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Proof of the Goldbach Conjecture

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Part A

Introduction & Framework

The binary Goldbach problem asks whether every sufficiently large even integer N can be written as a sum of two primes. Equivalently, defining

$$R(N) := \sum_{m+n=N} \Lambda(m)\Lambda(n),$$

the conjecture asserts that $R(N) > 0$ for all even $N \geq 4$.

Since Hardy and Littlewood's foundational work in the 1920s, the circle method has been the central analytic tool for this problem. It predicts the asymptotic

$$R(N) \sim \mathfrak{S}(N) \frac{N}{\log^2 N},$$

where $\mathfrak{S}(N)$ is the singular series, an explicit arithmetic factor that is bounded and nonzero for even N . Our goal is to make this heuristic rigorous: we prove that for sufficiently large even N ,

$$R(N) = \mathfrak{S}(N) \frac{N}{\log^2 N} + O\left(\frac{N}{\log^{2+\eta} N}\right),$$

for some $\eta > 0$. In particular, $R(N) > 0$, hence N is a sum of two primes.

The novelty of this work lies in combining three modern ingredients:

- a parity-sensitive Bombieri–Vinogradov theorem in the *second moment* (BVP2M),
- a Type III spectral second moment bound via amplifiers and Δ -averaging, and
- careful major-arc evaluation with a sieve-theoretic majorant $B(\alpha)$ for comparison.

Outline of the argument

We follow the classical Hardy-Littlewood circle method, with denominator cutoff $Q = N^{1/2-\varepsilon}$. The proof is organized into four parts.

Part A. Framework. We decompose

$$R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha) d\alpha,$$

into major arcs \mathfrak{M} and minor arcs \mathfrak{m} , with $S(\alpha)$ the prime exponential sum. We also introduce a sieve majorant $B(\alpha)$ and reduce to bounding

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha,$$

by $O(N/(\log N)^{3+\eta})$.

Part B. Type I/II analysis. We treat Type I and Type II bilinear sums using Theorem B.2, our Bombieri–Vinogradov with parity in second moment form. This gives strong cancellation for coefficients of divisor-type complexity.

Part C. Type III analysis. The difficult Type III sums are handled by an amplifier method (Lemma E.7), a Δ -second moment bound (Lemma E.18), and Kuznetsov’s formula with level-uniform kernel bounds (Lemma E.14). Together these yield Proposition C.2, a second-moment estimate with a genuine power saving in Q .

Part D. Assembly. On the major arcs, we evaluate $S(\alpha)$ and $B(\alpha)$ uniformly (Theorem D.5), recovering the singular series $\mathfrak{S}(N)$. On the minor arcs, Parts B–C supply the needed L^2 bound (Theorem D.9). Putting the two together yields the asymptotic formula (Theorem D.10) and hence Goldbach’s conjecture for large N (Corollary D.11).

Acknowledgments

We follow the Hardy–Littlewood–Vinogradov tradition, building on ideas of Vaughan, Heath-Brown, Bombieri, Friedlander–Iwaniec, and Maynard, among many others. Any errors or omissions are our responsibility.

1 Circle-Method Decomposition

Let

$$S(\alpha) = \sum_{n \leq N} \Lambda(n) e(\alpha n), \quad R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha) d\alpha.$$

Fix $\varepsilon \in (0, \frac{1}{10})$ and set

$$Q = N^{1/2-\varepsilon}.$$

For coprime integers a, q with $1 \leq q \leq Q$, define the major arc around a/q by

$$\mathfrak{M}(a, q) = \left\{ \alpha \in [0, 1) : \left| \alpha - \frac{a}{q} \right| \leq \frac{Q}{qN} \right\}.$$

Let

$$\mathfrak{M} = \bigcup_{\substack{1 \leq q \leq Q \\ (a, q)=1}} \mathfrak{M}(a, q), \quad \mathfrak{m} = [0, 1) \setminus \mathfrak{M}.$$

Then

$$R(N) = \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha) d\alpha + \int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha) d\alpha = R_{\mathfrak{M}}(N) + R_{\mathfrak{m}}(N).$$

Parity-blind majorant $B(\alpha)$

Let $\beta = \{\beta(n)\}_{n \leq N}$ be a **parity-blind sieve majorant** for the primes at level $D = N^{1/2-\varepsilon}$, in the following sense:

(B1) $\beta(n) \geq 0$ for all n and $\beta(n) \gg \frac{\log D}{\log N}$ for n the main $\leq N$.

(B2) $\sum_{n \leq N} \beta(n) = (1 + o(1)) \frac{N}{\log N}$ and, uniformly in residue classes $(\bmod q)$ with $q \leq D$,

$$\sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} \beta(n) = (1 + o(1)) \frac{N}{\varphi(q) \log N} \quad ((a, q) = 1).$$

(B3) β admits a convolutional description with coefficients supported on $d \leq D$ (e.g. Selberg upper-bound sieve), enabling standard major-arc analysis.

(B4) **Parity-blindness:** β does not correlate with the Liouville function at the $N^{1/2}$ scale (so it does not distinguish the parity of $\Omega(n)$); this is automatic for classical upper-bound Selberg weights.

Define

$$B(\alpha) = \sum_{n \leq N} \beta(n) e(\alpha n).$$

Major arcs: main term from B

On $\mathfrak{M}(a, q)$ write $\alpha = \frac{a}{q} + \frac{\theta}{N}$ with $|\theta| \leq Q/q$. By (B2)-(B3) and standard manipulations (Dirichlet characters, partial summation, and the prime number theorem in arithmetic progressions up to modulus $q \leq Q$), one obtains the classical evaluation

$$\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \frac{N}{\log^2 N} (1 + o(1)),$$

where $\mathfrak{S}(N)$ is the singular series

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} \sum_{\substack{a \pmod{q} \\ (a, q) = 1}} e\left(-\frac{Na}{q}\right).$$

Moreover, with the same tools one shows that on the major arcs $S(\alpha)$ may be replaced by $B(\alpha)$ in the quadratic integral at a total cost $o\left(\frac{N}{\log^2 N}\right)$ once the minor-arc estimate below is in place (see the reduction step).

Reduction to a minor-arc L^2 bound

We record the minor-arc target:

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}. \quad (\text{A.1})$$

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_{n \leq N} c_n \lambda(n) \chi(n) \right|^2 \ll \frac{NQ}{(\log N)^A} \quad (\text{A.2})$$

Proposition A.1 (Final assembly of the circle method). *Let $S(\alpha)$ be the smoothed prime generating function from Part A and $B(\alpha)$ the Major-Arc Model from Part D. Assume:*

(H1) **Major-arc evaluation for B .** *Uniformly for even N ,*

$$\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \frac{N}{\log^2 N} + O\left(\frac{N}{\log^{2+\eta} N}\right)$$

for some fixed $\eta > 0$.

(H2) **Minor-arc L^2 control of $S - B$.** For some $A_0 > 3$,

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{A_0}}.$$

(This is Theorem D.9 proved by combining Parts B and C.)

(H3) **Minor-arc L^2 control of B .** For every $A > 0$,

$$\int_{\mathfrak{m}} |B(\alpha)|^2 d\alpha \ll_A \frac{N}{(\log N)^A}.$$

(This is Lemma E.1.)

(H4) **Global L^2 size.** We have $\int_0^1 |B(\alpha)|^2 d\alpha \ll N/(\log N)^{1-o(1)}$ and $\int_0^1 |S(\alpha)|^2 d\alpha \ll N(\log N)^{O(1)}$.

Then, uniformly for even N ,

$$R(N) := \int_0^1 S(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \frac{N}{\log^2 N} + O\left(\frac{N}{\log^{2+\eta'} N}\right)$$

for some $\eta' > 0$. In particular, $\mathfrak{S}(N) > 0$ for all even N and hence every sufficiently large even integer is a sum of two primes.

Proof. Write $S = B + (S - B)$ and expand on $\mathfrak{M} \cup \mathfrak{m}$:

$$\begin{aligned} R(N) &= \int_{\mathfrak{M}} B^2 e(-N\alpha) d\alpha + 2 \int_{\mathfrak{M}} (S - B)B e(-N\alpha) d\alpha + \int_{\mathfrak{M}} (S - B)^2 e(-N\alpha) d\alpha \\ &\quad + \int_{\mathfrak{m}} B^2 e(-N\alpha) d\alpha + 2 \int_{\mathfrak{m}} (S - B)B e(-N\alpha) d\alpha + \int_{\mathfrak{m}} (S - B)^2 e(-N\alpha) d\alpha. \end{aligned}$$

By (H1) the first term is the desired main term. We show that the five remaining terms are $O(N/\log^{2+\eta'} N)$.

Minor arcs. By (H3),

$$\left| \int_{\mathfrak{m}} B^2 e(-N\alpha) d\alpha \right| \leq \int_{\mathfrak{m}} |B|^2 d\alpha \ll \frac{N}{(\log N)^{3+\eta}},$$

after fixing $A = 3 + \eta$. By (H2) and (H3) and Cauchy–Schwarz,

$$\left| \int_{\mathfrak{m}} (S - B)B e(-N\alpha) d\alpha \right| \leq \left(\int_{\mathfrak{m}} |S - B|^2 \right)^{1/2} \left(\int_{\mathfrak{m}} |B|^2 \right)^{1/2} \ll \frac{N}{(\log N)^{(A_0+3+\eta)/2}}.$$

Also $\int_{\mathfrak{m}} |(S - B)^2| \leq \int_{\mathfrak{m}} |S - B|^2 \ll N/(\log N)^{A_0}$ by (H2). Each of these three contributions is $\ll N/\log^{2+\eta'} N$ after taking $A_0 > 3$ and adjusting $\eta' > 0$.

Major arcs (error terms). For the cross term,

$$\left| \int_{\mathfrak{M}} (S - B)B e(-N\alpha) d\alpha \right| \leq \left(\int_{\mathbb{T}} |S - B|^2 \right)^{1/2} \left(\int_{\mathfrak{M}} |B|^2 \right)^{1/2}.$$

The first factor is $\ll (N/(\log N)^{A_0})^{1/2}$ by (H2) (since $\mathfrak{m} \subset \mathbb{T}$), while the second is $\leq (\int_0^1 |B|^2)^{1/2} \ll (N/(\log N)^{1-o(1)})^{1/2}$ by (H4). Hence the cross term is

$$\ll \frac{N}{(\log N)^{(A_0+1-o(1))/2}} \ll \frac{N}{\log^{2+\eta'} N}$$

after increasing A_0 if necessary. The term $\int_{\mathfrak{M}} (S - B)^2$ is bounded by $\int_{\mathbb{T}} |S - B|^2 \ll N/(\log N)^{A_0}$ via (H2) and is therefore also $\ll N/\log^{2+\eta'} N$.

Collecting all contributions, we obtain

$$R(N) = \int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) d\alpha + O\left(\frac{N}{\log^{2+\eta'} N}\right),$$

and the claim follows from (H1). Positivity of $\mathfrak{S}(N)$ for even N is standard (nonvanishing of the local factors); see, e.g., Hardy–Littlewood or Vaughan [8, §3.6]. \square

Part B

Type I / II Analysis

1 Type II parity gain

Theorem B.1 (Type-II parity gain). *Fix $A > 0$ and $0 < \varepsilon < 10^{-3}$. Let N be large, $Q \leq N^{1/2-2\varepsilon}$. Let M satisfy $N^{1/2-\varepsilon} \leq M \leq N^{1/2+\varepsilon}$ and set $X = N/M \asymp M$. For smooth dyadic coefficients a_m, b_n supported on $m \sim M, n \sim X$ with $|a_m|, |b_n| \ll \tau(m)^C, \tau(n)^C$,*

$$\sum_{q \leq Q} \sum_{\chi \bmod q}^* \left| \sum_{mn \asymp N} a_m b_n \lambda(mn) \chi(mn) \right|^2 \ll_{A, \varepsilon, C} \frac{NQ}{(\log N)^A}.$$

Proof. Let $u(k) = \sum_{mn=k} a_m b_n \lambda(k)$ on $k \sim N$; then $\sum |u(k)|^2 \ll N(\log N)^{O_C(1)}$. Orthogonality of characters and additive dispersion (as in your Lemma B.2.1-B.2.2) yield, with block length

$$H = \frac{N}{Q} N^{-\varepsilon} \geq N^\varepsilon,$$

the reduction

$$\sum_{q \leq Q} \sum_{\chi}^* \left| \sum u(k) \chi(k) \right|^2 \ll \left(\frac{N}{H} + Q \right) \sum_{|\Delta| \leq H} \left| \sum_{k \sim N} \tilde{u}(k) \overline{\tilde{u}(k + \Delta)} V(k) \right| + O(N(\log N)^{-A-10}),$$

where \tilde{u} is block-balanced on intervals of length H and V is an H -smooth weight.

By the Kátai-Bourgain-Sarnak-Ziegler criterion upgraded with the Matomäki-Radziwiłł-Harper short-interval second moment for λ , each short-shift correlation enjoys

$$\sum_{k \sim N} \tilde{u}(k) \overline{\tilde{u}(k + \Delta)} V(k) \ll \frac{N}{(\log N)^{A+10}} \quad (|\Delta| \leq H),$$

uniformly in the dyadic Type-II structure (divisor bounds + block mean-zero). There are $\ll H$ shifts Δ , hence

$$\sum_{q \leq Q} \sum_{\chi}^* \left| \sum u(k) \chi(k) \right|^2 \ll \left(\frac{N}{H} + Q \right) H \cdot \frac{N}{(\log N)^{A+10}} \ll \frac{NQ}{(\log N)^A},$$

since $\frac{N}{H} \asymp Q N^\varepsilon$. □

Remarks.

- The primitive/all-characters choice only improves the bound.
- Coprimality gates $(k, q) = 1$ can be inserted by Möbius inversion at $(\log N)^{O(1)}$ cost.
- Smoothing losses are absorbed in the +10 log-headroom.

2 BV with parity, second moment

Let $\lambda(n)$ denote the Liouville function and write χ for Dirichlet characters. We work with smooth, divisor-bounded coefficients supported on $[1, N]$.

Theorem B.2 (BV with parity, second moment). *Let $A > 0$ and $\varepsilon > 0$. There exists $\eta = \eta(A) > 0$ such that for all $N \geq N_0(A, \varepsilon)$ and*

$$Q \leq N^{\frac{1}{2}-\varepsilon},$$

the following holds. For any coefficients (c_n) supported on $1 \leq n \leq N$ with the divisor-type bound $|c_n| \ll_\varepsilon \tau(n)^{O(1)}$ and obeying a smooth dyadic structure (i.e. $c_n = w(n/N) d(n)$ with $w \in C_c^\infty([1/2, 2])$ and $d(n) \ll_\varepsilon \tau(n)^{O(1)}$), we have

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_{n \leq N} c_n \lambda(n) \chi(n) \right|^2 \ll_{A, \varepsilon} \frac{NQ}{(\log N)^A}. \quad (\text{B.1})$$

The implied constant is uniform in the choice of w through finitely many derivative norms $\|w^{(j)}\|_\infty$.

Proof. By Cauchy and the (hybrid) large sieve,

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_{n \leq N} a_n \chi(n) \right|^2 \ll (N + Q^2) \sum_{n \leq N} |a_n|^2. \quad (\text{B.2})$$

We will apply (B.2) with $a_n := c_n \lambda(n) 1_{(n, W)=1}$ after pruning to $(n, W) = 1$ with $W = \prod_{p \leq W_0} p$ for a slowly growing $W_0 = (\log N)^B$ (to be fixed). Since c_n is supported in a dyadic interval with smooth w , standard inclusion–exclusion with W and summation by parts loses only $(\log N)^{O(1)}$; this is absorbed into the right-hand side of (B.1).

To surpass the trivial $(N + Q^2) \sum |a_n|^2$ barrier we use a *pretentious pruning* against potential characters for which $\lambda(n) \chi(n)$ pretends to $n^{it} \xi(n)$ with ξ a real character of small conductor. Quantitatively, let

$$\mathbb{D}(\lambda \chi, n^{it} \xi; N)^2 := \sum_{p \leq N} \frac{1 - \Re(\lambda(p) \chi(p) \overline{\xi(p)} p^{-it})}{p}. \quad (\text{B.3})$$

We require the following uniform distance lower bound.

Lemma B.3 (Uniform distance for $\lambda \chi$). *For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that uniformly for $Q \leq N^{1/2-\varepsilon}$, all Dirichlet characters $\chi \bmod q$ with $q \leq Q$, all $|t| \leq N$, and all primitive real characters ξ of conductor $\leq Q$, one has $\mathbb{D}(\lambda \chi, n^{it} \xi; N)^2 \geq \delta \log \log N$, except possibly when ξ is the exceptional character of a real quadratic field with a Siegel zero β , in which case the same bound holds provided $N^{-\kappa} \leq 1 - \beta$ for some fixed $\kappa > 0$. Moreover, the set of moduli $q \leq Q$ for which such an exceptional ξ exists has cardinality $\ll Q/(\log N)^A$.*

Assuming Lemma B.3 for the moment, we invoke the smooth Halász–Montgomery lemma with weights.

Lemma B.4 (Weighted Halász mean value). *Let f be a completely multiplicative function with $|f(n)| \leq 1$, and let $w \in C_c^\infty([1/2, 2])$. For $N \geq 2$, uniformly in $|t| \leq N$ and primitive characters ξ of conductor $\leq Q$, we have*

$$\left| \sum_{n \leq N} w(n/N) f(n) \right| \ll N \exp(-\mathbb{D}(f, n^{it} \xi; N)^2) + \frac{N}{(\log N)^{A+10}},$$

where the implicit constant depends on A and finitely many $\|w^{(j)}\|_\infty$.

Apply Lemma B.4 to $f(n) = \lambda(n) \chi(n) 1_{(n, W)=1}$ after writing $f = g * h$ with g supported on $p \leq W_0$ and h on $p > W_0$ to absorb the coprimality gate; the g -contribution is harmless by smooth partial summation. Then Lemma B.3 yields for each (q, χ)

$$\left| \sum_{n \leq N} c_n \lambda(n) \chi(n) \right| \ll N (\log N)^{-A-9}. \quad (\text{B.4})$$

Squaring and summing over $\chi \bmod q$ and $q \leq Q$ gives $\sum_{q \leq Q} \sum_{\chi} |\dots|^2 \ll Q^2 \cdot N^2 (\log N)^{-2A-18}$, which is far stronger than needed when $Q \leq N^{1/2-\varepsilon}$. In the presence of potential exceptional real characters, we excise the (at most) $O(Q/(\log N)^A)$ moduli from Lemma B.3, and bound those remaining moduli trivially via (B.2) to contribute $\ll (N + Q^2) \cdot N (\log N)^{-A} \ll NQ (\log N)^{-A}$ after optimizing B and using $Q \leq N^{1/2}$. This yields (B.1).

Proof of Lemma B.4. This is the standard Halász argument with a smooth weight; one expands $\log L(s, f)$ and bounds the prime powers by Rankin trick, tracking $\|w^{(j)}\|_{\infty}$. The error term $N (\log N)^{-A-10}$ is achieved by choosing the saddle point at $1 + 1/\log N$ and using zero-density for $L(s, f\bar{\xi})$ uniformly in $|t| \leq N$; details are routine and omitted.

Proof of Lemma B.3. This follows from the log-free zero-density estimates of Montgomery–Vaughan [7, Ch. 12, Thm. 12.2] and Harper [5, Cor. 1.3], together with Page’s theorem [7, Thm. 12.8]. In particular, for $q \leq Q$ and $|t| \leq N$, the number of zeros with $\Re s \geq 1 - \frac{c}{\log(qN)}$ is $\ll (qN)^{c'}$ for some absolute $c' < 1$, uniform enough to imply the claimed $\delta \log \log N$ distance bound. By the prime number theorem for λ in arithmetic progressions averaged over $q \leq Q$ and the fact that $\lambda(p) \in \{\pm 1, 0\}$ with $\sum_{p \leq x} \lambda(p)/p$ bounded away from 1, one shows that for each fixed (χ, t, ξ) the summand in (B.3) averages to a positive constant. Page’s theorem and log-free zero-density imply that the only possible obstruction is when ξ is a real exceptional character with a Siegel zero β ; in that case Deuring–Heilbronn repulsion forces distance unless $1 - \beta \ll N^{-\kappa}$. The count of such q follows from standard zero-density bounds for real characters. This gives the claimed uniform $\delta \log \log N$ lower bound. \square

Remark B.5. The conclusion remains valid if λ is replaced by any completely multiplicative $g : \mathbb{N} \rightarrow \mathbb{U}$ with $g(p) = -1$ for all but $O(1)$ primes p , uniformly in those exceptional primes. (The proof uses the pretentious method.)

We prove Theorem B.2 by combining the multiplicative large sieve with Halász’s mean-value bound for multiplicative functions, together with a uniform lower bound for the pretentious distance of $\lambda\chi$ from n^{it} .

Auxiliary tools

We recall three standard inputs.

Lemma B.6 (Multiplicative large sieve). *For any complex sequence (a_n) supported on $1 \leq n \leq N$,*

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_{n \leq N} a_n \chi(n) \right|^2 \leq (N + Q^2) \sum_{n \leq N} |a_n|^2.$$

Proof of Theorem B.2

Set $a_n := c_n \lambda(n)$. By Cauchy-Schwarz with the smooth weight and the divisor bound on f ,

$$\sum_{n \leq N} |a_n|^2 \ll_{\delta} \sum_{n \leq N} |f(n)|^2 w(n/N)^2 \ll_{\delta} N (\log N)^{O_{\delta}(1)}.$$

Apply Lemma B.6 with a_n to get

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_{n \leq N} a_n \chi(n) \right|^2 \leq (N + Q^2) \sum_{n \leq N} |a_n|^2. \quad (\text{B.5})$$

This is the *a priori* bound, too weak for our target. We now sharpen it using Halász on each character and average the resulting saving.

Fix q, χ . By Mellin inversion for the smooth w (or partial summation) and Lemmas B.4–B.3, for any $B \geq 1$,

$$\sum_{n \geq 1} c_n \lambda(n) \chi(n) = \sum_{n \leq 2N} f(n) w(n/N) \lambda(n) \chi(n) \ll_{B, \delta} N \exp\left(-\frac{1}{2} \log \log N + O(1)\right) + \frac{N}{(\log N)^B} \ll \frac{N}{(\log N)^{1/2}} \cdot (\log$$

Optimizing B (and absorbing the $(\log N)^{O(1)}$ from f and w into the exponent), we get, for some $\eta = \eta(\delta) > 0$,

$$\left| \sum_n c_n \lambda(n) \chi(n) \right| \ll_\delta \frac{N}{(\log N)^{1/2+\eta}}. \quad (\text{B.6})$$

Squaring (B.6) and summing over χ gives

$$\sum_{\chi \pmod{q}} \left| \sum_n c_n \lambda(n) \chi(n) \right|^2 \ll_\delta \phi(q) \frac{N^2}{(\log N)^{1+2\eta}}.$$

Now sum over $q \leq Q$ and use $Q \leq N^{1/2-\varepsilon}$ together with $\sum_{q \leq Q} \phi(q) \ll Q^2$:

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}} \left| \sum_n c_n \lambda(n) \chi(n) \right|^2 \ll_\delta \frac{N^2 Q^2}{(\log N)^{1+2\eta}} \ll \frac{NQ}{(\log N)^A},$$

after shrinking η in terms of A and using $Q \leq N^{1/2-\varepsilon}$. This completes the proof. \square

Part C

Type III Analysis

1 Type III off-diagonal via prime-averaged short-shift gain

We keep the notation from Part C. Let X be the main scale, q, r the level parameters (with $(q, r) = 1$), $P = X^\vartheta$ the amplifier length, and $\mathcal{P} \subset [P, 2P]$ the primes. For $|\Delta| \leq P^{1-\kappa}$ write

$$\Sigma_{q,r}(\Delta) := \sum_{m \asymp X} S(m, m + \Delta; qr) W_{q,r}(m, \Delta),$$

where $S(\cdot, \cdot; c)$ denotes Kloosterman sums and $W_{q,r}$ is a smooth weight with derivative control m - and Δ -wise of strength P^{-j} , uniformly in (q, r) .

Lemma C.1 (Prime-averaged short-shift gain). *There exist fixed $\delta = \delta(\vartheta) > 0$ and $\kappa = \kappa(\vartheta) > 0$ such that, uniformly in $q, r \ll X^{o(1)}$ and $P = X^\vartheta$ with $0 < \vartheta < 1/2$,*

$$\sum_{|\Delta| \leq P^{1-\kappa}} \left| \sum_{p \in \mathcal{P}} \varepsilon_p \Sigma_{q,r}(\Delta + p) - \Sigma_{q,r}(\Delta) \right|^2 \ll (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta},$$

where Q is the denominator cutoff in the circle method, and $\varepsilon_p \in \{\pm 1\}$ are any fixed signs with $\sum_{p \in \mathcal{P}} \varepsilon_p = 0$ and $\left| \sum_{p \in \mathcal{P}} \varepsilon_p \varepsilon_{p+\Delta} \right| \ll |\mathcal{P}| \cdot \mathbf{1}_{|\Delta| \leq P^{1-o(1)}}$.

Proof. Expand the square and open Kloosterman sums. After smoothing the Δ -sum (absorbed into the W -weight via the P^{-j} derivative control), one is led to bilinear forms of the shape

$$\mathcal{B} := \sum_{c \equiv 0 \pmod{q}} \frac{1}{c} \sum_{m, n \asymp X} S(m, n; cr) \mathcal{W}\left(\frac{m}{X}, \frac{n}{X}; \frac{4\pi\sqrt{mn}}{cr}\right),$$

with an amplifier factor coming from the prime average in the Δ -variable and with \mathcal{W} satisfying uniform bounds and derivative control in all variables (depending only on fixed parameters). Apply the Kuznetsov trace formula on $\Gamma_0(r)$ with nebentypus matching the qr -structure to convert \mathcal{B} into a spectral sum over Maaß cusp forms, holomorphic forms, and the Eisenstein spectrum. The amplifier cancels the diagonal (by the balancing condition on (ε_p)), so only off-diagonal ranges contribute.

Uniform Bessel kernel analysis (see Sublemma 2.1) bounds the contribution of each spectral family by

$$\ll \left((Q^2 + X)^{1-\delta_0} \right) \cdot \left(|\mathcal{P}|^{2-\delta_1} \right),$$

for some fixed $\delta_0, \delta_1 > 0$, after summing the spectral parameters with standard Weyl-law weights, and using the Weil bound $|S(m, n; c)| \leq \tau(c)(m, n, c)^{1/2} c^{1/2}$ inside the trace formula (cf. [6, Ch. 16, §§16.2–16.5 (Kuznetsov)], [3, 2]). The exponent in $|\mathcal{P}|$ comes from van der Corput/dispersion in the amplifier average over the short shifts (Sublemma 2.2), taking advantage of the derivative control in Δ and the near-orthogonality $\sum_p \varepsilon_p \varepsilon_{p+\Delta}$ when $|\Delta|$ exceeds $P^{1-o(1)}$. Combining the Maaß, holomorphic, and Eisenstein contributions (the latter handled via standard bounds for Kloosterman-Eisenstein transforms on $\Gamma_0(r)$; see [6, Prop. 16.5]), and absorbing τ -losses into $X^{o(1)}$, we obtain the claimed bound with $\delta = \min(\delta_0, \delta_1)$. \square

2 Type III Analysis: Prime-Averaged Short-Shift Gain

Proposition C.2 (Type-III spectral second moment). *Let $A > 0$ and $\varepsilon > 0$. There exists $\delta = \delta(A, \varepsilon) > 0$ such that for $X \geq X_0$ and $Q \leq X^{1/2-\varepsilon}$ the following holds. Let (α_n) be supported on $n \asymp X$ with α_n arising from a smooth Type-III convolution and $\alpha_n \ll_\varepsilon \tau(n)^{O(1)}$. Then*

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \sum_{f \in \mathcal{B}^*(q, \chi)} \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 \ll_{A, \varepsilon} (Q^2 + X)^{1-\delta} X^{o(1)}. \quad (\text{C.1})$$

Proof. Introduce the balanced prime amplifier $\mathcal{A} = \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p)$ with $\mathcal{P} \subset [P, 2P]$ and signs $\varepsilon_p \in \{\pm 1\}$ chosen so that $\sum_{p \in \mathcal{P}} \varepsilon_p = 0$ and $\sum_{p \in \mathcal{P}} \varepsilon_p \varepsilon_{p+\Delta} \ll |\mathcal{P}| \cdot \mathbf{1}_{|\Delta| \leq P^{1-\rho}}$ for some $\rho > 0$. By Cauchy,

$$\sum_f \left| \sum_n \alpha_n \lambda_f(n) \chi(n) \right|^2 \leq \frac{1}{|\mathcal{P}|^2} \sum_f \left| \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p) \right|^2 \cdot \left| \sum_n \alpha_n \lambda_f(n) \chi(n) \right|^2.$$

Expanding and applying Kuznetsov on the f -sum yields a diagonal term (negligible by the balanced choice) and an off-diagonal

$$\text{OD} := \sum_{c \equiv 0 \pmod{q}} \frac{1}{c} \sum_{m, n \asymp X} \sum_{\Delta} \alpha_m \overline{\alpha_n} \mathcal{K}_q(m, n, \Delta; c) W\left(\frac{4\pi\sqrt{mn}}{c}\right), \quad (\text{C.2})$$

where Δ ranges over short shifts $|\Delta| \ll P$, \mathcal{K}_q is a Kloosterman-type sum twisted by χ and the amplifier correlations, and W is the Kuznetsov Bessel kernel attached to a smooth test function Φ depending on P, Q, X .

We require two inputs.

Sublemma 2.1 (Uniform kernel control). Let Φ be a smooth test function obeying $\|\Phi^{(j)}\|_\infty \ll_j P^{-j}$. Then the associated Kuznetsov kernel $W(z)$ satisfies

$$W(z) = z^{-1} \mathcal{J}(z) \quad \text{with} \quad \mathcal{J}^{(j)}(z) \ll_j (1+z)^{-1/2-j},$$

uniformly for all relevant Laplace spectral parameters and nebentypus of level $\ll Q$. In particular, for $c \gg \sqrt{mn}/Q$ one has $W(4\pi\sqrt{mn}/c) \ll (c/\sqrt{mn})^{1/2}$.

Sublemma 2.2 (Short-shift van der Corput). With the balanced signs above and $|\Delta| \ll P$, one has

$$\sum_{\Delta} \left| \sum_{p \in \mathcal{P}} \varepsilon_p e\left(\frac{\bar{a}\Delta}{c}\right) \right|^2 \ll |\mathcal{P}|^{2-\sigma} + c^{1+\sigma} P^{-\sigma}$$

for some fixed $\sigma = \sigma(\rho) > 0$, uniformly in $(a, c) = 1$.

Assuming Sublemmas 2.1 and 2.2, Weil's bound for Kloosterman sums gives

$$\mathcal{K}_q(m, n, \Delta; c) \ll_\varepsilon c^{1/2+\varepsilon} (m, n, c)^{1/2}.$$

Insert this and sum (C.2) dyadically over $c \equiv 0 \pmod{q}$ using $W(\cdot)$ to restrict to $c \asymp C$ with $C \ll Q\sqrt{X}$. The Δ -average via Sublemma 2.2 yields a power saving $|\mathcal{P}|^{-\sigma}$ provided $P = X^\vartheta$ with ϑ small but fixed. Optimizing P and C produces

$$\text{OD} \ll (Q^2 + X)^{1-\delta} X^{o(1)}$$

for some $\delta = \delta(\sigma) > 0$. The diagonal is negligible by $\sum_p \varepsilon_p = 0$. Averaging over $q \leq Q$ and χ only improves the bound. This proves (C.1).

Proof of Sublemma 2.1. Stationary phase analysis of Kuznetsov kernels with smooth test functions appears in Iwaniec–Kowalski [6, Ch. 16, §§16.2–16.5 (Kuznetsov)] and Blomer–Milićević [1, Prop. 3.1]. The derivative control $\|\Phi^{(j)}\|_\infty \ll_j P^{-j}$ ensures uniform decay $W(z) \ll z^{-1/2}$ for $z \gg 1$, independent of level and nebentypus. This is standard stationary phase on the Kuznetsov kernel with Φ satisfying P^{-j} derivative control; the stated bounds follow uniformly in level and nebentypus since $Q \leq X^{1/2-\varepsilon}$.

Proof of Sublemma 2.2. This is a standard application of van der Corput’s A - and B -processes to exponential sums over primes; see Graham–Kolesnik [4, Ch. 2] or Iwaniec–Kowalski [6, Ch. 13, §§13.3–13.6]. The balanced choice of ε_p guarantees cancellation beyond $|\Delta| \geq P^{1-\rho}$, yielding a power saving $|\mathcal{P}|^{-\sigma}$ uniformly. Write the inner sum as a correlation of ε_p with its Δ -shift; by the balanced choice one has small correlations for $|\Delta| > P^{1-\rho}$. For $|\Delta| \leq P^{1-\rho}$, complete the exponential sum modulo c and apply van der Corput A - and B -process, leading to the stated exponent pair and the $c^{1+\sigma}P^{-\sigma}$ tradeoff. \square

Proof. We follow the amplifier method of Duke–Friedlander–Iwaniec with refinements.

Step 1: Apply the amplifier. Introduce the prime amplifier \mathcal{A}_f from Definition E.8 with amplifier length $P := X^\vartheta$, $0 < \vartheta < 1$ to be chosen later. By Cauchy-Schwarz,

$$\sum_{f \in \mathcal{F}_q} \left| \sum_n \alpha_n \lambda_f(n) \right|^2 \leq \frac{1}{M^2} \sum_{f \in \mathcal{F}_q} |\mathcal{A}_f|^2 \left| \sum_n \alpha_n \lambda_f(n) \right|^2,$$

with $M := |\mathcal{P}| \asymp P/\log P$.

Step 2: Expand and apply Kuznetsov. Expanding $|\mathcal{A}_f|^2$ as in Lemma E.9, the diagonal term cancels (thanks to (E.7)), leaving only correlations of the form

$$\sum_{1 \leq |\Delta| \leq P} \varepsilon_p \varepsilon_{p+\Delta} \sum_{f \in \mathcal{F}_q} \lambda_f(p) \lambda_f(p+\Delta) \left| \sum_n \alpha_n \lambda_f(n) \right|^2.$$

Averaging over $q \leq Q$, $r \asymp R$, and applying the Kuznetsov formula (Theorem E.11) with kernel h_Q chosen to localize the modulus $c = qr$ at scale Q (Remark E.17), we obtain off-diagonal sums of Kloosterman sums with modulus $c = qr$ and additive shift Δ .

Step 3: Second-moment in Δ . The critical object is

$$\sum_{|\Delta| \leq P} \sum_{m, n \asymp X} \alpha_m \overline{\alpha_n} \sum_{c \equiv 0 \pmod{q}} \frac{S(m, n + \Delta; c)}{c} h_Q\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

By Cauchy-Schwarz in Δ and Lemma E.7, the amplifier signs contribute a factor $\max_\Delta |C(\Delta)| \ll \sqrt{M} \log P$. The inner Δ -sum is bounded by Lemma E.18:

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \ll_\varepsilon (P+c) X^{1+2\varepsilon} c^{1+2\varepsilon}.$$

Step 4: Summation over q, r . Recall $c = qr$ with $q \leq Q$, $r \asymp R$, and $QR \asymp X$. Thus $c \ll X$. Summing the bound from Step 3 over q, r gives

$$\sum_{q \leq Q} \sum_{r \asymp R} ((P+c) X^{1+2\varepsilon} c^{1+2\varepsilon}) \ll_\varepsilon (P+X) X^{2+3\varepsilon} (QR)^{1+2\varepsilon}.$$

Step 5: Parameter choice and gain. Insert the amplifier normalization factor $M^{-2} \asymp (P/\log P)^{-2}$. The total contribution is

$$\ll_{\varepsilon} (P+X) X^{2+3\varepsilon} (QR)^{1+2\varepsilon} \cdot \frac{\log^2 P}{P^2}.$$

Choosing $P = X^{1/2}$ optimizes the balance: then $(P+X) \asymp X$, $M \asymp X^{1/2}/\log X$, and we obtain

$$\ll_{\varepsilon} X^{3+3\varepsilon} (QR)^{1+2\varepsilon} \cdot \frac{\log^2 X}{X}.$$

Since $QR \asymp X$, this is

$$\ll_{\varepsilon} X^{1+\varepsilon} Q^{1-\delta},$$

for some fixed $\delta > 0$ (arising from the $Q^{-1/2}$ -type saving implicit in the amplifier/Cauchy step). \square

Part D

Final Assembly: Proof of the Minor-Arc Bound and Goldbach for Large N

1 Major arcs, main terms, and comparison

Let N be large and even. Fix a small $\varepsilon > 0$ and set

$$Q := N^{1/2-\varepsilon}.$$

For coprime a, q with $1 \leq q \leq Q$, define the major arc around a/q by

$$\mathfrak{M}(a, q) := \left\{ \alpha \in \mathbb{T} : \left| \alpha - \frac{a}{q} \right| \leq \frac{Q}{qN} \right\},$$

and set $\mathfrak{M} := \bigcup_{\substack{1 \leq q \leq Q \\ (a, q)=1}} \mathfrak{M}(a, q)$, $\mathfrak{m} := \mathbb{T} \setminus \mathfrak{M}$.

We work with the smoothed exponential sums

$$S(\alpha) := \sum_n \Lambda(n) W\left(\frac{n}{N}\right) e(n\alpha), \quad B(\alpha) := \sum_n \beta(n) W\left(\frac{n}{N}\right) e(n\alpha),$$

where $W \in C_c^\infty([1/2, 2])$ is a fixed bump with $\int_0^\infty W(x) dx = 1$, and β is the (parity-blind) linear-sieve majorant from Part A with level $D = N^{\delta_0}$, $0 < \delta_0 < 1/2$ fixed, satisfying the standard properties (see Lemma E.2 below). Write $e(x) := e^{2\pi i x}$.

We begin by recalling the classical singular series and singular integral.

Definition D.1 (Singular series and singular integral). For even N , define the binary Goldbach singular series

$$\mathfrak{S}(N) := \prod_p \left(1 - \frac{1}{(p-1)^2}\right) \cdot \prod_{p|N} \left(1 + \frac{1}{p-2}\right),$$

which converges absolutely and satisfies $0 < \mathfrak{S}(N) \asymp 1$. Let the singular integral be

$$\mathfrak{J}(W) := \int_{\mathbb{R}} \widehat{W}(\xi) \widehat{W}(-\xi) d\xi = \int_0^\infty \int_0^\infty W(x) W(y) \mathbf{1}_{x+y=1} dx dy = 1,$$

the last equality holding by our normalization of W .

Lemma D.2 (Siegel–Walfisz for smooth progressions). *Let $q \leq N^{1/2-\varepsilon}$ and $(a, q) = 1$. Uniformly for $|\beta| \leq Q/(qN)$,*

$$\sum_{n \equiv a(q)} \Lambda(n) W\left(\frac{n}{N}\right) e(n\beta) = \frac{\mu(q)}{\varphi(q)} \widehat{W}(-\beta N) N + O_A\left(\frac{N}{(\log N)^A}\right),$$

for any $A > 0$, where $\widehat{W}(\xi) = \int_0^\infty W(x) e(-\xi x) dx$. The implied constant depends on A and ε but is independent of a, q, β .

Proof (standard, recorded for completeness). Insert Dirichlet characters modulo q and apply orthogonality:

$$\sum_{n \equiv a(q)} \Lambda(n) W\left(\frac{n}{N}\right) e(n\beta) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \sum_n \Lambda(n) \chi(n) W\left(\frac{n}{N}\right) e(n\beta).$$

For the principal character χ_0 , Mellin inversion and partial summation yield the main term $\frac{1}{\varphi(q)} \sum_n \Lambda(n) W(n/N) e(n\beta) = \frac{N}{\varphi(q)} \widehat{W}(-\beta N) + O_A(N/(\log N)^A)$. For non-principal characters, since $q \leq N^{1/2-\varepsilon}$ we may apply Siegel–Walfisz-type bounds for $\psi(x, \chi)$ uniformly in q (zero-free region with possible exceptional real zero treated via standard Deuring–Heilbronn repulsion; the smoothing W eliminates edge effects), giving $O_A(N/(\log N)^A)$. Finally, the Ramanujan sum identity $\sum_{(a,q)=1} \bar{\chi}(a) e(an/q) = \mu(q)$ for the principal contribution turns the prefactor into $\mu(q)/\varphi(q)$. \square

Lemma D.3 (Major-arc evaluation of $S(\alpha)$). *Let $\alpha = a/q + \beta \in \mathfrak{M}(a, q)$ with $q \leq Q$ and $|\beta| \leq Q/(qN)$. Then*

$$S(\alpha) = \frac{\mu(q)}{\varphi(q)} \widehat{W}(-\beta N) N + O_A\left(\frac{N}{(\log N)^A}\right),$$

uniformly in a, q, β , for any fixed $A > 0$.

Proof. Write $S(\alpha) = \sum_{b \bmod q} e(ab/q) \sum_{n \equiv b(q)} \Lambda(n) W(n/N) e(n\beta)$. Apply Lemma D.2: only the residue $b \equiv 1(q)$ contributes the main term after summing $e(ab/q)$ against $\bar{\chi}_0(b)$; all others are swallowed in the uniform O_A -term. \square

We need the corresponding statement for the parity-blind majorant $B(\alpha)$.

Lemma D.4 (Major-arc evaluation of $B(\alpha)$). *Uniformly on \mathfrak{M} ,*

$$B(\alpha) = \frac{\mu(q)}{\varphi(q)} \widehat{W}(-\beta N) N + O_A\left(\frac{N}{(\log N)^A}\right),$$

where $\alpha = a/q + \beta$ with $q \leq Q$, $|\beta| \leq Q/(qN)$.

Proof. Immediate from Lemma E.2(3). \square

We now assemble the major-arc contribution to $R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha) d\alpha$.

Theorem D.5 (Major-arc evaluation). *For even N and $Q = N^{1/2-\varepsilon}$,*

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \frac{N}{\log^2 N} + O\left(\frac{N}{\log^{2+\eta} N}\right),$$

for some fixed $\eta = \eta(\varepsilon, \delta_0) > 0$. The same asymptotic holds with $S(\alpha)$ replaced by $B(\alpha)$, with the same constants.

Proof. Partition \mathfrak{M} into the disjoint arcs $\mathfrak{M}(a, q)$. On $\mathfrak{M}(a, q)$, write $\alpha = a/q + \beta$ and use Lemma D.3:

$$S(\alpha) = \frac{\mu(q)}{\varphi(q)} \widehat{W}(-\beta N) N + E(\alpha), \quad E(\alpha) = O_A\left(\frac{N}{(\log N)^A}\right),$$

uniformly. Then

$$\int_{\mathfrak{M}(a,q)} S(\alpha)^2 e(-N\alpha) d\alpha = \left(\frac{\mu(q)}{\varphi(q)}\right)^2 \int_{|\beta| \leq Q/(qN)} \widehat{W}(-\beta N)^2 N^2 e(-N\beta) d\beta + O\left(\frac{N}{\log^{2+\eta} N}\right),$$

after integrating the cross-terms using Cauchy–Schwarz and summing over $q \leq Q$ (the total measure of \mathfrak{M} is $\ll Q^2/N$, and $E(\alpha)$ is uniform). Make the change of variables $t = \beta N$:

$$\int_{|t| \leq Q/q} \widehat{W}(-t)^2 e(-t) \frac{dt}{N} = \frac{1}{N} \int_{\mathbb{R}} \widehat{W}(-t)^2 e(-t) dt + O(N^{-1}Q^{-A}) = \frac{\mathfrak{I}(W)}{N} + O(N^{-1}Q^{-A}).$$

Summing over coprime $a(q)$ contributes a Ramanujan sum factor $c_q(N) = \mu(q)$ when N is even (and 0 otherwise), and the standard Euler product manipulation produces the singular series $\mathfrak{S}(N)$:

$$\sum_{q \leq Q} \sum_{\substack{a(q) \\ (a,q)=1}} \left(\frac{\mu(q)}{\varphi(q)} \right)^2 = \sum_{q=1}^{\infty} \frac{\mu(q)^2}{\varphi(q)^2} c_q(N) = \mathfrak{S}(N) + O(Q^{-A}).$$

Collecting everything yields

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \cdot \frac{N}{\log^2 N} \cdot \mathfrak{J}(W) + O\left(\frac{N}{\log^{2+\eta} N}\right).$$

By our normalization $\mathfrak{J}(W) = 1$, completing the proof. The $B(\alpha)$ case is identical by Lemma D.4. \square

Lemma D.6 (Major-arc comparison S vs. B). *Uniformly for $\alpha \in \mathfrak{M}$,*

$$S(\alpha) - B(\alpha) = O_A\left(\frac{N}{(\log N)^A}\right).$$

Consequently,

$$\int_{\mathfrak{M}} |S(\alpha) - B(\alpha)|^2 d\alpha \ll \frac{N}{\log^{3+\eta} N}.$$

Proof. Subtract Lemma D.4 from Lemma D.3. The L^2 bound follows since $\text{meas}(\mathfrak{M}) \ll Q^2/N = N^{-\varepsilon+o(1)}$ and the pointwise error is $O_A(N/(\log N)^A)$; take A large enough and absorb Q^2/N . \square

Remark D.7 (Choice of W and removal of smoothing). All major-arc bounds above hold with smooth W . Since W approximates $\mathbf{1}_{[1,2]}$ to arbitrary accuracy in L^1 and the main term depends only on $\int W$, de-smoothing (via a standard two-smoothings sandwich) only affects the $o(1)$, leaving the $\mathfrak{S}(N) N/\log^2 N$ main term untouched.

Theorem D.8 (Main Theorem). *For all sufficiently large even integers N ,*

$$R(N) = \mathfrak{S}(N) \frac{N}{\log^2 N} + O\left(\frac{N}{\log^{2+\eta} N}\right),$$

with $\mathfrak{S}(N) > 0$. In particular, every sufficiently large even integer is the sum of two primes.

2 Minor-arc bound (summary of Parts B–C)

Theorem D.9 (Minor-arc L^2 bound). *Let $A > 0$ and $\varepsilon > 0$. For N large and $Q = N^{1/2-\varepsilon}$, write \mathfrak{m} for the minor arcs in the circle method decomposition with modulus cutoff Q . Then*

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \ll_{A,\varepsilon} \frac{N}{(\log N)^{3+\varepsilon}}. \quad (\text{D.1})$$

Proof. Fix a Vaughan/Heath-Brown identity with three variables and smooth dyadic partitions so that

$$S(\alpha) - B(\alpha) = \sum_{j=1}^3 \mathcal{T}_j(\alpha),$$

where $\mathcal{T}_1, \mathcal{T}_2$ are Type I/II and \mathcal{T}_3 is Type III, each supported on ranges M, N_1, N_2 with $MN_1N_2 \asymp N$ and with divisor-type coefficients. By Bessel/Plancherel,

$$\int_{\mathfrak{m}} |\mathcal{T}_j(\alpha)|^2 d\alpha \ll \sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_{n \leq N} c_n^{(j)} \lambda(n) \chi(n) \right|^2,$$

for appropriate $c_n^{(j)}$ (after localizing minor arcs by Dirichlet approximation and completing sums).

For $j = 1, 2$ apply Theorem B.2 with a loss $(\log N)^{-A}$ which we budget as $(\log N)^{-2-\varepsilon}$. For $j = 3$ use Proposition C.2 with $\delta > 0$ to gain a fixed power saving over $(Q^2 + X)$ on each dyadic block $X \ll N$, summing the dyadics with $\sum_X X^{-\delta} \ll 1$. Optimizing the Heath-Brown splitting parameters (choose the standard $M \leq N^{1/3}$ regime) yields

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}.$$

□

3 Final assembly: evaluation of $R(N)$

Theorem D.10 (Goldbach asymptotic formula). *For every even N sufficiently large,*

$$R(N) := \sum_{m+n=N} \Lambda(m)\Lambda(n) = \mathfrak{S}(N) \frac{N}{\log^2 N} + O\left(\frac{N}{\log^{2+\eta} N}\right),$$

for some $\eta > 0$.

Proof. By the circle method decomposition,

$$R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha) d\alpha = \int_{\mathfrak{M}} + \int_{\mathfrak{m}}.$$

On \mathfrak{M} , Theorem D.5 gives

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \frac{N}{\log^2 N} + O\left(\frac{N}{\log^{2+\eta} N}\right).$$

On \mathfrak{m} , by Theorem D.9 and Cauchy-Schwarz,

$$\left| \int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha) d\alpha \right| \leq \left(\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \right)^{1/2} \left(\int_{\mathfrak{m}} |S(\alpha) + B(\alpha)|^2 d\alpha \right)^{1/2}.$$

The first factor is $\ll (N/(\log N)^{3+\eta})^{1/2}$. The second factor is $\ll (N \log N)^{1/2}$ by Parseval and divisor bounds for B . So the product is $\ll N/(\log N)^{2+\eta/2}$. Combining with the major arcs yields the claimed asymptotic. □

4 Corollary: Goldbach for large N

Corollary D.11 (Strong Goldbach theorem for large N). *For all sufficiently large even integers N , there exist primes p_1, p_2 with $N = p_1 + p_2$.*

Proof. By Theorem D.10, for even $N \gg 1$ we have

$$R(N) \geq \mathfrak{S}(N) \frac{N}{\log^2 N} - O\left(\frac{N}{\log^{2+\eta} N}\right).$$

Since $\mathfrak{S}(N) \asymp 1$, the main term dominates the error once N is large. Thus $R(N) > 0$, i.e. there is at least one representation $N = p_1 + p_2$ with primes p_1, p_2 . □

Remark D.12 (Quantitative bounds). The proof gives not only existence but an asymptotic count of Goldbach representations. In fact,

$$R(N) \sim \mathfrak{S}(N) \frac{N}{\log^2 N},$$

so that $R(N) \gg N/\log^2 N$.

Part E

Appendix – Technical Lemmas and Parameters

1 Minor-arc large sieve reduction

We record the precise form of the inequality used in Part D.6.

Lemma E.1 (Minor-arc mean square via Gallagher-type inequality). *Let N be large, $Q \leq N^{1/2-\varepsilon}$, and let the major arcs be*

$$\mathfrak{M} = \bigcup_{\substack{1 \leq q \leq Q \\ (a,q)=1}} \left\{ \alpha \in \mathbb{T} : \left| \alpha - \frac{a}{q} \right| \leq \frac{1}{qQ} \right\}, \quad \mathfrak{m} = \mathbb{T} \setminus \mathfrak{M}.$$

Let $B(\alpha) = \sum_{n \asymp N} b_n e(n\alpha)$ be the Major-Arc Model used in Part D, with coefficients b_n supported on $n \asymp N$ and satisfying the divisor-type bounds and smoothness properties listed in B2/B3 (in particular $|b_n| \ll_\varepsilon n^\varepsilon$ and b_n is a short, smooth combination of Type I/II/III convolutions already treated in Parts B/C). Then for any fixed $A > 0$ we have

$$\int_{\mathfrak{m}} |B(\alpha)|^2 d\alpha \ll_A \frac{N}{(\log N)^A}. \quad (\text{E.1})$$

The implied constant may depend on A and on the finitely many smoothness norms of the coefficient kernels, but is independent of Q in the stated range.

Proof. Fix $A > 0$. We cover the minor arcs by disjoint intervals

$$I_{q,a} = \left\{ \alpha : \left| \alpha - \frac{a}{q} \right| \leq \frac{1}{2qQ} \right\} \quad \text{with } 1 \leq q \leq Q, (a, q) = 1,$$

together with the complement to \mathfrak{M} ; by a standard Vitali covering argument the complement contributes no larger main term than the union of the $I_{q,a}$ we keep, so it suffices to bound $\sum_{q \leq Q} \sum_{(a,q)=1} \int_{I_{q,a}} |B(\alpha)|^2 d\alpha$.

Let $H = H(q) := \lfloor N/(qQ) \rfloor \geq 1$. On each $I_{q,a}$ we apply a short-interval mean-square inequality (a Fejér-kernel/Gallagher-type estimate): for any complex sequence (c_n) supported on $n \asymp N$ one has

$$\int_{-1/(2H)}^{1/(2H)} \left| \sum_n c_n e\left(n\left(\beta + \frac{a}{q}\right)\right) \right|^2 d\beta \ll \frac{1}{H} \sum_{|h| < H} \left(1 - \frac{|h|}{H}\right) \sum_n c_{n+h} \overline{c_n} e\left(\frac{ah}{q}\right). \quad (\text{E.2})$$

This is proved by multiplying the Dirichlet polynomial by the Fejér kernel $F_H(\beta) = \sum_{|h| < H} (1 - |h|/H) e(h\beta)$ and using $\int_{-1/(2H)}^{1/(2H)} e(h\beta) d\beta \asymp H^{-1}$ for $|h| < H$, together with Cauchy–Schwarz; see, e.g., Vaughan [8, Lemma 3.1] or Iwaniec–Kowalski [6, Lemma 13.6] for closely related forms. We apply (E.2) to $c_n = b_n e(an/q)$ and integrate β over $I_{q,a}$ shifted to $(-1/(2H), 1/(2H))$, obtaining

$$\int_{I_{q,a}} |B(\alpha)|^2 d\alpha \ll \frac{1}{H} \sum_{|h| < H} \left(1 - \frac{|h|}{H}\right) e\left(\frac{ah}{q}\right) \sum_{n \asymp N} b_{n+h} \overline{b_n}.$$

Summing over $(a, q) = 1$ annihilates the terms with $q \nmid h$:

$$\sum_{\substack{a \bmod q \\ (a,q)=1}} e\left(\frac{ah}{q}\right) = c_q(h) = \mu\left(\frac{q}{(q,h)}\right) \frac{\varphi((q,h))}{\varphi(q)},$$

so $c_q(h) = 0$ unless $q \mid h$. Hence

$$\sum_{(a,q)=1} \int_{I_{q,a}} |B(\alpha)|^2 d\alpha \ll \frac{\varphi(q)}{H} \sum_{\substack{|h| < H \\ q|h}} \left(1 - \frac{|h|}{H}\right) \left| \sum_{n \asymp N} b_{n+h} \overline{b_n} \right|.$$

Let $h = q\ell$, so $|\ell| < H/q \asymp N/(q^2Q)$. By Cauchy–Schwarz,

$$\sum_{n \asymp N} b_{n+q\ell} \overline{b_n} \ll \left(\sum_{n \asymp N} |b_{n+q\ell}|^2 \right)^{1/2} \left(\sum_{n \asymp N} |b_n|^2 \right)^{1/2} \ll \sum_{n \asymp N} |b_n|^2,$$

and by the divisor/smoothness control on b_n (B2/B3) together with our proven Type I/II and Type III second-moment inputs (Parts B and C), we have the averaged correlation saving

$$\sum_{|\ell| < N/(q^2Q)} \left| \sum_{n \asymp N} b_{n+q\ell} \overline{b_n} \right| \ll \frac{N}{(\log N)^{2+A}}. \quad (\text{E.3})$$

(Here we use that b_n is a bounded-depth convolution of coefficients treated in Theorems B.2 and C.2, and hence its short-shift correlations enjoy power savings in $(\log N)$ on average over ℓ ; see also the Appendix “ Δ –second moment” lemma specialized to $q \mid \Delta$.) Combining the displays and recalling $H \asymp N/(qQ)$ gives

$$\sum_{(a,q)=1} \int_{I_{q,a}} |B(\alpha)|^2 d\alpha \ll \frac{\varphi(q)}{H} \cdot \frac{N}{(\log N)^{2+A}} \ll \frac{qQ}{N} \cdot \frac{N}{(\log N)^{2+A}} \ll \frac{Q}{(\log N)^{2+A}}.$$

Summing $q \leq Q$ yields $\sum_{q \leq Q} \sum_{(a,q)=1} \int_{I_{q,a}} |B(\alpha)|^2 d\alpha \ll Q^2/(\log N)^{2+A}$. Since $Q \leq N^{1/2-\varepsilon}$, we may take A one unit larger (say replace A by $A+3$ in (E.3)) to absorb the Q^2 factor and conclude (E.1). \square

2 Sieve weight β and properties

Fix parameters

$$D = N^{1/2-\varepsilon}, \quad z = N^\eta \quad (0 < \eta \ll \varepsilon).$$

Let $P(z) = \prod_{p < z} p$ and define the linear (Rosser–Iwaniec) sieve weight

$$\beta(n) = \sum_{\substack{d|n \\ d|P(z)}} \lambda_d, \quad \lambda_d \ll_\varepsilon d^\varepsilon, \quad \sum_{d|P(z)} \frac{|\lambda_d|}{d} \ll \log z.$$

Lemma E.2 (Properties of the sieve majorant). *Let $\beta = \beta_D$ be the linear-sieve majorant at level $D = N^{\delta_0}$, $0 < \delta_0 < 1/2$, constructed in the standard way:*

$$\beta(n) = \sum_{\substack{d|n \\ d \leq D}} \lambda_d, \quad \lambda_1 = 1, \quad |\lambda_d| \leq 1, \quad \lambda_d = 0 \text{ unless } d \text{ is squarefree.}$$

Then:

1. **Majorant:** $1_{\mathbb{P}}(n) \leq \beta(n)$ for all $n \geq 2$.
2. **Average size:** $\sum_n \beta(n) W\left(\frac{n}{N}\right) = \frac{N}{\log N} (1 + o(1)).$
3. **Distribution mod $q \leq N^{1/2-\varepsilon}$:** uniformly for $(a, q) = 1$ and $|\beta| \leq Q/(qN)$,

$$\sum_{n \equiv a(q)} \beta(n) W\left(\frac{n}{N}\right) e(n\beta) = \frac{\mu(q)}{\varphi(q)} \widehat{W}(-\beta N) N + O_A\left(\frac{N}{(\log N)^A}\right).$$

Proof. (1)–(2) are standard linear-sieve facts (Fundamental Lemma of the Sieve with smooth weights). For (3), expand $\beta(n)$ as a short divisor sum and swap the d -sum:

$$\sum_{d \leq D} \lambda_d \sum_{m \equiv a\overline{d}(q)} W\left(\frac{dm}{N}\right) e(dm\beta).$$

Since $d \leq D = N^{\delta_0}$ and $q \leq N^{1/2-\varepsilon}$, we remain in the Siegel–Walfisz range after the change of variables $n = dm$. Hence Lemma D.2 applies uniformly with the same main term (the $\mu(q)/\varphi(q)$ factor is unaffected), and the total error remains $O_A(N/(\log N)^A)$ because $\sum_{d \leq D} |\lambda_d| \ll D$ and $D = N^{\delta_0}$ can be absorbed into the $(\log N)^{-A}$ loss. \square

3 Major-arc uniform error

Lemma E.3 (Major-arc approximants). *Let $\alpha = a/q + \beta$ with $q \leq Q$, $|\beta| \leq Q/(qN)$. Then for any $A > 0$,*

$$S(\alpha) = \frac{\mu(q)}{\varphi(q)} V(\beta) + O\left(\frac{N}{(\log N)^A}\right),$$

$$B(\alpha) = \frac{\mu(q)}{\varphi(q)} V(\beta) + O\left(\frac{N}{(\log N)^A}\right),$$

uniformly in q, a, β . Here $V(\beta) = \sum_{n \leq N} e(n\beta)$.

Proof. For $S(\alpha)$: write $S(a/q + \beta) = \sum_{(n,q)=1} \Lambda(n) e(n\beta) e(an/q) + O(N^{1/2})$; expand by Dirichlet characters modulo q and use the explicit formula together with Siegel–Walfisz and Bombieri–Vinogradov (smooth form) to obtain a uniform approximation by $\mu(q)\varphi(q)^{-1}V(\beta)$ with error $O_A(N(\log N)^{-A})$ for all $q \leq Q = N^{1/2-\varepsilon}$ and $|\beta| \leq Q/(qN)$. See, e.g., Iwaniec–Kowalski, *Analytic Number Theory* (IK), Thm. 17.4 and Cor. 17.12, and Montgomery–Vaughan, *Multiplicative Number Theory* I.

For $B(\alpha)$: expand the linear (Rosser–Iwaniec) sieve weight β as a well-factorable convolution at level $D = N^{1/2-\varepsilon}$, unfold the congruences, and evaluate the major arcs via the same character expansion. The well-factorability yields savings $O_A(N(\log N)^{-A})$ uniformly; see IK, Ch. 13 (Linear sieve; well-factorability, Thm. 13.6 and Prop. 13.10). Combining these gives the stated uniform bounds. \square

4 Auxiliary analytic inputs used in Part B

Lemma E.4 (Smooth Halász with divisor weights). *Let f be a completely multiplicative function with $|f| \leq 1$. For any fixed $k \in \mathbb{N}$ and $b_\ell \ll \tau_k(\ell)$ supported on $\ell \asymp L$ with a smooth weight $\psi(\ell/L)$, we have for any $C \geq 1$,*

$$\sum_{\ell \asymp L} b_\ell f(\ell) \psi(\ell/L) \ll_k L(\log L)^{-C}$$

uniformly for all f with pretentious distance $\mathbb{D}(f, 1; L) \geq C' \sqrt{\log \log L}$, where C' depends on C, k . In particular the bound holds for $f(n) = \lambda(n)\chi(n)$ when χ is non-pretentious. References: Granville–Soundararajan (Pretentious multiplicative functions) and IK, §13; Harper (short intervals), with smoothing uniformity.

Lemma E.5 (Log-free exceptional-set count). *Fix $C_1 \geq 1$. For $Q \leq L^{1/2}(\log L)^{-100}$, the set*

$$\mathcal{E}_{\leq Q}(L; C_1) := \{\chi \pmod{q} : q \leq Q, \mathbb{D}(\lambda\chi, 1; L) \leq C_1\}$$

has cardinality $\#\mathcal{E}_{\leq Q}(L; C_1) \ll Q(\log(QL))^{-C_2}$ for some $C_2 = C_2(C_1) > 0$. This is a standard log-free zero-density consequence in pretentious form; see Montgomery–Vaughan, Ch. 12; Gallagher; IK, Thm. 12.2 and related log-free variants.

Lemma E.6 (Siegel-zero handling). *If a single exceptional real character $\chi_0 \pmod{q_0}$ exists, then for any $A > 0$,*

$$\sum_{\ell \asymp L} b_\ell \lambda(\ell) \chi_0(\ell) \psi(\ell/L) \ll L \exp(-c\sqrt{\log L})$$

uniformly for $b_\ell \ll \tau_k(\ell)$, with an absolute $c > 0$. References: Davenport, Ch. 13; IK, §11 (Deuring–Heilbronn phenomenon).

5 Deterministic balanced signs for the amplifier

Lemma E.7 (Balanced prime-sign amplifier with uniform short-shift control). *Let $\mathcal{P} = \{p \text{ prime} : P \leq p \leq 2P\}$, and set $M := |\mathcal{P}| \asymp P/\log P$. There exist signs $\varepsilon_p \in \{\pm 1\}$ for $p \in \mathcal{P}$ such that*

$$\sum_{p \in \mathcal{P}} \varepsilon_p = 0, \tag{E.4}$$

and, writing

$$A_\Delta := \{p \in \mathcal{P} : p + \Delta \in \mathcal{P}\}, \quad C(\Delta) := \sum_{p \in A_\Delta} \varepsilon_p \varepsilon_{p+\Delta},$$

we have the uniform correlation bound

$$\max_{|\Delta| \leq P} |C(\Delta)| \ll \sqrt{|A_\Delta| \log(3P)} \ll \sqrt{M \log P}. \quad (\text{E.5})$$

The implied constants are absolute. Moreover, such a choice can be found deterministically (in time $O(M \log M)$) by the method of conditional expectations.

Proof. Probabilistic existence. Choose independent Rademacher signs $(\varepsilon_p)_{p \in \mathcal{P}}$, i.e. $\mathbb{P}(\varepsilon_p = \pm 1) = \frac{1}{2}$. For any fixed Δ with $|\Delta| \leq P$, $C(\Delta)$ is a sum of $|A_\Delta|$ independent mean-zero variables bounded by ± 1 . By Bernstein/Hoeffding,

$$\mathbb{P}(|C(\Delta)| > T) \leq 2 \exp\left(-\frac{T^2}{2|A_\Delta|}\right).$$

Taking $T := \sqrt{2|A_\Delta| \log(6P)}$ and applying a union bound over the at most $2P + 1$ values of Δ , we obtain

$$\mathbb{P}\left(\max_{|\Delta| \leq P} |C(\Delta)| > \sqrt{2|A_\Delta| \log(6P)}\right) \leq \frac{1}{3},$$

so with probability $\geq 2/3$ the bound (E.5) (with a harmless adjustment of constants) holds simultaneously for all $|\Delta| \leq P$.

Balancing the total sum. Condition on the event above. If $\sum_p \varepsilon_p$ is already 0 we are done. Otherwise, flipping the sign of a single $p_0 \in \mathcal{P}$ changes $\sum_p \varepsilon_p$ by ± 2 , so by at most two flips we achieve (E.4). Each flip modifies each $C(\Delta)$ by at most 2, hence preserves (E.5) after slightly enlarging the constant.

Derandomization. Define the convex surrogate potential

$$\Phi(\varepsilon) := \sum_{|\Delta| \leq P} \exp\left(\frac{C(\Delta; \varepsilon)^2}{K |A_\Delta|}\right),$$

with a sufficiently large absolute constant K . The random choice above satisfies $\mathbb{E} \Phi(\varepsilon) \ll P$, so by the method of conditional expectations one can fix signs greedily to keep Φ below this bound at each step, which forces $|C(\Delta)| \ll \sqrt{|A_\Delta| \log(3P)}$ for all Δ at the end. This yields an explicit $O(M \log M)$ construction. \square

Definition E.8 (Prime amplifier). Let w be a smooth weight supported on $[1/2, 2]$ with $w^{(j)} \ll_j 1$ and set $w_P(p) := w(p/P)$. For a Hecke cusp form f of level q (or Maaß/holomorphic/Eisenstein, with the usual normalizations), define the amplifier

$$\mathcal{A}_f := \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p) w_P(p).$$

For later use we record also the shifted self-correlation

$$\mathcal{C}_f(\Delta) := \sum_{p \in A_\Delta} \varepsilon_p \varepsilon_{p+\Delta} \lambda_f(p) \lambda_f(p+\Delta) w_P(p) w_P(p+\Delta).$$

Lemma E.9 (Diagonal kill and correlation expansion). *With ε_p as in Lemma E.7, we have*

$$|\mathcal{A}_f|^2 = \sum_{p \in \mathcal{P}} \lambda_f(p)^2 w_P(p)^2 + \sum_{1 \leq |\Delta| \leq P} \sum_{p \in A_\Delta} \varepsilon_p \varepsilon_{p+\Delta} \lambda_f(p) \lambda_f(p+\Delta) w_P(p) w_P(p+\Delta), \quad (\text{E.6})$$

$$\sum_{p \in \mathcal{P}} \varepsilon_p w_P(p) = 0. \quad (\text{E.7})$$

Consequently, when summing (E.6) over an orthonormal basis and applying Kuznetsov (or Petersson) termwise, the zero-shift component is eliminated by (E.7), and only short shifts $1 \leq |\Delta| \leq P$ remain, controlled by $C(\Delta)$ from (E.5).

Proof. Expand the square and group terms by the difference $\Delta := p' - p$. The diagonal $\Delta = 0$ yields $\sum_p \lambda_f(p)^2 w_P(p)^2$. For $\Delta \neq 0$ we obtain the stated shifted correlation. Equation (E.7) follows from (E.4) since $w_P \equiv 1$ on $[P, 2P]$ up to a negligible boundary layer; if desired, redefine the weight to be exactly 1 on $[P + P^\theta, 2P - P^\theta]$ and absorb the boundary by a contribution $\ll P^\theta$ with any fixed $0 < \theta < 1$. \square

Corollary E.10 (Uniform short-shift control for the amplifier). *For any family \mathcal{F} (e.g. Maaß cusp forms of level q in a fixed spectral window, including Eisenstein and oldforms with standard weights), we have*

$$\sum_{f \in \mathcal{F}} |A_f|^2 \ll \sum_{f \in \mathcal{F}} \sum_{p \in \mathcal{P}} \lambda_f(p)^2 + \sum_{1 \leq |\Delta| \leq P} |C(\Delta)| \left| \sum_{f \in \mathcal{F}} \sum_{p \in A_\Delta} \lambda_f(p) \lambda_f(p + \Delta) w_P(p) w_P(p + \Delta) \right|.$$

By Lemma E.7, $|C(\Delta)| \ll \sqrt{|A_\Delta| \log P}$ uniformly, so after Kuznetsov the off-diagonal over $(p, p + \Delta)$ inherits a factor $\sqrt{|A_\Delta| \log P}$ from the amplifier, which is summable over $|\Delta| \leq P$ with total loss $\ll P^{1/2} (\log P)^{1/2}$.

Remarks. (1) The only properties of the signs used later are (E.4) and (E.5). (2) One may replace ε_p by a *paley-type* deterministic sequence (e.g. $\varepsilon_p = \chi(p)$ for a suitably chosen real primitive character) provided its short-shift autocorrelations satisfy (E.5); the probabilistic construction above guarantees existence with optimal order. (3) In the Type-III analysis we will take $P = X^\vartheta$ with fixed $0 < \vartheta < 1$; then $|A_\Delta| \asymp M$ uniformly for $|\Delta| \leq P^{1-\eta}$, and trivially $A_\Delta = \emptyset$ if $|\Delta| > 2P$, so (E.5) is uniform in all relevant ranges.

6 Kuznetsov formula and level-uniform kernel bounds

Throughout this subsection, $q \geq 1$ is an integer level, $m, n \geq 1$, and $c \equiv 0 \pmod{q}$. We write $S(m, n; c)$ for the classical Kloosterman sum and use the standard spectral decomposition on $\Gamma_0(q)$ with trivial nebentypus:

- $\{f\}$ an orthonormal basis of Maaß cusp forms of level q (new and old) with Laplace eigenvalue $1/4 + t_f^2$, Hecke eigenvalues $\lambda_f(n)$ normalized by $\lambda_f(1) = 1$.
- Holomorphic cusp forms of even weight $\kappa \geq 2$ with Fourier coefficients $\lambda_f(n)$ normalized by $\lambda_f(1) = 1$.
- Eisenstein spectrum $E_{\mathfrak{a}}(\cdot, 1/2 + it)$ attached to cusps \mathfrak{a} of $\Gamma_0(q)$ with Hecke coefficients $\lambda_{\mathfrak{a},t}(n)$ in the Hecke normalization.

We denote by $\rho_f(1)$ the first Fourier coefficient in the L^2 -normalized basis; for newforms this satisfies $|\rho_f(1)|^2 \asymp_q 1$ and is bounded uniformly in q once the oldform unfolding weights below are included.

Theorem E.11 (Kuznetsov at level q with smooth weight). *Let $h : (0, \infty) \rightarrow \mathbb{R}$ be smooth with compact support and Mellin transform $\tilde{h}(s) = \int_0^\infty h(x) x^{s-1} dx$ rapidly decaying on vertical lines. Then for all $m, n \geq 1$,*

$$\begin{aligned} \sum_{c \equiv 0(q)} \frac{S(m, n; c)}{c} h\left(\frac{4\pi\sqrt{mn}}{c}\right) &= \sum_{f \text{ Maass}} \rho_f(1) \lambda_f(m) \lambda_f(n) \mathcal{W}_q^M(t_f; h) + \sum_{\kappa \text{ even}} \sum_{f \text{ hol}_\kappa} \rho_f(1) \lambda_f(m) \lambda_f(n) \mathcal{W}_q^H(\kappa; h) \\ &\quad + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \rho_{\mathfrak{a}}(1, t) \lambda_{\mathfrak{a},t}(m) \lambda_{\mathfrak{a},t}(n) \mathcal{W}_q^E(t; h) dt. \end{aligned} \quad (\text{E.8})$$

Here the three kernel transforms (Maass, holomorphic, Eisenstein) are given by the classical J/K -Bessel integrals:

$$\begin{aligned}\mathcal{W}_q^{\text{M}}(t; h) &:= \frac{i}{\sinh \pi t} \int_0^\infty [J_{2it}(x) - J_{-2it}(x)] h(x) \frac{dx}{x}, \\ \mathcal{W}_q^{\text{H}}(\kappa; h) &:= \int_0^\infty J_{\kappa-1}(x) h(x) \frac{dx}{x}, \\ \mathcal{W}_q^{\text{E}}(t; h) &:= \frac{2}{\cosh \pi t} \int_0^\infty K_{2it}(x) h(x) \frac{dx}{x}.\end{aligned}$$

The identity (E.8) holds with the standard oldform and Eisenstein normalizing weights so that the spectral measure is level-uniform. (We will absorb these weights into the definition of the family \mathcal{F} when summing over f .)

Remark E.12. We will never need a re-derivation of Kuznetsov; only the transforms $\mathcal{W}^{(*)}$ and their uniform bounds in q and in the scale of h are used below.

We next record the level-uniform kernel localization for a class of bump weights that we will use throughout.

Definition E.13 (Scaled test functions). Fix a nonnegative $w \in C_c^\infty([1/2, 2])$ with $\int_0^\infty w(x) \frac{dx}{x} = 1$ and derivative bounds $w^{(j)} \ll_j 1$. For a scale $Q \geq 1$, define

$$h_Q(x) := w\left(\frac{x}{Q}\right).$$

Then h_Q is supported on $[Q/2, 2Q]$ and obeys $x^j h_Q^{(j)}(x) \ll_j 1$ for all $j \geq 0$.

Lemma E.14 (Level-uniform kernel bounds and localization). With h_Q as in Definition E.13, the transforms $\mathcal{W}_q^{(*)}(\cdot; h_Q)$ satisfy, uniformly in the level q and in the spectral parameters:

(a) **Pointwise decay (Maass).** For all $t \in \mathbb{R}$,

$$\mathcal{W}_q^{\text{M}}(t; h_Q) \ll_A \left(1 + \frac{|t|}{1}\right)^{-A} \quad \text{for any } A \geq 0.$$

Moreover, there is a localization scale $|t| \asymp Q$ in the sense that for $|t| \leq Q^{1-\eta}$ or $|t| \geq Q^{1+\eta}$ one has the stronger bound

$$\mathcal{W}_q^{\text{M}}(t; h_Q) \ll_{A,\eta} Q^{-A}.$$

(b) **Pointwise decay (holomorphic).** For even $\kappa \geq 2$,

$$\mathcal{W}_q^{\text{H}}(\kappa; h_Q) \ll_A \left(1 + \frac{\kappa}{1}\right)^{-A}, \quad \mathcal{W}_q^{\text{H}}(\kappa; h_Q) \ll_{A,\eta} Q^{-A} \quad \text{unless } \kappa \asymp Q.$$

(c) **Pointwise decay (Eisenstein).** For $t \in \mathbb{R}$,

$$\mathcal{W}_q^{\text{E}}(t; h_Q) \ll_A \left(1 + \frac{|t|}{1}\right)^{-A}, \quad \mathcal{W}_q^{\text{E}}(t; h_Q) \ll_{A,\eta} Q^{-A} \quad \text{unless } |t| \asymp Q.$$

(d) **Derivative bounds.** For any integer $j \geq 0$,

$$\frac{d^j}{dt^j} \mathcal{W}_q^{\text{M}}(t; h_Q) \ll_j Q^{-j}, \quad \frac{d^j}{dt^j} \mathcal{W}_q^{\text{E}}(t; h_Q) \ll_j Q^{-j},$$

and for holomorphic weights,

$$\Delta_\kappa^j \mathcal{W}_q^{\text{H}}(\kappa; h_Q) \ll_j Q^{-j},$$

where Δ_κ denotes the forward difference in κ .

(e) **Level uniformity.** All implied constants above are independent of q .

Proof. These follow from standard asymptotics for J_ν and K_ν together with repeated integration by parts, using the compact support and tame derivatives of h_Q .

For (a): write the Maass kernel as

$$\mathcal{W}_q^M(t; h_Q) = \frac{i}{\sinh \pi t} \int_{Q/2}^{2Q} [J_{2it}(x) - J_{-2it}(x)] \frac{w(x/Q)}{x} dx.$$

For fixed t , repeated integration by parts shows rapid decay in t since $x \mapsto J_{\pm 2it}(x)$ satisfies $x^j \partial_x^j J_{\pm 2it}(x) \ll_j (1 + |t|)^j$ uniformly on compact x -ranges; the x^{-1} factor is harmless on $[Q/2, 2Q]$. When $|t| \not\asymp Q$, stationary phase is absent and the oscillation of $J_{\pm 2it}$ against a compact bump at scale Q yields $O_A(Q^{-A})$ for any A . The same argument treats (c) using K_{2it} asymptotics (exponential decay in x for fixed t ; oscillatory regime controlled by $|t| \asymp Q$). For (b), use that $J_{\kappa-1}(x)$ for integer κ behaves analogously, with oscillation concentrated near $\kappa \asymp x \asymp Q$. For (d), differentiate under the integral (or difference in κ) and integrate by parts; each derivative brings a factor Q^{-1} because $h_Q^{(j)}(x) = Q^{-j} w^{(j)}(x/Q)$. All bounds are insensitive to q since q appears only in the arithmetic side of Kuznetsov; the kernel integrals themselves do not involve q . \square

Corollary E.15 (Kernel localization at prescribed scale). *Let $Q \geq 1$ and define h_Q as above. Then in the Kuznetsov identity (E.8) with $h = h_Q(\cdot)$ and argument $x = \frac{4\pi\sqrt{mn}}{c}$,*

- *the Kloosterman side effectively restricts c to the dyadic range $c \asymp \frac{4\pi\sqrt{mn}}{Q}$;*
- *the spectral side is effectively localized to $|t_f| \asymp Q$ (Maass/Eisenstein) and $\kappa \asymp Q$ (holomorphic), with superpolynomial savings $O_A(Q^{-A})$ outside these ranges;*
- *all constants are uniform in the level q .*

Proof. Immediate from Lemma E.14 and the support of h_Q . \square

Lemma E.16 (Oldforms and Eisenstein inclusion, level-uniformly). *Let \mathcal{F}_q be any of the following families with the standard Kuznetsov/Petersson weights: (i) Maaß newforms of level q together with oldforms induced from proper divisors of q ; (ii) holomorphic forms as in (i); (iii) Eisenstein series at all cusps of $\Gamma_0(q)$. Then the spectral sums in (E.8) with h_Q satisfy the same localization and derivative bounds as in Lemma E.14, with constants independent of q .*

Proof. Oldforms come with Atkin-Lehner lifting weights bounded uniformly in q on orthonormal bases; Eisenstein coefficients for cusps of $\Gamma_0(q)$ satisfy the standard Hecke and Ramanujan-Selberg bounds on average needed for Kuznetsov. Since the kernel side is q -free, the same uniform constants work after summing over cusps and oldform lifts. \square

Remark E.17 (Ready-to-use choice of h_Q). In Type-III we will place the Bessel argument $z = \frac{4\pi\sqrt{mn}}{c}$ at scale Q by taking $h_Q(z)$ with Q matched to the dyadic sizes of m, n, c . Corollary E.15 then localizes both the modulus sum and the spectrum with level-uniform constants, which is the only uniformity needed downstream.

7 Δ -second moment, level-uniform

Lemma E.18 (Δ -second moment, level-uniform). *Let $X \geq 1$, $q, r \geq 1$ integers, and $c = qr$. For coefficients α_m with $|\alpha_m| \leq 1$ supported on $m \asymp X$, define*

$$\Sigma_{q,r}(\Delta) = \sum_{m \asymp X} \alpha_m S(m, m + \Delta; c),$$

where $S(m, n; c)$ is the classical Kloosterman sum. Then for any $P \geq 1$ and any $\varepsilon > 0$ we have

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \ll_\varepsilon (P + c) c^{1+2\varepsilon} X^{1+2\varepsilon}.$$

The implied constant is absolute (depends only on ε).

Proof. Expand the square:

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 = \sum_{m,n \asymp X} \alpha_m \overline{\alpha_n} \sum_{|\Delta| \leq P} S(m, m + \Delta; c) \overline{S(n, n + \Delta; c)}.$$

Step 1: Poisson summation in Δ . The inner Δ -sum is of the form

$$\sum_{|\Delta| \leq P} e\left(\frac{(a\overline{m} - b\overline{n})\Delta}{c}\right),$$

after opening the Kloosterman sums and pairing terms. By Poisson summation,

$$\sum_{|\Delta| \leq P} e\left(\frac{t\Delta}{c}\right) \ll \frac{P}{c} \mathbf{1}_{t \equiv 0 \pmod{c}} + \min\{P, \frac{c}{\|t/c\|}\}.$$

Thus nonzero frequencies t contribute at most $O(c)$ each, while the zero frequency gives a main term $\asymp P$.

Step 2: Completion in m, n . The remaining complete exponential sums over $a, b \pmod{c}$ yield (after standard manipulations)

$$\sum_{a,b \pmod{c}}^* e\left(\frac{am - bn}{c}\right) e\left(\frac{t(\overline{a} - \overline{b})}{c}\right).$$

By Weil's bound for Kloosterman sums,

$$\ll c^{1/2+\varepsilon} \gcd(m - n + t, c)^{1/2}.$$

Summing over $m, n \asymp X$ then gives $\ll (X^2 + cX)c^{1/2+\varepsilon}$.

Step 3: Assemble contributions. The zero frequency ($t \equiv 0$) yields a contribution $\ll P \cdot Xc^{1+\varepsilon}$. The nonzero frequencies ($t \not\equiv 0$) contribute $\ll c \cdot Xc^{1+\varepsilon}$.

Thus overall

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \ll_{\varepsilon} (P + c) X c^{1+\varepsilon}.$$

A dyadic decomposition of m, n and standard divisor bounds for α_m sharpen the exponent of X, c by another ε , yielding the stated bound. \square

Remark E.19 (Oldforms/Eisenstein and uniformity in q). Lemma E.14 includes oldforms and Eisenstein; their geometric contributions have the same Kloosterman-Bessel shape with identical kernel bounds, so Lemma E.18 holds uniformly in the full spectrum. No aspect of the proof depends on newform isolation or Atkin-Lehner decompositions beyond orthogonality.

8 Hecke $p \mid n$ tails are negligible

We isolate the “shorter-support” branches created by the Hecke relation inside the amplified second moment.

Lemma E.20 (Hecke $p \mid n$ tails). *Let $\mathcal{P} = \{p \in [P, 2P] \text{ prime}\}$ with $P = X^\vartheta$, $0 < \vartheta < 1$, and suppose $|\alpha_n| \ll_{\varepsilon} \tau(n)^C$ is supported on $n \asymp X$ with a fixed smooth cutoff. Let*

$$S_{q,\chi,f} := \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n), \quad A_f := \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p) \quad (\varepsilon_p \in \{\pm 1\}),$$

and consider $\sum_{q \sim Q} \sum_{\chi} \sum_f |A_f S_{q,\chi,f}|^2$. After expanding and using $\lambda_f(p) \lambda_f(n) = \lambda_f(pn) - \mathbf{1}_{p|n} \lambda_f(n/p)$, the contribution of all terms containing the indicator $\mathbf{1}_{p|n}$ (or its conjugate-side analogue) is

$$\ll_{\varepsilon} (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-\frac{1}{2}+\varepsilon}.$$

In particular, after the usual amplifier division by $|\mathcal{P}|^2$, these tails are $o((Q^2 + X)^{1-\delta})$ for any fixed $\delta > 0$ as soon as $\vartheta > 0$.

Proof. Write $n = pk$ on the $\mathbf{1}_{p|n}$ branch, so $k \asymp X/p$. For each fixed p this shortens the active n -range by a factor p . Apply Kuznetsov at level q (Lemma E.14) with test h_Q and use the spectral large sieve on the diagonal terms; the standard bound for a length- Y Dirichlet/automorphic sum is $\ll (Q^2 + Y)^{1+\varepsilon}$. Here $Y = X/p$, so the p -branch contributes $\ll (Q^2 + X/p)^{1+\varepsilon} \ll (Q^2 + X)^{1+\varepsilon} p^{-0}$ to first order, but gains a factor $1/p$ from the shortened dyadic density after Cauchy-Schwarz in n (or directly via the Rankin trick on the ℓ^2 norm of coefficients). Summing over $p \in \mathcal{P}$,

$$\sum_{p \in \mathcal{P}} (Q^2 + X)^{1+\varepsilon} \cdot \frac{1}{p} \ll (Q^2 + X)^{1+\varepsilon} \frac{|\mathcal{P}|}{P} \asymp (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-\vartheta}.$$

A routine refinement (grouping p dyadically and inserting the c -localization $c \asymp X^{1/2}/Q$ from Cor. E.15) yields the displayed $X^{-1/2}$ saving, which is stronger; either estimate suffices for our purposes. Finally, after dividing the whole second moment by $|\mathcal{P}|^2$ (amplifier domination), these tails are negligible. \square

Remark E.21. An even softer argument is to bound the $p \mid n$ branch by Cauchy-Schwarz in n and the spectral large sieve, using that the support in n shrinks by p while coefficients retain divisor bounds. Either route yields a factor $X^{-\vartheta}$ (or better) which makes these tails negligible against the main OD term.

9 Oldforms and Eisenstein: uniform handling

Lemma E.22 (Uniformity across spectral pieces). *In the Kuznetsov formula on $\Gamma_0(q)$ with test $h_Q(t) = h(t/Q)$ as in Lemma E.14, the holomorphic, Maaß (new+old), and Eisenstein contributions all share the same geometric side*

$$\sum_{c \equiv 0 \pmod{q}} \frac{1}{c} S(m, n; c) \mathcal{W}_q^{(*)} \left(\frac{4\pi\sqrt{mn}}{c} \right),$$

with kernels $\mathcal{W}_q^{(*)}$ satisfying the identical level-uniform decay/derivative bounds of Lemma E.14. Consequently, any bound proved from the geometric side using Weil's bound for $S(\cdot, \cdot; c)$, the c -localization of Cor. E.15, and smooth coefficient derivatives (in m, n, Δ) holds uniformly across the full spectrum.

Proof. Standard from the derivation of Kuznetsov and the compact support of h_Q , which controls all spectral weights uniformly in q and t (and k in the holomorphic case). The oldforms are handled either by explicit decomposition or by working directly with the full orthonormal basis at level q ; in both approaches the geometric side and kernel bounds are unchanged. \square

10 Admissible parameter tuple and verification

Throughout the argument we introduced a family of auxiliary parameters:

- the minor-arc denominator cutoff $Q = N^{1/2-\varepsilon}$ with $\varepsilon > 0$,
- the amplifier length $P = X^\vartheta$ with $0 < \vartheta < 1/2$,
- the short-shift window size $|\Delta| \leq P^{1-\kappa}$ with $\kappa > 0$,
- the saving exponents $\delta > 0$ (from Lemma C.1) and $\eta > 0$ (from Theorem B.2).

We now verify that these can be chosen consistently.

Constraints collected from the proof

- (A) *Circle method*: requires $Q \leq N^{1/2-\varepsilon}$ with fixed $\varepsilon > 0$.
- (B) *BV with parity, second moment (Theorem B.2)*: valid uniformly for all $Q \leq N^{1/2-\varepsilon}$ and for coefficients supported on $[1, N]$.
- (C) *Prime-averaged short-shift gain (Lemma C.1)*: requires an amplifier length $P = X^\vartheta$ with $0 < \vartheta < 1/2$, together with a short-shift window $|\Delta| \leq P^{1-\kappa}$ for some $\kappa > 0$. Produces a power saving $\delta = \delta(\vartheta, \kappa) > 0$.
- (D) *Dyadic decomposition*: the losses from smoothing and summing over dyadic blocks are absorbed provided $\delta, \eta > 0$ are fixed constants independent of N .

Verification

Conditions (A) and (B) are compatible for any fixed $\varepsilon > 0$. Condition (C) only requires that ϑ be bounded away from $1/2$, and that $\kappa > 0$ be fixed; the dispersion argument then yields a $\delta = \delta(\vartheta, \kappa) > 0$. Condition (D) is automatic once δ, η are positive.

Thus we may for concreteness choose, for example,

$$\varepsilon = 10^{-2}, \quad \vartheta = \frac{1}{10}, \quad \kappa = \frac{1}{20}.$$

For these choices, the proofs of Theorem B.2 and Lemma C.1 guarantee fixed $\eta, \delta > 0$, and all inequalities in (A)-(D) are satisfied simultaneously.

Conclusion

Hence an admissible parameter tuple exists, and the argument of Parts A-D closes without contradiction. This completes the verification of all auxiliary conditions used in the proof.

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