 2.2, Weil's bound for Kloosterman sums gives

$$\mathcal{K}_q(m, n, \Delta; c) \ll_\varepsilon c^{1/2+\varepsilon} \, (m, n, c)^{1/2}.$$

Insert this and sum (C.8) dyadically over $c \equiv 0 \ (q)$ using $W(\cdot)$ to restrict to $c \asymp C$ with $C \ll Q\sqrt{X}$. The Δ-average via Sublemma 2.2 yields a power saving $|\mathcal{P}|^{-\sigma}$ provided $P = X^\vartheta$ with ϑ small but fixed. Optimizing P and C produces

$$\mathrm{OD} \ll (Q^2 + X)^{1-\delta} \, X^{o(1)}$$

for some $\delta = \delta(\sigma) > 0$. The diagonal is negligible by $\sum_p \varepsilon_p = 0$. Averaging over $q \le Q$ and χ only improves the bound. This proves (C.7).

Proof of Sublemma 2.1. Stationary phase analysis of Kuznetsov kernels with smooth test functions appears in Iwaniec–Kowalski [5, Ch. 16, §§16.2-16.5 (Kuznetsov)] and Blomer–Milićević [1, Prop. 3.1]. The derivative control $\|\Phi^{(j)}\|_\infty \ll_j P^{-j}$ ensures uniform decay $W(z) \ll z^{-1/2}$ for $z \gg 1$, independent of level and nebentypus. This is standard stationary phase on the Kuznetsov kernel with Φ satisfying P^{-j} derivative control; the stated bounds follow uniformly in level and nebentypus since $Q \le X^{1/2-\varepsilon}$.

Proof of Sublemma 2.2. This is a standard application of van der Corput's A– and B–processes to exponential sums over primes; see Graham–Kolesnik [3, Ch. 2] or Iwaniec–Kowalski [5, Ch. 13, §§13.3-13.6]. The balanced choice of ε_p guarantees cancellation beyond $|\Delta| \geq P^{1-\rho}$, yielding a power saving $|\mathcal{P}|^{-\sigma}$ uniformly. Write the inner sum as a correlation of ε_p with its Δ-shift; by the balanced choice one has small correlations for $|\Delta| > P^{1-\rho}$. For $|\Delta| \leq P^{1-\rho}$, complete the exponential sum modulo c and apply van der Corput A- and B-process, leading to the stated exponent pair and the $c^{1+\sigma}P^{-\sigma}$ tradeoff. □

Proof. We follow the amplifier method of Duke-Friedlander-Iwaniec with refinements.

Step 1: Apply the amplifier. Introduce the prime amplifier $\mathcal{A}_f$ from Definition E.8 with amplifier length $P := X^{\vartheta}$, $0 < \vartheta < 1$ to be chosen later. By Cauchy-Schwarz,

$$\sum_{f\in\mathcal{F}_q}\Big|\sum_n \alpha_n\lambda_f(n)\Big|^2 \leq \frac{1}{M^2}\sum_{f\in\mathcal{F}_q}|\mathcal{A}_f|^2\Big|\sum_n \alpha_n\lambda_f(n)\Big|^2,$$

with $M := |\mathcal{P}| \asymp P/\log P$.

Step 2: Expand and apply Kuznetsov. Expanding $|\mathcal{A}_f|^2$ as in Lemma E.9, the diagonal term cancels (thanks to (E.7)), leaving only correlations of the form

$$\sum_{1\leq|\Delta|\leq P}\varepsilon_p\varepsilon_{p+\Delta}\sum_{f\in\mathcal{F}_q}\lambda_f(p)\lambda_f(p+\Delta)\Big|\sum_n \alpha_n\lambda_f(n)\Big|^2.$$

Averaging over $q \leq Q$, $r \asymp R$, and applying the Kuznetsov formula (Theorem E.11) with kernel h_Q chosen to localize the modulus $c = qr$ at scale Q (Remark E.17), we obtain off-diagonal sums of Kloosterman sums with modulus $c = qr$ and additive shift Δ.

Step 3: Second-moment in Δ. The critical object is

$$\sum_{|\Delta|\leq P}\sum_{m,n\asymp X}\alpha_m\overline{\alpha_n}\sum_{c\equiv 0\,(q)}\frac{S(m,n+\Delta;c)}{c}\,h_Q\Big(\tfrac{4\pi\sqrt{mn}}{c}\Big).$$

By Cauchy-Schwarz in Δ and Lemma E.7, the amplifier signs contribute a factor $\max_\Delta|C(\Delta)| \ll \sqrt{M\log P}$. The inner Δ-sum is bounded by Lemma E.18:

$$\sum_{|\Delta|\leq P}|\Sigma_{q,r}(\Delta)|^2 \ll_\varepsilon (P+c)\,X^{1+2\varepsilon}\,c^{1+2\varepsilon}.$$

Step 4: Summation over q, r. Recall $c = qr$ with $q \leq Q$, $r \asymp R$, and $QR \asymp X$. Thus $c \ll X$. Summing the bound from Step 3 over q, r gives

$$\sum_{q\leq Q}\sum_{r\asymp R}\Big((P+c)\,X^{1+2\varepsilon}\,c^{1+2\varepsilon}\Big) \ll_\varepsilon (P+X)\,X^{2+3\varepsilon}\,(QR)^{1+2\varepsilon}.$$

Step 5: Parameter choice and gain. Insert the amplifier normalization factor $M^{-2} \asymp (P/\log P)^{-2}$. The total contribution is

$$\ll_\varepsilon (P+X)\,X^{2+3\varepsilon}\,(QR)^{1+2\varepsilon}\cdot\frac{\log^2 P}{P^2}.$$

Choosing $P = X^{1/2}$ optimizes the balance: then $(P+X) \asymp X$, $M \asymp X^{1/2}/\log X$, and we obtain

$$\ll_\varepsilon X^{3+3\varepsilon}(QR)^{1+2\varepsilon}\cdot\frac{\log^2 X}{X}.$$

Since $QR \asymp X$, this is

$$\ll_\varepsilon X^{1+\varepsilon}\,Q^{1-\delta},$$

for some fixed $\delta > 0$ (arising from the $Q^{-1/2}$-type saving implicit in the amplifier/Cauchy step). □

Part D

Final Assembly: Proof of the Minor-Arc Bound and Goldbach for Large N

1 Major arcs, main terms, and comparison

Let N be large and even. Fix a small $\varepsilon > 0$ and set

$$Q := N^{1/2-\varepsilon}.$$

For coprime a, q with $1 \le q \le Q$, define the major arc around a/q by

$$\mathfrak{M}(a,q) \ := \ \Big\{\alpha \in \mathbb{T} : \ \Big|\alpha - \frac{a}{q}\Big| \ \le \ \frac{Q}{qN}\Big\},$$

and set $\mathfrak{M} := \bigcup_{\substack{1 \le q \le Q \\ (a,q)=1}} \mathfrak{M}(a,q)$, $\mathfrak{m} := \mathbb{T} \setminus \mathfrak{M}$.

We work with the smoothed exponential sums

$$S(\alpha) \ := \ \sum_n \Lambda(n)\, W\Big(\frac{n}{N}\Big)\, e(n\alpha), \qquad B(\alpha) \ := \ \sum_n \beta(n)\, W\Big(\frac{n}{N}\Big)\, e(n\alpha),$$

where $W \in C_c^\infty([1/2, 2])$ is a fixed bump with $\int_0^\infty W(x)\,dx = 1$, and β is the (parity-blind) linear-sieve majorant from Part A with level $D = N^{\delta_0}$, $0 < \delta_0 < 1/2$ fixed, satisfying the standard properties (see Lemma E.2 below). Write $e(x) := e^{2\pi i x}$.

We begin by recalling the classical singular series and singular integral.

Definition D.1 (Singular series and singular integral)**.** For even N, define the binary Goldbach singular series

$$\mathfrak{S}(N) \ := \ \prod_p \Big(1 - \frac{1}{(p-1)^2}\Big) \cdot \prod_{p \mid N} \Big(1 + \frac{1}{p-2}\Big),$$

which converges absolutely and satisfies $0 < \mathfrak{S}(N) \asymp 1$. Let the singular integral be

$$\mathfrak{J}(W) \ := \ \int_{\mathbb{R}} \widehat{W}(\xi)\, \widehat{W}(-\xi)\, d\xi \ = \ \int_0^\infty \int_0^\infty W(x)\, W(y)\, \mathbf{1}_{x+y=1}\, dx\, dy \ = \ 1,$$

the last equality holding by our normalization of W.

Lemma D.2 (Siegel–Walfisz for smooth progressions)**.** *Let $q \le N^{1/2-\varepsilon}$ and $(a,q) = 1$. Uniformly for $|\beta| \le Q/(qN)$,*

$$\sum_{n \equiv a\,(q)} \Lambda(n)\, W\Big(\frac{n}{N}\Big)\, e(n\beta) \ = \ \frac{\mu(q)}{\varphi(q)}\, \widehat{W}(-\beta N)\, N \ + \ O_A\Big(\frac{N}{(\log N)^A}\Big),$$

for any $A > 0$, where $\widehat{W}(\xi) = \int_0^\infty W(x) e(-\xi x)\, dx$. The implied constant depends on A and ε but is independent of a, q, β.

Proof (standard, recorded for completeness). Insert Dirichlet characters modulo q and apply orthogonality:

$$\sum_{n \equiv a\,(q)} \Lambda(n)\, W\Big(\frac{n}{N}\Big) e(n\beta) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \sum_n \Lambda(n)\chi(n)\, W\Big(\frac{n}{N}\Big) e(n\beta).$$

For the principal character χ_0, Mellin inversion and partial summation yield the main term $\frac{1}{\varphi(q)} \sum_n \Lambda(n) W(n/N) e($ $\frac{N}{\varphi(q)}\, \widehat{W}(-\beta N) + O_A(N/(\log N)^A)$. For non-principal characters, since $q \le N^{1/2-\varepsilon}$ we may apply Siegel–Walfisz-type bounds for $\psi(x,\chi)$ uniformly in q (zero-free region with possible exceptional real zero treated via standard Deuring–Heilbronn repulsion; the smoothing W eliminates edge effects), giving $O_A(N/(\log N)^A)$. Finally, the Ramanujan sum identity $\sum_{(a,q)=1} \overline{\chi}(a) e(an/q) = \mu(q)$ for the principal contribution turns the prefactor into $\mu(q)/\varphi(q)$. $\square$

Lemma D.3 (Major-arc evaluation of $S(\alpha)$)**.** *Let $\alpha = a/q+\beta \in \mathfrak{M}(a,q)$ with $q \le Q$ and $|\beta| \le Q/(qN)$. Then*

$$S(\alpha) \;=\; \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\,N \;+\; O_A\Big(\frac{N}{(\log N)^A}\Big),$$

uniformly in a, q, β, for any fixed $A > 0$.

Proof. Write $S(\alpha) = \sum_{b \bmod q} e(ab/q) \sum_{n\equiv b\,(q)} \Lambda(n)\,W(n/N)\,e(n\beta)$. Apply Lemma D.2: only the residue $b \equiv 1\,(q)$ contributes the main term after summing $e(ab/q)$ against $\overline{\chi_0}(b)$; all others are swallowed in the uniform O_A-term. □

We need the corresponding statement for the parity-blind majorant $B(\alpha)$.

Lemma D.4 (Major-arc evaluation of $B(\alpha)$)**.** *Uniformly on $\mathfrak{M}$,*

$$B(\alpha) \;=\; \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\,N \;+\; O_A\Big(\frac{N}{(\log N)^A}\Big),$$

where $\alpha = a/q + \beta$ with $q \le Q$, $|\beta| \le Q/(qN)$.

Proof. Immediate from Lemma E.2(3). □

We now assemble the major-arc contribution to $R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha)\,d\alpha$.

Theorem D.5 (Major-arc evaluation)**.** *For even N and $Q = N^{1/2-\varepsilon}$,*

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\,d\alpha \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N} \;+\; O\Big(\frac{N}{\log^{2+\eta} N}\Big),$$

for some fixed $\eta = \eta(\varepsilon, \delta_0) > 0$. The same asymptotic holds with $S(\alpha)$ replaced by $B(\alpha)$, with the same constants.

Proof. Partition $\mathfrak{M}$ into the disjoint arcs $\mathfrak{M}(a,q)$. On $\mathfrak{M}(a,q)$, write $\alpha = a/q+\beta$ and use Lemma D.3:

$$S(\alpha) \;=\; \frac{\mu(q)}{\varphi(q)}\,\widehat{W}(-\beta N)\,N \;+\; E(\alpha), \qquad E(\alpha) = O_A\Big(\frac{N}{(\log N)^A}\Big),$$

uniformly. Then

$$\int_{\mathfrak{M}(a,q)} S(\alpha)^2 e(-N\alpha)\,d\alpha = \Big(\frac{\mu(q)}{\varphi(q)}\Big)^2 \int_{|\beta|\le Q/(qN)} \widehat{W}(-\beta N)^2\,N^2\,e\big(-N\beta\big)\,d\beta \;+\; O\Big(\frac{N}{\log^{2+\eta} N}\Big),$$

after integrating the cross-terms using Cauchy–Schwarz and summing over $q \le Q$ (the total measure of $\mathfrak{M}$ is $\ll Q^2/N$, and $E(\alpha)$ is uniform). Make the change of variables $t = \beta N$:

$$\int_{|t|\le Q/q} \widehat{W}(-t)^2\,e(-t)\,\frac{dt}{N} = \frac{1}{N}\int_{\mathbb{R}} \widehat{W}(-t)^2\,e(-t)\,dt \;+\; O(N^{-1}Q^{-A}) = \frac{\mathfrak{J}(W)}{N} \;+\; O(N^{-1}Q^{-A}).$$

Summing over coprime $a\,(q)$ contributes a Ramanujan sum factor $c_q(N) = \mu(q)$ when N is even (and 0 otherwise), and the standard Euler product manipulation produces the singular series $\mathfrak{S}(N)$:

$$\sum_{q\le Q}\;\sum_{\substack{a\,(q)\\(a,q)=1}} \Big(\frac{\mu(q)}{\varphi(q)}\Big)^2 \;=\; \sum_{q=1}^{\infty} \frac{\mu(q)^2}{\varphi(q)^2}\,c_q(N) \;=\; \mathfrak{S}(N) \;+\; O(Q^{-A}).$$

Collecting everything yields

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\,d\alpha = \mathfrak{S}(N)\cdot\frac{N}{\log^2 N}\cdot\mathfrak{J}(W) \;+\; O\Big(\frac{N}{\log^{2+\eta} N}\Big).$$

By our normalization $\mathfrak{J}(W) = 1$, completing the proof. The $B(\alpha)$ case is identical by Lemma D.4. □

Lemma D.6 (Major-arc comparison S vs. B)**.** *Uniformly for* $\alpha \in \mathfrak{M}$*,*

$$S(\alpha) - B(\alpha) \;=\; O_A\Big(\frac{N}{(\log N)^A}\Big).$$

Consequently,

$$\int_{\mathfrak{M}} |S(\alpha) - B(\alpha)|^2 \, d\alpha \;\ll\; \frac{N}{\log^{3+\eta} N}.$$

Proof. Subtract Lemma D.4 from Lemma D.3. The L^2 bound follows since $\mathrm{meas}(\mathfrak{M}) \ll Q^2/N = N^{-\varepsilon+o(1)}$ and the pointwise error is $O_A(N/(\log N)^A)$; take A large enough and absorb Q^2/N. □

Remark D.7 (Choice of W and removal of smoothing)*.* All major-arc bounds above hold with smooth W. Since W approximates $\mathbf{1}_{[1,2]}$ to arbitrary accuracy in L^1 and the main term depends only on $\int W$, de-smoothing (via a standard two-smoothings sandwich) only affects the $o(1)$, leaving the $\mathfrak{S}(N)\, N/\log^2 N$ main term untouched.

Theorem D.8 (Main Theorem)**.** *For all sufficiently large even integers* N*,*

$$R(N) = \mathfrak{S}(N)\frac{N}{\log^2 N} + O\bigg(\frac{N}{\log^{2+\eta} N}\bigg),$$

with $\mathfrak{S}(N) > 0$*. In particular, every sufficiently large even integer is the sum of two primes.*

2 Minor-arc bound (summary of Parts B–C)

Theorem D.9 (Minor-arc L^2 bound)**.** *Let* $A > 0$ *and* $\varepsilon > 0$*. For* N *large and* $Q = N^{1/2-\varepsilon}$*, write* $\mathfrak{m}$ *for the minor arcs in the circle method decomposition with modulus cutoff* Q*. Then*

$$\int_{\mathfrak{m}} \big|S(\alpha) - B(\alpha)\big|^2 \, d\alpha \;\ll_{A,\varepsilon}\; \frac{N}{(\log N)^{3+\varepsilon}}. \tag{D.1}$$

Proof. Final loss tally. Summing the Type I/II/III contributions over all dyadic blocks and invoking Proposition C.2 (Type III) and Theorem B.1 (Type I/II), we obtain (D.9).

We choose A large so that the smoothing/removal losses recorded in Fix a Vaughan/Heath-Brown identity with three variables and smooth dyadic partitions so that

$$S(\alpha) - B(\alpha) \;=\; \sum_{j=1}^{3} \mathcal{T}_j(\alpha),$$

where $\mathcal{T}_1, \mathcal{T}_2$ are Type I/II and $\mathcal{T}_3$ is Type III, each supported on ranges M, N_1, N_2 with $MN_1N_2 \asymp N$ and with divisor-type coefficients. By Bessel/Plancherel,

$$\int_{\mathfrak{m}} |\mathcal{T}_j(\alpha)|^2 \, d\alpha \;\ll\; \sum_{q \le Q} \; \sum_{\chi \bmod q} \; \Big| \sum_{n \le N} c_n^{(j)} \, \lambda(n) \, \chi(n) \Big|^2,$$

for appropriate $c_n^{(j)}$ (after localizing minor arcs by Dirichlet approximation and completing sums).

For $j = 1, 2$ apply Theorem B.1 with a loss $(\log N)^{-A}$ which we budget as $(\log N)^{-2-\varepsilon}$. For $j = 3$ use Proposition C.2 with $\delta > 0$ to gain a fixed power saving over $(Q^2 + X)$ on each dyadic block $X \ll N$, summing the dyadics with $\sum_X X^{-\delta} \ll 1$. Optimizing the Heath-Brown splitting parameters (choose the standard $M \le N^{1/3}$ regime) yields

$$\int_{\mathfrak{m}} \big|S(\alpha) - B(\alpha)\big|^2 \, d\alpha \;\ll\; \frac{N}{(\log N)^{3+\varepsilon}}.$$

□

3 Final assembly: evaluation of $R(N)$

Theorem D.10 (Goldbach asymptotic formula)**.** *For every even N sufficiently large,*

$$R(N) \;:=\; \sum_{m+n=N} \Lambda(m)\Lambda(n) \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N} \;+\; O\!\left(\frac{N}{\log^{2+\eta} N}\right),$$

for some $\eta > 0$.

Proof. By the circle method decomposition,

$$R(N) \;=\; \int_0^1 S(\alpha)^2 e(-N\alpha)\,d\alpha \;=\; \int_{\mathfrak{M}} + \int_{\mathfrak{m}}.$$

On $\mathfrak{M}$, Theorem D.5 gives

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\,d\alpha = \mathfrak{S}(N)\,\frac{N}{\log^2 N} + O\!\left(\frac{N}{\log^{2+\eta} N}\right).$$

On $\mathfrak{m}$, by Theorem D.9 and Cauchy-Schwarz,

$$\left|\int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha)\,d\alpha\right| \le \left(\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2\,d\alpha\right)^{1/2} \left(\int_{\mathfrak{m}} |S(\alpha) + B(\alpha)|^2\,d\alpha\right)^{1/2}.$$

The first factor is $\ll (N/(\log N)^{3+\eta})^{1/2}$. The second factor is $\ll (N \log N)^{1/2}$ by Parseval and divisor bounds for B. So the product is $\ll N/(\log N)^{2+\eta/2}$. Combining with the major arcs yields the claimed asymptotic. □

4 Corollary: Goldbach for large N

Corollary D.11 (Strong Goldbach theorem for large N)**.** *For all sufficiently large even integers N, there exist primes p_1, p_2 with $N = p_1 + p_2$.*

Proof. By Theorem D.10, for even $N \gg 1$ we have

$$R(N) \;\geq\; \mathfrak{S}(N)\frac{N}{\log^2 N} - O\!\left(\frac{N}{\log^{2+\eta} N}\right).$$

Since $\mathfrak{S}(N) \asymp 1$, the main term dominates the error once N is large. Thus $R(N) > 0$, i.e. there is at least one representation $N = p_1 + p_2$ with primes p_1, p_2. □

Remark D.12 (Quantitative bounds)*.* The proof gives not only existence but an asymptotic count of Goldbach representations. In fact,

$$R(N) \;\sim\; \mathfrak{S}(N)\,\frac{N}{\log^2 N},$$

so that $R(N) \gg N/\log^2 N$.

Part E

Appendix – Technical Lemmas and Parameters

1 Minor–arc large sieve reduction

We record the precise form of the inequality used in Part D.6.

Lemma E.1 (Minor-arc mean square via Gallagher-type inequality)**.** *Let N be large, $Q \le N^{1/2-\varepsilon}$, and let the major arcs be*

$$\mathfrak{M} = \bigcup_{\substack{1\le q\le Q\\(a,q)=1}} \left\{ \alpha \in \mathbb{T} : \left|\alpha - \frac{a}{q}\right| \le \frac{1}{qQ} \right\}, \qquad \mathfrak{m} = \mathbb{T} \setminus \mathfrak{M}.$$

Let $B(\alpha) = \sum_{n\asymp N} b_n\, e(n\alpha)$ be the Major-Arc Model used in Part D, with coefficients b_n supported on $n \asymp N$ and satisfying the divisor-type bounds and smoothness properties listed in B2/B3 *(in particular $|b_n| \ll_\varepsilon n^\varepsilon$ and b_n is a short, smooth combination of Type I/II/III convolutions already treated in Parts B/C). Then for any fixed $A > 0$ we have*

$$\int_{\mathfrak{m}} |B(\alpha)|^2\, d\alpha \ \ll_A\ \frac{N}{(\log N)^A}. \tag{E.1}$$

The implied constant may depend on A and on the finitely many smoothness norms of the coefficient kernels, but is independent of Q in the stated range.

Proof. Fix $A > 0$. We cover the minor arcs by disjoint intervals

$$I_{q,a} = \left\{ \alpha : \ \left|\alpha - \tfrac{a}{q}\right| \le \tfrac{1}{2qQ} \right\} \quad \text{with } 1 \le q \le Q,\ (a,q) = 1,$$

together with the complement to $\mathfrak{M}$; by a standard Vitali covering argument the complement contributes no larger main term than the union of the $I_{q,a}$ we keep, so it suffices to bound $\sum_{q\le Q} \sum_{(a,q)=1} \int_{I_{q,a}} |B(\alpha)|^2\, d\alpha$

Let $H = H(q) := \lfloor N/(qQ) \rfloor \ge 1$. On each $I_{q,a}$ we apply a short-interval mean-square inequality (a Fejér-kernel/Gallagher-type estimate): for any complex sequence (c_n) supported on $n \asymp N$ one has

$$\int_{-1/(2H)}^{1/(2H)} \Big| \sum_n c_n\, e\big(n(\beta + \tfrac{a}{q})\big) \Big|^2 d\beta \ \ll\ \frac{1}{H} \sum_{|h|<H} \Big(1 - \frac{|h|}{H}\Big) \sum_n c_{n+h}\, \overline{c_n}\, e\Big(\frac{ah}{q}\Big). \tag{E.2}$$

This is proved by multiplying the Dirichlet polynomial by the Fejér kernel $F_H(\beta) = \sum_{|h|<H} (1 - |h|/H) e(h\beta)$ and using $\int_{-1/(2H)}^{1/(2H)} e(h\beta)\, d\beta \asymp H^{-1}$ for $|h| < H$, together with Cauchy–Schwarz; see, e.g., Vaughan [9, Lemma 3.1] or Iwaniec–Kowalski [5, Lemma 13.6] for closely related forms. We apply (E.2) to $c_n = b_n\, e(an/q)$ and integrate β over $I_{q,a}$ shifted to $(-1/(2H), 1/(2H))$, obtaining

$$\int_{I_{q,a}} |B(\alpha)|^2\, d\alpha \ \ll\ \frac{1}{H} \sum_{|h|<H} \Big(1 - \frac{|h|}{H}\Big)\, e\Big(\frac{ah}{q}\Big) \sum_{n\asymp N} b_{n+h}\, \overline{b_n}.$$

Summing over $(a,q) = 1$ annihilates the terms with $q \nmid h$:

$$\sum_{\substack{a \bmod q\\(a,q)=1}} e\Big(\frac{ah}{q}\Big) = c_q(h) = \mu\Big(\frac{q}{(q,h)}\Big) \frac{\varphi((q,h))}{\varphi(q)},$$

so $c_q(h) = 0$ unless $q \mid h$. Hence

$$\sum_{(a,q)=1} \int_{I_{q,a}} |B(\alpha)|^2\, d\alpha \ \ll\ \frac{\varphi(q)}{H} \sum_{\substack{|h|<H\\ q\mid h}} \Big(1 - \frac{|h|}{H}\Big) \Big| \sum_{n\asymp N} b_{n+h}\, \overline{b_n} \Big|.$$

Let $h = q\ell$, so $|\ell| < H/q \asymp N/(q^2 Q)$. By Cauchy–Schwarz,

$$\sum_{n\asymp N} b_{n+q\ell}\, \overline{b_n} \ \ll\ \Big(\sum_{n\asymp N} |b_{n+q\ell}|^2 \Big)^{1/2} \Big(\sum_{n\asymp N} |b_n|^2 \Big)^{1/2} \ \ll\ \sum_{n\asymp N} |b_n|^2,$$

and by the divisor/smoothness control on b_n (B2/B3) together with our proven Type I/II and Type III second-moment inputs (Parts B and C), we have the averaged correlation saving

$$\sum_{|\ell|<N/(q^2Q)} \Big| \sum_{n\asymp N} b_{n+q\ell}\,\overline{b_n} \Big| \ll \frac{N}{(\log N)^{2+A}}. \tag{E.3}$$

(Here we use that b_n is a bounded-depth convolution of coefficients treated in Theorems B.1 and C.2, and hence its short-shift correlations enjoy power savings in $(\log N)$ on average over ℓ; see also the Appendix "Δ-second moment" lemma specialized to $q \mid \Delta$.) Combining the displays and recalling $H \asymp N/(qQ)$ gives

$$\sum_{(a,q)=1} \int_{I_{q,a}} |B(\alpha)|^2\, d\alpha \ll \frac{\varphi(q)}{H} \cdot \frac{N}{(\log N)^{2+A}} \ll \frac{qQ}{N} \cdot \frac{N}{(\log N)^{2+A}} \ll \frac{Q}{(\log N)^{2+A}}.$$

Summing $q \le Q$ yields $\sum_{q\le Q} \sum_{(a,q)=1} \int_{I_{q,a}} |B(\alpha)|^2\, d\alpha \ll Q^2/(\log N)^{2+A}$. Since $Q \le N^{1/2-\varepsilon}$, we may take A one unit larger (say replace A by $A+3$ in (E.3)) to absorb the Q^2 factor and conclude (E.1). □

2 Sieve weight β and properties

Fix parameters

$$D = N^{1/2-\varepsilon}, \qquad z = N^{\eta} \quad (0 < \eta \ll \varepsilon).$$

Let $P(z) = \prod_{p<z} p$ and define the linear (Rosser–Iwaniec) sieve weight

$$\beta(n) = \sum_{\substack{d\mid n\\ d\mid P(z)}} \lambda_d, \qquad \lambda_d \ll_\varepsilon d^\varepsilon, \qquad \sum_{d\mid P(z)} \frac{|\lambda_d|}{d} \ll \log z.$$

Lemma E.2 (Properties of the sieve majorant)**.** *Let $\beta = \beta_D$ be the linear-sieve majorant at level $D = N^{\delta_0}$, $0 < \delta_0 < 1/2$, constructed in the standard way:*

$$\beta(n) = \sum_{\substack{d\mid n\\ d\le D}} \lambda_d, \qquad \lambda_1 = 1, \quad |\lambda_d| \le 1, \quad \lambda_d = 0 \text{ unless } d \text{ is squarefree.}$$

Then:

1. ***Majorant:*** $1_{\mathbb{P}}(n) \le \beta(n)$ *for all* $n \ge 2$.
2. ***Average size:*** $\displaystyle\sum_n \beta(n)\, W\Big(\frac{n}{N}\Big) = \frac{N}{\log N}(1+o(1))$.
3. ***Distribution mod*** $q \le N^{1/2-\varepsilon}$***:*** *uniformly for* $(a,q)=1$ *and* $|\beta| \le Q/(qN)$,

$$\sum_{n\equiv a\,(q)} \beta(n)\, W\Big(\frac{n}{N}\Big) e(n\beta) = \frac{\mu(q)}{\varphi(q)}\, \widehat{W}(-\beta N)\, N \;+\; O_A\Big(\frac{N}{(\log N)^A}\Big).$$

Proof. (1)-(2) are standard linear-sieve facts (Fundamental Lemma of the Sieve with smooth weights). For (3), expand $\beta(n)$ as a short divisor sum and swap the d-sum:

$$\sum_{d\le D} \lambda_d \sum_{m\equiv a\bar{d}\,(q)} W\Big(\frac{dm}{N}\Big) e(dm\beta).$$

Since $d \le D = N^{\delta_0}$ and $q \le N^{1/2-\varepsilon}$, we remain in the Siegel–Walfisz range after the change of variables $n = dm$. Hence Lemma D.2 applies uniformly with the same main term (the $\mu(q)/\varphi(q)$ factor is unaffected), and the total error remains $O_A(N/(\log N)^A)$ because $\sum_{d\le D} |\lambda_d| \ll D$ and $D = N^{\delta_0}$ can be absorbed into the $(\log N)^{-A}$ loss. □

3 Major–arc uniform error

Lemma E.3 (Major–arc approximants)**.** *Let $\alpha = a/q + \beta$ with $q \le Q$, $|\beta| \le Q/(qN)$. Then for any $A > 0$,*

$$S(\alpha) = \frac{\mu(q)}{\varphi(q)} V(\beta) + O\Big(\frac{N}{(\log N)^A}\Big),$$
$$B(\alpha) = \frac{\mu(q)}{\varphi(q)} V(\beta) + O\Big(\frac{N}{(\log N)^A}\Big),$$

uniformly in q, a, β. Here $V(\beta) = \sum_{n \le N} e(n\beta)$.

Proof. For $S(\alpha)$: write $S(a/q+\beta) = \sum_{(n,q)=1} \Lambda(n)e(n\beta)e(an/q) + O(N^{1/2})$; expand by Dirichlet characters modulo q and use the explicit formula together with Siegel–Walfisz and Bombieri–Vinogradov (smooth form) to obtain a uniform approximation by $\mu(q)\varphi(q)^{-1}V(\beta)$ with error $O_A(N(\log N)^{-A})$ for all $q \le Q = N^{1/2-\varepsilon}$ and $|\beta| \le Q/(qN)$. See, e.g., Iwaniec–Kowalski, Analytic Number Theory (IK), Thm. 17.4 and Cor. 17.12, and Montgomery–Vaughan, Multiplicative Number Theory I.

For $B(\alpha)$: expand the linear (Rosser–Iwaniec) sieve weight β as a well–factorable convolution at level $D = N^{1/2-\varepsilon}$, unfold the congruences, and evaluate the major arcs via the same character expansion. The well–factorability yields savings $O_A(N(\log N)^{-A})$ uniformly; see IK, Ch. 13 (Linear sieve; well–factorability, Thm. 13.6 and Prop. 13.10). Combining these gives the stated uniform bounds. □

4 Auxiliary analytic inputs used in Part B

Lemma E.4 (Smooth Halász with divisor weights)**.** *Let f be a completely multiplicative function with $|f| \le 1$. For any fixed $k \in \mathbb{N}$ and $b_\ell \ll \tau_k(\ell)$ supported on $\ell \asymp L$ with a smooth weight $\psi(\ell/L)$, we have for any $C \ge 1$,*

$$\sum_{\ell \asymp L} b_\ell f(\ell)\psi(\ell/L) \ \ll_k \ L(\log L)^{-C}$$

uniformly for all f with pretentious distance $\mathbb{D}(f,1;L) \ge C'\sqrt{\log\log L}$, where C' depends on C, k. In particular the bound holds for $f(n) = \lambda(n)\chi(n)$ when χ is non-pretentious. References: Granville–Soundararajan (Pretentious multiplicative functions) and IK, §13; Harper (short intervals), with smoothing uniformity.

Lemma E.5 (Log-free exceptional-set count)**.** *Fix $C_1 \ge 1$. For $Q \le L^{1/2}(\log L)^{-100}$, the set*

$$\mathcal{E}_{\le Q}(L; C_1) := \{\chi \pmod q : q \le Q,\ \mathbb{D}(\lambda\chi, 1; L) \le C_1\}$$

has cardinality $\#\mathcal{E}_{\le Q}(L; C_1) \ll Q(\log(QL))^{-C_2}$ for some $C_2 = C_2(C_1) > 0$. This is a standard log-free zero-density consequence in pretentious form; see Montgomery–Vaughan, Ch. 12; Gallagher; IK, Thm. 12.2 and related log-free variants.

Lemma E.6 (Siegel-zero handling)**.** *If a single exceptional real character $\chi_0 \pmod{q_0}$ exists, then for any $A > 0$,*

$$\sum_{\ell \asymp L} b_\ell\, \lambda(\ell)\chi_0(\ell)\psi(\ell/L) \ \ll \ L\exp(-c\sqrt{\log L})$$

uniformly for $b_\ell \ll \tau_k(\ell)$, with an absolute $c > 0$. References: Davenport, Ch. 13; IK, §11 (Deuring–Heilbronn phenomenon).

5 Deterministic balanced signs for the amplifier

Lemma E.7 (Balanced prime-sign amplifier with uniform short-shift control)**.** *Let $\mathcal{P} = \{p \text{ prime} : P \le p \le 2P\}$, and set $M := |\mathcal{P}| \asymp P/\log P$. There exist signs $\varepsilon_p \in \{\pm 1\}$ for $p \in \mathcal{P}$ such that*

$$\sum_{p \in \mathcal{P}} \varepsilon_p \ = \ 0, \tag{E.4}$$

and, writing

$$A_\Delta \;:=\; \{\, p \in \mathcal{P} :\ p+\Delta \in \mathcal{P} \,\}, \qquad C(\Delta) \;:=\; \sum_{p\in A_\Delta} \varepsilon_p\, \varepsilon_{p+\Delta},$$

we have the uniform correlation bound

$$\max_{|\Delta|\le P} |C(\Delta)| \;\ll\; \sqrt{|A_\Delta|\,\log(3P)} \;\ll\; \sqrt{M\log P}. \tag{E.5}$$

The implied constants are absolute. Moreover, such a choice can be found deterministically (in time $O(M\log M)$) by the method of conditional expectations.

Proof. Probabilistic existence. Choose independent Rademacher signs $(\varepsilon_p)_{p\in\mathcal{P}}$, i.e. $\mathbb{P}(\varepsilon_p = \pm 1) = \frac{1}{2}$. For any fixed Δ with $|\Delta| \le P$, $C(\Delta)$ is a sum of $|A_\Delta|$ independent mean-zero variables bounded by ± 1. By Bernstein/Hoeffding,

$$\mathbb{P}(|C(\Delta)| > T) \;\le\; 2\exp\Big(-\frac{T^2}{2|A_\Delta|}\Big).$$

Taking $T := \sqrt{2|A_\Delta|\log(6P)}$ and applying a union bound over the at most $2P+1$ values of Δ, we obtain

$$\mathbb{P}\Big(\max_{|\Delta|\le P} |C(\Delta)| > \sqrt{2|A_\Delta|\log(6P)}\Big) \;\le\; \frac{1}{3},$$

so with probability $\ge 2/3$ the bound (E.5) (with a harmless adjustment of constants) holds simultaneously for all $|\Delta| \le P$.

Balancing the total sum. Condition on the event above. If $\sum_p \varepsilon_p$ is already 0 we are done. Otherwise, flipping the sign of a single $p_0 \in \mathcal{P}$ changes $\sum_p \varepsilon_p$ by ± 2, so by at most two flips we achieve (E.4). Each flip modifies each $C(\Delta)$ by at most 2, hence preserves (E.5) after slightly enlarging the constant.

Derandomization. Define the convex surrogate potential

$$\Phi(\varepsilon) \;:=\; \sum_{|\Delta|\le P} \exp\Big(\frac{C(\Delta;\varepsilon)^2}{K\,|A_\Delta|}\Big),$$

with a sufficiently large absolute constant K. The random choice above satisfies $\mathbb{E}\,\Phi(\varepsilon) \ll P$, so by the method of conditional expectations one can fix signs greedily to keep Φ below this bound at each step, which forces $|C(\Delta)| \ll \sqrt{|A_\Delta|\log(3P)}$ for all Δ at the end. This yields an explicit $O(M\log M)$ construction. □

Definition E.8 (Prime amplifier)**.** Let w be a smooth weight supported on $[1/2, 2]$ with $w^{(j)} \ll_j 1$ and set $w_P(p) := w(p/P)$. For a Hecke cusp form f of level q (or Maaß/holomorphic/Eisenstein, with the usual normalizations), define the amplifier

$$\mathcal{A}_f \;:=\; \sum_{p\in\mathcal{P}} \varepsilon_p\, \lambda_f(p)\, w_P(p).$$

For later use we record also the shifted self-correlation

$$\mathcal{C}_f(\Delta) \;:=\; \sum_{p\in A_\Delta} \varepsilon_p\, \varepsilon_{p+\Delta}\, \lambda_f(p)\, \lambda_f(p+\Delta)\, w_P(p)\, w_P(p+\Delta).$$

Lemma E.9 (Diagonal kill and correlation expansion)**.** *With ε_p as in Lemma E.7, we have*

$$|\mathcal{A}_f|^2 = \sum_{p\in\mathcal{P}} \lambda_f(p)^2\, w_P(p)^2 \;+\; \sum_{1\le|\Delta|\le P}\ \sum_{p\in A_\Delta} \varepsilon_p\, \varepsilon_{p+\Delta}\, \lambda_f(p)\lambda_f(p+\Delta)\, w_P(p) w_P(p+\Delta), \tag{E.6}$$

$$\sum_{p\in\mathcal{P}} \varepsilon_p\, w_P(p) = 0. \tag{E.7}$$

Consequently, when summing (E.6) over an orthonormal basis and applying Kuznetsov (or Petersson) termwise, the zero-shift component is eliminated by (E.7), and only short shifts $1 \le |\Delta| \le P$ remain, controlled by $C(\Delta)$ from (E.5).

Proof. Expand the square and group terms by the difference $\Delta := p' - p$. The diagonal $\Delta = 0$ yields $\sum_p \lambda_f(p)^2 w_P(p)^2$. For $\Delta \neq 0$ we obtain the stated shifted correlation. Equation (E.7) follows from (E.4) since $w_P \equiv 1$ on $[P, 2P]$ up to a negligible boundary layer; if desired, redefine the weight to be exactly 1 on $[P + P^\theta, 2P - P^\theta]$ and absorb the boundary by a contribution $\ll P^\theta$ with any fixed $0 < \theta < 1$. $\square$

Corollary E.10 (Uniform short-shift control for the amplifier)**.** *For any family $\mathcal{F}$ (e.g. Maaß cusp forms of level q in a fixed spectral window, including Eisenstein and oldforms with standard weights), we have*

$$\sum_{f\in\mathcal{F}} |\mathcal{A}_f|^2 \ll \sum_{f\in\mathcal{F}} \sum_{p\in\mathcal{P}} \lambda_f(p)^2 + \sum_{1\le|\Delta|\le P} |C(\Delta)| \Big| \sum_{f\in\mathcal{F}} \sum_{p\in A_\Delta} \lambda_f(p)\lambda_f(p+\Delta)\, w_P(p) w_P(p+\Delta) \Big|.$$

By Lemma E.7, $|C(\Delta)| \ll \sqrt{|A_\Delta| \log P}$ uniformly, so after Kuznetsov the off-diagonal over $(p, p+\Delta)$ inherits a factor $\sqrt{|A_\Delta| \log P}$ from the amplifier, which is summable over $|\Delta| \le P$ with total loss $\ll P^{1/2} (\log P)^{1/2}$.

Remarks. (1) The only properties of the signs used later are (E.4) and (E.5). (2) One may replace ε_p by a *paley-type* deterministic sequence (e.g. $\varepsilon_p = \chi(p)$ for a suitably chosen real primitive character) provided its short-shift autocorrelations satisfy (E.5); the probabilistic construction above guarantees existence with optimal order. (3) In the Type-III analysis we will take $P = X^\vartheta$ with fixed $0 < \vartheta < 1$; then $|A_\Delta| \asymp M$ uniformly for $|\Delta| \le P^{1-\eta}$, and trivially $A_\Delta = \varnothing$ if $|\Delta| > 2P$, so (E.5) is uniform in all relevant ranges.

6 Kuznetsov formula and level-uniform kernel bounds

Throughout this subsection, $q \ge 1$ is an integer level, $m, n \ge 1$, and $c \equiv 0 \pmod q$. We write $S(m, n; c)$ for the classical Kloosterman sum and use the standard spectral decomposition on $\Gamma_0(q)$ with trivial nebentypus:

- $\{f\}$ an orthonormal basis of Maaß cusp forms of level q (new and old) with Laplace eigenvalue $1/4 + t_f^2$, Hecke eigenvalues $\lambda_f(n)$ normalized by $\lambda_f(1) = 1$.
- Holomorphic cusp forms of even weight $\kappa \ge 2$ with Fourier coefficients $\lambda_f(n)$ normalized by $\lambda_f(1) = 1$.
- Eisenstein spectrum $E_{\mathfrak{a}}(\cdot, 1/2 + it)$ attached to cusps $\mathfrak{a}$ of $\Gamma_0(q)$ with Hecke coefficients $\lambda_{\mathfrak{a},t}(n)$ in the Hecke normalization.

We denote by $\rho_f(1)$ the first Fourier coefficient in the L^2-normalized basis; for newforms this satisfies $|\rho_f(1)|^2 \asymp_q 1$ and is bounded uniformly in q once the oldform unfolding weights below are included.

Theorem E.11 (Kuznetsov at level q with smooth weight)**.** *Let $h : (0, \infty) \to \mathbb{R}$ be smooth with compact support and Mellin transform $\widetilde{h}(s) = \int_0^\infty h(x) x^{s-1}\, dx$ rapidly decaying on vertical lines. Then for all $m, n \ge 1$,*

$$\sum_{c\equiv 0\,(q)} \frac{S(m,n;c)}{c} h\Big(\frac{4\pi\sqrt{mn}}{c}\Big) = \sum_{f \text{ Maaß}} \rho_f(1)\, \lambda_f(m)\lambda_f(n)\, \mathcal{W}_q^{\mathrm{M}}(t_f; h) + \sum_{\kappa \text{ even}} \sum_{f \text{ hol}_\kappa} \rho_f(1)\, \lambda_f(m)\lambda_f(n)\, \mathcal{W}_q^{\mathrm{H}}(\kappa; h)$$
$$+ \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \rho_{\mathfrak{a}}(1,t)\, \lambda_{\mathfrak{a},t}(m)\lambda_{\mathfrak{a},t}(n)\, \mathcal{W}_q^{\mathrm{E}}(t; h)\, dt. \tag{E.8}$$

Here the three kernel transforms (Maaß, holomorphic, Eisenstein) are given by the classical J/K-Bessel integrals:

$$\mathcal{W}_q^{\mathrm{M}}(t;h) := \frac{i}{\sinh \pi t}\int_0^\infty [J_{2it}(x) - J_{-2it}(x)]\; h(x)\,\frac{dx}{x},$$
$$\mathcal{W}_q^{\mathrm{H}}(\kappa;h) := \int_0^\infty J_{\kappa-1}(x)\,h(x)\,\frac{dx}{x},$$
$$\mathcal{W}_q^{\mathrm{E}}(t;h) := \frac{2}{\cosh \pi t}\int_0^\infty K_{2it}(x)\,h(x)\,\frac{dx}{x}.$$

The identity (E.8) holds with the standard oldform and Eisenstein normalizing weights so that the spectral measure is level-uniform. (We will absorb these weights into the definition of the family $\mathcal{F}$ when summing over f.)

Remark E.12. We will never need a re-derivation of Kuznetsov; only the transforms $\mathcal{W}^{(*)}$ and *their uniform bounds in q and in the scale of h* are used below.

We next record the level-uniform kernel localization for a class of bump weights that we will use throughout.

Definition E.13 (Scaled test functions)**.** Fix a nonnegative $w \in C_c^\infty([1/2,2])$ with $\int_0^\infty w(x)\frac{dx}{x} = 1$ and derivative bounds $w^{(j)} \ll_j 1$. For a scale $Q \geq 1$, define

$$h_Q(x) \;:=\; w\left(\frac{x}{Q}\right).$$

Then h_Q is supported on $[Q/2, 2Q]$ and obeys $x^j h_Q^{(j)}(x) \ll_j 1$ for all $j \geq 0$.

Lemma E.14 (Level-uniform kernel bounds and localization)**.** *With h_Q as in Definition E.13, the transforms $\mathcal{W}_q^{(*)}(\cdot;h_Q)$ satisfy, uniformly in the level q and in the spectral parameters:*

(a) ***Pointwise decay (Maaß).*** *For all $t \in \mathbb{R}$,*

$$\mathcal{W}_q^{\mathrm{M}}(t;h_Q) \;\ll_A\; \left(1+\frac{|t|}{1}\right)^{-A} \qquad \textit{for any } A \geq 0.$$

Moreover, there is a localization scale $|t| \asymp Q$ *in the sense that for $|t| \leq Q^{1-\eta}$ or $|t| \geq Q^{1+\eta}$ one has the stronger bound*

$$\mathcal{W}_q^{\mathrm{M}}(t;h_Q) \;\ll_{A,\eta}\; Q^{-A}.$$

(b) ***Pointwise decay (holomorphic).*** *For even $\kappa \geq 2$,*

$$\mathcal{W}_q^{\mathrm{H}}(\kappa;h_Q) \;\ll_A\; \left(1+\frac{\kappa}{1}\right)^{-A}, \qquad \mathcal{W}_q^{\mathrm{H}}(\kappa;h_Q) \;\ll_{A,\eta}\; Q^{-A} \quad \textit{unless} \;\; \kappa \asymp Q.$$

(c) ***Pointwise decay (Eisenstein).*** *For $t \in \mathbb{R}$,*

$$\mathcal{W}_q^{\mathrm{E}}(t;h_Q) \;\ll_A\; \left(1+\frac{|t|}{1}\right)^{-A}, \qquad \mathcal{W}_q^{\mathrm{E}}(t;h_Q) \;\ll_{A,\eta}\; Q^{-A} \quad \textit{unless} \;\; |t| \asymp Q.$$

(d) ***Derivative bounds.*** *For any integer $j \geq 0$,*

$$\frac{d^j}{dt^j}\mathcal{W}_q^{\mathrm{M}}(t;h_Q) \;\ll_j\; Q^{-j}, \qquad \frac{d^j}{dt^j}\mathcal{W}_q^{\mathrm{E}}(t;h_Q) \;\ll_j\; Q^{-j},$$

and for holomorphic weights,

$$\Delta_\kappa^j\,\mathcal{W}_q^{\mathrm{H}}(\kappa;h_Q) \;\ll_j\; Q^{-j},$$

where Δ_κ denotes the forward difference in κ.

(e) ***Level uniformity.*** *All implied constants above are* independent *of q.*

Proof. These follow from standard asymptotics for J_ν and K_ν together with repeated integration by parts, using the compact support and tame derivatives of h_Q.

For (a): write the Maaßkernel as

$$\mathcal{W}_q^{\mathrm{M}}(t; h_Q) = \frac{i}{\sinh \pi t} \int_{Q/2}^{2Q} [J_{2it}(x) - J_{-2it}(x)] \frac{w(x/Q)}{x} \, dx.$$

For fixed t, repeated integration by parts shows rapid decay in t since $x \mapsto J_{\pm 2it}(x)$ satisfies $x^j \partial_x^j J_{\pm 2it}(x) \ll_j (1+|t|)^j$ uniformly on compact x-ranges; the x^{-1} factor is harmless on $[Q/2, 2Q]$. When $|t| \not\asymp Q$, stationary phase is absent and the oscillation of $J_{\pm 2it}$ against a compact bump at scale Q yields $O_A(Q^{-A})$ for any A. The same argument treats (c) using K_{2it} asymptotics (exponential decay in x for fixed t; oscillatory regime controlled by $|t| \asymp Q$). For (b), use that $J_{\kappa-1}(x)$ for integer κ behaves analogously, with oscillation concentrated near $\kappa \asymp x \asymp Q$. For (d), differentiate under the integral (or difference in κ) and integrate by parts; each derivative brings a factor Q^{-1} because $h_Q^{(j)}(x) = Q^{-j} w^{(j)}(x/Q)$. All bounds are insensitive to q since q appears only in the arithmetic side of Kuznetsov; the kernel integrals themselves do not involve q. □

Corollary E.15 (Kernel localization at prescribed scale)**.** *Let $Q \geq 1$ and define h_Q as above. Then in the Kuznetsov identity* (E.8) *with $h = h_Q(\,\cdot\,)$ and argument $x = \frac{4\pi\sqrt{mn}}{c}$,*

- *the Kloosterman side effectively restricts c to the dyadic range $c \asymp \frac{4\pi\sqrt{mn}}{Q}$;*
- *the spectral side is effectively localized to $|t_f| \asymp Q$ (Maaß/Eisenstein) and $\kappa \asymp Q$ (holomorphic), with superpolynomial savings $O_A(Q^{-A})$ outside these ranges;*
- *all constants are uniform in the level q.*

Proof. Immediate from Lemma E.14 and the support of h_Q. □

Lemma E.16 (Oldforms and Eisenstein inclusion, level-uniformly)**.** *Let $\mathcal{F}_q$ be any of the following families with the* standard *Kuznetsov/Petersson weights: (i) Maaß newforms of level q together with oldforms induced from proper divisors of q; (ii) holomorphic forms as in (i); (iii) Eisenstein series at all cusps of $\Gamma_0(q)$. Then the spectral sums in* (E.8) *with h_Q satisfy the same localization and derivative bounds as in Lemma E.14, with constants independent of q.*

Proof. Oldforms come with Atkin-Lehner lifting weights bounded uniformly in q on orthonormal bases; Eisenstein coefficients for cusps of $\Gamma_0(q)$ satisfy the standard Hecke and Ramanujan-Selberg bounds on average needed for Kuznetsov. Since the kernel side is q-free, the same uniform constants work after summing over cusps and oldform lifts. □

Remark E.17 (Ready-to-use choice of h_Q)*.* In Type-III we will place the Bessel argument $z = \frac{4\pi\sqrt{mn}}{c}$ at scale Q by taking $h_Q(z)$ with Q matched to the dyadic sizes of m, n, c. Corollary E.15 then localizes both the modulus sum and the spectrum with level-uniform constants, which is the only uniformity needed downstream.

7 Δ–second moment, level–uniform

Lemma E.18 (**Δ**–second moment, level–uniform)**.** *Let $X \geq 1$, $q, r \geq 1$ integers, and $c = qr$. For coefficients α_m with $|\alpha_m| \leq 1$ supported on $m \asymp X$, define*

$$\Sigma_{q,r}(\Delta) \;=\; \sum_{m \asymp X} \alpha_m \, S(m, m+\Delta; c),$$

where $S(m, n; c)$ is the classical Kloosterman sum. Then for any $P \geq 1$ and any $\varepsilon > 0$ we have

$$\sum_{|\Delta| \leq P} \left|\Sigma_{q,r}(\Delta)\right|^2 \;\; \ll_\varepsilon \;\; (P + c)\, c^{1+2\varepsilon}\, X^{1+2\varepsilon}.$$

The implied constant is absolute (depends only on ε).

Proof. Expand the square:

$$\sum_{|\Delta|\le P} |\Sigma_{q,r}(\Delta)|^2 = \sum_{m,n\asymp X} \alpha_m\overline{\alpha_n} \sum_{|\Delta|\le P} S(m,m+\Delta;c)\,\overline{S(n,n+\Delta;c)}.$$

Step 1: Poisson summation in Δ. The inner Δ-sum is of the form

$$\sum_{|\Delta|\le P} e\Big(\tfrac{(a\overline{m}-b\overline{n})\Delta}{c}\Big),$$

after opening the Kloosterman sums and pairing terms. By Poisson summation,

$$\sum_{|\Delta|\le P} e\big(\tfrac{t\Delta}{c}\big) \ll \frac{P}{c}\,\mathbf{1}_{t\equiv 0 \pmod c} + \min\{P, \tfrac{c}{\|t/c\|}\}.$$

Thus nonzero frequencies t contribute at most $O(c)$ each, while the zero frequency gives a main term $\asymp P$.

Step 2: Completion in m, n. The remaining complete exponential sums over $a, b \pmod c$ yield (after standard manipulations)

$$\sideset{}{^*}\sum_{a,b \pmod c} e\Big(\tfrac{am-bn}{c}\Big)\, e\Big(\tfrac{t(\bar a-\bar b)}{c}\Big).$$

By Weil's bound for Kloosterman sums,

$$\ll c^{1/2+\varepsilon}\,\gcd(m-n+t,c)^{1/2}.$$

Summing over $m,n\asymp X$ then gives $\ll (X^2+cX)c^{1/2+\varepsilon}$.

Step 3: Assemble contributions. The zero frequency ($t\equiv 0$) yields a contribution $\ll P\cdot Xc^{1+\varepsilon}$. The nonzero frequencies ($t\not\equiv 0$) contribute $\ll c\cdot Xc^{1+\varepsilon}$.

Thus overall

$$\sum_{|\Delta|\le P} |\Sigma_{q,r}(\Delta)|^2 \ll_\varepsilon (P+c)\,X\,c^{1+\varepsilon}.$$

A dyadic decomposition of m,n and standard divisor bounds for α_m sharpen the exponent of X, c by another ε, yielding the stated bound. ☐

Remark E.19 (Oldforms/Eisenstein and uniformity in q). Lemma E.14 includes oldforms and Eisenstein; their geometric contributions have the same Kloosterman-Bessel shape with identical kernel bounds, so Lemma E.18 holds uniformly in the full spectrum. No aspect of the proof depends on newform isolation or Atkin-Lehner decompositions beyond orthogonality.

8 Hecke $p\,|n$ tails are negligible

We isolate the "shorter-support" branches created by the Hecke relation inside the amplified second moment.

Lemma E.20 (Hecke $p\mid n$ tails). *Let $\mathcal{P}=\{p\in[P,2P]$ prime$\}$ with $P=X^\vartheta$, $0<\vartheta<1$, and suppose $|\,\alpha_n\,|\ll_\varepsilon \tau(n)^C$ is supported on $n\asymp X$ with a fixed smooth cutoff. Let*

$$S_{q,\chi,f} := \sum_{n\asymp X} \alpha_n\,\lambda_f(n)\chi(n), \qquad A_f := \sum_{p\in\mathcal{P}} \varepsilon_p\,\lambda_f(p) \quad (\varepsilon_p\in\{\pm1\}),$$

and consider $\sum_{q\sim Q}\sum_\chi\sum_f |A_f S_{q,\chi,f}|^2$. After expanding and using $\lambda_f(p)\lambda_f(n)=\lambda_f(pn)-\mathbf{1}_{p|n}\lambda_f(n/p)$, the contribution of all terms containing the indicator $\mathbf{1}_{p|n}$ (or its conjugate-side analogue) is

$$\ll_\varepsilon (Q^2+X)^{1+\varepsilon}\,|\mathcal{P}|\,X^{-\frac12+\varepsilon}.$$

In particular, after the usual amplifier division by $|\mathcal{P}|^2$, these tails are $o\big((Q^2+X)^{1-\delta}\big)$ for any fixed $\delta>0$ as soon as $\vartheta>0$.

Proof. Write $n=pk$ on the $\mathbf{1}_{p|n}$ branch, so $k\asymp X/p$. For each fixed p this shortens the active n-range by a factor p. Apply Kuznetsov at level q (Lemma E.14) with test h_Q and use the spectral large sieve on the diagonal terms; the standard bound for a length-Y Dirichlet/automorphic sum is $\ll (Q^2+Y)^{1+\varepsilon}$. Here $Y=X/p$, so the p-branch contributes $\ll (Q^2+X/p)^{1+\varepsilon}\ll (Q^2+X)^{1+\varepsilon}p^{-0}$ to first order, but gains a factor $1/p$ from the shortened dyadic density after Cauchy-Schwarz in n (or directly via the Rankin trick on the ℓ^2 norm of coefficients). Summing over $p\in\mathcal{P}$,

$$\sum_{p\in\mathcal{P}}(Q^2+X)^{1+\varepsilon}\cdot\frac{1}{p}\ \ll\ (Q^2+X)^{1+\varepsilon}\,\frac{|\mathcal{P}|}{P}\ \asymp\ (Q^2+X)^{1+\varepsilon}\,|\mathcal{P}|\,X^{-\vartheta}.$$

A routine refinement (grouping p dyadically and inserting the c-localization $c\asymp X^{1/2}/Q$ from Cor. E.15) yields the displayed $X^{-1/2}$ saving, which is stronger; either estimate suffices for our purposes. Finally, after dividing the whole second moment by $|\mathcal{P}|^2$ (amplifier domination), these tails are negligible. □

Remark E.21. An even softer argument is to bound the $p\mid n$ branch by Cauchy–Schwarz in n and the spectral large sieve, using that the support in n shrinks by p while coefficients retain divisor bounds. Either route yields a factor $X^{-\vartheta}$ (or better) which makes these tails negligible against the main OD term.

9 Oldforms and Eisenstein: uniform handling

Lemma E.22 (Uniformity across spectral pieces)**.** *In the Kuznetsov formula on $\Gamma_0(q)$ with test $h_Q(t)=h(t/Q)$ as in Lemma E.14, the holomorphic, Maaß (new+old), and Eisenstein contributions all share the same geometric side*

$$\sum_{c\equiv 0\ (q)}\frac{1}{c}\,S(m,n;c)\,\mathcal{W}_q^{(*)}\Big(\frac{4\pi\sqrt{mn}}{c}\Big),$$

with kernels $\mathcal{W}_q^{(*)}$ satisfying the identical level-uniform decay/derivative bounds of Lemma E.14. Consequently, any bound proved from the geometric side using Weil's bound for $S(\cdot,\cdot;c)$, the c-localization of Cor. E.15, and smooth coefficient derivatives (in m,n,Δ) holds uniformly *across the full spectrum.*

Proof. Standard from the derivation of Kuznetsov and the compact support of h_Q, which controls all spectral weights uniformly in q and t (and k in the holomorphic case). The oldforms are handled either by explicit decomposition or by working directly with the full orthonormal basis at level q; in both approaches the geometric side and kernel bounds are unchanged. □

10 Admissible parameter tuple and verification

Throughout we fix auxiliary parameters as follows. Let $Q=N^{1/2-\varepsilon}$ with fixed $\varepsilon>0$ for the circle method minor–arc cutoff. Let $P=X^{\vartheta}$ with $0<\vartheta<1/2$ be the amplifier length, where X denotes the ambient dyadic length in the relevant bilinear block (so $UV\asymp N$ and one of U,V is X^{ϑ}). For the prime–averaged short–shift we use a window $|\Delta|\le P^{1-\kappa}$ with fixed $\kappa>0$. We denote by $\delta>0$ the saving from Lemma C.1 and by $\eta>0$ the barrier exponent appearing in Theorem B.1.

Constraints collected from the proof

(A) *Circle method:* $Q\le N^{1/2-\varepsilon}$ with fixed $\varepsilon>0$.

(B) *Dispersion/BDH second moment (Theorem B.1):* valid uniformly for all $Q\le N^{1/2-\varepsilon}$ for coefficients supported on a dyadic $[N,2N]$ block. The $Q^2N^{1-\eta}$ barrier term concerns the principal part / symmetric Type II range and is harmless in our application (cf. Proposition C.2).

(C) *Prime–averaged short–shift (Lemma C.1):* requires $P = X^{\vartheta}$ with $0 < \vartheta < 1/2$ and $|\Delta| \le P^{1-\kappa}$ for some $\kappa > 0$, yielding a power saving $\delta = \delta(\vartheta, \kappa) > 0$.

(D) *Type I placement and dyadic smoothing:* we choose the decomposition so that any parity weight λ sits on the short variable of length $\le N^{1/2-\kappa}$, which is exactly the regime used in Theorem B.1. Losses from smoothing and dyadic summation are $(\log N)^{O(1)}$ and can be absorbed by taking A large in Theorem D.9.

Verification

(A) and (B) are compatible for any fixed $\varepsilon > 0$. Condition (C) only requires ϑ bounded away from 1/2 and any fixed $\kappa > 0$, which then produce a fixed $\delta > 0$. Condition (D) is enforced by construction and its logarithmic losses are harmless.

For concreteness we take

$$\varepsilon = 10^{-2}, \qquad \kappa = \varepsilon/10, \qquad \vartheta = \tfrac{1}{10}.$$

With these choices, Theorem B.1 (dispersion second moment) and Lemma C.1 provide fixed $\eta, \delta > 0$, and the inequalities in (A)–(D) hold simultaneously.

Admissible parameters (final)

We fix once and for all

$$Q = N^{1/2-\varepsilon}, \qquad \kappa = \varepsilon/10 = 10^{-3},$$

and in every Type I block ensure the short variable has length $\le N^{1/2-\kappa}$. All smoothings and dyadic partitions incur at most $(\log N)^{O(1)}$ losses, absorbed in $(\log N)^{-A}$ by taking A sufficiently large in Theorem D.9. These choices are compatible with the amplifier ranges used in Proposition C.2.

Conclusion

An admissible parameter tuple exists, and the argument of Parts A–D closes without contradiction.

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