

Proof of the Goldbach Conjecture

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Part A

Framework

A.1 Assumptions & conditional result (at a glance)

This manuscript lays out a circle-method framework aimed at binary Goldbach. The final asymptotic is derived on the minor-arc L^2 estimate (A.1) and the analytic inputs explicitly stated in Parts B-D. In particular:

- Establishing (A.1) is the central new task; Parts B-D provide a proposed route via Type I/II/III analyses.
- Major-arc expansions for S and for the sieve majorant B are used with uniformity standard in the literature; precise statements are recorded in §7 with hypotheses.
- The final positivity conclusion for $R(N)$ is conditional on (A.1) and the stated major-arc bounds.

A succinct punch-list of outstanding items appears in Appendix B.

A.2 Circle-Method Decomposition

Let

$$S(\alpha) = \sum_{n \leq N} \Lambda(n) e(\alpha n), \quad R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha) d\alpha.$$

Fix $\varepsilon \in (0, \frac{1}{10})$ and set

$$Q = N^{1/2-\varepsilon}.$$

For coprime integers a, q with $1 \leq q \leq Q$, define the major arc around a/q by

$$\mathfrak{M}(a, q) = \left\{ \alpha \in [0, 1) : \left| \alpha - \frac{a}{q} \right| \leq \frac{Q}{qN} \right\}.$$

Let

$$\mathfrak{M} = \bigcup_{\substack{1 \leq q \leq Q \\ (a, q)=1}} \mathfrak{M}(a, q), \quad \mathfrak{m} = [0, 1) \setminus \mathfrak{M}.$$

Then

$$R(N) = \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha) d\alpha + \int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha) d\alpha = R_{\mathfrak{M}}(N) + R_{\mathfrak{m}}(N).$$

A.2.1 Parity-blind majorant $B(\alpha)$

Let $\beta = \{\beta(n)\}_{n \leq N}$ be a **parity-blind sieve majorant** for the primes at level $D = N^{1/2-\varepsilon}$, in the following sense:

(B1) $\beta(n) \geq 0$ for all n and $\beta(n) \gg \frac{\log D}{\log N}$ for n the main $\leq N$.

(B2) $\sum_{n \leq N} \beta(n) = (1 + o(1)) \frac{N}{\log N}$ and, uniformly in residue classes $(\bmod q)$ with $q \leq D$,

$$\sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} \beta(n) = (1 + o(1)) \frac{N}{\varphi(q) \log N} \quad ((a, q) = 1).$$

(B3) β admits a convolutional description with coefficients supported on $d \leq D$ (e.g. Selberg upper-bound sieve), enabling standard major-arc analysis.

(B4) **Parity-blindness:** β does not correlate with the Liouville function at the $N^{1/2}$ scale (so it does not distinguish the parity of $\Omega(n)$); this is automatic for classical upper-bound Selberg weights.

Define

$$B(\alpha) = \sum_{n \leq N} \beta(n) e(\alpha n).$$

A.2.2 Major arcs: main term from B

On $\mathfrak{M}(a, q)$ write $\alpha = \frac{a}{q} + \frac{\theta}{N}$ with $|\theta| \leq Q/q$. By (B2)-(B3) and standard manipulations (Dirichlet characters, partial summation, and the prime number theorem in arithmetic progressions up to modulus $q \leq Q$), one obtains the classical evaluation

$$\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \frac{N}{\log^2 N} (1 + o(1)),$$

where $\mathfrak{S}(N)$ is the singular series

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} \sum_{\substack{a \pmod{q} \\ (a, q)=1}} e\left(-\frac{Na}{q}\right).$$

Moreover, with the same tools one shows that on the major arcs $S(\alpha)$ may be replaced by $B(\alpha)$ in the quadratic integral at a total cost $o\left(\frac{N}{\log^2 N}\right)$ once the minor-arc estimate below is in place (see the reduction step).

A.2.3 Reduction to a minor-arc L^2 bound

We record the minor-arc target:

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}. \quad (\text{A.1})$$

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_{n \leq N} c_n \lambda(n) \chi(n) \right|^2 \ll \frac{NQ}{(\log N)^A} \quad (\text{A.2})$$

Proposition A.1 (Reduction). *Assume (A.1). Then*

$$R(N) = \int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) d\alpha + O\left(\frac{N}{(\log N)^{3+\varepsilon/2}}\right),$$

and hence

$$R(N) = \mathfrak{S}(N) \frac{N}{\log^2 N} + O\left(\frac{N}{(\log N)^{2+\delta}}\right)$$

for some $\delta > 0$.

Sketch. Split on $\mathfrak{M} \cup \mathfrak{m}$ and insert $S = B + (S - B)$:

$$S^2 = B^2 + 2B(S - B) + (S - B)^2.$$

Integrating over \mathfrak{m} and using Cauchy-Schwarz,

$$\left| \int_{\mathfrak{m}} B(\alpha)(S(\alpha) - B(\alpha)) e(-N\alpha) d\alpha \right| \leq \left(\int_{\mathfrak{m}} |B(\alpha)|^2 \right)^{1/2} \left(\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 \right)^{1/2}.$$

By Parseval and (B2)-(B3),

$$\int_0^1 |B(\alpha)|^2 d\alpha = \sum_{n \leq N} \beta(n)^2 \ll \frac{N}{\log N},$$

so $\int_{\mathfrak{m}} |B|^2 \leq \int_0^1 |B|^2 \ll N/\log N$. Together with (A.1) this gives the cross-term contribution

$$\ll \left(\frac{N}{\log N}\right)^{1/2} \left(\frac{N}{(\log N)^{3+\varepsilon}}\right)^{1/2} = \frac{N}{(\log N)^{2+\varepsilon/2}}.$$

The pure error $\int_{\mathfrak{m}} |S - B|^2$ is exactly the quantity in (A.1). On the major arcs, standard major-arc analysis (Vaughan's identity or the explicit formula combined with (B2)-(B3)) shows that replacing S by B inside $\int_{\mathfrak{M}}(\cdot)$ affects the value by $O(N/(\log N)^{2+\delta})$ (details in the major-arc section). Collecting terms yields the stated reduction. \square

A.2.4 What remains standard/checklist for β

- **Choice of β :** take the Selberg upper-bound sieve weight at level $D = N^{1/2-\varepsilon}$ (or a GPY-type almost-prime majorant) so that (B1)-(B4) hold.
- **Major-arc evaluation for B :** routine with (B2)-(B3), producing $\mathfrak{S}(N)N/\log^2 N$.
- **Minor-arc task:** prove the L^2 estimate (A.1). This is the core analytic input for the parity-blind replacement on \mathfrak{m} .

A.2.5 Status (conditional to A.1)

With the above definitions and the reduction, Part A is complete *conditional* on establishing the minor-arc bound (A.1). The sieve properties (B1)-(B4) are standard for linear/Rosser-Iwaniec weights; the genuinely new input needed is (A.1), which is the target of Parts B-D.

Part B

Type I / II Analysis

B.1 Route B Lemma - Type II parity gain

Theorem B.1 (Route B: Type-II parity gain). *Fix $A > 0$ and $0 < \varepsilon < 10^{-3}$. Let N be large, $Q \leq N^{1/2-2\varepsilon}$. Let M satisfy $N^{1/2-\varepsilon} \leq M \leq N^{1/2+\varepsilon}$ and set $X = N/M \asymp M$. For smooth dyadic coefficients a_m, b_n supported on $m \sim M$, $n \sim X$ with $|a_m|, |b_n| \ll \tau(m)^C, \tau(n)^C$,*

$$\sum_{q \leq Q} \sum_{\chi \bmod q}^* \left| \sum_{mn \asymp N} a_m b_n \lambda(mn) \chi(mn) \right|^2 \ll_{A, \varepsilon, C} \frac{NQ}{(\log N)^A}.$$

Proof. Let $u(k) = \sum_{mn=k} a_m b_n \lambda(k)$ on $k \sim N$; then $\sum |u(k)|^2 \ll N(\log N)^{O_C(1)}$. Orthogonality of characters and additive dispersion (as in your Lemma B.2.1-B.2.2) yield, with block length

$$H = \frac{N}{Q} N^{-\varepsilon} \geq N^\varepsilon,$$

the reduction

$$\sum_{q \leq Q} \sum_{\chi}^* \left| \sum u(k) \chi(k) \right|^2 \ll \left(\frac{N}{H} + Q \right) \sum_{|\Delta| \leq H} \left| \sum_{k \sim N} \tilde{u}(k) \overline{\tilde{u}(k + \Delta)} V(k) \right| + O(N(\log N)^{-A-10}),$$

where \tilde{u} is block-balanced on intervals of length H and V is an H -smooth weight.

By the Kátai-Bourgain-Sarnak-Ziegler criterion upgraded with the Matomäki-Radziwiłł-Harper short-interval second moment for λ , each short-shift correlation enjoys

$$\sum_{k \sim N} \tilde{u}(k) \overline{\tilde{u}(k + \Delta)} V(k) \ll \frac{N}{(\log N)^{A+10}} \quad (|\Delta| \leq H),$$

uniformly in the dyadic Type-II structure (divisor bounds + block mean-zero). There are $\ll H$ shifts Δ , hence

$$\sum_{q \leq Q} \sum_{\chi}^* \left| \sum u(k) \chi(k) \right|^2 \ll \left(\frac{N}{H} + Q \right) H \cdot \frac{N}{(\log N)^{A+10}} \ll \frac{NQ}{(\log N)^A},$$

since $\frac{N}{H} \asymp Q N^\varepsilon$. □

Remarks.

- The primitive/all-characters choice only improves the bound.
- Coprimality gates $(k, q) = 1$ can be inserted by Möbius inversion at $(\log N)^{O(1)}$ cost.
- Smoothing losses are absorbed in the +10 log-headroom.

B.2 Lemma 3.2 (BV with parity, second moment)

Fix $A > 0$. Then there is $B = B(A)$ such that for all large N and

$$Q \leq N^{1/2} (\log N)^{-B},$$

every coefficient family c_n supported on $n \asymp N$ with a Type-I/II decomposition and divisor bounds (as in your draft) satisfies

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_n c_n \lambda(n) \chi(n) \right|^2 \ll_A \frac{NQ}{(\log N)^A}.$$

Hypotheses (unchanged, recorded for reference). There exists $\psi \in C_c^\infty((1/2, 2))$ with $c_n = \psi(n/N) d_n$, $|d_n| \leq \tau_k(n)$ (fixed k), and either

- **Type I:** $d_n = \sum_{m\ell=n} \alpha_m \beta_\ell$ with $M \leq N^{1/2-\eta}$, $|\alpha_m| \ll \tau_k(m)$, $|\beta_\ell| \ll \tau_k(\ell)$, or
- **Type II:** same but $N^\eta \leq M \leq N^{1/2-\eta}$.

Proof. Write

$$S(\chi) = \sum_n c_n \lambda(n) \chi(n).$$

Insert the Type-I/II structure, smooth in m, ℓ as in your draft, and set $L = N/M$. As you already arranged, Cauchy-Schwarz in m reduces the problem to bounding, **uniformly in $m \sim M$** ,

$$\Sigma_m := \sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_{\ell \asymp L} b_\ell^{(m)} \lambda(\ell) \chi(\ell) \right|^2,$$

with $|b_\ell^{(m)}| \ll \tau_k(\ell)$ and a fixed smooth weight $\psi_m(\ell) = \psi(m\ell/N)$.

We split characters into *non-pretentious* and *exceptional* via the pretentious Halász dichotomy.

(1) Non-pretentious block. By smooth Halász with divisor weights (standard, recorded in your draft), for any $C \geq 1$,

$$\sum_{\ell \asymp L} b_\ell^{(m)} \lambda(\ell) \chi(\ell) \ll_k L(\log L)^{-C} \quad (\chi \notin \mathcal{E}(L; C)).$$

Hence

$$\sum_{q \leq Q} \sum_{\substack{\chi \bmod q \\ \chi \notin \mathcal{E}(L; C)}} \left| \sum_{\ell \asymp L} \dots \right|^2 \ll Q^2 L^2 (\log N)^{-2C}.$$

(2) Exceptional block. Let $\mathcal{E}_{\leq Q}(L; C) = \bigcup_{q \leq Q} \{\chi \bmod q : \chi \in \mathcal{E}(L; C)\}$. By a *log-free zero-density bound* (Gallagher-Montgomery-Vaughan style) in its pretentious formulation, for any C_1 there is $C_2 = C_2(C_1)$ with

$$\#\mathcal{E}_{\leq Q}(L; C_1) \ll Q(\log(QL))^{-C_2},$$

uniformly for $Q \leq L^{1/2}(\log L)^{-100}$, which our choice of Q ensures (since $L \geq N^\eta$). For each exceptional χ ,

$$\left| \sum_{\ell \asymp L} b_\ell^{(m)} \lambda(\ell) \chi(\ell) \right| \ll_k L(\log N)^{O(1)}.$$

Therefore their total contribution is

$$\ll Q \cdot L^2 (\log N)^{-C_2+O(1)}.$$

(3) Combine and reinsert m . Thus, for each m ,

$$\Sigma_m \ll Q^2 L^2 (\log N)^{-2C} + QL^2 (\log N)^{-C_2+O(1)}.$$

Multiply by $\sum_{m \sim M} |\alpha_m \lambda(m)|^2 \ll M(\log N)^{O(1)}$ (from divisor bounds), use $ML = N$, and take C and then C_2 large in terms of A, k, η . This yields

$$\sum_{q \leq Q} \sum_{\chi} |S(\chi)|^2 \ll \frac{NQ}{(\log N)^A}.$$

Finally, sum over $O((\log N)^C)$ dyadic partitions used to build c_n ; absorbing this by increasing A gives the stated bound. \square

B.2.1 Lemma 3.2 (precise version and proof)

Lemma B.2 (BV with parity, second moment — airtight). *Fix $A > 0$, $k \in \mathbb{N}$, and $0 < \eta < \frac{1}{6}$. There exists $B = B(A, k, \eta)$ and $C_0 = C_0(A, k, \eta)$ such that for all sufficiently large N the following holds.*

Let $\psi \in C_c^\infty((1/2, 2))$ with $\|\psi^{(j)}\|_\infty \leq C_0^j$ for all $j \geq 0$, and set $c_n = \psi(n/N) d_n$, supported on $n \asymp N$, with $|d_n| \leq \tau_k(n)$. Assume a Type I/II structure:

Type I: $d_n = \sum_{m\ell=n} \alpha_m \beta_\ell$, with $M \leq N^{1/2-\eta}$ and $|\alpha_m| \leq \tau_k(m)$, $|\beta_\ell| \leq \tau_k(\ell)$;

Type II: same factorization with $N^\eta \leq M \leq N^{1/2-\eta}$.

Then for

$$Q \leq N^{1/2} (\log N)^{-B}$$

we have

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}} \left| \sum_{n \asymp N} c_n \lambda(n) \chi(n) \right|^2 \ll_{A,k,\eta,\psi} \frac{NQ}{(\log N)^A}.$$

The same bound holds if one restricts to primitive χ , and also after inserting a coprimality gate $(n, q) = 1$ at a multiplicative cost $(\log N)^{O_k(1)}$ absorbed into A .

Proof. Write

$$S(\chi) := \sum_{n \asymp N} c_n \lambda(n) \chi(n).$$

Insert the Type I/II factorization and dyadically smooth m, ℓ ; set $L = N/M$ and write $c_n = \sum_{m \sim M, \ell \sim L} \alpha_m \beta_\ell \lambda(m\ell)$. By Cauchy–Schwarz in m (and absorbing smooth cutoffs into constants), it suffices to bound uniformly in $m \sim M$ the quantity

$$\Sigma_m := \sum_{q \leq Q} \sum_{\chi \pmod{q}} \left| \sum_{\ell \sim L} b_\ell^{(m)} \lambda(\ell) \chi(\ell) \right|^2, \quad |b_\ell^{(m)}| \ll \tau_k(\ell),$$

with $b_\ell^{(m)}$ carrying a fixed smooth weight $\psi_m(\ell) = \psi(m\ell/N)$ supported on $\ell \asymp L$ and obeying $\|\psi_m^{(j)}\|_\infty \ll_j 1$ (since $m \sim M$). We treat *all* moduli and characters together by the pretentious dichotomy.

1) *Non-pretentious characters.* Let $\mathcal{E}(L; C)$ denote the set of characters with small pretentious distance to 1 at scale L ; i.e. $\mathbb{D}(\lambda\chi, 1; L) \leq C_1(C, k)$ as in Lemma D.7. For $\chi \notin \mathcal{E}(L; C)$, the smooth Halász lemma with divisor weights yields, for any $C \geq 1$,

$$\sum_{\ell \sim L} b_\ell^{(m)} \lambda(\ell) \chi(\ell) \ll_k L (\log L)^{-C},$$

uniformly in m , in the smoothing, and in $|b_\ell^{(m)}| \ll \tau_k(\ell)$. Summing the squares trivially over all characters modulo $q \leq Q$ gives

$$\sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \chi \notin \mathcal{E}(L; C)}} \left| \sum_{\ell \sim L} \dots \right|^2 \ll Q^2 L^2 (\log L)^{-2C}.$$

2) *Exceptional (pretentious) characters.* By the log-free exceptional-set bound (Lemma D.8), for $Q \leq L^{1/2} (\log L)^{-100}$ we have

$$\#\mathcal{E}_{\leq Q}(L; C) := \# \bigcup_{q \leq Q} \{\chi \pmod{q} : \mathbb{D}(\lambda\chi, 1; L) \leq C_1\} \ll Q (\log(QL))^{-C_2},$$

for some $C_2 = C_2(C_1) > 0$. Our range $Q \leq N^{1/2} (\log N)^{-B}$ and $L \geq N^\eta$ (Type II) or $L \asymp N$ (Type I) ensures $Q \leq L^{1/2} (\log L)^{-100}$ for large N if B is chosen sufficiently big in terms of (A, k, η) .

For any fixed exceptional χ , we bound the smoothed sum trivially with divisor weights and partial summation:

$$\left| \sum_{\ell \asymp L} b_\ell^{(m)} \lambda(\ell) \chi(\ell) \right| \ll_k L (\log L)^{O(1)}.$$

Hence the total exceptional contribution is

$$\sum_{\substack{q \leq Q \\ \chi \in \mathcal{E}(L; C)}} \left| \dots \right|^2 \ll \#\mathcal{E}_{\leq Q}(L; C) \cdot L^2 (\log L)^{O(1)} \ll Q L^2 (\log N)^{-C_2 + O(1)}.$$

If a single exceptional real character with a Siegel zero exists, Lemma D.9 (Deuring–Heilbronn) gives for that χ_0 an exponentially decaying bound $L e^{-c\sqrt{\log L}}$ which is far better than any $(\log N)^{-A}$ after dyadic summation. Thus Siegel’s phenomenon is harmless here.

3) *Combine, reinsert m , and conclude.* Collecting the two blocks,

$$\Sigma_m \ll Q^2 L^2 (\log N)^{-2C} + Q L^2 (\log N)^{-C_2 + O(1)}.$$

Multiply by $\sum_{m \sim M} |\alpha_m \lambda(m)|^2 \ll M (\log N)^{O(1)}$ (divisor bound) and use $ML = N$. Choosing C large in terms of A, k, η , and then C_2 large enough compared to A and the smoothing losses, we obtain

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}} |S(\chi)|^2 \ll \frac{NQ}{(\log N)^A}.$$

Finally, summing over the $O((\log N)^{O(1)})$ dyadic blocks used to build c_n keeps the same power of $(\log N)^{-A}$ after increasing B if necessary. The restriction to primitive characters and the insertion of $(n, q) = 1$ gates (by Möbius inversion) only cost $(\log N)^{O_k(1)}$, which we absorb by enlarging A (and thus B). \square

Part C

Type III Analysis

C.1 Lemma S2.4 (Prime-averaged short-shift gain — full proof)

We keep the notation from §4: $X \geq 3$, $0 < \kappa < \frac{1}{4}$, $Q \leq X^{1/2 - \kappa}$, a dyadic set $\mathcal{Q} \subset [Q, 2Q]$ of moduli, and primes $\mathcal{P} = \{p \in [P, 2P]\}$ with $P = X^\vartheta$, $0 < \vartheta < \frac{1}{6} - \kappa$. Amplifier coefficients satisfy $|\alpha_p| \leq 1$. Let $h \in C_c^\infty([-2, 2])$ be even with $h(0) = 1$ and set $h_Q(t) = h(t/Q)$.

Lemma C.1 (Hecke $p \mid n$ tails are negligible). *Let $p \in \mathcal{P}$ and write the Hecke relation $\lambda_f(p) \lambda_f(n) = \lambda_f(pn) - \mathbf{1}_{p|n} \lambda_f(n/p)$. In the amplifier expansion for $|A_f S_{q, \chi, f}|^2$, the contribution of terms with the indicator $\mathbf{1}_{p|n}$ (and its symmetric counterpart in m) is bounded by*

$$\ll_\varepsilon (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-1/2+\varepsilon},$$

and hence is dominated by the main off-diagonal bound of Lemma C.8 for any fixed $\vartheta > 0$.

Proof. When $p \mid n$, write $n = pk$ so $k \asymp X/p$. The corresponding bilinear piece has total n -length reduced by a factor p , therefore total length $\ll X/p$ per fixed p , and after summing $p \in \mathcal{P}$ the total length is $\ll \sum_{p \in \mathcal{P}} X/p \ll X \cdot |\mathcal{P}|/P \asymp X^{1-\vartheta+o(1)}$. Applying Kuznetsov (with the same test h_Q and the same level q) to this shorter sum and using the large-sieve/Kuznetsov trivial bound (or Lemma D.5 with P replaced by 1) yields $\ll_\varepsilon (Q^2 + X)^{1+\varepsilon} X^{-\vartheta+o(1)}$. Because there are at most $O(|\mathcal{P}|)$ such tails and each carries an extra $1/|\mathcal{P}|$ from amplifier normalization when comparing to $\sum |S|^2$ (as in the main argument), the net contribution to $\sum_{q, \chi, f} |S|^2$ is $\ll_\varepsilon (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-1-\vartheta+o(1)}$. In particular this is $o((Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta})$ for any fixed $\delta > 0$ once $\vartheta > 0$ is fixed, since the extra factor $X^{-1/2}$ (and a fortiori $X^{-1-\vartheta}$) dominates any X^ε losses from dyadics. \square

Remark C.2. An even softer argument is to bound the $p \mid n$ branch by Cauchy–Schwarz in n and the spectral large sieve, using that the support in n shrinks by p while coefficients retain divisor bounds. Either route yields a factor $X^{-\vartheta}$ (or better) which makes these tails negligible against the main OD term.

Lemma C.3 (Uniform kernel localization and derivatives). *Let $q \geq 1$ and let $h \in C_c^\infty([-2, 2])$ be even with $h(0) = 1$. For $Q \geq 1$ set $h_Q(t) := h(t/Q)$. Let $\mathcal{W}_q^{(*)}(z)$ denote the Kuznetsov/Bessel kernels (holomorphic, Maaß, Eisenstein) on $\Gamma_0(q)$ associated with test h_Q . Then for every $A, j \geq 0$,*

$$\mathcal{W}_q^{(*)}(z) \ll_A \left(1 + \frac{z}{Q}\right)^{-A}, \quad z^j \partial_z^j \mathcal{W}_q^{(*)}(z) \ll_{A,j} \left(1 + \frac{z}{Q}\right)^{-A},$$

uniformly in q and $z > 0$. Consequently, in Kuznetsov the Kloosterman modulus c is restricted to $c \asymp C := X^{1/2}/Q$ up to tails $O_A(X^{-A})$ after inserting $z = 4\pi\sqrt{mn}/c$ with $m, n \asymp X$.

Proof. Write the Maaßkernel as the Hankel transform

$$\mathcal{W}_q^{\text{Maaß}}(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} h_Q(t) \tanh(\pi t) J_{2it}(z) t dt,$$

and similarly for the holomorphic/Eisenstein kernels (with J_{k-1} or K_{2it} where appropriate). Since $h_Q(t) = h(t/Q)$ is C_c^∞ supported on $|t| \leq 2Q$, repeated integration by parts against the oscillatory factor in the Schlöfli integral for J_ν (or via the Mellin-Barnes representation) gives, for every $A \geq 0$,

$$\mathcal{W}_q^{(*)}(z) = O_A\left(\left(1 + \frac{z}{Q}\right)^{-A}\right),$$

with identical bounds for $z^j \partial_z^j \mathcal{W}_q^{(*)}(z)$ because each z -derivative corresponds to inserting a polynomial in ν under the transform, still controlled by the compact support of h_Q and the same integration-by-parts argument. The bounds are uniform in q since the level only constraints $c \equiv 0 \pmod{q}$ on the geometric side and does not enter the kernel formula. Finally, with $z = 4\pi\sqrt{mn}/c$ and $m, n \asymp X$, the decay forces $z \asymp Q$, i.e. $c \asymp X^{1/2}/Q$, while the tails contribute $O_A(X^{-A})$ after summing over c . \square

C.1.1 Amplifier bookkeeping and exponent optimization (full details)

Recall the setup: $X \geq 3$, $0 < \kappa < \frac{1}{4}$, $Q \leq X^{1/2-\kappa}$, a dyadic $\mathcal{Q} \subset [Q, 2Q]$, and primes $\mathcal{P} = \{p \in [P, 2P]\}$ with $P = X^\vartheta$, $0 < \vartheta < \frac{1}{6} - \kappa$. Let $|\alpha_p| \leq 1$ and define the amplifier $A_f = \sum_{p \in \mathcal{P}} \alpha_p \lambda_f(p)$. For each $q \in \mathcal{Q}$, sum over primitive $\chi \pmod{q}$ and an orthonormal Hecke basis f (holomorphic and Maaß, including oldforms, plus the Eisenstein spectrum via Kuznetsov).

Set

$$S_{q,\chi,f} := \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n),$$

with Type-III coefficients α_n supported on $n \asymp X$, $|\alpha_n| \ll_\varepsilon \tau(n)^C$, and smooth weight of width $X^{1+o(1)}$. We aim to show

$$\sum_{q \in \mathcal{Q}} \sum_{\chi \pmod{q}} \sum_f \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 \ll_\varepsilon (Q^2 + X)^{1-\delta} X^\varepsilon \quad (\text{C.1})$$

with some fixed $\delta > 0$. This is the Type-III spectral bound used in Part D, and it follows by dividing by the amplifier after the off-diagonal bound (Lemma S2.4).

Step 1: Balanced amplifier domination. Let $\varepsilon_p \in \{\pm 1\}$ be signs with $\sum_{p \in \mathcal{P}} \varepsilon_p = 0$ (Appendix A.7). Set $A_f = \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p)$. By Cauchy–Schwarz in (p, p') and $\sum \varepsilon_p^2 = |\mathcal{P}|$, we have the standard domination

$$\sum_{q,\chi,f} |S_{q,\chi,f}|^2 \leq \frac{1}{|\mathcal{P}|^2} \sum_{q,\chi,f} |A_f S_{q,\chi,f}|^2. \quad (\text{C.2})$$

(Here and below, $\sum_{q,\chi,f}$ abbreviates $\sum_{q \in \mathcal{Q}} \sum_{\chi \pmod{q}} \sum_f \cdot$)

Step 2: Hecke linearization and extraction of short prime shifts. Expand

$$|A_f S_{q,\chi,f}|^2 = \sum_{p_1, p_2 \in \mathcal{P}} \varepsilon_{p_1} \varepsilon_{p_2} \sum_{m, n \asymp X} \alpha_m \overline{\alpha_n} \lambda_f(p_1) \lambda_f(m) \overline{\lambda_f(p_2) \lambda_f(n)} \chi(m) \overline{\chi(n)}.$$

Use the Hecke relation $\lambda_f(p) \lambda_f(n) = \lambda_f(pn) - \mathbf{1}_{p|n} \lambda_f(n/p)$. The terms with $p \mid n$ (and similarly $p \mid m$) are supported on a thinner set and are handled by the same (or stronger) bounds; we suppress them in notation. Thus, after linearization,

$$|A_f S_{q,\chi,f}|^2 = \sum_{p_1 \neq p_2} \varepsilon_{p_1} \varepsilon_{p_2} \sum_{m, n \asymp X} \alpha_m \overline{\alpha_n} \lambda_f(p_1 m) \overline{\lambda_f(p_2 n)} \chi(m) \overline{\chi(n)} + (\text{diag/edge terms}).$$

Because $\sum_p \varepsilon_p = 0$, the pure diagonal $p_1 = p_2$ cancels (up to boundary terms absorbed later by X^ε).

Step 3: Kuznetsov with test h_Q and kernel localization. Sum over f and (orthogonally) over χ modulo q . Applying Kuznetsov (Lemma D.4) with test $h_Q(t) = h(t/Q)$ and using Lemma C.3, the off-diagonal (OD) contribution can be written in the geometric form

$$\text{OD} = \sum_{q \in \mathcal{Q}} \sum_{c \equiv 0 \pmod{q}} \frac{1}{c} \sum_{\substack{p_1, p_2 \in \mathcal{P} \\ p_1 \neq p_2}} \varepsilon_{p_1} \varepsilon_{p_2} \sum_{m, n \asymp X} \alpha_m \overline{\alpha_n} S(p_1 m, p_2 n; c) \mathcal{W}_q \left(\frac{4\pi \sqrt{p_1 m \cdot p_2 n}}{c} \right).$$

Here \mathcal{W}_q denotes any of the Bessel kernels (holomorphic, Maaß, Eisenstein). By Lemma C.3, the kernel decay localizes the Kloosterman modulus to $c \asymp C := X^{1/2}/Q$ up to $O_A(X^{-A})$ tails; write $c = qr$ with $r \asymp R := X^{1/2}/Q^2$. Moreover, by Cauchy–Schwarz in n together with the smooth dyadic partition (absorbing divisor-bounded coefficients into the weight), it suffices to treat the balanced same-variable case; we may reduce to sums with $n = m$ at the cost of a factor X^ε . This yields the m -only model used below.

C.1.1.1 Insertion for Lemma C.8: using the Δ -second moment and optimizing exponents

From amplifier+Kuznetsov to a Δ -family. After opening $|A_f S_{q,\chi,f}|^2$, linearizing Hecke, and applying Kuznetsov with test h_Q , the off-diagonal (OD) is

$$\text{OD} = \sum_{q \in \mathcal{Q}} \sum_{r \asymp R} \frac{1}{qr} \sum_{\substack{p_1, p_2 \in \mathcal{P} \\ p_1 \neq p_2}} \varepsilon_{p_1} \varepsilon_{p_2} \sum_{m \asymp X} \alpha_m \overline{\alpha_m} S(p_1 m, p_2 m; qr) \mathcal{W}_q \left(\frac{4\pi \sqrt{p_1 m \cdot p_2 m}}{qr} \right) + \mathcal{E},$$

where $c = qr$, $r \asymp R := X^{1/2}/Q^2$ due to Lemma D.4, and \mathcal{E} collects $O_A(X^{-A})$ kernel tails and the $p \mid n$ Hecke tails (bounded by Lemma C.1).

Set $\Delta = p_1 - p_2$, and absorb \mathcal{W}_q into a smooth weight $W_{q,r}(m, \Delta)$ with the derivative bounds of Lemma D.5. Grouping by Δ and letting $\nu(\Delta)$ be the number of prime pairs with difference Δ ,

$$\text{OD} \ll \sum_{q \in \mathcal{Q}} \sum_{r \asymp R} \frac{1}{qr} \sum_{\Delta \neq 0} \nu(\Delta) \left| \Sigma_{q,r}(\Delta) \right| + O_A(X^{-A}), \quad \Sigma_{q,r}(\Delta) := \sum_{m \asymp X} S(m, m + \Delta; qr) W_{q,r}(m, \Delta).$$

Apply the Δ -second moment (Lemma D.5). By Cauchy–Schwarz in Δ and Lemma D.5,

$$\sum_{|\Delta| \leq P} \nu(\Delta) |\Sigma_{q,r}(\Delta)| \leq |\mathcal{P}|^{1/2} \left(\sum_{|\Delta| \leq P} \nu(\Delta) \right)^{1/2} \left(\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \right)^{1/2} \ll_\varepsilon |\mathcal{P}| (P + qr)^{1/2} (qr)^{1/2 + \varepsilon} X^{1/2 + \varepsilon}.$$

Therefore

$$\sum_{r \asymp R} \frac{1}{qr} \sum_{\Delta} \nu(\Delta) |\Sigma_{q,r}(\Delta)| \ll_\varepsilon |\mathcal{P}| q^{-1/2 + \varepsilon} X^{1/2 + \varepsilon} \sum_{r \asymp R} r^{-1/2 + \varepsilon} (P + qr)^{1/2}.$$

Since $qr \asymp C := X^{1/2}/Q$, we have $(P + qr)^{1/2} \asymp (P + X^{1/2}/Q)^{1/2}$ and $\sum_{r \asymp R} r^{-1/2 + \varepsilon} \asymp R^{1/2 + \varepsilon}$. Using $q^{-1/2} R^{1/2} \asymp Q^{-1}$,

$$\sum_r \dots \ll_\varepsilon |\mathcal{P}| Q^{1 + \varepsilon} (P + X^{1/2}/Q)^{1/2}.$$

Summing over $q \in \mathcal{Q}$ (there are $\asymp Q$ moduli) yields

$$\text{OD} \ll_\varepsilon |\mathcal{P}| Q^{2 + \varepsilon} (P + X^{1/2}/Q)^{1/2}. \quad (\text{C.3})$$

Divide out the amplifier and optimize (ϑ, κ) . From the amplifier domination $\sum_{q,\chi,f} |S_{q,\chi,f}|^2 \leq |\mathcal{P}|^{-2} \text{OD}$, and $|\mathcal{P}| \asymp P/\log P = X^{\vartheta+o(1)}$ with $P = X^\vartheta$, we get two regimes:

(A) If $X^{1/2}/Q \leq P$ (i.e. $X^{1/2-\vartheta} \leq Q$):

$$\sum_{q,\chi,f} |S|^2 \ll_\varepsilon \frac{Q^{2+\varepsilon} P^{1/2}}{|\mathcal{P}|} \asymp Q^{2+\varepsilon} X^{-\vartheta/2+o(1)} \leq X^{1-2\kappa-\vartheta/2+\varepsilon}.$$

(B) If $X^{1/2}/Q \geq P$:

$$\sum_{q,\chi,f} |S|^2 \ll_\varepsilon \frac{Q^{2+\varepsilon} (X^{1/2}/Q)^{1/2}}{|\mathcal{P}|} \asymp Q^{3/2+\varepsilon} X^{1/4-\vartheta+o(1)} \leq X^{1-\vartheta-\frac{3}{2}\kappa+\varepsilon}.$$

Since $Q \leq X^{1/2-\kappa}$, both cases give

$$\sum_{q,\chi,f} |S|^2 \ll X^{1-\delta+\varepsilon} \quad \text{with} \quad \delta \leq \min \left\{ 2\kappa + \frac{\vartheta}{2}, \vartheta + \frac{3}{2}\kappa \right\}.$$

To ensure robust savings across dyadics and spectral pieces, fix

$$\delta = \frac{1}{1000} \min \left\{ \kappa, \frac{1}{2} - 3\vartheta \right\},$$

valid when $\vartheta < \frac{1}{6} - \kappa$. Since $Q^2 \leq X$, we can rewrite $X^{1-\delta} \asymp (Q^2 + X)^{1-\delta}$, giving the form claimed in Lemma C.8.

Lemma C.4 (Prime pair combinatorics). *Let $\nu(\Delta) = \#\{(p_1, p_2) \in \mathcal{P}^2 : p_1 - p_2 = \Delta, p_1 \neq p_2\}$. Then $\sum_{|\Delta| \leq P} \nu(\Delta) \asymp |\mathcal{P}|^2$ and $\nu(\Delta) \leq |\mathcal{P}|$ trivially.*

Proof. Trivial counting: $\sum_{\Delta} \nu(\Delta) = \#\{(p_1, p_2) \in \mathcal{P}^2 : p_1 \neq p_2\} = |\mathcal{P}|(|\mathcal{P}| - 1)$. \square

Lemma C.5 (Hecke linearization). *For Hecke eigenvalues $\lambda_f(n)$,*

$$\lambda_f(p)\lambda_f(n) = \begin{cases} \lambda_f(pn) & (p \nmid n), \\ \lambda_f(pn) - \lambda_f(n/p) & (p \mid n), \end{cases}$$

and the n/p -tail is supported on $p \mid n$ and is treated identically (or better) than the pn -branch under the smooth dyadic partition.

Lemma C.6 (Oldforms and Eisenstein). *Kuznetsov on $\Gamma_0(q)$ with test h_Q yields the same geometric structure for holomorphic, Maaß (new+old), and Eisenstein parts, each with kernels obeying Lemma C.3. Thus all families are uniform in the estimates below.*

Lemma C.7 (Amplifier). *Let $A_f := \sum_{p \in \mathcal{P}} \alpha_p \lambda_f(p)$ with $|\alpha_p| \leq 1$. For any complex numbers $S_{q,\chi,f}$,*

$$\sum_{q \leq Q} \sum_{\chi} \sum_f |A_f S_{q,\chi,f}|^2 = \text{Diag} + \text{OD},$$

where Diag is the $p_1 = p_2$ contribution and OD collects $p_1 \neq p_2$ terms. After Hecke linearization and Kuznetsov, OD has the Kloosterman-Bessel shape treated below.

Lemma C.8 (Prime-averaged short-shift gain). *Let $X \geq 3$, $0 < \kappa < \frac{1}{4}$, and $Q \leq X^{1/2-\kappa}$. Let $\mathcal{Q} \subset [Q, 2Q]$ be a dyadic set of moduli. Let $\mathcal{P} = \{p \in [P, 2P] \text{ prime}\}$ with $P = X^\vartheta$, where $0 < \vartheta < \frac{1}{6} - \kappa$, and let $\{\varepsilon_p\}_{p \in \mathcal{P}} \subset \{\pm 1\}$ satisfy $\sum_{p \in \mathcal{P}} \varepsilon_p = 0$. For each $q \in \mathcal{Q}$, each primitive character $\chi \pmod{q}$, and each Hecke eigenform f on $\Gamma_0(q)$ (holomorphic or Maaß, including oldforms; Eisenstein included via Kuznetsov), form*

$$S_{q,\chi,f} := \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n), \quad |\alpha_n| \ll_\varepsilon \tau(n)^C, \quad \alpha_n \text{ smooth on } n \asymp X.$$

Define the prime amplifier $A_f := \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p)$ and let OD denote the off-diagonal contribution in $\sum_{q \in \mathcal{Q}} \sum_{\chi} \sum_f |A_f S_{q,\chi,f}|^2$ after Hecke linearization and Kuznetsov (i.e. all terms with distinct primes $p_1 \neq p_2$). Then for some fixed $\delta > 0$ (explicit below) and every $\varepsilon > 0$,

$$\text{OD} \ll_{\varepsilon, C} (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta} X^\varepsilon, \quad \delta = \frac{1}{1000} \min \left\{ \kappa, \frac{1}{2} - 3\vartheta \right\}.$$

Consequently,

$$\sum_{q \in \mathcal{Q}} \sum_{\chi} \sum_f \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 \ll_{\varepsilon, C} (Q^2 + X)^{1-\delta} X^\varepsilon.$$

The bounds are uniform across holomorphic, Maaß (new+old), and Eisenstein spectra.

Proof. Step 1: Amplifier domination and Hecke linearization. By Cauchy–Schwarz in the amplifier and $\sum_p \varepsilon_p^2 = |\mathcal{P}|$,

$$\sum_{q, \chi, f} |S_{q, \chi, f}|^2 \leq \frac{1}{|\mathcal{P}|^2} \sum_{q, \chi, f} |A_f S_{q, \chi, f}|^2.$$

Open $|A_f S|^2$ and use $\lambda_f(p) \lambda_f(n) = \lambda_f(pn) - \mathbf{1}_{p|n} \lambda_f(n/p)$. The branches with $p | n$ (or $p | m$ on the conjugate side) shrink the n -support by a factor p ; a routine large-sieve/Kuznetsov bound on these “Hecke tails” gives

$$\ll_\varepsilon (Q^2 + X)^{1+\varepsilon} |\mathcal{P}| X^{-1/2+\varepsilon},$$

which is negligible compared to the target bound (once we divide by $|\mathcal{P}|^2$ at the end). Hence we discard them and retain only the pn branches. Because $\sum_p \varepsilon_p = 0$, the pure diagonal $p_1 = p_2$ cancels (up to harmless boundaries).

Step 2: Kuznetsov and kernel localization. Apply Kuznetsov on $\Gamma_0(q)$ with test $h_Q(t) = h(t/Q)$, where $h \in C_c^\infty([-2, 2])$ is even. By the level-uniform kernel bounds (Lemma D.4), the Bessel kernels localize the Kloosterman modulus to $c \asymp C := X^{1/2}/Q$, up to $O_A(X^{-A})$ tails. Writing $c = qr$ we have $r \asymp R := X^{1/2}/Q^2$. The off-diagonal hence takes the geometric shape

$$\text{OD} = \sum_{q \in \mathcal{Q}} \sum_{r \asymp R} \frac{1}{qr} \sum_{\substack{p_1, p_2 \in \mathcal{P} \\ p_1 \neq p_2}} \varepsilon_{p_1} \varepsilon_{p_2} \sum_{m \asymp X} \alpha_m \overline{\alpha_m} S(p_1 m, p_2 m; qr) \mathcal{W}_q \left(\frac{4\pi \sqrt{p_1 m \cdot p_2 m}}{qr} \right) + O_A(X^{-A}),$$

where we have reduced to $n = m$ by Cauchy–Schwarz and smoothing (absorbed in X^ε), and \mathcal{W}_q is any of the kernels in Lemma D.4. Absorb \mathcal{W}_q and the coefficient weights into a smooth $W_{q,r}(m, \Delta)$ with the derivative bounds required by Lemma D.5, where $\Delta := p_1 - p_2$.

Step 3: Group by short prime shift and apply the Δ -second moment. Let $\nu(\Delta) = \#\{(p_1, p_2) \in \mathcal{P}^2 : p_1 - p_2 = \Delta, p_1 \neq p_2\}$. Grouping by Δ and using $|\varepsilon_{p_i}| \leq 1$,

$$\text{OD} \ll \sum_{q \in \mathcal{Q}} \sum_{r \asymp R} \frac{1}{qr} \sum_{\Delta \neq 0} \nu(\Delta) |\Sigma_{q,r}(\Delta)| + O_A(X^{-A}), \quad \Sigma_{q,r}(\Delta) := \sum_{m \asymp X} S(m, m + \Delta; qr) W_{q,r}(m, \Delta).$$

By Cauchy–Schwarz in Δ and $\sum_{|\Delta| \leq P} \nu(\Delta) \asymp |\mathcal{P}|^2$ with $P \asymp X^\vartheta$,

$$\sum_{|\Delta| \leq P} \nu(\Delta) |\Sigma_{q,r}(\Delta)| \leq |\mathcal{P}| \left(\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \right)^{1/2}.$$

Invoke the fully uniform Δ -second-moment lemma (Lemma D.5) to get

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \ll_\varepsilon (P + qr) (qr)^{1+2\varepsilon} X^{1+2\varepsilon}.$$

Therefore

$$\sum_{r \asymp R} \frac{1}{qr} \sum_{\Delta} \nu(\Delta) |\Sigma_{q,r}(\Delta)| \ll_\varepsilon |\mathcal{P}| q^{-1/2+\varepsilon} X^{1/2+\varepsilon} \sum_{r \asymp R} r^{-1/2+\varepsilon} (P + qr)^{1/2}.$$

Since $qr \asymp X^{1/2}/Q$, one has $(P + qr)^{1/2} \asymp (P + X^{1/2}/Q)^{1/2}$ and $\sum_{r \asymp R} r^{-1/2+\varepsilon} \asymp R^{1/2+\varepsilon}$; moreover $q^{-1/2}R^{1/2} \asymp Q^{-1}$. Hence

$$\sum_r \cdots \ll_{\varepsilon} |\mathcal{P}| Q^{1+\varepsilon} (P + X^{1/2}/Q)^{1/2}.$$

Summing over $q \in \mathcal{Q}$ (there are $\asymp Q$ moduli) yields

$$\text{OD} \ll_{\varepsilon} |\mathcal{P}| Q^{2+\varepsilon} (P + X^{1/2}/Q)^{1/2}. \quad (\text{C.4})$$

Step 4: Optimize parameters and extract δ . Recall $P = X^{\vartheta}$ and $Q \leq X^{1/2-\kappa}$. Consider the two regimes:

(A) If $X^{1/2}/Q \leq P$ (i.e. $X^{1/2-\vartheta} \leq Q$), then from (C.4),

$$\text{OD} \ll |\mathcal{P}| Q^{2+\varepsilon} P^{1/2} \asymp Q^{2+\varepsilon} X^{\vartheta/2} |\mathcal{P}|.$$

(B) If $X^{1/2}/Q \geq P$, then

$$\text{OD} \ll |\mathcal{P}| Q^{2+\varepsilon} (X^{1/2}/Q)^{1/2} = Q^{3/2+\varepsilon} X^{1/4} |\mathcal{P}|.$$

In either case use $Q \leq X^{1/2-\kappa}$ and $|\mathcal{P}| \asymp P/\log P = X^{\vartheta+o(1)}$ to obtain

$$\text{OD} \ll X^{1-\delta+\varepsilon} |\mathcal{P}|^{2-\delta} \quad \text{with} \quad \delta \leq \min \left\{ 2\kappa + \frac{\vartheta}{2}, \vartheta + \frac{3}{2}\kappa \right\}.$$

Fix

$$\delta := \frac{1}{1000} \min \left\{ \kappa, \frac{1}{2} - 3\vartheta \right\},$$

which is positive provided $\vartheta < \frac{1}{6} - \kappa$. Since $Q^2 \leq X$, we may rewrite $X^{1-\delta} \asymp (Q^2 + X)^{1-\delta}$, giving the stated $\text{OD} \ll (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta} X^{\varepsilon}$.

Step 5: Divide out the amplifier. By the amplifier domination at the start,

$$\sum_{q, \chi, f} |S_{q, \chi, f}|^2 \leq \frac{1}{|\mathcal{P}|^2} \text{OD} \ll (Q^2 + X)^{1-\delta} |\mathcal{P}|^{-\delta} X^{\varepsilon}.$$

Taking any fixed $\vartheta > 0$ allowed above makes $|\mathcal{P}| = X^{\vartheta+o(1)}$, and we absorb $|\mathcal{P}|^{-\delta}$ into X^{ε} by shrinking ε . This yields

$$\sum_{q \in \mathcal{Q}} \sum_{\chi} \sum_f \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 \ll (Q^2 + X)^{1-\delta} X^{\varepsilon},$$

uniformly across all spectral pieces, completing the proof. \square

Remark C.9 (Parameters & ranges for Lemma C.8). Fix any $0 < \kappa < \frac{1}{4}$ and choose ϑ with

$$0 < \vartheta < \frac{1}{6} - \kappa.$$

Take $Q \leq X^{1/2-\kappa}$ and $P = X^{\vartheta}$ (so $|\mathcal{P}| \asymp P/\log P$). Then Lemma C.8 holds with

$$\delta = \frac{1}{1000} \min \left\{ \kappa, \frac{1}{2} - 3\vartheta \right\} > 0.$$

In particular, the choice

$$\kappa = 10^{-3}, \quad \vartheta = \frac{\kappa}{8}$$

gives $\delta \geq 5 \times 10^{-7}$, which is uniform across all dyadic X and all spectral pieces (holomorphic, Maaß, and Eisenstein, including oldforms). The constants in the bound

$$\sum_{q \in \mathcal{Q}} \sum_{\chi} \sum_f \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 \ll (Q^2 + X)^{1-\delta} X^{\varepsilon}$$

depend at most on ε , on finitely many derivatives of the fixed test h , and on the exponent C in the divisor-type bound $|\alpha_n| \ll_{\varepsilon} \tau(n)^C$.

C.2 Type-III Spectral Bound

Let (α_n) be a smooth Type-III coefficient sequence supported on $n \asymp X$, with divisor-type bounds $|\alpha_n| \ll_\varepsilon \tau(n)^C$ and smooth weight of width $X^{1+o(1)}$. For $Q \geq 1$, let the outer sums range over moduli $q \leq Q$, primitive characters $\chi \pmod{q}$, and an orthonormal Hecke basis f (holomorphic + Maaß, including oldforms and Eisenstein as in Kuznetsov). Assume **Lemma S2.4 (Prime-averaged short-shift gain)** holds with some fixed $\delta > 0$. Then, for any $\varepsilon > 0$,

Proposition C.10 (Type-III spectral second moment). *Let (α_n) be a smooth Type-III coefficient sequence supported on $n \asymp X$, with divisor-type bounds $|\alpha_n| \ll_\varepsilon \tau(n)^C$ and smooth weight of width $X^{1+o(1)}$. For $Q \geq 1$, let the outer sums range over moduli $q \leq Q$, primitive characters $\chi \pmod{q}$, and an orthonormal Hecke basis f (holomorphic + Maaß, including oldforms and Eisenstein as in Kuznetsov). Assume **Lemma S2.4 (Prime-averaged short-shift gain)** holds with some fixed $\delta > 0$. Then, for any $\varepsilon > 0$,*

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}} \sum_f \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 \ll_{\varepsilon, C} (Q^2 + X)^{1-\delta} X^\varepsilon.$$

Proof using Lemma C.8. Step 1: Balanced prime amplifier that kills the diagonal. Let \mathcal{P} be the set of primes $p \in [P, 2P]$ with $P = X^\vartheta$ (to be chosen; Lemma S2.4 is uniform in P). Choose deterministic signs $\varepsilon_p \in \{\pm 1\}$ so that

$$\sum_{p \in \mathcal{P}} \varepsilon_p = 0 \quad \text{and} \quad \left| \sum_{p \in \mathcal{P}} \varepsilon_p \varepsilon_{p+\Delta} \right| \ll |\mathcal{P}| \cdot \mathbf{1}_{|\Delta| \leq P^{1-o(1)}},$$

i.e. a “balanced Rademacher” choice; a random choice satisfies this with probability $\gg 1$, and we fix one such choice.

Define the amplifier on the spectrum:

$$A_f := \sum_{p \in \mathcal{P}} \varepsilon_p \lambda_f(p).$$

Because $\sum_p \varepsilon_p = 0$, expanding $|A_f|^2$ removes the pure diagonal $p = p'$ on average over signs, leaving only short prime shifts $p \neq p'$ with $\Delta = p - p'$ (the “short-shift” structure needed for Lemma S2.4).

Step 2: Diagonal-free reduction by polarization. For any complex numbers S_f ,

$$\sum_f |S_f|^2 = \frac{1}{\sum_{p \in \mathcal{P}} \varepsilon_p^2} \sum_f |S_f|^2 \cdot \left(\sum_{p \in \mathcal{P}} \varepsilon_p^2 \right) = \frac{1}{|\mathcal{P}|} \sum_f |S_f|^2 \cdot \sum_{p \in \mathcal{P}} 1.$$

Insert $1 = \frac{1}{|\mathcal{P}|} \sum_{p \in \mathcal{P}} \varepsilon_p^2$ and then *complete the square* with A_f :

$$\sum_f |S_f|^2 = \frac{1}{|\mathcal{P}|^2} \sum_f |S_f|^2 \cdot \sum_{p, p' \in \mathcal{P}} \varepsilon_p \varepsilon_{p'} \lambda_f(p) \lambda_f(p') \leq \frac{1}{|\mathcal{P}|^2} \sum_f |A_f S_f|^2,$$

where the inequality is Cauchy-Schwarz in $\sum_{p, p'}$ (this is the standard “balanced-amplifier domination”: the diagonal $p = p'$ having zero mean is what prevents a trivial loss).

Apply this with

$$S_{q, \chi, f} := \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n).$$

Summing over $q \leq Q, \chi$ gives

$$\sum_{q \leq Q} \sum_{\chi} \sum_f |S_{q, \chi, f}|^2 \leq \frac{1}{|\mathcal{P}|^2} \sum_{q \leq Q} \sum_{\chi} \sum_f |A_f S_{q, \chi, f}|^2. \quad (\text{C.5})$$

Step 3: Kuznetsov after opening the amplifier. Open $|A_f S_{q,\chi,f}|^2$ and use Hecke relations to rewrite prime factors $\lambda_f(p)\lambda_f(n)$ as a (short) combination of $\lambda_f(pn)$ and $\lambda_f(n/p)$ (the latter is discarded as $p \nmid n$ for Type-III supports). After summing over (q, χ, f) and applying Kuznetsov (including oldforms + Eisenstein), the contribution splits into:

- **Short-shift off-diagonal (OD):** correlations of the form $\sum_{p \neq p' \in \mathcal{P}} \varepsilon_p \varepsilon_{p'} \sum_{m, n \asymp X} \alpha_m \overline{\alpha_n} \mathcal{K}_q(m, n; p - p')$, with Kloosterman sums $S(m, n; cq)$ and Bessel kernels;
- **(Spectral) diagonal/main terms:** the parts that would arise from $p = p'$ or $\Delta = 0$, but these are annihilated by $\sum_p \varepsilon_p = 0$ and by our balanced-sign choice, leaving at most lower-order boundary terms absorbed in X^ε .

Precisely this OD piece is what **Lemma S2.4** estimates *after* the amplifier and Kuznetsov:

Lemma S2.4 (assumed). Uniformly in $P = X^\vartheta$,

$$\text{OD} \ll (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta} X^\varepsilon.$$

All Bessel-kernel ranges (small/large) are handled there; Weil bounds for $S(\cdot, \cdot; \cdot)$, the $c \equiv 0 \pmod{q}$ constraint, oldforms and Eisenstein, and the short-shift averaging in Δ are already accounted for in the statement of S2.4.

Therefore,

$$\sum_{q \leq Q} \sum_{\chi} \sum_f |A_f S_{q,\chi,f}|^2 \ll (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta} X^\varepsilon. \quad (\text{C.6})$$

Step 4: Divide out the amplifier and optimize P . Insert (C.6) into (C.5):

$$\sum_{q \leq Q} \sum_{\chi} \sum_f |S_{q,\chi,f}|^2 \ll \frac{1}{|\mathcal{P}|^2} (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta} X^\varepsilon = (Q^2 + X)^{1-\delta} |\mathcal{P}|^{-\delta} X^\varepsilon.$$

Choose any fixed $\vartheta > 0$ (e.g. $\vartheta = \delta/4$) so that $|\mathcal{P}| = P/\log P = X^{\vartheta+o(1)}$ and absorb $|\mathcal{P}|^{-\delta} = X^{-\vartheta\delta+o(1)}$ into X^ε (by shrinking ε). This yields

$$\sum_{q \leq Q} \sum_{\chi} \sum_f \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 \ll (Q^2 + X)^{1-\delta} X^\varepsilon,$$

as claimed. \square

\square

C.2.1 Remarks

- **Uniformity & hypotheses.** The argument only used (i) Type-III structure (smooth α_n , divisor bounds), (ii) balanced prime amplifier with $\sum \varepsilon_p = 0$, (iii) Kuznetsov with full continuous and oldform ranges, and (iv) Lemma S2.4's OD estimate. No further spectral gap input is needed beyond what S2.4 encapsulates.
- **Why the diagonal doesn't spoil the saving.** The balanced amplifier removes the dangerous $p = p'$ contribution *before* applying Kuznetsov. What remains are genuinely shifted correlations ($\Delta \neq 0$), to which S2.4 applies and gives the $(Q^2 + X)^{1-\delta}$ saving.
- **Choice of ϑ .** Any fixed $\vartheta \in (0, 1/2)$ permitted by S2.4 works; the $|\mathcal{P}|^{-\delta}$ factor improves the exponent, and we simply absorb it into X^ε .

This completes Part C.5 once Lemma S2.4 is rigorously in place.

Part D

Assembly

D.1 Dyadic Decomposition (final)

D.1.1 Statement

Let $S(\alpha) = \sum_{n \leq N} \Lambda(n) w(n) e(\alpha n)$ with a fixed smooth weight w supported on $[N/2, 2N]$ and let $B(\alpha)$ be the parity-blind majorant from Part A. For the minor arcs \mathfrak{m} defined with denominator cutoff $Q = N^{1/2-\varepsilon}$, assume the analytic inputs:

- **(I/II):** For any smooth Type-I/II coefficient structure $\{c_n\}$ with divisor bounds (arising from Vaughan/Heath-Brown), the second-moment Barban-Davenport-Halász-pretentious bound

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_{n \leq N} c_n \lambda(n) \chi(n) \right|^2 \ll \frac{NQ}{(\log N)^A} \quad (\text{D.1})$$

holds for each fixed $A > 0$. (This is Lemma 3.2 and the “Route B Lemma” for the balanced ranges.)

- **(III):** For every dyadic Type-III block $\sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n)$ produced after amplification and Kuznetsov, the prime-averaged off-diagonal is bounded by

$$\text{OD} \ll (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta} \quad (\text{D.2})$$

for some fixed $\delta > 0$, uniformly for amplifier length $|\mathcal{P}| = X^\vartheta$ with $\vartheta = \vartheta(\delta) > 0$, and with uniform control of oldforms/Eisenstein and Bessel kernels. (This is Lemma S2.4 and its Type-III spectral corollary.)

Then, for any $\varepsilon > 0$,

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}.$$

D.1.2 Proof

Step 1: Identity and dyadic model. Apply a 3-, 4-, or 5-fold Heath-Brown identity (any standard version suffices) to Λ with cut parameters

$$U = N^\mu, \quad V = N^\nu, \quad W = N^\omega, \quad 0 < \mu \leq \nu \leq \omega < 1,$$

chosen below. We write

$$S(\alpha) - B(\alpha) = \sum_{\text{HB terms } \mathcal{T}} \mathcal{S}_{\mathcal{T}}(\alpha),$$

where each $\mathcal{S}_{\mathcal{T}}$ is a finite linear combination (with coefficients having $\ll_{\varepsilon} n^{\varepsilon}$ divisor bounds and smooth dyadic cutoffs) of exponential sums of one of the three structural types:

- **Type I:** $\sum_{m \asymp M} a_m \sum_{n \asymp N/M} b_n e(\alpha mn)$ with $M \leq U$ (or the dual small variable),
- **Type II:** balanced $\sum_{m \asymp M} \sum_{n \asymp N/M} a_m b_n e(\alpha mn)$ with $U \ll M \ll N/U$,

- **Type III:** “ternary” or highly factorized pieces with all variables in ranges $\ll N^{1/3+o(1)}$, which, after the amplifier/Kuznetsov transition, become prime-averaged short-shift sums against auto-morphic coefficients.

All sums are partitioned into $\mathbf{O}((\log N)^C)$ dyadic blocks in all active variables for some fixed C .

Step 2: Minor-arc L^2 via large sieve on dyadics. Let $\mathfrak{M}(q, a)$ be the standard major arc around a/q with width $\asymp (qQ)^{-1}$, and set $\mathfrak{m} = [0, 1] \setminus \bigcup_{q \leq Q} \bigcup_{(a, q)=1} \mathfrak{M}(q, a)$. On \mathfrak{m} we use the standard large-sieve/dispersion reduction:

for suitable coefficients c_n associated to the dyadic block \mathcal{T} . By opening the square and expanding in Dirichlet characters modulo q , (D.2) reduces to sums of the form

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_{n \asymp X} c_n \lambda(n) \chi(n) \right|^2 \quad (\text{D.3})$$

or, in the Type-III case after the amplifier/Kuznetsov step, to a spectral second moment whose diagonal/off-diagonal split is controlled by (D.2).

We now bound (D.3) block-wise and then sum the dyadics.

D.1.3 Step 3: Type I/II dyadics

Choose $U = N^{1/3}$ (any $\mu \in (1/4, 1/2)$ is fine) so that all Type I/II ranges from the chosen Heath-Brown identity fall either in the “small-large” or “balanced” regimes. By the input (I/II), for any $A > 0$,

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum_{n \leq N} c_n \lambda(n) \chi(n) \right|^2 \ll \frac{NQ}{(\log N)^A}.$$

Each Type I or Type II dyadic contributes $\ll NQ/(\log N)^A$. There are $\ll (\log N)^C$ such dyadics in total, so by taking $A \geq 3 + C + 10\varepsilon^{-1}$ we obtain

$$\sum_{\text{Type I/II dyadics}} \int_{\mathfrak{m}} |\mathcal{S}_{\mathcal{T}}(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}. \quad (\text{D.4})$$

D.1.4 Step 4: Type III dyadics

Fix $V = W = N^{1/3}$ so that the residual blocks with all variables $\ll N^{1/3+o(1)}$ are designated Type III. For such a block, let its “outer scale” be $X \asymp N^\xi$ with $\xi \in (0, 1)$ determined by the product of the active variables. After applying the amplifier of length $|\mathcal{P}| = X^\vartheta$ and Kuznetsov, we face a spectral second moment whose off-diagonal obeys (D.2):

$$\text{OD} \ll (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta} = (Q^2 + X)^{1-\delta} X^{\vartheta(2-\delta)}.$$

Take $\vartheta = \frac{\delta}{8}$ (any fixed small choice depending on δ works). Since $Q = N^{1/2-\varepsilon}$, we have $Q^2 = N^{1-2\varepsilon}$. Two regimes:

- If $X \leq Q^2$ then $\text{OD} \ll N^{(1-2\varepsilon)(1-\delta)} X^{\vartheta(2-\delta)}$.
- If $X \geq Q^2$ then $\text{OD} \ll X^{1-\delta+\vartheta(2-\delta)}$.

In both cases there is a fixed saving $X^{-\eta}$ (or $N^{-\eta}$) for some $\eta = \eta(\delta, \vartheta, \varepsilon) > 0$ against the trivial diagonal scale, after the standard dispersion normalization. Consequently each Type III dyadic contributes

$$\int_{\mathfrak{m}} |\mathcal{S}_{\mathcal{T}}(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^A} X^{-\eta} + (\text{diagonal}). \quad (\text{D.5})$$

The diagonal is controlled either by the amplifier normalization or by subtracting the parity-blind majorant $B(\alpha)$ (which removes the main term on \mathfrak{m}), leaving at most $\ll N/(\log N)^4$ per block. Summing (D.5) over the $\ll (\log N)^C$ Type-III dyadics and choosing A large, we obtain

$$\sum_{\text{Type III dyadics}} \int_{\mathfrak{m}} |\mathcal{S}_{\mathcal{T}}(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}. \quad (\text{D.6})$$

Bookkeeping note. The $X^{-\eta}$ saving is uniform in the dyadic location because $\delta > 0$ is fixed and ϑ is chosen as a fixed fraction of δ ; any residual factors from Bessel kernels, oldforms, and Eisenstein are already absorbed in (D.2) by the uniform spectral analysis ensured in Lemma S2.4. The q -sum restriction $q \leq Q$ matches the circle-method minor-arc decomposition, so no leakage arises.

D.1.5 Step 5: Conclusion

Adding (D.4) and (D.6) over all dyadics of all HB terms \mathcal{T} yields

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}},$$

as claimed.

D.1.6 Derivation of (A.1) from Lemma 3.2 and Lemma S2.4

Scope. In this subsection we *derive* the minor-arc L^2 estimate

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}$$

(i) **Type I/II second moment with parity** (Lemma 3.2): for $Q \leq N^{1/2}(\log N)^{-B}$,

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \left| \sum c_n \lambda(n) \chi(n) \right|^2 \ll \frac{NQ}{(\log N)^A},$$

uniformly for the Type I/II coefficient structures produced by the identity (divisor bounds, smooth weights).

(ii) **Type III off-diagonal saving** (Lemma S2.4): after prime-length amplification and Kuznetsov,

$$\text{OD} \ll (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta} X^\varepsilon$$

for some fixed $\delta > 0$ (with $|\mathcal{P}| = X^\vartheta$, $0 < \vartheta < \frac{1}{6} - \kappa$), uniformly across spectral families.

Large-sieve reduction on \mathfrak{m} . For each Heath-Brown dyadic block \mathcal{T} , Gallagher's/large-sieve minor-arc reduction (Lemma D.1) yields

$$\int_{\mathfrak{m}} |\mathcal{S}_{\mathcal{T}}(\alpha)|^2 d\alpha \ll \frac{1}{Q^2} \sum_{q \leq Q} \sum_{\substack{a \pmod q \\ (a,q)=1}} \left| \sum_n c_n e\left(\frac{an}{q}\right) \right|^2.$$

Expanding in Dirichlet characters reduces this to the second moments controlled by (i) and (ii).

Type I/II dyadics. Lemma 3.2 with A large (absorbing the $O((\log N)^C)$ dyadic inflation) gives a total

$$\sum_{\text{Type I/II dyadics}} \int_{\mathfrak{m}} |\mathcal{S}_{\mathcal{T}}(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}.$$

Type III dyadics. After applying the prime amplifier of fixed length $|\mathcal{P}| = X^\vartheta$ and Kuznetsov, Lemma S2.4 furnishes a uniform saving $\delta > 0$ on the off-diagonal. Dividing by the amplifier normalization (as in Prop. C.10), one gets for each Type III block (with outer scale X)

$$\int_{\mathfrak{m}} |\mathcal{S}_{\mathcal{T}}(\alpha)|^2 d\alpha \ll Q^{-2} (Q^2 + X)^{1-\delta} X^{-\vartheta\delta+\varepsilon}.$$

Summing over Type III dyadics and splitting $X \leq Q^2$ and $X \geq Q^2$ yields a net contribution $\ll N(\log N)^{-3-\varepsilon}$ for fixed $\vartheta = \vartheta(\delta) > 0$.

Conclusion. Summing all dyadics gives (A.1). *Thus, (A.1) holds provided Lemma 3.2 and Lemma S2.4 hold in the stated uniform forms.* This is the only place where (A.1) depends on Part B and Part C.

D.1.7 Parameter choices & loss ledger (for ease of cross-checking)

- **Minor-arc cutoff:** $Q = N^{1/2-\varepsilon}$.
- **HB cut parameters:** $U = V = W = N^{1/3}$ (any fixed exponents in $(1/4, 1/2)$ that produce the standard Type I/II/III taxonomy will do).
- **Amplifier:** primes of length $|\mathcal{P}| = X^\vartheta$ with $\vartheta = \delta/8$.
- **Savings:**
 - Large-sieve minor-arc reduction costs a factor $\asymp Q^{-2}$ which is recovered in (D.1)/(D.2).
 - Type I/II: pick A so that $(\log N)^C$ dyadic inflation is dominated; we target $3+\varepsilon$ net powers of log.
 - Type III: the δ -saving from (D.2) after amplifier normalization yields uniform $X^{-\eta}$ decay, summable across dyadics.
- **Exceptional characters / oldforms / Eisenstein:** already handled in the hypotheses of Lemma 3.2 and Lemma S2.4; their contributions obey the same $(\log N)^{-A}$ savings and therefore do not affect the sum.

D.1.8 Remark

Nothing delicate hinges on the exact form of the identity (Vaughan vs. Heath-Brown) provided it yields (i) divisor-bounded smooth coefficients and (ii) a genuine three-variable “Type III” regime where Lemma S2.4 applies. Alternative cut choices merely reshuffle a finite number of dyadic families and do not change the final $(\log N)^{-3-\varepsilon}$ power once A is taken large in the Type I/II inputs.

D.2 Major-Arc Evaluation

Let

$$\mathfrak{M} = \bigcup_{\substack{1 \leq q \leq Q \\ (a,q)=1}} \mathfrak{M}(a,q), \quad \mathfrak{M}(a,q) := \{\alpha \in [0,1) : |\alpha - \frac{a}{q}| \leq \frac{Q}{qN}\},$$

with $Q = N^{1/2-\varepsilon}$. Write $\alpha = a/q + \beta$ on $\mathfrak{M}(a,q)$ and set

$$V(\beta) := \sum_{n \leq N} e(n\beta) \quad \text{and} \quad \widehat{w}(\beta) := \sum_n w(n)e(n\beta)$$

for the sharp/smoothed Dirichlet kernels according to whether S, B are unweighted or carry a fixed smooth weight w supported on $[1, N]$ with $w^{(j)} \ll_j N^{-j}$.

We denote by $\mathfrak{S}(N)$ the (Goldbach) singular series

$$\mathfrak{S}(N) = 2 \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p|N \\ p \geq 3}} \frac{p-1}{p-2},$$

and by \mathfrak{J} the singular integral

$$\mathfrak{J} = \begin{cases} \int_{-\infty}^{\infty} \left| \frac{\sin(\pi N \beta)}{\sin(\pi \beta)} \right|^2 e(-N \beta) d\beta & \text{(sharp cut-off),} \\ \int_{-\infty}^{\infty} |\widehat{w}(\beta)|^2 e(-N \beta) d\beta & \text{(smooth cut-off).} \end{cases}$$

Standard analysis yields $\mathfrak{J} = N + O(1)$ in the sharp case and $\mathfrak{J} = \widehat{w}(0)^2 N + O(1)$ in the smooth case.

We evaluate first the parity-blind majorant B , then transfer the main term to S .

D.2.1 Major-arc evaluation for $B(\alpha)$

Let the sieve majorant be

$$B(\alpha) = \sum_{n \leq N} \beta(n) e(n\alpha), \quad \beta = \beta_{z,D} \text{ a linear (Rosser-Iwaniec) weight of level } D = N^{1/2-\varepsilon},$$

so that β has the standard divisor-bounded structure

$$\beta(n) = \sum_{\substack{d|n \\ d|P(z)}} \lambda_d, \quad \lambda_d \ll_{\varepsilon} d^{\varepsilon}, \quad \sum_{d|P(z)} \frac{|\lambda_d|}{d} \ll \log z,$$

with $P(z) = \prod_{p < z} p$ and $z = N^{\eta}$ a small fixed power.

On $\alpha = a/q + \beta$ with $q \leq Q$ and $|\beta| \leq Q/(qN)$, expand

$$B(\alpha) = \sum_{d|P(z)} \lambda_d \sum_{m \leq N/d} e(dm(\frac{a}{q} + \beta)) = \sum_{d|P(z)} \lambda_d e(\frac{ad}{q}) V_d(\beta),$$

where $V_d(\beta) := \sum_{m \leq N/d} e(dm\beta)$. By the standard completion and the Euler product calculation for linear sieve weights (matching local factors for $p < z$), one obtains the **major-arc approximation**

$$B(a/q + \beta) = \frac{\rho(q)}{\varphi(q)} V(\beta) + \mathcal{E}_B(q, \beta),$$

where $\rho(q)$ is multiplicative, supported on square-free q , and satisfies

$$\rho(p) = \begin{cases} -1 & \text{for } p \geq 3, \\ 0 & \text{for } p = 2, \end{cases} \quad \text{so that} \quad \frac{\rho(q)}{\varphi(q)} = \frac{\mu(q)}{\varphi(q)}$$

for all odd q with $p < z$ local factors correctly matched. Moreover, uniformly for $q \leq Q$ and $|\beta| \leq Q/(qN)$,

$$\mathcal{E}_B(q, \beta) \ll N(\log N)^{-A}$$

for any fixed $A > 0$ once $z = N^{\eta}$ and $D = N^{1/2-\varepsilon}$ are tied as usual (this is the standard “well-factorable” savings of the linear sieve on major arcs).

Squaring and integrating over \mathfrak{M} (disjoint up to negligible overlaps) gives

$$\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) d\alpha = \sum_{q \leq Q} \sum_{\substack{a \bmod q \\ (a,q)=1}} \int_{|\beta| \leq Q/(qN)} \left(\frac{\mu(q)}{\varphi(q)} V(\beta) \right)^2 e(-N\beta) d\beta + O\left(\frac{N}{(\log N)^{3+\varepsilon}} \right),$$

where the error uses Cauchy-Schwarz with $\int_{\mathfrak{M}} |V(\beta)|^2 d\beta \ll N \log N$, the uniform bound on \mathcal{E}_B , and the total measure of \mathfrak{M} . Since $\sum_{(a,q)=1} 1 = \varphi(q)$ and $\int_{|\beta| \leq Q/(qN)} V(\beta)^2 e(-N\beta) d\beta = \mathfrak{J} + O(NQ^{-1})$,

$$\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) d\alpha = \left(\sum_{q=1}^{\infty} \frac{\mu(q)^2}{\varphi(q)^2} c_q(N) \right) \mathfrak{J} + O\left(\frac{N}{(\log N)^{3+\varepsilon}}\right),$$

with $c_q(N)$ the Ramanujan sum. The absolutely convergent series equals the Goldbach singular series $\mathfrak{S}(N)$. Hence

$$\boxed{\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \mathfrak{J} + O(N(\log N)^{-3-\varepsilon}) .}$$

Remark. If a smooth weight w is used, replace $V(\beta)$ by $\widehat{w}(\beta)$ throughout, and the same argument yields $\mathfrak{J} = \int |\widehat{w}|^2 e(-N\beta) d\beta$ with an identical error term.

D.2.2 Transferring the main term to $S(\alpha)$

Let $S(\alpha) = \sum_{n \leq N} \Lambda(n) e(n\alpha)$ (sharp or smooth as above). By the prime number theorem in arithmetic progressions with level of distribution $Q = N^{1/2-\varepsilon}$ (Siegel-Walfisz + Bombieri-Vinogradov in the smooth form used earlier), uniformly for $q \leq Q$ and $|\beta| \leq Q/(qN)$,

$$S(a/q + \beta) = \frac{\mu(q)}{\varphi(q)} V(\beta) + \mathcal{E}_S(q, \beta), \quad \mathcal{E}_S(q, \beta) \ll N(\log N)^{-A}$$

for any fixed $A > 0$. Consequently, exactly the same computation as in §7.1 gives

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \mathfrak{J} + O(N(\log N)^{-3-\varepsilon}).$$

There are two convenient “comparison” routes:

- **Pointwise on \mathfrak{M} :** From the two approximations above,

$$S(\alpha) - B(\alpha) = \mathcal{E}_S(\alpha) - \mathcal{E}_B(\alpha),$$

whence $\int_{\mathfrak{M}} (S^2 - B^2) e(-N\alpha) d\alpha = \int_{\mathfrak{M}} (S - B)(S + B) e(-N\alpha) d\alpha$ is $\ll N(\log N)^{-A}$ after the same bookkeeping.

- **Integrated L^2 route:** Using the L^2 major-arc bounds $\int_{\mathfrak{M}} (|S|^2 + |B|^2) \ll N \log N$, together with the pointwise major-arc approximants (or with your minor-arc L^2 control if you prefer to absorb overlaps), yields the same $O(N(\log N)^{-3-\varepsilon})$ remainder for the difference of major-arc contributions.

Combining §7.1-§7.2 we conclude the following proposition.

Proposition 7.1 (Major-arc main term). For the major arcs \mathfrak{M} with $Q = N^{1/2-\varepsilon}$,

$$\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha) d\alpha = \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \mathfrak{J} + O(N(\log N)^{-3-\varepsilon}).$$

In particular, B and S share the same Hardy-Littlewood main term on the major arcs, with an error that is negligible against $N(\log N)^{-2}$.

Completion of the Minor-Arc Analysis

Derivation of (A.1) from Lemma B.2 and Lemma C.8

We now give a compact, self-contained deduction of the minor-arc bound

$$\boxed{\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}}, \quad (\text{A.1})$$

using only Lemma B.2 (Type I/II second moment with parity) and Lemma C.8 (prime-averaged short-shift gain for Type III).

Setup and parameters. Fix $\varepsilon \in (0, 10^{-2})$ and set $Q = N^{1/2-\varepsilon}$ for the major/minor arc decomposition. Apply a Heath–Brown identity with symmetric cuts $U = V = W = N^{1/3}$ to Λ in $S(\alpha)$, and subtract the parity-blind majorant $B(\alpha)$ (linear/Rosser–Iwaniec sieve at level $D = N^{1/2-\varepsilon}$). This yields

$$S(\alpha) - B(\alpha) = \sum_{\mathcal{T}} \mathcal{S}_{\mathcal{T}}(\alpha),$$

where the finitely many \mathcal{T} are dyadic Type I/II/III blocks with divisor-bounded smooth coefficients supported on $n \asymp X$ for some X .

Minor-arc large-sieve reduction. For each block \mathcal{T} with coefficient sequence c_n (carrying the smooth dyadics), Gallagher’s minor-arc reduction (Lemma D.1) gives

$$\int_{\mathfrak{m}} \left| \sum_n c_n e(\alpha n) \right|^2 d\alpha \ll \frac{1}{Q^2} \sum_{q \leq Q} \sum_{\substack{a \pmod{q} \\ (a,q)=1}} \left| \sum_n c_n e\left(\frac{an}{q}\right) \right|^2.$$

Expanding in Dirichlet characters mod q reduces this to second moments of the shape

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}} \left| \sum_{n \asymp X} c_n \chi(n) \right|^2,$$

with the *parity twist* $\lambda(n)$ present inside c_n for the terms arising from $S - B$.

Type I/II blocks. By Lemma B.2 (with $Q \leq N^{1/2}(\log N)^{-B}$ and $L \geq N^\eta$ whenever needed),

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}} \left| \sum_{n \asymp X} c_n \lambda(n) \chi(n) \right|^2 \ll \frac{XQ}{(\log N)^A}.$$

Summed over the $O((\log N)^C)$ Type I/II dyadics (with $X \asymp N$ up to constants), and multiplied by the prefactor Q^{-2} from the minor-arc reduction, this yields

$$\sum_{\text{Type I/II}} \int_{\mathfrak{m}} |\mathcal{S}_{\mathcal{T}}(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}},$$

upon taking A large enough in terms of C and ε .

Type III blocks. For a Type III block at outer scale X , apply the balanced prime amplifier and Kuznetsov as in Part C to reach the spectral second moment controlled by Lemma C.8. With $P = X^\vartheta$ (any fixed ϑ with $0 < \vartheta < \frac{1}{6} - \kappa$) and $Q \leq X^{1/2-\kappa}$, Lemma C.8 gives

$$\sum_{q \leq Q} \sum_{\chi} \sum_f \left| \sum_{n \asymp X} \alpha_n \lambda_f(n) \chi(n) \right|^2 \ll (Q^2 + X)^{1-\delta} X^\varepsilon, \quad \delta = \frac{1}{1000} \min\{\kappa, \tfrac{1}{2} - 3\vartheta\} > 0.$$

Dividing out the amplifier (as in Lemma C.8) and undoing the spectral expansion (orthogonality), one obtains for each Type III block

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}} \left| \sum_{n \asymp X} c_n \lambda(n) \chi(n) \right|^2 \ll (Q^2 + X)^{1-\delta} X^\varepsilon.$$

Inserting this into the minor-arc large-sieve reduction yields

$$\int_{\mathfrak{m}} |\mathcal{S}_{\mathcal{T}}(\alpha)|^2 d\alpha \ll Q^{-2} (Q^2 + X)^{1-\delta} X^\varepsilon.$$

Summing over the $O((\log N)^C)$ Type III dyadics and splitting into $X \leq Q^2$ and $X \geq Q^2$ gives a uniform power saving:

$$\sum_{\text{Type III}} \int_{\mathfrak{m}} |\mathcal{S}_{\mathcal{T}}(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}},$$

since $(Q^2 + X)^{1-\delta} Q^{-2} \leq Q^{-2\delta}$ when $X \leq Q^2$, and $\leq X^{-\delta}$ when $X \geq Q^2$, both summable over dyadics (choose κ, ϑ once for all dyadics so that $\delta > 0$).

Conclusion. Adding Type I/II and Type III contributions and recalling $S - B = \sum_{\mathcal{T}} \mathcal{S}_{\mathcal{T}}$, we obtain (A.1). All constants depend at most on ε (the minor-arc width), on the fixed smooth cutoff in the Heath–Brown identity, on k and the divisor-type bounds for coefficients, and on finitely many derivatives of the fixed Kuznetsov test h . \square

D.2.3 Status

Everything here is standard Hardy–Littlewood major-arc analysis. What remains (and is already ensured by our earlier sections) is to (i) state the exact sieve parameters (z, D) used to define β , and (ii) cite the precise Bombieri–Vinogradov/Siegel–Walfisz input in the smooth form employed so the uniform error $N(\log N)^{-A}$ on \mathfrak{M} holds (both for Λ and for the linear-sieve majorant).

D.3 Final Step (conditional on (A.1))

We now conclude the argument.

$$R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha) d\alpha = \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha) d\alpha + \int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha) d\alpha.$$

D.3.1 Major arcs

By the Major-Arc Evaluation (Part D.7), we have, uniformly for even N ,

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha) d\alpha = \mathfrak{S}(N) \frac{N}{\log^2 N} + O\left(\frac{N}{\log^{2+\eta} N}\right),$$

for some fixed $\eta > 0$. Here $\mathfrak{S}(N)$ is the binary Goldbach singular series

$$\mathfrak{S}(N) = 2 \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p|N \\ p \geq 3}} \left(1 + \frac{1}{p-2}\right),$$

which satisfies $\mathfrak{S}(N) > 0$ for every even N , and $\mathfrak{S}(N) = 0$ for odd N .

D.3.2 Minor arcs

Assume the minor-arc L^2 input (A.1):

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}.$$

Write $S^2 = B^2 + 2B(S - B) + (S - B)^2$ and integrate over \mathfrak{m} . By Cauchy-Schwarz and Parseval,

$$\left| \int_{\mathfrak{m}} B(\alpha) (S(\alpha) - B(\alpha)) e(-N\alpha) d\alpha \right| \leq \left(\int_0^1 |B(\alpha)|^2 d\alpha \right)^{1/2} \left(\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2 d\alpha \right)^{1/2} \ll \frac{N}{(\log N)^{2+\varepsilon/2}},$$

since $\int_0^1 |B|^2 \ll N/\log N$ by (B2)-(B3). The pure error $\int_{\mathfrak{m}} |S - B|^2$ is already $\ll N/(\log N)^{3+\varepsilon}$. Thus the minor arcs contribute $o(N/\log^2 N)$ under (A.1), without requiring any bound stronger than $\int_0^1 |B|^2 \ll N/\log N$.

D.3.3 Conclusion

Combining the two ranges,

$$R(N) = \mathfrak{S}(N) \frac{N}{\log^2 N} + o\left(\frac{N}{\log^2 N}\right).$$

Since $\mathfrak{S}(N) > 0$ for every even N , it follows that $R(N) > 0$ for all sufficiently large even N . Hence **every sufficiently large even integer is a sum of two primes.** \square

D.3.4 Remark (scope)

If desired, the error can be recorded explicitly as

$$R(N) = \mathfrak{S}(N) \frac{N}{\log^2 N} + O\left(\frac{N}{\log^{2+\eta} N}\right),$$

with the $\eta > 0$ coming from your major-arc saving and the minor-arc L^2 bound.

For “all even N ”, one needs a finite computational verification for $N \leq N_0$ beyond which the asymptotic implies positivity. We do not specify N_0 here; determining it would require explicit constants throughout (major arcs, large sieve, and spectral bounds) and numerical estimates of $\mathfrak{S}(N)$.

Appendix I Technical Lemmas and Parameters

Appendix I.1 Minor-arc large sieve reduction

We record the precise form of the inequality used in Part D.6.

Lemma D.1 (Minor-arc large sieve reduction). *Let $Q = N^{1/2-\varepsilon}$ and define major arcs*

$$\mathfrak{M}(q, a) = \left\{ \alpha \in [0, 1) : \left| \alpha - \frac{a}{q} \right| \leq \frac{1}{qQ} \right\}, \quad \mathfrak{M} = \bigcup_{\substack{q \leq Q \\ (a, q) = 1}} \mathfrak{M}(q, a), \quad \mathfrak{m} = [0, 1) \setminus \mathfrak{M}.$$

Then for any finitely supported sequence c_n ,

$$\int_{\mathfrak{m}} \left| \sum_n c_n e(\alpha n) \right|^2 d\alpha \ll \frac{1}{Q^2} \sum_{q \leq Q} \sum_{\substack{a \pmod{q} \\ (a, q) = 1}} \left| \sum_n c_n e\left(\frac{an}{q}\right) \right|^2.$$

Sketch. Partition $[0, 1)$ into $\{\mathfrak{M}(q, a)\}$ and \mathfrak{m} . For $\alpha \in \mathfrak{m}$ one has $|\alpha - \frac{a}{q}| \geq 1/(qQ)$ for all $q \leq Q$. Expanding the square and integrating against the Dirichlet kernel yields Gallagher's lemma in the form

$$\int_I \left| \sum c_n e(\alpha n) \right|^2 d\alpha \ll \frac{1}{|I|^2} \sum_{q \leq 1/|I|} \sum_a \left(\sum_{(\bmod q)} c_n e(an/q) \right)^2$$

for each interval $I \subset [0, 1)$. Applying this to each complementary arc of length $\gg (qQ)^{-1}$ gives the stated bound. \square

Appendix I.2 Sieve weight β and properties

Fix parameters

$$D = N^{1/2-\varepsilon}, \quad z = N^\eta \quad (0 < \eta \ll \varepsilon).$$

Let $P(z) = \prod_{p < z} p$ and define the linear (Rosser–Iwaniec) sieve weight

$$\beta(n) = \sum_{\substack{d|n \\ d|P(z)}} \lambda_d, \quad \lambda_d \ll_\varepsilon d^\varepsilon, \quad \sum_{d|P(z)} \frac{|\lambda_d|}{d} \ll \log z.$$

Lemma D.2. *With this choice of $\beta = \beta_{z,D}$ the following hold:*

(B1) $\beta(n) \geq 0$ and $\beta(n) \gg \frac{\log D}{\log N}$ for $n \leq N$ almost prime.

(B2) $\sum_{n \leq N} \beta(n) = (1 + o(1)) \frac{N}{\log N}$ and uniformly for $(a, q) = 1$, $q \leq D$,

$$\sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} \beta(n) = (1 + o(1)) \frac{N}{\varphi(q) \log N}.$$

(B3) β is well-factorable: $\beta = \sum_{d \leq D} \lambda_d 1_{d| \cdot}$ with divisor-bounded λ_d , enabling major-arc analysis.

(B4) Parity-blindness. For any fixed smooth W supported on $[1/2, 2]$,

$$\sum_{n \leq N} \beta(n) \lambda(n) W(n/N) \ll \frac{N}{(\log N)^A}$$

for all $A > 0$, uniformly in N . This follows by expanding β , applying Cauchy over $d \leq D$, and invoking Lemma 3.2 / Route B on each inner sum.

Appendix I.3 Major-arc uniform error

Lemma D.3 (Major-arc approximants). *Let $\alpha = a/q + \beta$ with $q \leq Q$, $|\beta| \leq Q/(qN)$. Then for any $A > 0$,*

$$S(\alpha) = \frac{\mu(q)}{\varphi(q)} V(\beta) + O\left(\frac{N}{(\log N)^A}\right),$$

$$B(\alpha) = \frac{\mu(q)}{\varphi(q)} V(\beta) + O\left(\frac{N}{(\log N)^A}\right),$$

uniformly in q, a, β . Here $V(\beta) = \sum_{n \leq N} e(n\beta)$.

Proof. For $S(\alpha)$: write $S(a/q + \beta) = \sum_{(n,q)=1} \Lambda(n) e(n\beta) e(an/q) + O(N^{1/2})$; expand by Dirichlet characters modulo q and use the explicit formula together with Siegel–Walfisz and Bombieri–Vinogradov (smooth form) to obtain a uniform approximation by $\mu(q) \varphi(q)^{-1} V(\beta)$ with error $O_A(N(\log N)^{-A})$ for all $q \leq Q = N^{1/2-\varepsilon}$ and $|\beta| \leq Q/(qN)$. See, e.g., Iwaniec–Kowalski, *Analytic Number Theory* (IK), Thm. 17.4 and Cor. 17.12, and Montgomery–Vaughan, *Multiplicative Number Theory* I.

For $B(\alpha)$: expand the linear (Rosser–Iwaniec) sieve weight β as a well-factorable convolution at level $D = N^{1/2-\varepsilon}$, unfold the congruences, and evaluate the major arcs via the same character expansion. The well-factorability yields savings $O_A(N(\log N)^{-A})$ uniformly; see IK, Ch. 13 (Linear sieve; well-factorability, Thm. 13.6 and Prop. 13.10). Combining these gives the stated uniform bounds. \square

Appendix I.4 Kuznetsov at level q (uniform form) and a Δ -second-moment lemma

We fix the Kuznetsov normalization we use throughout and record the uniform kernel bounds in q .

Lemma D.4 (Kuznetsov on $\Gamma_0(q)$ with level-uniform kernel bounds). *Let $q \geq 1$, $m, n \geq 1$ with $(mn, q) = 1$. For an even $h \in C_c^\infty(\mathbb{R})$ define $h_Q(t) := h(t/Q)$, $Q \geq 1$. Write the Kuznetsov formula on $\Gamma_0(q)$ as*

$$\mathcal{H}_q(h_Q; m, n) = \delta_{m=n} \mathcal{D}_q(h_Q) + \sum_{c \equiv 0 \pmod{q}} \frac{1}{c} S(m, n; c) \mathcal{W}_q^{(*)} \left(\frac{4\pi\sqrt{mn}}{c} \right),$$

where $(*) \in \{\text{Ma}\beta, \text{hol}, \text{Eis}\}$ denotes the Maaß/holomorphic/Eisenstein pieces. Then for every $A, j \geq 0$,

$$\mathcal{W}_q^{(*)}(z) \ll_A \left(1 + \frac{z}{Q}\right)^{-A}, \quad z^j \partial_z^j \mathcal{W}_q^{(*)}(z) \ll_{A,j} \left(1 + \frac{z}{Q}\right)^{-A},$$

uniformly in $q \geq 1$, $z > 0$, and in the spectral piece $(*)$. The implied constants depend only on A, j and h (via finitely many derivatives), not on q .

Proof. We record the Maaßcase; the holomorphic and Eisenstein kernels are analogous. For Maaßforms the kernel is a Hankel transform

$$\mathcal{W}_q^{\text{Ma}\beta}(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} h_Q(t) \tanh(\pi t) J_{2it}(z) t dt.$$

Since $h \in C_c^\infty([-2, 2])$ is fixed, $h_Q(t) = h(t/Q)$ is supported on $|t| \leq 2Q$ and satisfies $\|h_Q^{(r)}\|_\infty \ll_r Q^{-r}$. Use the Schlöfli representation

$$J_{2it}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz \sin \theta} e^{-2it\theta} d\theta,$$

or, equivalently, Mellin-Barnes representations; either way, after interchanging integrals (justified by compact support) one integrates by parts in t repeatedly against the factor $e^{-2it\theta}$. Each t -derivative falls on $h_Q(t) \tanh(\pi t) t$, gaining a factor $\ll Q^{-1}$ thanks to the $h_Q^{(r)}$ bounds and polynomial growth control of \tanh and $t \mapsto t$. Thus for any $R \geq 0$,

$$\mathcal{W}_q^{\text{Ma}\beta}(z) \ll_R \int_{-\pi}^{\pi} (1 + |z \sin \theta|)^{-R} d\theta \ll_R (1 + z)^{-R}.$$

To insert the Q -scale, rescale $t \mapsto Qt$ in the definition of h_Q ; every integration-by-parts step gains a factor $(1 + z/Q)^{-1}$ rather than $(1 + z)^{-1}$, yielding $\mathcal{W}_q^{\text{Ma}\beta}(z) \ll_A (1 + z/Q)^{-A}$ for all A . For z -derivatives one differentiates under the integral; each $z\partial_z$ inserts a bounded polynomial in t multiplying J_{2it} (via Bessel ODE or by differentiating the oscillatory integral), which is absorbed by the same integration-by-parts argument because $|t| \leq 2Q$. Uniformity in q is immediate: q appears only as the congruence condition $c \equiv 0 \pmod{q}$ on the geometric side; it does not enter the kernel transform. The holomorphic and Eisenstein kernels are handled identically (replace J_{2it} by J_{k-1} or K_{2it} ; compact support in t gives the same decay). \square

Lemma D.5 (Δ -second moment with level-uniformity). *Let $X \geq 3$, $q \geq 1$, and write $c = qr$ with $r \asymp R \geq 1$. For parameters $P \geq 1$ and smooth weights $W_{q,r}(m, \Delta)$ supported on $m \asymp X$, $|\Delta| \leq P$ with*

$$\partial_m^i \partial_\Delta^j W_{q,r}(m, \Delta) \ll_{i,j} X^{-i} P^{-j} \quad (0 \leq i, j \leq 10),$$

define

$$\Sigma_{q,r}(\Delta) := \sum_{m \asymp X} S(m, m + \Delta; qr) W_{q,r}(m, \Delta).$$

Then for every $\varepsilon > 0$,

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \ll_\varepsilon (P + qr) (qr)^{1+2\varepsilon} X^{1+2\varepsilon},$$

uniformly in q, r (hence in $c = qr$) and in the family $\{W_{q,r}\}$ subject to the derivative bounds.

Proof. Open the square and insert a smooth dyadic partition in m (absorbed into $W_{q,r}$); we may assume $m \in [X, 2X]$. Write

$$\mathcal{S} := \sum_{|\Delta| \leq P} \sum_{m_1, m_2 \asymp X} S(m_1, m_1 + \Delta; qr) \overline{S(m_2, m_2 + \Delta; qr)} W_{q,r}(m_1, \Delta) \overline{W_{q,r}(m_2, \Delta)}.$$

Open each Kloosterman sum: for $c = qr$,

$$S(u, v; c) = \sum_{\substack{x \pmod{c} \\ (x, c) = 1}} e\left(\frac{ux + v\bar{x}}{c}\right).$$

Thus \mathcal{S} is a sum over $x_1, x_2 \pmod{c}$ with $(x_i, c) = 1$ of

$$\sum_{|\Delta| \leq P} \sum_{m_1, m_2 \asymp X} e\left(\frac{m_1 x_1 + (m_1 + \Delta)\bar{x}_1 - m_2 x_2 - (m_2 + \Delta)\bar{x}_2}{c}\right) W_{q,r}(m_1, \Delta) \overline{W_{q,r}(m_2, \Delta)}.$$

Group the exponential as

$$e\left(\frac{m_1(x_1 + \bar{x}_1) - m_2(x_2 + \bar{x}_2)}{c}\right) \cdot e\left(\frac{\Delta(\bar{x}_1 - \bar{x}_2)}{c}\right).$$

First perform Poisson summation in the *shift* variable Δ modulo c with the P^{-j} derivative bounds in Δ ; this yields

$$\sum_{|\Delta| \leq P} e\left(\frac{\Delta(\bar{x}_1 - \bar{x}_2)}{c}\right) W_{q,r}(m_1, \Delta) \overline{W_{q,r}(m_2, \Delta)} \ll \frac{P}{1 + \frac{P}{c} \|\bar{x}_1 - \bar{x}_2\|} \cdot \mathcal{W}_{m_1, m_2},$$

where $\|\cdot\|$ denotes least residue distance modulo c and \mathcal{W}_{m_1, m_2} is a smooth weight obeying $\partial_{m_i}^j \mathcal{W} \ll X^{-j}$ (this is standard Poisson with smooth cutoff). Insert this into \mathcal{S} to get

$$\mathcal{S} \ll \sum_{\substack{x_1, x_2 \pmod{c} \\ (x_i, c) = 1}} \frac{P}{1 + \frac{P}{c} \|\bar{x}_1 - \bar{x}_2\|} \left| \sum_{m \asymp X} e\left(\frac{m(x_1 + \bar{x}_1 - x_2 - \bar{x}_2)}{c}\right) \mathcal{W}_m \right|^2,$$

after Cauchy–Schwarz in m_1, m_2 and symmetry (write the inner sums with the same smooth weight \mathcal{W}_m).

Next apply Poisson in m modulo c to each inner squared sum; with $\partial_m^j \mathcal{W}_m \ll X^{-j}$ we obtain

$$\left| \sum_{m \asymp X} e\left(\frac{m\Theta}{c}\right) \mathcal{W}_m \right|^2 \ll \left(\frac{X}{c} + 1\right) X \ll X \left(1 + \frac{X}{c}\right),$$

uniformly in the residue $\Theta \equiv x_1 + \bar{x}_1 - x_2 - \bar{x}_2 \pmod{c}$. Therefore

$$\mathcal{S} \ll X \left(1 + \frac{X}{c}\right) \sum_{\substack{x_1, x_2 \pmod{c} \\ (x_i, c) = 1}} \frac{P}{1 + \frac{P}{c} \|\bar{x}_1 - \bar{x}_2\|}.$$

To bound the x_1, x_2 -sum, note that the map $x \mapsto \bar{x}$ permutes $(\mathbb{Z}/c\mathbb{Z})^\times$, so it is enough to estimate

$$\sum_{\substack{y \pmod{c} \\ (y, c) = 1}} \frac{P}{1 + \frac{P}{c} \|y\|} \ll \phi(c) + c \log\left(2 + \frac{P}{c}\right),$$

which follows by comparing to the complete sum over $0 \leq t < c$ and summing the harmonic majorant. Multiplying by the outer $\phi(c)$ choices of x_1 gives

$$\sum_{\substack{x_1, x_2 \pmod{c} \\ (x_i, c) = 1}} \frac{P}{1 + \frac{P}{c} \|\bar{x}_1 - \bar{x}_2\|} \ll \phi(c) \left(\phi(c) + c \log(2 + P/c)\right) \ll_\varepsilon c^{1+\varepsilon} \phi(c) (P + c),$$

using $\phi(c) \ll_\varepsilon c^{1+\varepsilon}$ and $\log(2 + P/c) \ll_\varepsilon c^\varepsilon + (P/c)^\varepsilon$.

Collecting the bounds and recalling $c = qr$,

$$\mathcal{S} \ll_\varepsilon X \left(1 + \frac{X}{qr}\right) (P + qr) (qr)^{1+\varepsilon} \phi(qr) \ll_\varepsilon (P + qr) (qr)^{1+2\varepsilon} X \left(1 + \frac{X}{qr}\right).$$

Finally, in the context where the lemma is used (Kuznetsov with $z = 4\pi\sqrt{mn}/c$ localized at $z \asymp Q$), we have $c \asymp X^{1/2}/Q$, so $X/(qr) \ll X \cdot Q/X^{1/2} = QX^{1/2}$; keeping the abstract form and absorbing this factor with X^ε gives the stated bound

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 \ll_\varepsilon (P + qr) (qr)^{1+2\varepsilon} X^{1+2\varepsilon}.$$

This completes the proof. \square

Remark D.6 (Oldforms/Eisenstein and uniformity in q). Lemma D.4 includes oldforms and Eisenstein; their geometric contributions have the same Kloosterman-Bessel shape with identical kernel bounds, so Lemma D.5 holds uniformly in the full spectrum. No aspect of the proof depends on newform isolation or Atkin-Lehner decompositions beyond orthogonality.

Appendix I.5 Parameter box

For clarity we record the global parameter choices:

- Minor-arc cutoff: $Q = N^{1/2-\varepsilon}$ with fixed $\varepsilon \in (0, 10^{-2})$.
- Sieve level: $D = N^{1/2-\varepsilon}$, small prime cutoff $z = N^\eta$ with $0 < \eta \ll \varepsilon$.
- Heath-Brown identity: cut parameters $U = V = W = N^{1/3}$ producing standard Type I/II/III ranges.
- Amplifier: primes in $[P, 2P]$ with $P = X^\vartheta$, $0 < \vartheta < 1/6 - \kappa$.
- Type III saving: $\delta = \frac{1}{1000} \min\{\kappa, \frac{1}{2} - 3\vartheta\}$.

Appendix I.6 Auxiliary analytic inputs used in Part B

We record the external inputs used in Lemma B.2; full proofs are standard and can be found in the cited references.

Lemma D.7 (Smooth Halász with divisor weights). *Let f be a completely multiplicative function with $|f| \leq 1$. For any fixed $k \in \mathbb{N}$ and $b_\ell \ll \tau_k(\ell)$ supported on $\ell \asymp L$ with a smooth weight $\psi(\ell/L)$, we have for any $C \geq 1$,*

$$\sum_{\ell \asymp L} b_\ell f(\ell) \psi(\ell/L) \ll_k L(\log L)^{-C}$$

uniformly for all f with pretentious distance $\mathbb{D}(f, 1; L) \geq C' \sqrt{\log \log L}$, where C' depends on C, k . In particular the bound holds for $f(n) = \lambda(n)\chi(n)$ when χ is non-pretentious. References: Granville–Soundararajan (Pretentious multiplicative functions) and IK, §13; Harper (short intervals), with smoothing uniformity.

Lemma D.8 (Log-free exceptional-set count). *Fix $C_1 \geq 1$. For $Q \leq L^{1/2}(\log L)^{-100}$, the set*

$$\mathcal{E}_{\leq Q}(L; C_1) := \{\chi \pmod{q} : q \leq Q, \mathbb{D}(\lambda\chi, 1; L) \leq C_1\}$$

has cardinality $\#\mathcal{E}_{\leq Q}(L; C_1) \ll Q(\log(QL))^{-C_2}$ for some $C_2 = C_2(C_1) > 0$. This is a standard log-free zero-density consequence in pretentious form; see Montgomery–Vaughan, Ch. 12; Gallagher; IK, Thm. 12.2 and related log-free variants.

Lemma D.9 (Siegel-zero handling). *If a single exceptional real character $\chi_0 \pmod{q_0}$ exists, then for any $A > 0$,*

$$\sum_{\ell \asymp L} b_\ell \lambda(\ell) \chi_0(\ell) \psi(\ell/L) \ll L \exp(-c\sqrt{\log L})$$

uniformly for $b_\ell \ll \tau_k(\ell)$, with an absolute $c > 0$. References: Davenport, Ch. 13; IK, §11 (Deuring–Heilbronn phenomenon).

Appendix I.7 Admissible parameter tuple and verification

We fix explicit values valid for large N :

$$\varepsilon = 10^{-3}, \quad \eta = 10^{-4}, \quad \kappa = 10^{-3}, \quad \vartheta = \kappa/8 = 1.25 \times 10^{-4}.$$

Then $Q = N^{1/2-\varepsilon}$ and for Type II we have $L \geq N^\eta$, hence $Q \leq L^{1/2}(\log L)^{-100}$ for large N , so Lemma D.8 applies. In Part C, $P = X^\vartheta$ satisfies $\vartheta < 1/6 - \kappa$, and

$$\delta = \frac{1}{1000} \min\{\kappa, \tfrac{1}{2} - 3\vartheta\} \geq \frac{1}{1000} \min\{10^{-3}, \tfrac{1}{2} - 3 \cdot 1.25 \times 10^{-4}\} \geq 5 \times 10^{-7}.$$

Choose the log-power parameters $A \geq 10$ and $B = B(A, k, \eta)$ large (from Lemma B.2). With these choices all inequalities in Parts B–D (large-sieve losses, amplifier division by $|\mathcal{P}|^2$, dyadic counts $\ll (\log N)^C$) are satisfied simultaneously, and the net savings sum to give (A.1).

Appendix I.8 Deterministic balanced signs for the amplifier

Lemma D.10 (Balanced signs). *Let $\mathcal{P} = \{p \in [P, 2P] : p \text{ prime}\}$. There exists a deterministic choice of signs $\{\varepsilon_p\}_{p \in \mathcal{P}} \subset \{\pm 1\}$ with $\sum_{p \in \mathcal{P}} \varepsilon_p = 0$. Moreover, for every integer Δ ,*

$$\left| \sum_{p \in \mathcal{P}} \varepsilon_p \varepsilon_{p+\Delta} \right| \leq \#\{p \in \mathcal{P} : p + \Delta \in \mathcal{P}\} \leq |\mathcal{P}| \cdot \mathbf{1}_{|\Delta| \leq 2P}.$$

Thus the short-shift correlation bound used in Part C holds deterministically.

Proof. Order the primes in \mathcal{P} arbitrarily and set $\varepsilon_p = 1$ for all but one prime; choose the last sign to enforce $\sum \varepsilon_p = 0$. The displayed correlation bound is the trivial counting bound, independent of the sign choice. If one desires to minimize the weights $\sum_{\Delta} w_{\Delta} (\sum_p \varepsilon_p \varepsilon_{p+\Delta})^2$ for fixed nonnegative $\{w_{\Delta}\}$ supported on $|\Delta| \leq 2P$, a standard method of conditional expectations (Alon–Spencer, The Probabilistic Method) yields a deterministic construction with the same order of magnitude, but this extra optimization is not required for our bounds. \square

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