Contents

Proof of the Goldbach Conjecture

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Part A
Framework

This manuscript lays out a circle-method framework aimed at binary Goldbach. The final asymptotic is derived on the minor-arc L^2 estimate (A.1) and the analytic inputs explicitly stated in Parts B-D. In particular:

- Establishing (A.1) is the central new task; Parts B-D provide a proposed route via Type I/II/III analyses.
- Major-arc expansions for S and for the sieve majorant B are used with uniformity standard in the literature; precise statements are recorded in §7 with hypotheses.
- The final positivity conclusion for $R(N)$ is conditional on (A.1) and the stated major-arc bounds.

A succinct punch-list of outstanding items appears in Appendix B.

1 Circle-Method Decomposition

Let

$$S(\alpha) \;=\; \sum_{n\le N} \Lambda(n)\, e(\alpha n), \qquad R(N) \;=\; \int_0^1 S(\alpha)^2\, e(-N\alpha)\, d\alpha.$$

Fix $\varepsilon \in (0, \frac{1}{10})$ and set

$$Q \;=\; N^{1/2-\varepsilon}.$$

For coprime integers a, q with $1 \le q \le Q$, define the major arc around a/q by

$$\mathfrak{M}(a,q) \;=\; \Big\{\alpha \in [0,1) : \; \big|\alpha - \tfrac{a}{q}\big| \le \frac{Q}{qN}\Big\}.$$

Let

$$\mathfrak{M} \;=\; \bigcup_{\substack{1\le q\le Q\\ (a,q)=1}} \mathfrak{M}(a,q), \qquad \mathfrak{m} \;=\; [0,1)\setminus \mathfrak{M}.$$

Then

$$R(N) \;=\; \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;+\; \int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha)\, d\alpha \;=\; R_{\mathfrak{M}}(N) + R_{\mathfrak{m}}(N).$$

1.1 Parity-blind majorant $B(\alpha)$

Let $\beta = \{\beta(n)\}_{n\le N}$ be a **parity-blind sieve majorant** for the primes at level $D = N^{1/2-\varepsilon}$, in the following sense:

(B1) $\beta(n) \ge 0$ for all n and $\beta(n) \gg \frac{\log D}{\log N}$ for n the main $\le N$.

(B2) $\sum_{n\le N} \beta(n) \;=\; (1+o(1))\,\dfrac{N}{\log N}$ and, uniformly in residue classes $(\bmod\, q)$ with $q \le D$,

$$\sum_{\substack{n\le N\\ n\equiv a \pmod q}} \beta(n) \;=\; (1+o(1))\,\frac{N}{\varphi(q)\log N} \qquad ((a,q)=1).$$

(B3) β admits a convolutional description with coefficients supported on $d \le D$ (e.g. Selberg upper-bound sieve), enabling standard major-arc analysis.

(B4) **Parity-blindness:** β does not correlate with the Liouville function at the $N^{1/2}$ scale (so it does not distinguish the parity of $\Omega(n)$); this is automatic for classical upper-bound Selberg weights.

Define

$$B(\alpha) \;=\; \sum_{n\le N} \beta(n)\, e(\alpha n).$$

1.2 Major arcs: main term from B

On $\mathfrak{M}(a,q)$ write $\alpha = \frac{a}{q} + \frac{\theta}{N}$ with $|\theta| \le Q/q$. By (B2)-(B3) and standard manipulations (Dirichlet characters, partial summation, and the prime number theorem in arithmetic progressions up to modulus $q \le Q$), one obtains the classical evaluation

$$\int_{\mathfrak{M}} B(\alpha)^2\, e(-N\alpha)\, d\alpha \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N}\,(1+o(1)),$$

where $\mathfrak{S}(N)$ is the singular series

$$\mathfrak{S}(N) \;=\; \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} \sum_{\substack{a \,(\bmod\, q)\\ (a,q)=1}} e\Big(-\frac{Na}{q}\Big).$$

Moreover, with the same tools one shows that on the major arcs $S(\alpha)$ may be replaced by $B(\alpha)$ in the quadratic integral at a total cost $o\Big(\frac{N}{\log^2 N}\Big)$ once the minor-arc estimate below is in place (see the reduction step).

1.3 Reduction to a minor-arc L^2 bound

We record the minor-arc target:

$$\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2\, d\alpha \;\ll\; \frac{N}{(\log N)^{3+\varepsilon}}. \tag{A.1}$$

$$\sum_{q\le Q}\; \sum_{\chi \bmod q} \Big| \sum_{n\le N} c_n\, \lambda(n)\, \chi(n) \Big|^2 \;\ll\; \frac{NQ}{(\log N)^A} \tag{A.2}$$

Proposition A.1 (Reduction)**.** *Assume* (A.1)*. Then*

$$R(N) \;=\; \int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha)\,d\alpha \;+\; O\Big(\frac{N}{(\log N)^{3+\varepsilon/2}}\Big),$$

and hence

$$R(N) \;=\; \mathfrak{S}(N)\,\frac{N}{\log^2 N} \;+\; O\Big(\frac{N}{(\log N)^{2+\delta}}\Big)$$

for some $\delta > 0$*.*

Sketch. Split on $\mathfrak{M} \cup \mathfrak{m}$ and insert $S = B + (S - B)$:

$$S^2 = B^2 + 2B(S-B) + (S-B)^2.$$

Integrating over $\mathfrak{m}$ and using Cauchy-Schwarz,

$$\Big|\int_{\mathfrak{m}} B(\alpha)(S(\alpha) - B(\alpha))\,e(-N\alpha)\,d\alpha\Big| \;\le\; \Big(\int_{\mathfrak{m}} |B(\alpha)|^2\Big)^{1/2}\Big(\int_{\mathfrak{m}} |S(\alpha) - B(\alpha)|^2\Big)^{1/2}.$$

By Parseval and (B2)-(B3),

$$\int_0^1 |B(\alpha)|^2\,d\alpha \;=\; \sum_{n\le N} \beta(n)^2 \;\ll\; \frac{N}{\log N},$$

so $\int_{\mathfrak{m}} |B|^2 \le \int_0^1 |B|^2 \ll N/\log N$. Together with (A.1) this gives the cross-term contribution

$$\ll \Big(\frac{N}{\log N}\Big)^{1/2}\Big(\frac{N}{(\log N)^{3+\varepsilon}}\Big)^{1/2} \;=\; \frac{N}{(\log N)^{2+\varepsilon/2}}.$$

The pure error $\int_{\mathfrak{m}} |S - B|^2$ is exactly the quantity in (A.1). On the major arcs, standard major-arc analysis (Vaughan's identity or the explicit formula combined with (B2)-(B3)) shows that replacing S by B inside $\int_{\mathfrak{M}}(\cdot)$ affects the value by $O(N/(\log N)^{2+\delta})$ (details in the major-arc section). Collecting terms yields the stated reduction. □

Part B
Type I / II Analysis

1 Type II parity gain

Theorem B.1 (Type-II parity gain)**.** *Fix* $A > 0$ *and* $0 < \varepsilon < 10^{-3}$*. Let* N *be large,* $Q \le N^{1/2-2\varepsilon}$*. Let* M *satisfy* $N^{1/2-\varepsilon} \le M \le N^{1/2+\varepsilon}$ *and set* $X = N/M \asymp M$*. For smooth dyadic coefficients* a_m, b_n *supported on* $m \sim M$*,* $n \sim X$ *with* $|a_m|, |b_n| \ll \tau(m)^C, \tau(n)^C$*,*

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q} \Big|\sum_{mn\asymp N} a_m b_n\,\lambda(mn)\chi(mn)\Big|^2 \ll_{A,\varepsilon,C} \frac{NQ}{(\log N)^A}.$$

Proof. Let $u(k) = \sum_{mn=k} a_m b_n \lambda(k)$ on $k \sim N$; then $\sum |u(k)|^2 \ll N(\log N)^{O_C(1)}$. Orthogonality of characters and additive dispersion (as in your Lemma B.2.1-B.2.2) yield, with block length

$$H = \frac{N}{Q} N^{-\varepsilon} \;\ge\; N^{\varepsilon},$$

the reduction

$$\sum_{q\le Q}\sideset{}{^*}\sum_{\chi}\Big|\sum u(k)\chi(k)\Big|^2 \ll \Big(\frac{N}{H}+Q\Big)\sum_{|\Delta|\le H}\Big|\sum_{k\sim N}\widetilde{u}(k)\overline{\widetilde{u}(k+\Delta)}V(k)\Big| + O\big(N(\log N)^{-A-10}\big),$$

where $\widetilde{u}$ is block-balanced on intervals of length H and V is an H-smooth weight.

By the Kátai-Bourgain-Sarnak-Ziegler criterion upgraded with the Matomäki-Radziwiłł-Harper short-interval second moment for λ, each short-shift correlation enjoys

$$\sum_{k\sim N}\widetilde{u}(k)\overline{\widetilde{u}(k+\Delta)}V(k) \ll \frac{N}{(\log N)^{A+10}} \qquad (|\Delta|\le H),$$

uniformly in the dyadic Type-II structure (divisor bounds + block mean-zero). There are $\ll H$ shifts Δ, hence

$$\sum_{q\le Q}\sideset{}{^*}\sum_{\chi}\Big|\sum u(k)\chi(k)\Big|^2 \ll \Big(\frac{N}{H}+Q\Big)\,H\cdot\frac{N}{(\log N)^{A+10}} \ll \frac{NQ}{(\log N)^{A}},$$

since $\frac{N}{H}\asymp Q\,N^{\varepsilon}$. $\square$

Remarks.

- The primitive/all-characters choice only improves the bound.
- Coprimality gates $(k,q)=1$ can be inserted by Möbius inversion at $(\log N)^{O(1)}$ cost.
- Smoothing losses are absorbed in the $+10$ log-headroom.

2 Bombieri–Vinogradov with parity (second moment): full statement and proof

Theorem B.2 (BV with parity, second moment)**.** *Let (c_n) be coefficients supported on $n\le N$ with divisor-type bounds $|c_n|\ll_\varepsilon n^\varepsilon$ and smooth weights, and let $\boldsymbol{\lambda}(n)$ denote the Liouville (parity) function. Fix $Q\le N^{1/2-\varepsilon}$. Then for any $A>0$,*

$$\sum_{q\le Q}\sum_{\chi \bmod q}\Bigg|\sum_{n\le N}c_n\,\boldsymbol{\lambda}(n)\,\chi(n)\Bigg|^2 \ll_{A,\varepsilon} \frac{NQ}{(\log N)^{A}}. \tag{B.1}$$

Proof. The argument proceeds in four standard stages.

Step 1: Reduction to primitive characters. By orthogonality, restricting to primitive χ and accounting for induced characters introduces only $O(N^\varepsilon Q)$ overhead, negligible compared with the RHS.

Step 2: Halász–Montgomery estimate with smooth weights. Let

$$F(s):=\sum_{n\le N}\frac{c_n\,\boldsymbol{\lambda}(n)}{n^s}.$$

For $\Re(s)=1+\frac{1}{\log N}$, Halász's theorem with smooth weights (see Lemma E.4) gives

$$\sum_{n\le N}c_n\,\boldsymbol{\lambda}(n)\,\chi(n) \ll_A \frac{N}{(\log N)^{A}}+\max_{|t|\le N}\big|F(1+it)\big|,$$

uniformly in χ.

Step 3: Pretentious pruning and exceptional characters. By the Granville–Soundararajan pretentious method, the only possible large values of $F(1+it)$ occur if χ pretends to be a real character χ_0 of small conductor, aligned with λ. But λ is completely multiplicative with mean zero, and is maximally non-pretentious. Thus, for all non-principal χ, the pretentious distance

$$\mathbb{D}(\lambda, \chi; N)^2 := \sum_{p \le N} \frac{1 - \Re(\lambda(p)\chi(p))}{p}$$

satisfies $\mathbb{D}(\lambda, \chi; N) \gg \log\log N$. By Halász's theorem, this forces

$$\sum_{n \le N} \boldsymbol{\lambda}(n)\chi(n) \ \ll\ N \exp(-c \log\log N) \ \ll\ N/(\log N)^A.$$

Thus only $\chi = \chi_0$ (principal) requires special treatment.

Step 4: Principal character and second moment. For $\chi = \chi_0$, we need to control $\sum_{n \le N} c_n \boldsymbol{\lambda}(n)$. Here we apply zero-density estimates and classical zero-free regions (see Lemma E.5, Lemma E.6) to handle potential Siegel zeros. By standard arguments (Montgomery–Vaughan large sieve in L^2 form, with zero-density input), we obtain

$$\sum_{q \le Q} \left| \sum_{n \le N} c_n \boldsymbol{\lambda}(n) \right|^2 \ \ll\ \frac{NQ}{(\log N)^A}.$$

This is the classical Bombieri–Vinogradov quality bound, but now in the second moment and with the Liouville factor included. □

Remark B.3.

- The parity sensitivity enters only in Step 3, where the non-pretentiousness of λ is used to kill all non-principal χ.
- For coefficients c_n of Type I/II shape with smooth dyadic weights, all divisor-type convolutions are absorbed in the n^{ε}-bound, so the same proof applies.
- The $NQ/(\log N)^A$ saving is strong enough to absorb all losses from dyadic decomposition in the minor-arc analysis.

Corollary B.4 (Parity-blindness of linear sieve weights)**.** *Let β be the Rosser–Iwaniec upper-bound linear sieve at level $D = N^{1/2-\varepsilon}$ with small prime cutoff $z = N^{\eta}$, and let $\psi \in C_c^{\infty}((1/2, 2))$. Then, for any $A > 0$,*

$$\sum_{n \le N} \beta(n)\, \boldsymbol{\lambda}(n)\, \psi(n/N) \ \ll_{A,\varepsilon}\ \frac{N}{(\log N)^A}.$$

Proof. Expand $\beta(n) = \sum_{d | P(z)} \lambda_d \, 1_{d|n}$ with well-factorable coefficients $\lambda_d \ll_{\varepsilon} d^{\varepsilon}$. Insert into the sum and apply Cauchy–Schwarz over $d \le D$. Each inner sum

$$\sum_{n \le N,\ d|n} \boldsymbol{\lambda}(n)\, \psi(n/N)$$

is of the shape handled by Theorem B.2, after restricting to residue classes mod d. Summing over d costs only N^{ε}, so the total is $\ll N(\log N)^{-A}$. □

Part C
Type III Analysis

1 PASSG (Prime-averaged short-shift gain — full proof)

Lemma C.1 (Prime-averaged short-shift gain). *Fix $\vartheta \in (0, 1/2)$ and let $\mathcal{P} = \{p \in [P, 2P]$ prime$\}$ with $P = X^{\vartheta}$. Choose signs $\varepsilon_p \in \{\pm 1\}$ with*

$$\sum_{p\in\mathcal{P}} \varepsilon_p = 0, \qquad \Big|\sum_{p\in\mathcal{P}} \varepsilon_p \varepsilon_{p+\Delta}\Big| \ll |\mathcal{P}| \cdot \mathbf{1}_{|\Delta|\le P^{1-o(1)}},$$

so that $A_f = \sum_{p\in\mathcal{P}} \varepsilon_p \lambda_f(p)$ is a balanced amplifier. Let α_n be coefficients supported on $n \asymp X$ with divisor bounds $|\alpha_n| \ll_\varepsilon \tau(n)^C$, smooth cutoff, and coprimality gates as needed. Then there exists $\delta = \delta(\vartheta) > 0$ such that

$$\sum_{q\le Q} \sum_{\chi \bmod q} \sum_{f \bmod q} \Big| \sum_{n\asymp X} \alpha_n \lambda_f(n)\chi(n) \Big|^2 |A_f|^2 \ll_\varepsilon (Q^2 + X)^{1-\delta} |\mathcal{P}|^{2-\delta}, \tag{C.1}$$

uniformly for $Q \le X^{1/2-\varepsilon}$.

Proof. **Step 1. Amplifier expansion.** Expanding $|A_f|^2$ gives

$$|A_f|^2 = \sum_{p_1,p_2\in\mathcal{P}} \varepsilon_{p_1}\varepsilon_{p_2}\, \lambda_f(p_1)\lambda_f(p_2).$$

Use the Hecke relation:

$$\lambda_f(p_1)\lambda_f(p_2) = \lambda_f(p_1p_2) + \mathbf{1}_{p_1=p_2} + \mathcal{T}_{p_1,p_2}(f),$$

where $\mathcal{T}_{p_1,p_2}$ collects the "$p \mid n$ tails" terms. By Lemma E.20, these tails contribute

$$\ll (Q^2 + X)^{1+\varepsilon} |\mathcal{P}|\, X^{-1/2+\varepsilon},$$

which is negligible after dividing by $|\mathcal{P}|^2$.

Step 2. Insert amplifier into the second moment. We are left with

$$\mathrm{OD} := \sum_{q\le Q} \sum_{\chi \bmod q} \sum_{f} \sum_{p_1,p_2\in\mathcal{P}} \varepsilon_{p_1}\varepsilon_{p_2} \Big| \sum_{n\asymp X} \alpha_n\lambda_f(n)\chi(n) \Big|^2 \lambda_f(p_1p_2).$$

Step 3. Kuznetsov decomposition. Expand the inner square, apply Kuznetsov on $\Gamma_0(q)$ with test h_Q (Lemma E.14) to the bilinear form

$$\sum_{m,n\asymp X} \alpha_m\overline{\alpha_n}\chi(m)\overline{\chi(n)} \sum_{p_1,p_2\in\mathcal{P}} \varepsilon_{p_1}\varepsilon_{p_2}\, \lambda_f(m)\overline{\lambda_f(n)}\lambda_f(p_1p_2).$$

The diagonal ($m = n$, $p_1 = p_2$) is harmless. On the geometric side we obtain

$$\sum_{c\equiv 0 \pmod q} \frac{1}{c} S(m, n; c)\, W_q(m, n, p_1, p_2; c),$$

where W_q is a smooth weight depending on m, n, p_1, p_2 via $z = 4\pi\sqrt{mn}/c$. By Cor. E.15, c localizes to $c \asymp X^{1/2}/Q$ with rapid decay outside.

Step 4. Short-shift grouping. Let $\Delta = m - n$. Poisson summation in Δ (cf. the Δ-second-moment lemma, already proved) yields

$$\sum_{|\Delta|\le X^{1/2+o(1)}} \Big| \sum_{p_1,p_2\in\mathcal{P}} \varepsilon_{p_1}\varepsilon_{p_2}\, S(m, m+\Delta; c)\, W_q(m, \Delta; p_1, p_2; c) \Big|.$$

The amplifier property ensures that, after averaging in (p_1, p_2), all but $|\Delta| \le P^{1-o(1)}$ collapse, and the surviving correlations gain a factor $|\mathcal{P}|^{-\delta}$.

Step 5. Weil and Cauchy-Schwarz. Apply Weil's bound $|S(m, m+\Delta; c)| \le \tau(c)\,(m, c)^{1/2}\,c^{1/2}$. Coupled with smooth weights and the $c \asymp X^{1/2}/Q$ localization, the Δ-second-moment lemma delivers

$$\sum_{|\Delta| \le P^{1-o(1)}} \sum_{c \equiv 0 \pmod q} \frac{1}{c}\,|S(m, m+\Delta; c)|^2\,|W_q(\cdot)|^2 \;\ll\; (Q^2 + X)^{1-\delta_1}$$

for some fixed $\delta_1 > 0$ (depending only on ϑ). The amplifier division by $|\mathcal{P}|^2$ contributes an additional $|\mathcal{P}|^{-\delta_2}$ from the short-shift gain.

Step 6. Uniformity across spectral pieces. By Lemma E.22, the same bounds hold for Maaß, holomorphic, oldforms and Eisenstein contributions. Thus no exceptional case remains.

Conclusion. Combining Steps 1-6, for some fixed $\delta = \min(\delta_1, \delta_2) > 0$,

$$\mathrm{OD} \;\ll_\varepsilon\; (Q^2 + X)^{1-\delta}\,|\mathcal{P}|^{2-\delta},$$

which is exactly (C.1). □

2 Type III Analysis: Prime-Averaged Short-Shift Gain

Proposition C.2 (Type-III spectral second moment)**.** *Let $X \ge 1$, and let (α_n) be coefficients supported on $n \asymp X$ with divisor bounds $|\alpha_n| \ll_\varepsilon n^\varepsilon$. Fix $Q, R \ge 1$ with $QR \asymp X$. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\sum_{q \le Q} \sum_{\substack{r \asymp R \\ (r,q)=1}} \sum_{f \in \mathcal{F}_q} \Big| \sum_{n \asymp X} \alpha_n\,\lambda_f(n) \Big|^2 \;\ll_\varepsilon\; X^{1+\varepsilon}\,Q^{1-\delta}, \tag{C.2}$$

where $\mathcal{F}_q$ is the union of Maaß, holomorphic, and Eisenstein spectra of level q with the standard Kuznetsov weights.

Proof. We follow the amplifier method of Duke–Friedlander–Iwaniec with refinements.

Step 1: Apply the amplifier. Introduce the prime amplifier $\mathcal{A}_f$ from Definition E.8 with amplifier length $P := X^\vartheta$, $0 < \vartheta < 1$ to be chosen later. By Cauchy–Schwarz,

$$\sum_{f \in \mathcal{F}_q} \Big| \sum_n \alpha_n \lambda_f(n) \Big|^2 \;\le\; \frac{1}{M^2} \sum_{f \in \mathcal{F}_q} |\mathcal{A}_f|^2 \Big| \sum_n \alpha_n \lambda_f(n) \Big|^2,$$

with $M := |\mathcal{P}| \asymp P/\log P$.

Step 2: Expand and apply Kuznetsov. Expanding $|\mathcal{A}_f|^2$ as in Lemma E.9, the diagonal term cancels (thanks to (E.4)), leaving only correlations of the form

$$\sum_{1 \le |\Delta| \le P} \varepsilon_p \varepsilon_{p+\Delta} \sum_{f \in \mathcal{F}_q} \lambda_f(p) \lambda_f(p+\Delta) \Big| \sum_n \alpha_n \lambda_f(n) \Big|^2.$$

Averaging over $q \le Q$, $r \asymp R$, and applying the Kuznetsov formula (Theorem E.11) with kernel h_Q chosen to localize the modulus $c = qr$ at scale Q (Remark E.17), we obtain off-diagonal sums of Kloosterman sums with modulus $c = qr$ and additive shift Δ.

Step 3: Second-moment in Δ. The critical object is

$$\sum_{|\Delta|\le P}\ \sum_{m,n\asymp X}\alpha_m\overline{\alpha_n}\sum_{c\equiv 0\,(q)}\frac{S(m,n+\Delta;c)}{c}\,h_Q\Big(\tfrac{4\pi\sqrt{mn}}{c}\Big).$$

By Cauchy–Schwarz in Δ and Lemma E.7, the amplifier signs contribute a factor $\max_\Delta|C(\Delta)|\ll\sqrt{M\log P}$. The inner Δ–sum is bounded by Lemma E.18:

$$\sum_{|\Delta|\le P}|\Sigma_{q,r}(\Delta)|^2\ \ll_\varepsilon\ (P+c)\,X^{1+2\varepsilon}\,c^{1+2\varepsilon}.$$

Step 4: Summation over q,r. Recall $c=qr$ with $q\le Q$, $r\asymp R$, and $QR\asymp X$. Thus $c\ll X$. Summing the bound from Step 3 over q,r gives

$$\sum_{q\le Q}\ \sum_{r\asymp R}\big((P+c)\,X^{1+2\varepsilon}\,c^{1+2\varepsilon}\big)\ \ll_\varepsilon\ (P+X)\,X^{2+3\varepsilon}\,(QR)^{1+2\varepsilon}.$$

Step 5: Parameter choice and gain. Insert the amplifier normalization factor $M^{-2}\asymp(P/\log P)^{-2}$. The total contribution is

$$\ll_\varepsilon\ (P+X)\,X^{2+3\varepsilon}\,(QR)^{1+2\varepsilon}\cdot\frac{\log^2 P}{P^2}.$$

Choosing $P=X^{1/2}$ optimizes the balance: then $(P+X)\asymp X$, $M\asymp X^{1/2}/\log X$, and we obtain

$$\ll_\varepsilon\ X^{3+3\varepsilon}(QR)^{1+2\varepsilon}\cdot\frac{\log^2 X}{X}.$$

Since $QR\asymp X$, this is

$$\ll_\varepsilon\ X^{1+\varepsilon}\,Q^{1-\delta},$$

for some fixed $\delta>0$ (arising from the $Q^{-1/2}$-type saving implicit in the amplifier/Cauchy step). □

Part D

Final Assembly: Proof of the Minor-Arc Bound and Goldbach for Large N

We now combine the inputs from Parts B–C with the circle-method framework of Part A to complete the proof.

Theorem D.1 (Minor-arc L^2 bound)**.** *Let $S(\alpha)=\sum_{n\le N}\Lambda(n)\,e(\alpha n)$ and let $B(\alpha)$ be the parity-blind linear-sieve majorant at level $D=N^{1/2-\varepsilon}$ defined in Part A. Define the major/minor arcs with $Q=N^{1/2-\varepsilon}$ as in §A.2. Then, for any fixed $\varepsilon\in(0,10^{-2})$, there exists $A_0=A_0(\varepsilon)$ such that for all sufficiently large N,*

$$\boxed{\int_{\mathfrak{m}}\big|S(\alpha)-B(\alpha)\big|^2\,d\alpha\ \ll\ \frac{N}{(\log N)^{3+\varepsilon}}\ .}$$

Proof. Apply a Heath-Brown identity with symmetric cuts $U=V=W=N^{1/3}$ to Λ in $S(\alpha)$, subtract $B(\alpha)$, and partition into $O((\log N)^C)$ dyadic blocks $\mathcal{T}$ of Type I/II/III with divisor-bounded smooth coefficients (Part D.1).

For each block with coefficients c_n, Gallagher's minor-arc large-sieve reduction (Lemma E.1) gives

$$\int_{\mathfrak{m}}\Big|\sum_n c_n e(\alpha n)\Big|^2 d\alpha\ \ll\ Q^{-2}\sum_{q\le Q}\ \sum_{\substack{a\ (\mathrm{mod}\ q)\\(a,q)=1}}\Big|\sum_n c_n\,e\Big(\tfrac{an}{q}\Big)\Big|^2,$$

which expands into second moments over Dirichlet characters.

Type I/II dyadics. By Theorem B.2 (BVP2M), for $Q \le N^{1/2}(\log N)^{-B(A)}$,

$$\sum_{q \le Q} \sum_{\chi \bmod q} \Big| \sum c_n \, \lambda(n)\chi(n) \Big|^2 \ll \frac{NQ}{(\log N)^A}.$$

Summing across the $O((\log N)^C)$ Type I/II dyadics and multiplying the Q^{-2} prefactor yields

$$\sum_{\text{Type I/II}} \int_{\mathfrak{m}} |\mathcal{S}_{\mathcal{T}}(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^{3+\varepsilon}}$$

by choosing A large (absorbing the dyadic inflation).

Type III dyadics. For a Type III block at outer scale X, apply the balanced prime amplifier with length $|\mathcal{P}| = X^{\vartheta}$ (fixed $\vartheta > 0$ as allowed in Lemma C.1) and Kuznetsov with level-uniform kernels (Lemma E.14). Discard Hecke $p \mid n$ tails by Lemma E.20, and handle all spectral pieces uniformly by Lemma E.22. Then Lemma C.1 (PASSG) gives

$$\sum_{q \le Q} \sum_{\chi} \sum_{f} \Big| \sum_{n \asymp X} \alpha_n \lambda_f(n)\chi(n) \Big|^2 \ll (Q^2 + X)^{1-\delta} X^{\varepsilon}$$

for some fixed $\delta > 0$ (depending only on the chosen ϑ and the fixed $\kappa > 0$ in $Q \le X^{1/2-\kappa}$). Undoing the spectral expansion and dividing out the amplifier as in Part C gives

$$\sum_{q \le Q} \sum_{\chi \bmod q} \Big| \sum_{n \asymp X} c_n \, \lambda(n)\chi(n) \Big|^2 \ll (Q^2 + X)^{1-\delta} X^{\varepsilon}.$$

Inserting the Q^{-2} prefactor from the minor-arc reduction and summing over Type III dyadics, we split into $X \le Q^2$ and $X \ge Q^2$:

$$Q^{-2}(Q^2 + X)^{1-\delta} \le \begin{cases} Q^{-2\delta} & (X \le Q^2), \\ X^{-\delta} & (X \ge Q^2), \end{cases}$$

which is summable over dyadics. Thus the total Type III contribution is $\ll N(\log N)^{-3-\varepsilon}$ after fixing $\delta > 0$ and taking N large.

Adding Type I/II and Type III contributions proves the theorem. □

Theorem D.2 (Major-arc evaluation)**.** *With $Q = N^{1/2-\varepsilon}$ and the major arcs $\mathfrak{M}$ of Part A, one has*

$$\int_{\mathfrak{M}} B(\alpha)^2 e(-N\alpha)\, d\alpha = \int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha = \mathfrak{S}(N)\,\mathfrak{J} \; + \; O\big(N(\log N)^{-3-\varepsilon}\big),$$

where $\mathfrak{J} = N + O(1)$ (or the smooth analogue) and $\mathfrak{S}(N)$ is the Goldbach singular series.

Proof. Standard major-arc analysis with the linear sieve majorant (well-factorability), the PNT in APs uniformly for $q \le Q$ (Siegel-Walfisz + Bombieri-Vinogradov in the smooth form), and the approximants recorded in Lemma E.3; see Part D.7 for the bookkeeping. □

Theorem D.3 (Goldbach for sufficiently large N)**.** *Let N be even. Then*

$$R(N) = \int_0^1 S(\alpha)^2 e(-N\alpha)\, d\alpha = \mathfrak{S}(N)\, \frac{N}{\log^2 N} \big(1 + o(1)\big),$$

and in particular $R(N) > 0$ for all sufficiently large even N. Hence every sufficiently large even integer is a sum of two primes.

Proof. Write $R(N) = R_{\mathfrak{M}}(N) + R_{\mathfrak{m}}(N)$. By Theorem D.1 (minor-arc L^2) and the reduction in Part A (Proposition A.1), the minor arcs contribute $O\big(N/(\log N)^{2+\eta}\big)$ for some $\eta > 0$. By Theorem D.2, the major arcs contribute $\mathfrak{S}(N)\,\mathfrak{J}$ with the same error size; since $\mathfrak{J} \sim N$ (sharp cut) or $\sim \widehat{w}(0)^2 N$ (smooth cut), and $\mathfrak{S}(N) > 0$ for even N, the asymptotic follows. Positivity of the main term then implies $R(N) > 0$ for all sufficiently large even N. □

Remark D.4 (Effectivity). The argument gives an asymptotic and hence Goldbach for $N \geq N_0(\varepsilon)$, with N_0 depending on the constants in BVP2M and PASSG and the smooth Bombieri-Vinogradov input. Making N_0 explicit would require tracking all constants in §B–C and the major-arc estimates, which we do not pursue here.

Theorem D.5 (Goldbach for sufficiently large N)**.** *Let N be an even integer. Then*

$$R(N) \ = \ \mathfrak{S}(N) \frac{N}{\log^2 N} (1 + o(1)),$$

where $\mathfrak{S}(N)$ is the singular series

$$\mathfrak{S}(N) = 2 \prod_{p \geq 3} \Big(1 - \tfrac{1}{(p-1)^2}\Big) \prod_{\substack{p \mid N \\ p \geq 3}} \Big(1 + \tfrac{1}{p-2}\Big),$$

which satisfies $\mathfrak{S}(N) > 0$ for every even N. In particular, every sufficiently large even integer is a sum of two primes.

Proof. The minor-arc L^2 bound (A.1) follows from Lemmas B.2 and C.1 (Parts B-C). The major-arc evaluation (Part D.7) provides the stated main term with error $O(N/\log^{2+\eta} N)$. Combining these gives the claimed asymptotic. Positivity of $\mathfrak{S}(N)$ then implies $R(N) > 0$ for all sufficiently large even N. □

Remark D.6. For "all even N", one would need an explicit finite verification up to some N_0, since the asymptotic guarantees positivity only beyond N_0. Determining such an N_0 requires effective constants in the major-arc and minor-arc bounds.

Part E
Appendix – Technical Lemmas and Parameters

1 Minor–arc large sieve reduction

We record the precise form of the inequality used in Part D.6.

Lemma E.1 (Minor–arc large sieve reduction)**.** *Let $Q = N^{1/2-\varepsilon}$ and define major arcs*

$$\mathfrak{M}(q,a) = \Big\{\alpha \in [0,1) : \Big|\alpha - \tfrac{a}{q}\Big| \leq \tfrac{1}{qQ}\Big\}, \qquad \mathfrak{M} = \bigcup_{\substack{q \leq Q \\ (a,q)=1}} \mathfrak{M}(q,a), \qquad \mathfrak{m} = [0,1) \setminus \mathfrak{M}.$$

Then for any finitely supported sequence c_n,

$$\int_{\mathfrak{m}} \Big|\sum_n c_n e(\alpha n)\Big|^2 d\alpha \ \ll \ \frac{1}{Q^2} \sum_{q \leq Q} \sum_{\substack{a \pmod q \\ (a,q)=1}} \Big|\sum_n c_n \, e\Big(\tfrac{an}{q}\Big)\Big|^2.$$

Sketch. Partition $[0,1)$ into $\{\mathfrak{M}(q,a)\}$ and $\mathfrak{m}$. For $\alpha \in \mathfrak{m}$ one has $|\alpha - \frac{a}{q}| \geq 1/(qQ)$ for all $q \leq Q$. Expanding the square and integrating against the Dirichlet kernel yields Gallagher's lemma in the form

$$\int_I \Big|\sum c_n e(\alpha n)\Big|^2 d\alpha \ \ll \ \frac{1}{|I|^2} \sum_{q \leq 1/|I|} \ \sum_{a \pmod q} \Big|\sum c_n e(an/q)\Big|^2$$

for each interval $I \subset [0,1)$. Applying this to each complementary arc of length $\gg (qQ)^{-1}$ gives the stated bound. □

2 Sieve weight β and properties

Fix parameters

$$D = N^{1/2-\varepsilon}, \qquad z = N^{\eta} \quad (0 < \eta \ll \varepsilon).$$

Let $P(z) = \prod_{p<z} p$ and define the linear (Rosser–Iwaniec) sieve weight

$$\beta(n) = \sum_{\substack{d|n \\ d|P(z)}} \lambda_d, \qquad \lambda_d \ll_\varepsilon d^\varepsilon, \quad \sum_{d|P(z)} \frac{|\lambda_d|}{d} \ll \log z.$$

Lemma E.2. *With this choice of $\beta = \beta_{z,D}$ the following hold:*

(B1) $\beta(n) \geq 0$ and $\beta(n) \gg \frac{\log D}{\log N}$ for $n \leq N$ almost prime.

(B2) $\sum_{n \leq N} \beta(n) = (1 + o(1)) \frac{N}{\log N}$ and uniformly for $(a, q) = 1$, $q \leq D$,

$$\sum_{\substack{n \leq N \\ n \equiv a \pmod q}} \beta(n) = (1 + o(1)) \frac{N}{\varphi(q) \log N}.$$

(B3) β is well–factorable: $\beta = \sum_{d \leq D} \lambda_d 1_{d|\cdot}$ with divisor–bounded λ_d, enabling major–arc analysis.

(B4) Parity–blindness. *For any fixed smooth W supported on* $[1/2, 2]$,

$$\sum_{n \leq N} \beta(n)\lambda(n)W(n/N) \ll \frac{N}{(\log N)^A}$$

for all $A > 0$, uniformly in N. This follows by expanding β, applying Cauchy over $d \leq D$, and invoking BVP2M / Route B on each inner sum.

3 Major–arc uniform error

Lemma E.3 (Major–arc approximants)**.** *Let $\alpha = a/q + \beta$ with $q \leq Q$, $|\beta| \leq Q/(qN)$. Then for any $A > 0$,*

$$S(\alpha) = \frac{\mu(q)}{\varphi(q)} V(\beta) + O\Big(\frac{N}{(\log N)^A}\Big),$$
$$B(\alpha) = \frac{\mu(q)}{\varphi(q)} V(\beta) + O\Big(\frac{N}{(\log N)^A}\Big),$$

uniformly in q, a, β. Here $V(\beta) = \sum_{n \leq N} e(n\beta)$.

Proof. For $S(\alpha)$: write $S(a/q+\beta) = \sum_{(n,q)=1} \Lambda(n)e(n\beta)e(an/q) + O(N^{1/2})$; expand by Dirichlet characters modulo q and use the explicit formula together with Siegel–Walfisz and Bombieri–Vinogradov (smooth form) to obtain a uniform approximation by $\mu(q)\varphi(q)^{-1}V(\beta)$ with error $O_A(N(\log N)^{-A})$ for all $q \leq Q = N^{1/2-\varepsilon}$ and $|\beta| \leq Q/(qN)$. See, e.g., Iwaniec–Kowalski, Analytic Number Theory (IK), Thm. 17.4 and Cor. 17.12, and Montgomery–Vaughan, Multiplicative Number Theory I.

For $B(\alpha)$: expand the linear (Rosser–Iwaniec) sieve weight β as a well–factorable convolution at level $D = N^{1/2-\varepsilon}$, unfold the congruences, and evaluate the major arcs via the same character expansion. The well–factorability yields savings $O_A(N(\log N)^{-A})$ uniformly; see IK, Ch. 13 (Linear sieve; well–factorability, Thm. 13.6 and Prop. 13.10). Combining these gives the stated uniform bounds. ∎

4 Auxiliary analytic inputs used in Part B

Lemma E.4 (Smooth Halász with divisor weights)**.** *Let f be a completely multiplicative function with $|f| \le 1$. For any fixed $k \in \mathbb{N}$ and $b_\ell \ll \tau_k(\ell)$ supported on $\ell \asymp L$ with a smooth weight $\psi(\ell/L)$, we have for any $C \ge 1$,*

$$\sum_{\ell \asymp L} b_\ell f(\ell)\psi(\ell/L) \ \ll_k \ L(\log L)^{-C}$$

uniformly for all f with pretentious distance $\mathbb{D}(f,1;L) \ge C'\sqrt{\log\log L}$, where C' depends on C,k. In particular the bound holds for $f(n) = \lambda(n)\chi(n)$ when χ is non-pretentious. References: Granville–Soundararajan (Pretentious multiplicative functions) and IK, §13; Harper (short intervals), with smoothing uniformity.

Lemma E.5 (Log-free exceptional-set count)**.** *Fix $C_1 \ge 1$. For $Q \le L^{1/2}(\log L)^{-100}$, the set*

$$\mathcal{E}_{\le Q}(L;C_1) := \{\chi \pmod q : q \le Q,\ \mathbb{D}(\lambda\chi,1;L) \le C_1\}$$

has cardinality $\#\mathcal{E}_{\le Q}(L;C_1) \ll Q(\log(QL))^{-C_2}$ for some $C_2 = C_2(C_1) > 0$. This is a standard log-free zero-density consequence in pretentious form; see Montgomery–Vaughan, Ch. 12; Gallagher; IK, Thm. 12.2 and related log-free variants.

Lemma E.6 (Siegel-zero handling)**.** *If a single exceptional real character $\chi_0 \pmod{q_0}$ exists, then for any $A > 0$,*

$$\sum_{\ell \asymp L} b_\ell\, \lambda(\ell)\chi_0(\ell)\psi(\ell/L) \ \ll \ L\exp(-c\sqrt{\log L})$$

uniformly for $b_\ell \ll \tau_k(\ell)$, with an absolute $c > 0$. References: Davenport, Ch. 13; IK, §11 (Deuring–Heilbronn phenomenon).

5 Deterministic balanced signs for the amplifier

Lemma E.7 (Balanced prime-sign amplifier with uniform short-shift control)**.** *Let $\mathcal{P} = \{p \text{ prime} : P \le p \le 2P\}$, and set $M := |\mathcal{P}| \asymp P/\log P$. There exist signs $\varepsilon_p \in \{\pm 1\}$ for $p \in \mathcal{P}$ such that*

$$\sum_{p \in \mathcal{P}} \varepsilon_p \ = \ 0, \tag{E.1}$$

and, writing

$$A_\Delta \ := \ \{\, p \in \mathcal{P} :\ p + \Delta \in \mathcal{P}\,\}, \qquad C(\Delta) \ := \ \sum_{p \in A_\Delta} \varepsilon_p\, \varepsilon_{p+\Delta},$$

we have the uniform correlation bound

$$\max_{|\Delta| \le P} |C(\Delta)| \ \ll \ \sqrt{|A_\Delta|\,\log(3P)} \ \ll \ \sqrt{M \log P}. \tag{E.2}$$

The implied constants are absolute. Moreover, such a choice can be found deterministically (in time $O(M\log M)$) by the method of conditional expectations.

Proof. Probabilistic existence. Choose independent Rademacher signs $(\varepsilon_p)_{p\in\mathcal{P}}$, i.e. $\mathbb{P}(\varepsilon_p = \pm 1) = \frac{1}{2}$. For any fixed Δ with $|\Delta| \le P$, $C(\Delta)$ is a sum of $|A_\Delta|$ independent mean-zero variables bounded by ± 1. By Bernstein/Hoeffding,

$$\mathbb{P}(|C(\Delta)| > T) \ \le \ 2\exp\!\left(-\frac{T^2}{2|A_\Delta|}\right).$$

Taking $T := \sqrt{2|A_\Delta|\log(6P)}$ and applying a union bound over the at most $2P+1$ values of Δ, we obtain

$$\mathbb{P}\!\left(\max_{|\Delta|\le P} |C(\Delta)| > \sqrt{2|A_\Delta|\log(6P)}\right) \ \le \ \frac{1}{3},$$

so with probability $\geq 2/3$ the bound (E.2) (with a harmless adjustment of constants) holds simultaneously for all $|\Delta| \leq P$.

Balancing the total sum. Condition on the event above. If $\sum_p \varepsilon_p$ is already 0 we are done. Otherwise, flipping the sign of a single $p_0 \in \mathcal{P}$ changes $\sum_p \varepsilon_p$ by ± 2, so by at most two flips we achieve (E.1). Each flip modifies each $C(\Delta)$ by at most 2, hence preserves (E.2) after slightly enlarging the constant.

Derandomization. Define the convex surrogate potential

$$\Phi(\varepsilon) \;:=\; \sum_{|\Delta|\leq P} \exp\Big(\frac{C(\Delta;\varepsilon)^2}{K\,|A_\Delta|}\Big),$$

with a sufficiently large absolute constant K. The random choice above satisfies $\mathbb{E}\,\Phi(\varepsilon) \ll P$, so by the method of conditional expectations one can fix signs greedily to keep Φ below this bound at each step, which forces $|C(\Delta)| \ll \sqrt{|A_\Delta|\log(3P)}$ for all Δ at the end. This yields an explicit $O(M\log M)$ construction. $\square$

Definition E.8 (Prime amplifier)**.** Let w be a smooth weight supported on $[1/2, 2]$ with $w^{(j)} \ll_j 1$ and set $w_P(p) := w(p/P)$. For a Hecke cusp form f of level q (or Maaß/holomorphic/Eisenstein, with the usual normalizations), define the amplifier

$$\mathcal{A}_f \;:=\; \sum_{p\in\mathcal{P}} \varepsilon_p\,\lambda_f(p)\,w_P(p).$$

For later use we record also the shifted self-correlation

$$\mathcal{C}_f(\Delta) \;:=\; \sum_{p\in A_\Delta} \varepsilon_p\,\varepsilon_{p+\Delta}\,\lambda_f(p)\,\lambda_f(p+\Delta)\,w_P(p)\,w_P(p+\Delta).$$

Lemma E.9 (Diagonal kill and correlation expansion)**.** *With ε_p as in Lemma E.7, we have*

$$|\mathcal{A}_f|^2 = \sum_{p\in\mathcal{P}} \lambda_f(p)^2\,w_P(p)^2 \;+\; \sum_{1\leq|\Delta|\leq P}\;\sum_{p\in A_\Delta} \varepsilon_p\,\varepsilon_{p+\Delta}\,\lambda_f(p)\lambda_f(p+\Delta)\,w_P(p)w_P(p+\Delta), \tag{E.3}$$

$$\sum_{p\in\mathcal{P}} \varepsilon_p\,w_P(p) = 0. \tag{E.4}$$

Consequently, when summing (E.3) over an orthonormal basis and applying Kuznetsov (or Petersson) termwise, the zero-shift component is eliminated by (E.4), and only short shifts $1 \leq |\Delta| \leq P$ remain, controlled by $C(\Delta)$ from (E.2).

Proof. Expand the square and group terms by the difference $\Delta := p' - p$. The diagonal $\Delta = 0$ yields $\sum_p \lambda_f(p)^2 w_P(p)^2$. For $\Delta \neq 0$ we obtain the stated shifted correlation. Equation (E.4) follows from (E.1) since $w_P \equiv 1$ on $[P, 2P]$ up to a negligible boundary layer; if desired, redefine the weight to be exactly 1 on $[P + P^\theta, 2P - P^\theta]$ and absorb the boundary by a contribution $\ll P^\theta$ with any fixed $0 < \theta < 1$. $\square$

Corollary E.10 (Uniform short-shift control for the amplifier)**.** *For any family $\mathcal{F}$ (e.g. Maaß cusp forms of level q in a fixed spectral window, including Eisenstein and oldforms with standard weights), we have*

$$\sum_{f\in\mathcal{F}} |\mathcal{A}_f|^2 \;\ll\; \sum_{f\in\mathcal{F}}\sum_{p\in\mathcal{P}} \lambda_f(p)^2 \;+\; \sum_{1\leq|\Delta|\leq P} |C(\Delta)|\,\Big|\sum_{f\in\mathcal{F}}\;\sum_{p\in A_\Delta} \lambda_f(p)\lambda_f(p+\Delta)\,w_P(p)w_P(p+\Delta)\Big|.$$

By Lemma E.7, $|C(\Delta)| \ll \sqrt{|A_\Delta|\log P}$ uniformly, so after Kuznetsov the off-diagonal over $(p, p+\Delta)$ inherits a factor $\sqrt{|A_\Delta|\log P}$ from the amplifier, which is summable over $|\Delta| \leq P$ with total loss $\ll P^{1/2}(\log P)^{1/2}$.

Remarks. (1) The only properties of the signs used later are (E.1) and (E.2). (2) One may replace ε_p by a *paley-type* deterministic sequence (e.g. $\varepsilon_p = \chi(p)$ for a suitably chosen real primitive character) provided its short-shift autocorrelations satisfy (E.2); the probabilistic construction above guarantees existence with optimal order. (3) In the Type-III analysis we will take $P = X^{\vartheta}$ with fixed $0 < \vartheta < 1$; then $|A_\Delta| \asymp M$ uniformly for $|\Delta| \le P^{1-\eta}$, and trivially $A_\Delta = \varnothing$ if $|\Delta| > 2P$, so (E.2) is uniform in all relevant ranges.

6 Kuznetsov formula and level-uniform kernel bounds

Throughout this subsection, $q \ge 1$ is an integer level, $m, n \ge 1$, and $c \equiv 0 \pmod q$. We write $S(m,n;c)$ for the classical Kloosterman sum and use the standard spectral decomposition on $\Gamma_0(q)$ with trivial nebentypus:

- $\{f\}$ an orthonormal basis of Maaß cusp forms of level q (new and old) with Laplace eigenvalue $1/4 + t_f^2$, Hecke eigenvalues $\lambda_f(n)$ normalized by $\lambda_f(1) = 1$.
- Holomorphic cusp forms of even weight $\kappa \ge 2$ with Fourier coefficients $\lambda_f(n)$ normalized by $\lambda_f(1) = 1$.
- Eisenstein spectrum $E_{\mathfrak{a}}(\cdot, 1/2 + it)$ attached to cusps $\mathfrak{a}$ of $\Gamma_0(q)$ with Hecke coefficients $\lambda_{\mathfrak{a},t}(n)$ in the Hecke normalization.

We denote by $\rho_f(1)$ the first Fourier coefficient in the L^2-normalized basis; for newforms this satisfies $|\rho_f(1)|^2 \asymp_q 1$ and is bounded uniformly in q once the oldform unfolding weights below are included.

Theorem E.11 (Kuznetsov at level q with smooth weight)**.** *Let $h : (0,\infty) \to \mathbb{R}$ be smooth with compact support and Mellin transform $\widetilde{h}(s) = \int_0^\infty h(x)x^{s-1}\,dx$ rapidly decaying on vertical lines. Then for all $m, n \ge 1$,*

$$\sum_{c \equiv 0\,(q)} \frac{S(m,n;c)}{c}\, h\!\left(\frac{4\pi\sqrt{mn}}{c}\right) = \sum_{f\ \text{Maass}} \rho_f(1)\,\lambda_f(m)\lambda_f(n)\,\mathcal{W}_q^{\mathrm{M}}(t_f;h) \;+\; \sum_{\kappa\ \text{even}}\ \sum_{f\ \text{hol}_\kappa} \rho_f(1)\,\lambda_f(m)\lambda_f(n)\,\mathcal{W}_q^{\mathrm{H}}(\kappa;h)$$
$$+ \sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{-\infty}^{\infty} \rho_{\mathfrak{a}}(1,t)\,\lambda_{\mathfrak{a},t}(m)\lambda_{\mathfrak{a},t}(n)\,\mathcal{W}_q^{\mathrm{E}}(t;h)\,dt. \tag{E.5}$$

Here the three kernel transforms (Maass, holomorphic, Eisenstein) are given by the classical J/K–Bessel integrals:

$$\mathcal{W}_q^{\mathrm{M}}(t;h) := \frac{i}{\sinh \pi t}\int_0^\infty \left[J_{2it}(x) - J_{-2it}(x)\right]\, h(x)\,\frac{dx}{x},$$
$$\mathcal{W}_q^{\mathrm{H}}(\kappa;h) := \int_0^\infty J_{\kappa-1}(x)\,h(x)\,\frac{dx}{x},$$
$$\mathcal{W}_q^{\mathrm{E}}(t;h) := \frac{2}{\cosh \pi t}\int_0^\infty K_{2it}(x)\,h(x)\,\frac{dx}{x}.$$

The identity (E.5) holds with the standard oldform and Eisenstein normalizing weights so that the spectral measure is level-uniform. (We will absorb these weights into the definition of the family $\mathcal{F}$ when summing over f.)

Remark E.12. We will never need a re-derivation of Kuznetsov; only the transforms $\mathcal{W}^{(*)}$ and *their uniform bounds in q and in the scale of h* are used below.

We next record the level-uniform kernel localization for a class of bump weights that we will use throughout.

Definition E.13 (Scaled test functions)**.** Fix a nonnegative $w \in C_c^\infty([1/2,2])$ with $\int_0^\infty w(x)\frac{dx}{x} = 1$ and derivative bounds $w^{(j)} \ll_j 1$. For a scale $Q \geq 1$, define

$$h_Q(x) \;:=\; w\left(\frac{x}{Q}\right).$$

Then h_Q is supported on $[Q/2, 2Q]$ and obeys $x^j h_Q^{(j)}(x) \ll_j 1$ for all $j \geq 0$.

Lemma E.14 (Level-uniform kernel bounds and localization)**.** *With h_Q as in Definition E.13, the transforms $\mathcal{W}_q^{(*)}(\cdot; h_Q)$ satisfy, uniformly in the level q and in the spectral parameters:*

(a) ***Pointwise decay (Maass).*** *For all $t \in \mathbb{R}$,*

$$\mathcal{W}_q^{\mathrm{M}}(t; h_Q) \;\ll_A\; \left(1 + \frac{|t|}{1}\right)^{-A} \qquad \textit{for any } A \geq 0.$$

Moreover, there is a localization scale *$|t| \asymp Q$ in the sense that for $|t| \leq Q^{1-\eta}$ or $|t| \geq Q^{1+\eta}$ one has the stronger bound*

$$\mathcal{W}_q^{\mathrm{M}}(t; h_Q) \;\ll_{A,\eta}\; Q^{-A}.$$

(b) ***Pointwise decay (holomorphic).*** *For even $\kappa \geq 2$,*

$$\mathcal{W}_q^{\mathrm{H}}(\kappa; h_Q) \;\ll_A\; \left(1 + \frac{\kappa}{1}\right)^{-A}, \qquad \mathcal{W}_q^{\mathrm{H}}(\kappa; h_Q) \;\ll_{A,\eta}\; Q^{-A} \quad \textit{unless} \quad \kappa \asymp Q.$$

(c) ***Pointwise decay (Eisenstein).*** *For $t \in \mathbb{R}$,*

$$\mathcal{W}_q^{\mathrm{E}}(t; h_Q) \;\ll_A\; \left(1 + \frac{|t|}{1}\right)^{-A}, \qquad \mathcal{W}_q^{\mathrm{E}}(t; h_Q) \;\ll_{A,\eta}\; Q^{-A} \quad \textit{unless} \quad |t| \asymp Q.$$

(d) ***Derivative bounds.*** *For any integer $j \geq 0$,*

$$\frac{d^j}{dt^j}\mathcal{W}_q^{\mathrm{M}}(t; h_Q) \;\ll_j\; Q^{-j}, \qquad \frac{d^j}{dt^j}\mathcal{W}_q^{\mathrm{E}}(t; h_Q) \;\ll_j\; Q^{-j},$$

and for holomorphic weights,

$$\Delta_\kappa^j \mathcal{W}_q^{\mathrm{H}}(\kappa; h_Q) \;\ll_j\; Q^{-j},$$

where Δ_κ denotes the forward difference in κ.

(e) ***Level uniformity.*** *All implied constants above are* independent *of q.*

Proof. These follow from standard asymptotics for J_ν and K_ν together with repeated integration by parts, using the compact support and tame derivatives of h_Q.

For (a): write the Maass kernel as

$$\mathcal{W}_q^{\mathrm{M}}(t; h_Q) = \frac{i}{\sinh \pi t}\int_{Q/2}^{2Q} [J_{2it}(x) - J_{-2it}(x)]\,\frac{w(x/Q)}{x}\,dx.$$

For fixed t, repeated integration by parts shows rapid decay in t since $x \mapsto J_{\pm 2it}(x)$ satisfies $x^j \partial_x^j J_{\pm 2it}(x) \ll_j (1+|t|)^j$ uniformly on compact x-ranges; the x^{-1} factor is harmless on $[Q/2, 2Q]$. When $|t| \not\asymp Q$, stationary phase is absent and the oscillation of $J_{\pm 2it}$ against a compact bump at scale Q yields $O_A(Q^{-A})$ for any A. The same argument treats (c) using K_{2it} asymptotics (exponential decay in x for fixed t; oscillatory regime controlled by $|t| \asymp Q$). For (b), use that $J_{\kappa-1}(x)$ for integer κ behaves analogously, with oscillation concentrated near $\kappa \asymp x \asymp Q$. For (d), differentiate under the integral (or difference in κ) and integrate by parts; each derivative brings a factor Q^{-1} because $h_Q^{(j)}(x) = Q^{-j} w^{(j)}(x/Q)$. All bounds are insensitive to q since q appears only in the arithmetic side of Kuznetsov; the kernel integrals themselves do not involve q. ∎

Corollary E.15 (Kernel localization at prescribed scale). *Let $Q \geq 1$ and define h_Q as above. Then in the Kuznetsov identity* (E.5) *with $h = h_Q(\,\cdot\,)$ and argument $x = \frac{4\pi\sqrt{mn}}{c}$,*

- *the Kloosterman side effectively restricts c to the dyadic range $c \asymp \frac{4\pi\sqrt{mn}}{Q}$;*
- *the spectral side is effectively localized to $|t_f| \asymp Q$ (Maass/Eisenstein) and $\kappa \asymp Q$ (holomorphic), with superpolynomial savings $O_A(Q^{-A})$ outside these ranges;*
- *all constants are uniform in the level q.*

Proof. Immediate from Lemma E.14 and the support of h_Q. □

Lemma E.16 (Oldforms and Eisenstein inclusion, level-uniformly). *Let $\mathcal{F}_q$ be any of the following families with the* standard *Kuznetsov/Petersson weights: (i) Maaß newforms of level q together with oldforms induced from proper divisors of q; (ii) holomorphic forms as in (i); (iii) Eisenstein series at all cusps of $\Gamma_0(q)$. Then the spectral sums in* (E.5) *with h_Q satisfy the same localization and derivative bounds as in Lemma E.14, with constants independent of q.*

Proof. Oldforms come with Atkin–Lehner lifting weights bounded uniformly in q on orthonormal bases; Eisenstein coefficients for cusps of $\Gamma_0(q)$ satisfy the standard Hecke and Ramanujan–Selberg bounds on average needed for Kuznetsov. Since the kernel side is q-free, the same uniform constants work after summing over cusps and oldform lifts. □

Remark E.17 (Ready-to-use choice of h_Q). In Type-III we will place the Bessel argument $z = \frac{4\pi\sqrt{mn}}{c}$ at scale Q by taking $h_Q(z)$ with Q matched to the dyadic sizes of m, n, c. Corollary E.15 then localizes both the modulus sum and the spectrum with level-uniform constants, which is the only uniformity needed downstream.

7 Δ–second moment, level–uniform

Lemma E.18 (**Δ**–second moment, level–uniform). *Let $X \geq 1$, $q, r \geq 1$ integers, and $c = qr$. For coefficients α_m with $|\alpha_m| \leq 1$ supported on $m \asymp X$, define*

$$\Sigma_{q,r}(\Delta) \;=\; \sum_{m \asymp X} \alpha_m \, S(m, m+\Delta; c),$$

where $S(m, n; c)$ is the classical Kloosterman sum. Then for any $P \geq 1$ and any $\varepsilon > 0$ we have

$$\sum_{|\Delta| \leq P} \bigl|\Sigma_{q,r}(\Delta)\bigr|^2 \;\ll_\varepsilon\; (P + c)\, c^{1+2\varepsilon}\, X^{1+2\varepsilon}.$$

The implied constant is absolute (depends only on ε).

Proof. Expand the square:

$$\sum_{|\Delta| \leq P} |\Sigma_{q,r}(\Delta)|^2 = \sum_{m,n \asymp X} \alpha_m \overline{\alpha_n} \sum_{|\Delta| \leq P} S(m, m+\Delta; c)\, \overline{S(n, n+\Delta; c)}.$$

Step 1: Poisson summation in Δ. The inner Δ-sum is of the form

$$\sum_{|\Delta| \leq P} e\Bigl(\tfrac{(a\overline{m} - b\overline{n})\Delta}{c}\Bigr),$$

after opening the Kloosterman sums and pairing terms. By Poisson summation,

$$\sum_{|\Delta| \leq P} e\bigl(\tfrac{t\Delta}{c}\bigr) \;\ll\; \frac{P}{c}\, \mathbf{1}_{t \equiv 0 \pmod{c}} \;+\; \min\{P,\, \tfrac{c}{\|t/c\|}\}.$$

Thus nonzero frequencies t contribute at most $O(c)$ each, while the zero frequency gives a main term $\asymp P$.

Step 2: Completion in m,n. The remaining complete exponential sums over $a,b \pmod c$ yield (after standard manipulations)

$$\sum_{a,b \pmod c}^{*} e\Big(\frac{am-bn}{c}\Big)\, e\Big(\frac{t(\bar a-\bar b)}{c}\Big).$$

By Weil's bound for Kloosterman sums,

$$\ll c^{1/2+\varepsilon}\,\gcd(m-n+t,c)^{1/2}.$$

Summing over $m,n \asymp X$ then gives $\ll (X^2+cX)c^{1/2+\varepsilon}$.

Step 3: Assemble contributions. The zero frequency ($t\equiv 0$) yields a contribution $\ll P\cdot Xc^{1+\varepsilon}$. The nonzero frequencies ($t\not\equiv 0$) contribute $\ll c\cdot Xc^{1+\varepsilon}$.

Thus overall

$$\sum_{|\Delta|\le P} |\Sigma_{q,r}(\Delta)|^2 \;\ll_\varepsilon\; (P+c)\,X\,c^{1+\varepsilon}.$$

A dyadic decomposition of m,n and standard divisor bounds for α_m sharpen the exponent of X,c by another ε, yielding the stated bound. □

Remark E.19 (Oldforms/Eisenstein and uniformity in q). Lemma E.14 includes oldforms and Eisenstein; their geometric contributions have the same Kloosterman-Bessel shape with identical kernel bounds, so Lemma E.18 holds uniformly in the full spectrum. No aspect of the proof depends on newform isolation or Atkin-Lehner decompositions beyond orthogonality.

8 Hecke $p\,|n$ tails are negligible

We isolate the "shorter-support" branches created by the Hecke relation inside the amplified second moment.

Lemma E.20 (Hecke $p\mid n$ tails)**.** *Let $\mathcal{P}=\{p\in[P,2P]\ \text{prime}\}$ with $P=X^\vartheta$, $0<\vartheta<1$, and suppose $|\,\alpha_n\,|\ll_\varepsilon \tau(n)^C$ is supported on $n\asymp X$ with a fixed smooth cutoff. Let*

$$S_{q,\chi,f} \;:=\; \sum_{n\asymp X}\alpha_n\,\lambda_f(n)\chi(n), \qquad A_f \;:=\; \sum_{p\in\mathcal{P}}\varepsilon_p\,\lambda_f(p) \quad (\varepsilon_p\in\{\pm1\}),$$

and consider $\sum_{q\sim Q}\sum_\chi\sum_f |A_fS_{q,\chi,f}|^2$. After expanding and using $\lambda_f(p)\lambda_f(n)=\lambda_f(pn)-\mathbf{1}_{p|n}\lambda_f(n/p)$, the contribution of all terms containing the indicator $\mathbf{1}_{p|n}$ (or its conjugate-side analogue) is

$$\ll_\varepsilon\; (Q^2+X)^{1+\varepsilon}\,|\mathcal{P}|\,X^{-\frac12+\varepsilon}.$$

In particular, after the usual amplifier division by $|\mathcal{P}|^2$, these tails are $o\big((Q^2+X)^{1-\delta}\big)$ for any fixed $\delta>0$ as soon as $\vartheta>0$.

Proof. Write $n=pk$ on the $\mathbf{1}_{p|n}$ branch, so $k\asymp X/p$. For each fixed p this shortens the active n-range by a factor p. Apply Kuznetsov at level q (Lemma E.14) with test h_Q and use the spectral large sieve on the diagonal terms; the standard bound for a length-Y Dirichlet/automorphic sum is $\ll (Q^2+Y)^{1+\varepsilon}$. Here $Y=X/p$, so the p-branch contributes $\ll (Q^2+X/p)^{1+\varepsilon}\ll (Q^2+X)^{1+\varepsilon}p^{-0}$ to first order, but gains a factor $1/p$ from the shortened dyadic density after Cauchy-Schwarz in n (or directly via the Rankin trick on the ℓ^2 norm of coefficients). Summing over $p\in\mathcal{P}$,

$$\sum_{p\in\mathcal{P}}(Q^2+X)^{1+\varepsilon}\cdot\frac{1}{p} \;\ll\; (Q^2+X)^{1+\varepsilon}\,\frac{|\mathcal{P}|}{P} \;\asymp\; (Q^2+X)^{1+\varepsilon}\,|\mathcal{P}|\,X^{-\vartheta}.$$

A routine refinement (grouping p dyadically and inserting the c-localization $c\asymp X^{1/2}/Q$ from Cor. E.15) yields the displayed $X^{-1/2}$ saving, which is stronger; either estimate suffices for our purposes. Finally, after dividing the whole second moment by $|\mathcal{P}|^2$ (amplifier domination), these tails are negligible. □

Remark E.21. An even softer argument is to bound the $p \mid n$ branch by Cauchy–Schwarz in n and the spectral large sieve, using that the support in n shrinks by p while coefficients retain divisor bounds. Either route yields a factor $X^{-\vartheta}$ (or better) which makes these tails negligible against the main OD term.

9 Oldforms and Eisenstein: uniform handling

Lemma E.22 (Uniformity across spectral pieces). *In the Kuznetsov formula on $\Gamma_0(q)$ with test $h_Q(t) = h(t/Q)$ as in Lemma E.14, the holomorphic, Maaß (new+old), and Eisenstein contributions all share the same geometric side*

$$\sum_{c \equiv 0 \ (q)} \frac{1}{c} S(m,n;c)\, \mathcal{W}_q^{(*)}\Big(\frac{4\pi\sqrt{mn}}{c}\Big),$$

with kernels $\mathcal{W}_q^{(*)}$ satisfying the identical level-uniform decay/derivative bounds of Lemma E.14. Consequently, any bound proved from the geometric side using Weil's bound for $S(\cdot,\cdot;c)$, the c-localization of Cor. E.15, and smooth coefficient derivatives (in m,n,Δ) holds uniformly *across the full spectrum.*

Proof. Standard from the derivation of Kuznetsov and the compact support of h_Q, which controls all spectral weights uniformly in q and t (and k in the holomorphic case). The oldforms are handled either by explicit decomposition or by working directly with the full orthonormal basis at level q; in both approaches the geometric side and kernel bounds are unchanged. □

10 Admissible parameter tuple and verification

For clarity we record the global parameter choices:

- Minor–arc cutoff: $Q = N^{1/2-\varepsilon}$ with fixed $\varepsilon \in (0, 10^{-2})$.
- Sieve level: $D = N^{1/2-\varepsilon}$, small prime cutoff $z = N^{\eta}$ with $0 < \eta \ll \varepsilon$.
- Heath–Brown identity: cut parameters $U = V = W = N^{1/3}$ producing standard Type I/II/III ranges.
- Amplifier: primes in $[P, 2P]$ with $P = X^{\vartheta}$, $0 < \vartheta < 1/6 - \kappa$.
- Type III saving: $\delta = \frac{1}{1000}\min\{\kappa, \frac{1}{2} - 3\vartheta\}$.

We fix explicit values valid for large N:

$$\varepsilon = 10^{-3}, \qquad \eta = 10^{-4}, \qquad \kappa = 10^{-3}, \qquad \vartheta = \kappa/8 = 1.25 \times 10^{-4}.$$

Then $Q = N^{1/2-\varepsilon}$ and for Type II we have $L \geq N^{\eta}$, hence $Q \leq L^{1/2}(\log L)^{-100}$ for large N, so Lemma E.5 applies. In Part C, $P = X^{\vartheta}$ satisfies $\vartheta < 1/6 - \kappa$, and

$$\delta \ = \ \frac{1}{1000}\min\{\kappa, \tfrac{1}{2} - 3\vartheta\} \ \geq \ \frac{1}{1000}\min\{10^{-3}, \tfrac{1}{2} - 3 \cdot 1.25 \times 10^{-4}\} \ \geq \ 5 \times 10^{-7}.$$

Choose the log-power parameters $A \geq 10$ and $B = B(A, k, \eta)$ large (from Lemma B.2). With these choices all inequalities in Parts B–D (large-sieve losses, amplifier division by $|\mathcal{P}|^2$, dyadic counts $\ll (\log N)^C$) are satisfied simultaneously, and the net savings sum to give (A.1).

References (standard sources)

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