# Cheatsheet Analysis 2

Vinzenz Wolf

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## 1 Differential equations

**Ordinary Differential Equation (ODE)**

In general an **ordinary** differential equation (ODE) relates a function $f(x)$ at $x$ to the values of its derivatives at $x$. I.e. it's an equation of the Form

$$F(x, f(x), f'(x), f''(x), ..., f^{(n)}(x)) = 0$$

The order of the diff. equation is the highest order of derivative that appears in the equation.
A partial diff. equation is a diff. equation for a function of several variables. (It involves "partial derivatives").

$f'(x+2) = f(x)$ is not an **ordinary** differential equation.

**Linear ODE**

A linear ODE of order $k$ on $I$, is an equation of the form

$$y^{(k)} + a_{k-1}(x)y^{(k-1)} + ... + a_1(x)y' + a_0(x)y = b(x)$$

where $b, a_1, ..., a_{k-1}$ are continous functions of $x$ defined on $I$ with values in $\mathbb{C}$.
If $b(x) = 0, \forall x \in I$, we call the ODE **homogeneous** and otherwise **inhomogeneous**.

### Recognising a linear ODE

- no coefficients before the highest order derivative (**excluding constants**)
- alle coefficients are continous functions
- no products of $y$ and it's derivatives
- $y$ and all of it's derivatives occur with the power one
- Neither $y$ nor it's derivatives are *inside* another function.

**Solutions of Linear ODE's**

Let $I \subset \mathbb{R}$ open interval, $k \geq 1, k \in \mathbb{N}$.

$$y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_0 y = b$$

is a linear ODE over $I$ with continous coefficients.
Then

1. The set of solutions $S_0$ for the associated **homogeneous** ODE (when $b = 0$), is a vector space of dimension $k$.
2. For any initial conditions (i.e. any choice of $x_0 \in I$ and $(y_0, ..., y_{k-1}) \in \mathbb{C}^k$) there exists an **unique** solution $f \in S_0$ s.t. $f(x_0) = y_0, f'(x_0) = y_1, ..., f^{(k-1)}(x_0) = y_{k-1}$.
3. For any arbitrary $b(x)$, the set of solutions of the ODE is
$$S_b = \{f + f_p \mid f \in S_0\}$$
where $f_p$ is a **paritcular** solution of the ODE.
4. For any initial condition there is a unique condition there is a unique solution $f \in S_b$.

$S_b$ is **not** a Vector Space! (It's an affine Space.)

### 1.1 Linear ODE's of order 1

$I \subset \mathbb{R}$ be an open interval.
We consider the diff. equation of the form

$$y' = f(x)y + g(x)$$

1. (Homogeneous solution)
$$y_h = c \cdot e^{\int f(x)dx} \quad c \in \mathbb{R}$$
2. (Particular solution)
$$y_p = y_h \cdot \int \frac{g(x)}{y_h} dx$$

### 1.2 Variation of parameters

We assume that the particular solution is of the form $f_p = K(x)e^{-A(x)}$ for a function $K : I \to \mathbb{C}$. Then we can insert our guess into the ODE and see what it forces $K$ to satisfy. We get

$$\begin{aligned} b(x) &= (K(x)e^{-A(x)})' + a(x)(K(x)e^{-A(x)}) \\ b(x) &= K'(x)e^{-A(x)} - a(x)K(x)e^{-A(x)} + a(x)K(x)e^{-A(x)} \\ b(x) &= K'(x)e^{-A(x)} \\ K'(x) &= b(x)e^{A(x)} \end{aligned}$$

and thus

$$K(x) = \int_{x_0}^{x} b(t)e^{A(t)}\, dt$$

Therefore we get

$$f_p = \left(\int_{x_0}^{x} b(t)e^{A(t)}\, dt\right) \cdot e^{-A(t)}$$

The method with the "Integration factor" gives the same particular solution!

### 1.3 Educated Guess for constant coefficients

If $b(x)$ is of a specific form, we try following $f_p$, where we insert the $f_p$ into the ODE, which gives us a system of equations for the constants:

| $b(x)$ | Ansatz |
|---|---|
| $a \cdot e^{\alpha x}$ | $b \cdot e^{\alpha x}$ |
| $a\sin(\beta x)$ | $c\sin(\beta x) + d\cos(\beta x)$ |
| $b\cos(\beta x)$ | $c\sin(\beta x) + d\cos(\beta x)$ |
| $ae^{\alpha x}\sin(\beta x)$ | $e^{\alpha x}\left(c\sin(\beta x) + d\cos(\beta x)\right)$ |
| $be^{\alpha x}\cos(\beta x)$ | $e^{\alpha x}\left(c\sin(\beta x) + d\cos(\beta x)\right)$ |
| $P_n(x) \cdot e^{\alpha x}$ | $R_n(x) \cdot e^{\alpha x}$ |
| $P_n(x) \cdot e^{\alpha x}\sin(\beta x)$ | $e^{\alpha x}\left(R_n(x)\sin(\beta x) + S_n(x)\cos(\beta x)\right)$ |
| $P_n(x) \cdot e^{\alpha x}\cos(\beta x)$ | $e^{\alpha x}\left(R_n(x)\sin(\beta x) + S_n(x)\cos(\beta x)\right)$ |

$P_n, R_n$ and $S_n$ are Polynomials of degree $n$.

1. If $b(x)$ is a linear combination of any of the base functions, try that linear combination of 'Ansatz' functions.
2. If $\alpha/\beta$ from any of the 'Ansatz' functions is a root of the companion Polynomial of the ODE with multiplicity $j$, then we try the same 'Ansatz' as shown in the table but multiply it with a Polynomial of degree $j$.

### 1.4 Linear ODE's with constant coefficients

We consider an ODE of the form

$$y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_1 y' + a_0 y = b(x)$$

We search for a homogeneous solution of the form $e^{\lambda x}$. Now we can solve the characteristic polynomial:

$$\begin{aligned} P(\lambda) = e^{\lambda x}\left(\lambda^k + a_{k-1}\lambda^{k-1} + \ldots + a_0\right) = 0 \\ \implies 0 = \lambda^k + a_{k-1}\lambda^{k-1} + \ldots + a_0 \end{aligned}$$

- The roots of $P(\lambda)$ are the Eigenvalues $\lambda_i$, with corresponding multiplicity $m_r$. Thus the functions $f_{i,r} : x \to x^r e^{\lambda_i x}, 0 \leq r < m_r$ span the Vector Space $S_0$.
- If $\lambda = \beta + \gamma i$ is a complex of $P(\lambda)$, then the complex conjugation, i.e. $\bar{\lambda} = \beta - \gamma i$ is also a root. Thus $f_1 = e^{\lambda x}$ and $f_2 = e^{\bar{\lambda} x}$ are solutions to the homogeneous equation.
- We realize that $f_1 = e^{\lambda x} = e^{\beta x}(\cos(\gamma x) + i\sin(\gamma x))$ and $f_2 = e^{\bar{\lambda} x} = e^{\beta x}(\cos(\gamma x) - i\sin(\gamma x))$.
- We can thus replace $f_1$ and $f_2$ by $\tilde{f}_1 = e^{\beta x}\cos(\gamma x)$ and $\tilde{f}_2 = e^{\beta x}\sin(\gamma x)$. (Note that $f_1 = \tilde{f}_1 + i\tilde{f}_2$ and $f_2 = \tilde{f}_1 - i\tilde{f}_2$)
- Note that we are often only interested in finding real-valued solutions if the coefficients are all real valued.

- If $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = 0$ only has real coefficients, every pair of complex conjugated roots $\beta_j \pm \gamma_j i$ with multiplicity $m_j$ leads to a solution
$$x^l e^{\beta_j x}\Big(\cos(\gamma_j x) \pm i \sin(\gamma_j x)\Big) \quad \text{for } 0 \le l < m_j$$
of which then the real part can be extracted.

To find a particular solution $f_p$ we can as in the general case use **Varation of parameters** or **Educated guess**. We will now show an simple example with 2 basis functions:
Consider the Linear ODE $y'' + ya_1y' + a_0y = b$

(1) Assume the space of homogeneous solutions $S_0$ is spanned by $f_1, f_2$, i.e. $f_0 = f_1 + f_2$ is also a solution

(2) Now we try $f_p = z_1(x)f_1 + z_2(x)f_2$

(3) We first insert $f_p$ into the ODE and we require the additional constraint that $z_1'(x)f_1 + z_2'(x)f_2 = 0$ to find a concrete solution.

Therefore we get the following system of equations:
$$z_1'(x)f_1 + z_2'(x)f_2 = 0$$
$$z_1'(x)f_1' + z_2'(x)f_2' = b(x)$$

We can solve this as follows:
$$W = f_1f_2' - f_2f_1' \neq 0$$
$$\Rightarrow z_1' = \frac{-f_2b}{W}, z_2' = \frac{f_1b}{W}$$
$$\Rightarrow f_p = \left(\int \frac{-f_2b}{W}dt\right) f_1 + \left(\int \frac{f_1b}{W}dt\right) f_2$$

## 1.5 Seperation of Variables

Consider a differential equation of the form
$$y'(x) = b(x)g(y)$$
Assume $g(y(x)) \neq 0$. If $\exists y_0$ s.t. $g(y_0(x)) = 0$ then $y = y_0$ is a solution.
$$\frac{y'(x)}{g(y(x))} = b(x)$$
$$\int \frac{y'(x)}{g(y(x))}\, dx = \int b(x)\, dx$$

Applying substitution with $u = y(x)$ we obtain
$$\int \frac{1}{g(u)}\, du = \int b(x)\, dx$$
We can then determine both integrals and solve for $u = y$.

# 2 Derivations in $\mathbb{R}^n$

**Monomial in $\mathbb{R}^n$**

A Monomial of degree $e$ is a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ :
$$(x_1, \ldots, x_n) \mapsto \alpha x_1^{d_1} \cdot \ldots \cdot x_n^{d_n}$$
$$e = d_1 + \ldots + d_n$$
$\rightarrow$ i.e. a Polynomial that only has one term.

**Polynomial in $\mathbb{R}^n$**

A Polynomial with $n$ variables of degree $d$ is a finite sum of Monomials of degree $e \le d$.

## 2.1 Continuity

**Definition Continuity**

Let $f : \underline{X} \subset \mathbb{R}^n \to \mathbb{R}^m$ and $x_0 \in \underline{X}$.
$f$ is continous in $x_0$, if one of the following conditions is fulfilled:

1. $\forall \epsilon > 0\ \exists \delta > 0$ such that for all $x \in \underline{X}$
$$||x - x_0|| < \delta \implies ||f(x) - f(x_0)|| < \epsilon$$
2. $\forall$ sequences $(x_k)$ in $X$ with $\lim_{k\to\infty} x_k = x_0$ we have
$$\lim_{k\to\infty} f(x_k) = f\left(\lim_{k\to\infty} x_k\right) = f(x_0)$$

$f$ is continous on $\underline{X} \iff f$ is continous at every point $x_0 \in \underline{X}$.
In Addition we have the following:

1. Cartesian product of continous functions is continous.
2. $f : \mathbb{R}^n \mapsto \mathbb{R}^m$
$(x_1, \ldots, x_n) \mapsto (f_1(x), \ldots, f_m(x))$
is continous $\iff f_i : \mathbb{R}^n \to \mathbb{R}$ continous $\forall i = 1, \ldots, m$.
3. Linear Maps $x \mapsto Ax$ are continous.
4. Finite sums and products of continous functions are continous.
5. Functions with seperated Variables are continous if each factor is continous. (i.e. $f(x_1, ..., x_n) = f_1(x_1)f_2(x_2) \cdot \ldots \cdot f_n(x_n)$ is continous if $f_1, f_2, ..., f_n$ are continous.)
6. In particular Polynomials are continous.
7. The composition of continous functions is continous.
8. If $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is continous. For an arbitrary fixed $y_0 \in \mathbb{R}$ we can define $g_{y_0}(x) := f(x, y_0)$. Since $g_{y_0}$ is a composition of continous functions it is also continous.
9. Warning! The converse is not true. $g_{y_0}$ continous for all $y_0 \in \mathbb{R}$ does **not** imply that $f$ is continous!

**Sandwich-Lemma**

If $f, g, h : \mathbb{R}^n \to \mathbb{R}$ are functions with $f(x) < g(x) < h(x)$ $\forall x \in \mathbb{R}^n$. Let $a \in \mathbb{R}^n$:
$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \implies \lim_{x\to a} g(x) = L$$

## 2.2 Properties of sets

A set $\underline{X} \subset \mathbb{R}^n$ is

- **bounded**, if the set $\{||x|| \mid x \in \underline{X}\}$ is bounded in $\mathbb{R}$(i.e. $\exists K \ge 0, \forall x \in \underline{X} : ||x|| \le K$).
- **closed**, if every sequence $(x_k)_{k\in\mathbb{N}} \subset \underline{X}$, that converges to some Vector $y \in \mathbb{R}^n$, we have $y \in \underline{X}$ (i.e. limits of sequences in $X$ are also in $X$).
- **compact**, if its closed and bounded.
- **open** if, for any $x = (x_1, x_2, ..., x_n) \in \underline{X}$, there exists $\delta > 0$ such that the set
$$\{y = (y_1, ..., y_n) \in \mathbb{R}^n \mid |x_i - y_i| < \delta, \forall 1 \le i \le n\}$$
is contained in $\underline{X}$.
- **convex**, if $\forall x, y \in \underline{X} : \lambda x + (1 - \lambda)y \in \underline{X}, \forall 0 \le \lambda \le 1$ (the line segment between $x, y$ is contained in $\underline{X}$).
- **open**, if and only if the complement $Y = \mathbb{R}^n \setminus \underline{X}$ is **closed**. (Equivalent definition)

**Important examples:**

- $(a, b) \subset \mathbb{R}$ is open.
- $[a, b) \subset \mathbb{R}$ is neither open nor closed.
- $\mathbb{R}^n$ and $\emptyset$ are both open and closed. There exists no other set in $\mathbb{R}^n$ which is both open and closed.
- If $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ are both bounded (rsp. closed/compact) then $X \times Y \subseteq \mathbb{R}^{n+m}$ is bounded (rsp. closed/compact)
- In particular the cartesian product of compact intervals $I_i \in \mathbb{R}$: $I_1 \times I_2 \times ... \times I_n = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_i \in I_i\}$ is compact (i.e. closed and bounded).
- Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be continous. Then for every closed(/open) set $Y \subseteq \mathbb{R}^m$, the set $f^{-1}(Y)$ is closed(/open).

## 2.3 Partial Derivatives

**Partial Derivative**

To find the partial derivative of $f : \underline{X} \subset \mathbb{R}^n \to \mathbb{R}$ (whereby $\underline{X}$ open) with respect to $x_j, 1 \le j \le n$ at a point $x_0 \in \underline{X}$ we define:
$$\partial_j f(x_0) = \frac{\partial f}{\partial x_j}(x_0) = \lim_{h\to 0} \frac{f(x_0 + h \cdot e_j) - f(x_0)}{h}$$
where $e_j$ is the $j$-th canonical basis vector of $\mathbb{R}^n$.

For $f:\mathbb{R}^n \to \mathbb{R}^m, x_0 \in \mathbb{R}^n$ we have

$$\frac{\partial f(x_0)}{\partial x_j} := \begin{pmatrix} \frac{\partial}{\partial x_j} f_1(x_0) \\ \vdots \\ \frac{\partial}{\partial x_j} f_m(x_0) \end{pmatrix}$$

Partial derivatives have following properties:

1. $\partial_j(f+g) = \partial_j(f) + \partial_j g$
2. $\partial_j(f \cdot g) = \partial_j(f) \cdot g + \partial_j(g) \cdot f$
3. $\partial_j(f/g) = \frac{\partial_j(f)\cdot g - \partial_j(g)\cdot f}{g^2}$ for $g \neq 0$

**Jacobi-Matrix**

Let $f: \underline{X} \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\underline{X}$ an open set. The Jacobi-Matrix is the $m \times n$ Matrix:

$$J_f = \left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1 \leq j \leq n \\ 1 \leq i \leq m}}$$

## 2.4 Differentiability

$$\begin{array}{ccc}
\boxed{f \text{ stetig diffbar}} & \Longleftrightarrow & \boxed{\text{Alle } \partial_j f_i \text{ stetig}} \\
\Downarrow & & \Downarrow \\
\boxed{f \text{ diffbar}} & \overset{\Longrightarrow}{\not\Longleftarrow} & \boxed{\text{Alle } \partial_j f_i \text{ existieren}} \\
\Downarrow & & \\
\boxed{f \text{ stetig}} & \not\swarrow \text{ siehe Bsp. 3.3.8(4) oder 3.4.17} &
\end{array}$$

**Differentiability**

Let $\underline{X} \subset \mathbb{R}^n$ be open, $x_0 \in \underline{X}$. We have $f: \underline{X} \to \mathbb{R}^m$.
We say that $f$ is **differentiable** at $x_0$, with the differential $u$, if there exists a **linear map** $u: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{f(x) - f(x_0) - u\cdot(x-x_0)}{||x-x_0||} = 0$$

We denote $u = df(x_0) = d_{x_0} f$.

- If $f$ is differentiable at all points $x_0 \in \underline{X}$, then $f$ is differentiable on $\underline{X}$.
- Having all partial derivatives defined is not sufficient to conclude Differentiability.
- If all partial derivatives are defined and continous, then $f$ is differentiable.
- The composition of differentiable functions is differentiable.

**Conclusions from Differentiability**

If $f, g$ are differentiable in $x_0 \in \underline{X}$ we have:

1. $f$ is continous in $x_0$
2. $f$ has all partial derivatives at $x_0$ and the matrix of the linear map $df(x_0): x \mapsto Ax$ is given by the **Jacobi-Matrix** of $f$ at $x_0$, i.e. $A = J_f(x_0)$
3. $d(f+g)(x_0) = df(x_0) + dg(x_0)$
4. If $m = 1$, then $f \cdot g$ is differentiable. If additionally $g \neq 0$, then $f/g$ is also differentiable. (Product rule and Quotient rule apply)
5. (Chain rule): Let $\underline{X} \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ be open.
   If $f: \underline{X} \to Y, g: Y \to \mathbb{R}^p$ are both differentiable, we have $d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0)$. Furthermore
   $$J_{g\circ f}(x_0) = J_g(f(x_0)) \cdot J_f(x_0)$$
   Therefore
   $$d(g\circ f)(x_0): \underline{X} \to \mathbb{R}^p \qquad x \mapsto J_g(f(x_0)) \cdot J_f(x_0) \cdot x$$

**Tangent Space**

The **tangent space** at $x_0$ of $f$ is given by the graph of the affine linear map $g(x) = f(x_0) + df(x_0)(x - x_0)$.
I.e.

$$\{(x, g(x)) \in \mathbb{R}^n \times \mathbb{R}^m \mid g(x) = f(x_0) + df(x_0)(x-x_0)\}$$

For two variables we have:

$$z(x,y) = f(x_0,y_0) + \partial_x f(x_0,y_0)\cdot(x-x_0) + \partial_y f(x_0,y_0)\cdot(y-y_0)$$

**Change of variables (Bijection)**

Let $\underline{X} \subset \mathbb{R}^n$ be open and $f: \underline{X} \to \mathbb{R}^n$ differentiable. $f$ is a **change of variable** around $x_0 \in \underline{X}$ if there exists a radius $r > 0$ such that the Ball

$$B_r(x_0) := \{x \in \mathbb{R}^n \mid ||x - x_0|| < r\}$$

has the property that the Image $Y = f(B_r(x_0))$ is open and there exists a differentiable map $g: Y \to B_r(x_0)$ such that $f \circ g = g \circ f = \text{id}$.
We find that if $\det(J_f(x_0)) \neq 0$ (i.e. $J_f(x_0)$ is invertible), then $f$ is a change of variables around $x_0$. Moreover the Jacobian of the inverse map $g$ is determined by

$$J_g(f(x_0)) = J_f(x_0)^{-1}$$

(Analog to the fact that a function $h: I \subseteq \mathbb{R} \to \mathbb{R}$ is bijective from $I$ to its image if $h' > 0$ or $h' < 0$)

## 2.5 Higher derivatives

### Notation for higher partial derivatives

For a function $f: \mathbb{R}^n \to \mathbb{R}^t$ we denote higher order partial derivatives with the following:
First let $m = (m_1, m_2, ..., m_n)$ and $|m| = m_1 + m_2 + ... + m_n$.
We write

$$\frac{\partial^{|m|} f_j}{\partial x_1^{m_1} \partial x_2^{m_2} \cdot \ldots \cdot \partial x_n^{m_n}} = \partial_{x^m}^{|m|} f_j, \; 1 \leq j \leq t$$

## 2.6 Taylor polynomials

Let $k \leq 1$ and $f: \underline{X} \mapsto \mathbb{R}$ be a funciton of class $C^k$ on $\underline{X}$, and fix $x_0 \in \underline{X}$. The $k$-th Taylor polynomial of $f$ at the point $x_0$ is the polynomial in $n$ variables of degree $\leq k$ given by

$$T_k f(y; x_0) = f(x_0) + \sum_{i=1}^{n} \partial_i f(x_0) \cdot y_i + \ldots$$

$$+ \sum_{m_1+\cdots+m_n=k} \frac{1}{m_1! \cdots m_n!} \frac{\partial^k f(x_0)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} \cdot y_1^{m_1} \cdots y_n^{m_n}$$

Using our new notation for higher-order derivatives we can denote the Taylor polynomial by

$$T_k(y; x_0) = \sum_{|m| \leq k} \frac{1}{m!} \partial_{x^m}^{|m|} f(x_0) \cdot y^m$$

with $m! = m_1! m_2! \cdot \ldots \cdot m_n!$

### Examples

$$T_1 f(\vec{x}; x_0) := f(x_0) + (\nabla f(x_0))^{\mathrm{T}} \cdot \vec{x}$$

$$T_2 f(\vec{x}; x_0) := T_1 f + \frac{1}{2} \cdot \vec{x}^{\top} \cdot \text{Hess}_f(x_0) \cdot \vec{x}$$

## 2.7 Extrema

### Local Extrema

Let $f: \underline{X} \subset \mathbb{R}^n \mapsto \mathbb{R}$ be differentiable and $\underline{X}$ open.
Then $x_0 \in \underline{X}$ is a **local Maximum (Minimum)** if there exists an $r > 0, r \in \mathbb{R}$ and $B_{x_0}(r) = \{x \in \mathbb{R}^n \mid ||x - x_0|| < r\} \subset \underline{X}$ such that:

$$\forall x \in B_{x_0}(r): f(x) \leq (\geq) f(x_0)$$

If $x_0 \in \underline{X}$ is a local extrema, we additionally have $\nabla f(x_0) = 0$.

### Critical point

A point $x_0 \in \underline{X}$ with $\nabla f(x_0) = 0$ is a **critical point**.
Critical points are candidates for local extrema.
If additionally $\det(\text{Hess}_f(x_0)) \neq 0$, then $x_0$ is a **non-degenerate** critical point.

### Saddle point

If a critical point is neither a maximum nor a minimum, we call it a **saddle point**.

### Global Extrema
Let $f : K \mapsto \mathbb{R}$ and $K$ compact, then a global extrema of $f$ exists and is either at a point $x_0$ in the interior of $K$ or on the boundary of $K$. To determine such an global extrema we split $K$ into it's interior $\underline{X}$ and the boundary $B$.
First we determine the critical points of $\underline{X}$. To determine the Maximas/Minimas $B$, we will need Knowledge from Analysis I (redefine the boundary as a union of sets dependend on 1 variable, i.e. Line-segments).

### Testing critical points
Let $f : \underline{X} \subseteq \mathbb{R}^n \mapsto \mathbb{R}, \underline{X}$ open and $f \in C^2$. Let $x_0$ be a **non-degenerate critical point** of $f$. Then we:

1. $\text{Hess}_f(x_0)$ pos. def. $\implies$ $x_0$ is a local Minimum.
2. $\text{Hess}_f(x_0)$ neg. def. $\implies$ $x_0$ is a local Maximum.
3. $\text{Hess}_f(x_0)$ indefinite $\implies$ $x_0$ is a saddle point.

If $x_0$ is a **degenerate critical point**, we can't conclude anything in general. In such a case we would have to verify the signs in the neighborhood of $x_0$. (Not much information found on how to do that in multi-variable calculus)

## 2.8 Definite
A symmetric (non-singular) matrix $A$, $\det A \neq 0$ is

- **positive definite** $\iff$ all Eigenvalues are positive $\iff$ all principal minors of $A$ are positive
- **negativ definite** $\iff$ all E.V. are negative $\iff$ $-A$ is positive definite.
- **indefinite** if it has positive and negative Eigenvalues.

Eigenvalues can be found with the characterstic polynomial:

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$
$$\Rightarrow ad - (a+d)\lambda + \lambda^2 - bc = 0$$

For non-symmetric Matrices we have to test for all Vectors $v$, if $v^\top A v > 0$ (rsp. $< 0$).

### 3D Determinant

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\cdot e\cdot i + b\cdot f\cdot g + c\cdot d\cdot h - g\cdot e\cdot c - h\cdot f\cdot a - i\cdot d\cdot b$$

**Principal Minor**

The $k$-th leading principal minor of $A$ is given by

$$M_k = \det\left((A)_{1:k,1:k}\right)$$

# 3 Integrals in $\mathbb{R}^n$

## 3.1 Simple Integrals
For $f : \mathbb{R} \mapsto \mathbb{R}^n$ we define the integral of $f$ as

$$\int_a^b f(t)dt = \begin{pmatrix} \int_a^b f_1(t)dt \\ \vdots \\ \int_a^b f_n(t)dt \end{pmatrix}$$

## 3.2 Line Integrals (Path Integrals)

**A parameterized curve**

A parameterized curve in $\mathbb{R}^n$ is a continous map $\gamma : [a,b] \mapsto \mathbb{R}^n$ that is piecewise in $C^1$, i.e. $\exists k > 1, k \in \mathbb{N}$ and a partition $a = t_0 < t_1 < \ldots < t_k = b$, such that

$$\gamma|_{[t_{i-1},t_i]} \in C^1, \forall 1 \leq i \leq k.$$

A parameterized curve does not have to be injective.

**Useful trick:**
In general if $\gamma : [a,b] \to \mathbb{R}^n (t \mapsto \gamma(t))$ is a curve, then $\alpha : [a,b] \to \mathbb{R}^n$ with $\alpha(t) := \gamma(b+a-t)$ traces the same curve in the opposite direction.

### Length of a parameterized curve
Let $\gamma : [a,b] \to \mathbb{R}^n$ be an **injective** and $\gamma \in C^1$. The length of the curve $\gamma(t)$ can be found by

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt,$$

where $|.|$ is the Euclidean norm.

**Line(Path) Integrals**

Let $\gamma : [a,b] \mapsto \mathbb{R}^n$ be a parameterized curve and $\underline{X} \subset \mathbb{R}^n$ a set containing the image of $\gamma$, and let $f : \underline{X} \to \mathbb{R}^n$ be a continous function. The **Line Integral (Path Integral)** is defined as

$$\int_\gamma f(s)\, ds = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

For Notation $s$ represents $\gamma(t)$ and $ds$ represents $\gamma'(t)\, dt$.

Line Integrals have following properties:

1. It is independent of orientation preserving reparameterization of the curve:
$$\begin{aligned} &\gamma : [a,\ b] \mapsto \mathbb{R}^n \\ &\sigma : [c,\ d] \mapsto [a,\ b],\ \sigma'(t) > 0\ \forall t \in (c,\ d) \\ &\tilde{\gamma} : [c,\ d] \mapsto \mathbb{R}^n \\ &\tilde{\gamma} = \gamma \circ \sigma = \gamma(\sigma) \\ &\Rightarrow \int_\gamma f(s)\ ds = \int_{\tilde{\gamma}} f(s)\ ds \end{aligned}$$
2. We have $\gamma_1 : [a,\ b] \to \underline{X} \subset \mathbb{R}^n, \gamma_2 : [c,\ d] \to \underline{X}$ with $\gamma_1(b) = \gamma_2(c)$. We can now concatenate these 2 curves to $\gamma_1 + \gamma_2$
$$\gamma_1 + \gamma_2 := \begin{cases} \gamma_1(t) & t \in [a,\ b] \\ \gamma_2(t-b+c) & t \in [b,\ b+(d-c)] \end{cases}$$
$$\int_{\gamma_1+\gamma_2} f(s)\ ds = \int_{\gamma_1} f(s)\ ds + \int_{\gamma_2} f(s)\ ds$$
3. Let $\gamma : [a,b] \mapsto \mathbb{R}^n$ be a path and $-\gamma : [a,b] \to \mathbb{R}^n$ the same path in the opposite direction (i.e. $(-\gamma)(t) = \gamma(a+b-t)$). Then we have
$$\int_{-\gamma} f(s)ds = -\int_\gamma f(s)ds$$

## 3.3 Potential
A differentiable function $g : \underline{X} \subset \mathbb{R}^n \mapsto \mathbb{R}$ with $\nabla g = f, f : \underline{X} \mapsto \mathbb{R}^n$ is called a **potential** for $f$. This can be used as follows:

$$\begin{aligned} \int_\gamma f\ ds &= \int_a^b f(\gamma(t)) \cdot \gamma'(t)\ dt \\ &= \int_a^b \nabla g(\gamma(t)) \cdot \gamma'(t)\ dt \\ &= \int_a^b \frac{d}{dt}(g \circ \gamma)\ dt \\ &= (g\circ\gamma)(b) - (g\circ\gamma)(a) \end{aligned}$$

For a function $f : \mathbb{R}^n \to \mathbb{R}^n$ there is not always a potential $g$! And continuity of $f$ is not sufficient for the existence of a potential for $f$. (Counterexample: $f(x,y) = (2xy^2, 2x)$)

## 3.4 Conservative Vector fields

$$\begin{tikzcd} f = \nabla g \arrow[r, Leftrightarrow] & f \text{ konservativ} \arrow[r, Leftrightarrow] \arrow[d, shift left, Rightarrow] & \oint_\gamma f(s)\cdot d\vec{s} = 0 \\ & J_f \text{ symmetrisch} \arrow[u, shift left, Rightarrow, "\text{falls } U \text{ sternförmig}"] \arrow[r, Leftrightarrow, "n=3"'] & \text{curl } f = 0 \end{tikzcd}$$

**Conservative Vector fields**

$f : \underline{X} \to \mathbb{R}^n$ continous Vector field. If for any $x_1, x_2 \in \underline{X}$ the line integrals $\int_\gamma f\, ds$ for any curve between $x_1, x_2$ are equal, $f$ is called **conservative**.

Let $\underline{X}$ be open and a path-connected subset of $\mathbb{R}^n$. Let $f : \underline{X} \subset \mathbb{R}^n \mapsto \mathbb{R}^n$ be a continous vector field. The following are equivalent:

1. $f$ is the gradient of a function $g : \underline{X} \to \mathbb{R}$, i.e. $f = \nabla g$.
2. The line integral of $f$ is independent of the path between any 2 points.
3. The line integral of $f$ along any closed path is always 0. (A closed path $\gamma : [a,b] \to \underline{X}$ fulfills $\gamma(a) = \gamma(b)$)

We additionally have this necessary but not sufficient condition:

$$f \text{ is conservative} \implies \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad \forall i,\ j$$

**Path-connected set**

Let $\underline{X} \subset \mathbb{R}^n$ be open. $\underline{X}$ is path-connected, if for every pair of points $x, y \in \underline{X}$ there exists a parameterized curve $\gamma : [0,1] \mapsto \underline{X}$, such that $\gamma(0) = x, \gamma(1) = y$.

**Starshaped set**

A subset $\underline{X} \subset \mathbb{R}^n$ is starshaped if there $\exists x_0 \in \underline{X}$ such that, $\forall x \in \underline{X}$ the line segment joining $x_0$ to $x$ is contained in $\underline{X}$.

$$\underline{X} \text{ convex} \implies \underline{X} \text{ starshaped}$$

If $\underline{X}$ is a starshaped, open subset of $\mathbb{R}^n$ and $f \in C^1$ a vector field, we have:

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad \forall i, j \quad \Rightarrow \quad f \text{ is conservative}$$

For $f : \underline{X} \subset \mathbb{R}^3 \to \mathbb{R}^3, f \in C^1$ we also have:

$$\text{curl}(f) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad f \text{ is conservative}$$

(As above only for $f : \underline{X} \subset \mathbb{R}^3 \to \mathbb{R}^3$)
curl($f$) is defined as

$$\text{curl}(f) := \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

Divergence of a vector Field is defined as:

$$\text{div} F = \nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

From series we know that:

$$\text{div}(\text{curl}(F)) = 0$$

## 3.5 Riemann Integral in $\mathbb{R}^n$

**Cuboid / box in $\mathbb{R}^n$**

A **cuboid** or box $Q \subset \mathbb{R}^n$ is a set of the form

$$Q = [a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n], \quad a_k, b_k \in \mathbb{R}$$

The **volume function** vol($\cdot$) assigns to each cuboid a real number by the rule

$$\text{vol}(Q) = \prod_{k=1}^{n} (b_k - a_k) = \mu(Q).$$

For a given cuboid $Q$ we call $\mathcal{P} = \{Q_1, Q_2, ..., Q_m\}$ a **partition** of $Q$ if the $Q_k$ are cuboids such that

$$1. Q = \bigcup_{k=1}^{m} Q_k$$

$$2. \forall\ 1 \leq k, l \leq m : \text{Int}(Q_k) \cap \text{Int}(Q_l) = \emptyset$$

Let $f : Q \subset \mathbb{R}^n \to \mathbb{R}$ for some cuboid $Q$. Let $\mathcal{P} = \{Q_1, ..., Q_m\}$ be a partition of $Q$. We define the upper resp. lower sum of $f$ with respect to $\mathcal{P}$ by

$$U(f, \mathcal{P}) = \sum_{k=1}^{m} \sup_{x \in Q_k} f(x) \cdot \text{vol}(Q_k),$$

$$L(f, \mathcal{P}) = \sum_{k=1}^{m} \inf_{x \in Q_k} f(x) \cdot \text{vol}(Q_k).$$

Accordingly we define the upper resp. lower sum of $f$ by

$$\underline{I}(f) = \inf\{U(f, \mathcal{P}) \mid \mathcal{P} \text{ is a partition of } Q\}$$

$$\overline{I}(f) = \sup\{L(f, \mathcal{P}) \mid \mathcal{P} \text{ is a partition of } Q\}$$

If the lower and upper sum of $f$ are equal we say that $f$ is Riemann integrable and write

$$\int_Q f\,dx = \underline{I}(f) = \overline{I}(f).$$

### 3.5.1 General sets

Let $\underline{X} \subset \mathbb{R}^n$ be some bounded set and let $f : \underline{X} \to \mathbb{R}$. We can define an indicator function by

$$\mathbb{1}_{\underline{X}}(x) = \begin{cases} 1, & x \in \underline{X} \\ 0, & \text{else} \end{cases}, (\mathbb{1}_{\underline{X}} f)(x) \begin{cases} f(x), & x \in \underline{X} \\ 0, & \text{else} \end{cases}.$$

Now given some cuboid $Q$, s.t. $\underline{X} \subseteq Q$ we define

$$\int_{\underline{X}} f\,dx = \int_Q (\mathbb{1}_{\underline{X}} f)(x)\,dx$$

provided the latter integral exists.
Note that in general $(\mathbb{1}_{\underline{X}} f)(x)$ is not continous on $Q$. But if $\underline{X}$ is *Jordan-measurable*(i.e. $\mathbb{1}_{\underline{X}}(x)$ is integrable) and $f$ is continous on $\underline{X}$, then one can prove that $(\mathbb{1}_{\underline{X}} f)(x)$ is integrable on $Q$.

### 3.5.2 Properties of the Integral

Let $f, g : Q \subset \mathbb{R}^n \to \mathbb{R}$:

1. If $f$ is continous and bounded on $Q$, then $f$ is integrable.
2. If $f, g$ integrable, $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is integrable:
$$\int_Q \alpha f + \beta g\,dx = \alpha \int_Q f\,dx + \beta \int_Q g\,dx$$
3. If $\forall x \in Q : f(x) \leq g(x)$, then:
$$\int_Q f(x)\,dx \leq \int_Q g(x)\,dx$$
4. If $f(x) \geq 0$, then: $\int_Q f(x)\,dx \geq 0$
5. Triangle inequality:
$$\left| \int_Q f(x)\,dx \right| \leq \int_Q |f(x)|\,dx \leq (\sup_Q |f|)(\text{vol} Q)$$

**Fubini's Theorem**

Let $f : \underline{X} \subset \mathbb{R}^n \to \mathbb{R}$ and $n = n_1 + n_2$, $n_1, n_2 \geq 1$
For $x \in \mathbb{R}^n$ write $x = (x_1, x_2)$ with $x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$.
We define

$$\underline{X}_{x_1} := \{x_2 \in \mathbb{R}^{n_2} \mid (x_1, x_2) \in \underline{X}\} \qquad \subset \mathbb{R}^{n_2}$$

$$\underline{X}_1 := \{x_1 \in \mathbb{R}^{n_1} \mid \underline{X}_{x_1} \neq \emptyset\} \qquad \subset \mathbb{R}^{n_1}$$

If $g(x_1) := \int_{\underline{X}_{x_1}} f((x_1, x_2))\,dx_2$ is continous on $\underline{X}_1$ then

$$\int_{\underline{X}} f(x)\,dx = \int_{\underline{X}_1} g(x_1)\,dx_1 = \int_{\underline{X}_1} \left( \int_{\underline{X}_{x_1}} f((x_1, x_2))\,dx_2 \right) dx_1$$

More concretely for a cuboid $Q = [a_1, b_1] \times ... \times [a_n, b_n]$ and $f : Q \to \mathbb{R}$ integrable we have:

$$\int_Q f\,dx = \left( \int_{a_1}^{b_1} \left( \int_{a_2}^{b_2} ... \left( \int_{a_n}^{b_n} f(x)\,dx_n \right) ... \right) dx_1 \right)$$

For a general set
$A := \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ g(x) \leq y \leq h(x)\} \subset \mathbb{R}^2$:

$$\int_A f\,dx\,dy = \int_a^b \left( \int_{g(x)}^{h(x)} f(x, y)\,dy \right) dx$$

### Remark

Note that $g(x_1)$ might not be continous on $\underline{X}_1$.
E.g. $\underline{X} = [0, 1] \times [0, 2] \cup [1, 2] \times [0, 1]$ and $f = 1$.
$\int_{\underline{X}} 1\,dx\,dy$ exists.

We then have $\underline{X}_1 = [0, 2]$ and $\underline{X}_x = \begin{cases} [0, 2] & x \leq 1 \\ [0, 1] & 1 \leq x \leq 2 \end{cases}$

$$g(x) = \int_{\underline{X}_x} 1\,dy = \begin{cases} 2 & x \leq 1 \\ 1 & 1 \leq x \leq 2 \end{cases}$$

is not continous.
We can still solve this problem by seperating $\underline{X}$ into $A = [0, 1] \times [0, 2]$ and $B = [1, 2] \times [0, 1]$. Note that $A \cap B \neq \emptyset$, i.e. there's a segment we integrate twice. But this is negligible in a similar fashion to the 1D case when we split an integral.

### 3.5.3 Negligible Sets

1. Let $1 \leq m \leq n$ be an integer. A **parameterized** $m$**-set** in $\mathbb{R}^n$ is a continuous map $f : [a_1, b_1] \times ... \times [a_m, b_m] \to \mathbb{R}^n$ which is in $C^1$ on $]a_1, b_1[\times ... \times]a_m, b_m[$.
2. A subset $Y \subset \mathbb{R}^n$ is **negligible** if there $\exists k >= 1$ with parameterized $m_i$-sets $f_i : X_i \to \mathbb{R}^n$, with $1 \leq i \leq k$ and $m_i < n$, such that
$$Y \subset \bigcup_{i=1}^{k} f(X_i)$$
3. (Integral on negligible sets) For $X \subset \mathbb{R}^n$ compact. $X$ negligible. For any continous function on $X$ we have $\int_X f(x)\,dx = 0$.
4. (Domain additivity) If $X = A_1 \cup A_2$, $A_1, A_2$ compact then for $f : X \to \mathbb{R}$:
$$\int_X f\,dx = \int_{A_1} f\,dx + \int_{A_2} f\,dx - \int_{A_1 \cap A_2} f\,dx$$
Of course if $A_1 \cap A_2$ is negligible we can disregard that integral.

## 3.6 Change of Variables

Let $\underline{X}, Y$ compact, $f : Y \subset \mathbb{R}^n \to \mathbb{R}$ be continous.
Suppose we have $\gamma : \underline{X} \to Y$ where $\underline{X} = \underline{X}_0 \cup B, Y = Y_0 \cup C$ ($B, C$ boundary of $\underline{X}, Y$ resp.).
Suppose $\gamma : \underline{X}_0 \to Y_0$ is $C^1$ bijective and $\det(J_\gamma(x)) \neq 0, \forall x \in \underline{X}_0$.
Then we have the following:

$$\int_Y f(y)\,dy = \int_{\underline{X}} f(\gamma(x)) |\det J_\gamma(x)|\,dx$$

1. Polar coordinates:
$$\gamma(r,\theta) = (r\cos(\theta), r\sin(\theta))$$
with $dx\,dy = r\,dr\,d\theta$
2. Ellipse with: $(x,y) = (a \cdot r \cdot cos(\theta), b \cdot r \cdot sin(\theta))$ with $dx\,dy = a \cdot b \cdot r\,dr\,d\theta$
3. Cylindrical coordinates:
$$\gamma(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z)$$
with $dx\,dy\,dz = r\,dr\,d\theta\,dz$
4. Spherical coordinates:
$$a \leq x^2, y^2, z^2 \leq b$$
$$\gamma(r,\theta,\varphi) = (r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi))$$
with $dx\,dy\,dz = \int_{\sqrt{a}}^{\sqrt{b}} \int_o^{2\pi} \int_o^{\pi} r^2 \sin(\varphi)\,d\varphi\,d\theta\,dr$

**Don't forget the determinant of the Jacobi-Matrix!**

## 3.7 Green's theorem

Green's theorem only concerns itself with functions from a 2-dimensional to a 2-dimensional space.

**Green's theorem**

Let $\overline{X} \subset \mathbb{R}^2$ be compact with a boundary $\partial\overline{X} = \bigcup_{i=1}^n \gamma_i$ that is the union of finitely many simple closed parameterized curves. Assume that
$$\gamma_i : [a_i, b_i] \to \mathbb{R}^2$$
has the property that $\overline{X}$ lies always "to the left" of the tangent vector $\gamma_i'(t)$ based at $\gamma_i(t)$.
Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ with $f = (f_1, f_2)$ in $C^1$ and $\overline{X} \subset U$. Then we have
$$\int_{\partial\overline{X}} f(x)\,ds = \int\int_{\overline{X}} \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\,dx\,dy$$

Note that from $f$ in $C^1$ it follows, that $\mathrm{curl}(f) = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}$ is integrable.
For any curve $\gamma : [a,b] \to \mathbb{R}^2$

- **simple** means that there exists no $s, t \in ]a,b[, s \neq t$ such that $\gamma(s) = \gamma(t)$.
- **closed** means: $\gamma(a) = \gamma(b)$.

**To calculate the surface of $\overline{X}$ with Green's Theorem we use a vector field $f$ with curl$(f) = 1$. For instance:**
$$f = (0, x) \textbf{ or } f = (-y, 0)$$

### Parametrization of common planes

- General Ellipse Equation around $(x_0, y_0)$:
$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$
can be parameterized by
$$f(t) = (x_0 + a \cdot \cos(2\pi t), y_0 + b \cdot \sin(2\pi t)), t \in [0, 2\pi]$$
- Any curve given by an explicit equation
$$y = f(x), f : [a,b] \to \mathbb{R},$$
can be parameterized trivially by $(x,y) = (t, f(t))$ for some $t \in [a,b]$.

## 3.8 Center of Gravity

The center of gravity $\overline{x}$ of a compact set $U$ can be calculated as follows:
$$\overline{x}_i = \frac{1}{\mathrm{vol}(U)} \int_U x_i dx$$
The Volume of a sphere:
$$V = \frac{4}{3} \cdot \pi \cdot r^3$$

# 4 Topics from Analysis I

**Partial Integration**

$$\int f'(x)g(x)\ \mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\ \mathrm{d}x$$

- In general: Choose to derive Polynomials (as $g(x)$), for periodic functions ($\sin, \cos, e^x$,...) choose to integrate (as $f'(x)$)
- It can be necessary to multiply with 1, to be able to use partial Integration (e.g. for $\int \log(x)\ \mathrm{d}x$)
- There exist combinations, where partial integration will always circle back to the original function (e.g. $\int e^x \cos(x)\,dx$). In such cases treat the integral as an unknown and solve for it (indirect computation of the integral).

**Substitution**

To calculate $\int_a^b f(g(x))\ \mathrm{d}x$: Replace $g(x)$ by $u$ and integrate $\int_{g(a)}^{g(b)} f(u) \frac{du}{g'(x)}$.

- $g'(x)$ has to be elimnated otherwise useless.
- Don't forget to change the boundaries of the integral.
- Alternatively one could compute the improper integral and then resubstitute $u$ by $g(x)$.
- One can also use the theorem in the other direction. In essence $\int_a^b f(u)\,du = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(x))g'(x)\,dx$.

**Partial fraction decomposition**

Let $p(x), q(x)$ be 2 Polynomials. $\int \frac{p(x)}{q(x)}$ can be computed as follows:

1. If $\deg(p) \geq \deg(q)$, we do a Polynomdivision. This leads to the Integral $\int a(x) + \frac{r(x)}{q(x)}$.
2. Find the roots of $q(x)$.
3. Per root: Create one partial fraction.
   - non-repeating, real: $x_1 \to \frac{A}{x-x_1}$
   - multiplicity $n$, real: $x_1 \to \frac{A_1}{x-x_1} + \ldots + \frac{A_r}{(x-x_1)^r}$
   - non-repeating, complex: $x^2 + px + q \to \frac{Ax+B}{x^2+px+q}$
   - multiplicity $n$, complex: $x^2+px+q \to \frac{A_1x+b_1}{x^2+px+q} + \ldots$
4. Determine the parameters $A_1, \ldots, A_n$ (rsp. $B_1, \ldots, B_n$). (Multiply both sides of the equation with $q(x)$ and then solve for the coefficients. Due to the powers of $x$, you will have $n$ equations for $n$ unkown parameters).

# 5 Trigonometric identities

### Doubled angles

- $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
- $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 1 - 2\sin^2(\alpha)$
- $\tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$

### Addition

- $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$

### Subtraction

- $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
- $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
- $\tan(\alpha-\beta) = \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$

### Multiplication

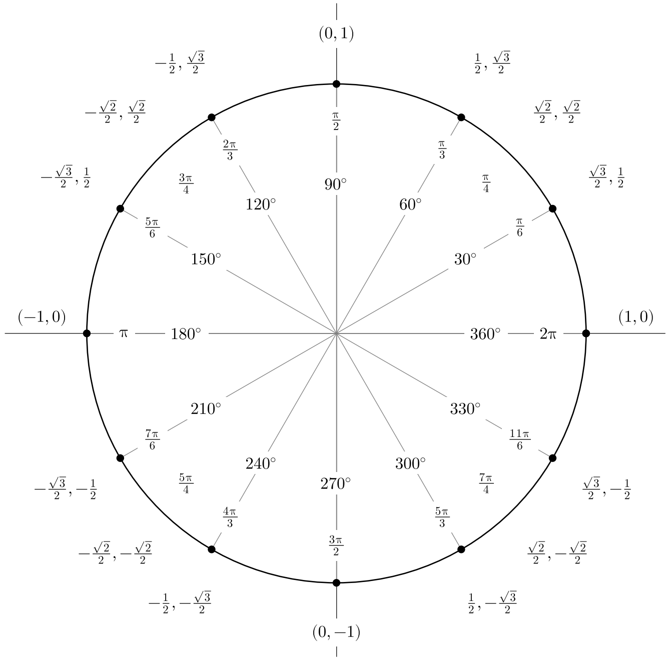
- $\sin(\alpha)\sin(\beta) = -\frac{\cos(\alpha+\beta)-\cos(\alpha-\beta)}{2}$
- $\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$
- $\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$

### Powers

- $\sin^2(\alpha) = \frac{1}{2}(1-\cos(2\alpha))$
- $\sin^3(\alpha) = (3\sin(\alpha) - \sin(3\alpha))/4$
- $\cos^2(\alpha) = \frac{1}{2}(1+\cos(2\alpha))$
- $\cos^3(\alpha) = (3\cos(\alpha) - \cos(3\alpha))/4$
- $\tan^2(\alpha) = \frac{1-\cos(2\alpha)}{1+\cos(2\alpha)}$
- $\sin^2(\alpha)\cos^2(\alpha) = (1-\cos(4\alpha))/8$

### Divers

- $\sin^2(\alpha) + \cos^2(\alpha) = 1$
- $\cosh^2(\alpha) - \sinh^2(\alpha) = 1$
- $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$ und $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$




### Taylor expansions 1D at $0$:

- $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$
- $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $e^{-x} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^n}{n!}$
- $\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$
- $\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$
- $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+1}}{n+1}$
- $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$
- $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

# 6 Tables

### Derivations

| $\mathbf{F(x)}$ | $\mathbf{f(x)}$ | $\mathbf{f'(x)}$ |
|---|---|---|
| $\frac{x^{-a+1}}{-a+1}$ | $\frac{1}{x^a}$ | $\frac{a}{x^{a+1}}$ |
| $\frac{x^{a+1}}{a+1}$ | $x^a \ (a \neq 1)$ | $a \cdot x^{a-1}$ |
| $\frac{1}{k\ln(a)} a^{kx}$ | $a^{kx}$ | $ka^{kx}\ln(a)$ |
| $\ln\|x\|$ | $\frac{1}{x}$ | $-\frac{1}{x^2}$ |
| $\frac{2}{3}x^{3/2}$ | $\sqrt{x}$ | $\frac{1}{2\sqrt{x}}$ |
| $-\cos(x)$ | $\sin(x)$ | $\cos(x)$ |
| $\sin(x)$ | $\cos(x)$ | $-\sin(x)$ |
| $\frac{1}{2}(x - \frac{1}{2}\sin(2x))$ | $\sin^2(x)$ | $2\sin(x)\cos(x)$ |
| $\frac{1}{2}(x + \frac{1}{2}\sin(2x))$ | $\cos^2(x)$ | $-2\sin(x)\cos(x)$ |
| $-\ln\|\cos(x)\|$ | $\tan(x)$ | $\frac{1}{\cos^2(x)}$ <br> $1 + \tan^2(x)$ |
| $\cosh(x)$ | $\sinh(x)$ | $\cosh(x)$ |
| $\log(\cosh(x))$ | $\tanh(x)$ | $\frac{1}{\cosh^2(x)}$ |
| $\ln\|\sin(x)\|$ | $\cot(x)$ | $-\frac{1}{\sin^2(x)}$ |
| $\frac{1}{c} \cdot e^{cx}$ | $e^{cx}$ | $c \cdot e^{cx}$ |
| $x(\ln\|x\| - 1)$ | $\ln\|x\|$ | $\frac{1}{x}$ |
| $\frac{1}{2}(\ln(x))^2$ | $\frac{\ln(x)}{x}$ | $\frac{1-\ln(x)}{x^2}$ |
| $\frac{x}{\ln(a)}(\ln\|x\| - 1)$ | $\log_a\|x\|$ | $\frac{1}{\ln(a)x}$ |

### Further derivations

| $\mathbf{F(x)}$ | $\mathbf{f(x)}$ |
|---|---|
| $\arcsin(x)$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\arccos(x)$ | $\frac{-1}{\sqrt{1-x^2}}$ |
| $\arctan(x)$ | $\frac{1}{1+x^2}$ |
| $x^x \ (x > 0)$ | $x^x \cdot (1 + \ln x)$ |

### Integrals

| $\mathbf{f(x)}$ | $\mathbf{F(x)}$ |
|---|---|
| $\int f'(x)f(x) \ \mathrm{d}x$ | $\frac{1}{2}(f(x))^2$ |
| $\int \frac{f'(x)}{f(x)} \ \mathrm{d}x$ | $\ln\|f(x)\|$ |
| $\int_{-\infty}^{\infty} e^{-x^2} \ \mathrm{d}x$ | $\sqrt{\pi}$ |
| $\int (ax+b)^n \ \mathrm{d}x$ | $\frac{1}{a(n+1)}(ax+b)^{n+1}$ |
| $\int x(ax+b)^n \ \mathrm{d}x$ | $\frac{(ax+b)^{n+2}}{(n+2)a^2} - \frac{b(ax+b)^{n+1}}{(n+1)a^2}$ |
| $\int (ax^p+b)^n x^{p-1} \ \mathrm{d}x$ | $\frac{(ax^p+b)^{n+1}}{ap(n+1)}$ |
| $\int (ax^p+b)^{-1} x^{p-1} \ \mathrm{d}x$ | $\frac{1}{ap}\ln\|ax^p+b\|$ |
| $\int \frac{ax+b}{cx+d} \ \mathrm{d}x$ | $\frac{ax}{c} - \frac{ad-bc}{c^2}\ln\|cx+d\|$ |
| $\int \frac{1}{x^2+a^2} \ \mathrm{d}x$ | $\frac{1}{a}\arctan\frac{x}{a}$ |
| $\int \frac{1}{x^2-a^2} \ \mathrm{d}x$ | $\frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\|$ |
| $\int \sqrt{a^2+x^2} \ \mathrm{d}x$ | $\frac{x}{2}f(x) + \frac{a^2}{2}\ln(x + f(x))$ |

# 7 Sample tasks

Some tasks and Multiple Choice from the homework and previous exams.
Solutions are either my own, solutions published by the course (homeworks, offical exam solutions) or solutions published in the VIS comsol. Many thanks to all contributors!

## 7.1 Multiple Choice

Let $F(x,y) = (F_1, F_2)(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ and let $\gamma$ be the standard parametrization of the unit circle centered at the origin and oriented counter-clockwise. Since $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$, by Green's theorem it holds that $\int_\gamma F \cdot ds = 0$.
(A) true
**(B) false**
Not all partial derivatives are continous, i.e. $F \notin C^1$, which is a necessary condition for Green's theorem.

---

The following sets are:

- $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 2022\}$
  **bounded**, closed, compact
- $\{(x,y,0) \in \mathbb{R}^3 \mid (x,y) \in \mathbb{R}'2\}$
  bounded, **closed**, compact
- $\mathbb{N} \times \mathbb{N} \subseteq \mathbb{R}^2$
  bounded, **closed**, compact
- $B = \{(\sin(1/t), \cos(1/t)) \in \mathbb{R}^2 \mid t \in ]0, 1/2\pi[\}$
  **bounded, closed, compact**

We find that

$$\{(\sin(t), \cos(t)) \in \mathbb{R}^2 \mid t \in [2\pi, 4\pi]\} = B$$

It's easy to see that, this newly defined set is closed and bounded, which therefore implies that $B$ is closed and bounded (and therefore compact).

- $[-1,1]^2 \subseteq \mathbb{R}^2$
  **bounded, closed, compact**
- $\{(x, y, x^2 - y^2) \in \mathbb{R}^3 \mid (x, y) \in [0,1]^2\}$
  **bounded, closed, compact**

---

If $f$ is a vector field of class $C^1$ on $\mathbb{R}^2 - \{0\}$ and for all closed curves $\gamma : [0,1] \to \mathbb{R}^2 - \{0\}$, then $f$ is conservative.
**(A) true**
(B) false
Proof by contradiction: Assume $f$ is a non-conservative vector field of class $C'$ on $\mathbb{R} - \{0\}$ and $\int_\gamma f \cdot ds = 0$ for all closed $\gamma : [0,1] \to \mathbb{R}^2 - \{0\}$. Since $f$ is non-conservative, there exists parameterized curve $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^2 - \{0\}$ from $a$ to $b$, where $a, b \in \mathbb{R}^2$, such that:

$$\int_{\gamma_1} f \cdot ds \neq \int_{\gamma_2} f \cdot ds$$

We let $\gamma'$ be the closed curve formed by $\gamma_1$ and the inversed curve of $\gamma_2$:

$$\gamma' = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le 1/2 \\ \gamma_2(2(1-t)) & \text{if } 1/2 \le t \le 1 \end{cases}$$

We have:

$$\int_{\gamma'} f \cdot ds = \int_{\gamma_1} f \cdot ds - \int_{\gamma_2} f \cdot ds \neq 0$$

This contradicts our assumption, $f$ must be conservative.

---

The function $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x, y) = |xy|$ is differentiable at $(0,0)$.
**(A) true**
(B) false
Since $f(x, y)$ is constant, if we approach $(0,0)$ along the $x$ or $y$-Axis, we know that the differential at $(0,0)$ should be $J_f = (0,0)$, if it exists.
We now use the definition of Differentiability(section 2.5), with $u = J_f(0,0)$:

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\neq(0,0)}} \frac{f(x,y) - f(0,0) - (0,0) \cdot ((x,y) - (0,0))}{||(x,y) - (0,0)||}$$
$$= \lim_{\substack{x\to(0,0)\\x\neq(0,0)}} \frac{f(x,y)}{||(x,y)||}$$
$$= \lim_{\substack{x\to(0,0)\\x\neq(0,0)}} \frac{|xy|}{\sqrt{x^2+y^2}}$$

Since $\frac{|y|}{\sqrt{x^2+y^2}} \le 1$, we have $0 \le \frac{|xy|}{\sqrt{x^2+y^2}} \le |x|$. By Sandwich-Theorem therefore since $|x| \to 0$ for $(x, y) \to (0,0)$, we have

$$\lim_{\substack{x\to(0,0)\\x\neq(0,0)}} \frac{|xy|}{\sqrt{x^2+y^2}} = 0.$$

---

The limit $\lim_{(x,y)\to(0,0)} \cos(\frac{x^2}{x^2+|y|})$ exists and is finite.
(A) true
**(B) false**
Observe that for $y = x^2$, we find the limit: $\lim_{x\to 0} \cos\left(\frac{x^2}{2x^2}\right) = \cos\left(\frac{1}{2}\right)$, while if $y = 0$, we have: $\lim_{x\to 0} \cos\left(\frac{x^2}{x^2}\right) = \cos(1)$.

---

Which of the following is the tangent plane of the ellipsoid

$$2x^2 + 2y^2 + \frac{1}{4}z^2 = 1,$$

which is parallel to the plane $x + y + z = 1$?

☐ $x + y + z = 0$

☐ $x + y + z = k$ for $k \in \{\pm\frac{2}{\sqrt{5}}\}$

☑ $x + y + z = k$ for $k \in \{\pm\sqrt{5}\}$

☐ $x + y + z = k$ for $k \in \{\pm 1\}$

We can describe the points on the Ellipsoid, as a level set of $f(x, y, z) = 2x^2 + 2y^2 + \frac{z^2}{4}$. I.e. the points on the Ellipsoid are $L = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 1\}$.
Since for a level set, the gradient is always the normal vector, we can look for a point, where the gradient is parallel to the normal vector of the plane $(1,1,1)$.
Thus we have

$$\nabla f(x, y, z) = \left(4x, 4y, \frac{z}{2}\right) = a(1,1,1)$$

for a real number $a$. We deduce that $x = y = \frac{a}{4}$ and $z = 2a$.
By substituting these values into the equation for the level set, we get

$$f(x, y, z) = \frac{a^2}{8} + \frac{a^2}{8} + a^2 = 1 \implies a_\pm = \pm\frac{2}{\sqrt{5}}$$

We therefore find $(x, y, z)_+ = (\frac{1}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, \frac{4}{\sqrt{5}})$ and $(x, y, z)_- = (-\frac{1}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}}, -\frac{4}{\sqrt{5}})$. We insert these points into the equation $x + y + z = k$ and find that $k_\pm = \pm\sqrt{5}$.

## 7.2 Other tasks

Compute the value $f(2\pi)$ where $f$ is the unique function on $[\pi, 2\pi]$ such that

$$xf'(x) = f(x)x^2 \sin(x) \text{ for } \pi \le x \le 2\pi$$

and $f(\pi) = 0$.

If we rewrite the equation to $f'(x) - \frac{1}{x}f(x) = x\sin(x)$, we find that it's a linear ODE of the form $y' + a(x)y = b(x)$.
Using Variation of parameters (section 1.2), we find that the solution is of the form $f_p = K(x)e^{-A(x)}$, where $A(x) = -\ln(x)$ is the primitive of $a(x) = -\frac{1}{x}$.

We now have $f_p = K(x)e^{-(-\ln(x))} = K(x) \cdot x$.

$$\begin{aligned} K(x) &= \int_{x_0}^x x \cdot \sin(x) \cdot e^{-\ln(x)}\, dx = \int_{x_0}^x x \cdot \sin(x) \cdot \frac{1}{x}\, dx = \int_{x_0}^x \sin(x)\, dx \\ &= -\cos(x) + C,\ C \in \mathbb{R} \\ &\implies f_p = (-\cos(x) + C)x \end{aligned}$$

From $f(\pi) = 0$ we know that $(-\cos(\pi) + C)\pi = 0 \implies C = -1$.
Therefore $f_p = (-\cos(x) - 1)x$ and $f(2\pi) = -4\pi$.

---

We introduce the following definition: Let $k \ge 0$ be a real number. A function $f : ]0, \infty[^3 \to \mathbb{R}$ is said to be *homogeneous of degree* $k$ if $f(tx, ty, tz) = t^k f(x, y, z)$ for all $x, y$ and $z$ strictly positive.

Let $f$ be of class $C^1$ on $]0, \infty[^3$. For a fixed $(x, y, z) \in ]0, \infty[^3$, $k \ge 0$, and $t > 0$, define

$$g(t) = f(tx, ty, tz) - t^k f(x, y, z) = f(h(t)) - t^k f(x, y, z), h(t) = (tx, ty, tz).$$

(a) Show that $g$ is differentiable on $]0, \infty[$ and that $g'(t) = x(\partial_x f)(tx, ty, tz) + y(\partial_y f)(tx, ty, tz) + z(\partial_z f)(tx, ty, tz) - kt^{k-1} f(x, y, z)$
Since $h$ and $f$ are differentiable, their composition is differentiable.
For the derivation of $g$ we use the chain rule for the left part:

$$\begin{aligned} g'(t) &= \partial f(h(t)) \cdot \partial h(t) - kt^{k-1} f(x, y, z) \\ &= (\nabla f(h(t)))^T \cdot (x, y, z) - kt^{k-1} f(x, y, z) \\ &= \left(\frac{\partial}{\partial x} f(h(t)), \frac{\partial}{\partial y} f(h(t)), \frac{\partial}{\partial z} f(h(t))\right)^T (x, y, z) \\ &\quad - kt^{k-1} f(x, y, z) \end{aligned}$$

(b) Deduce that if $f$ is homogeneous of degree $k$, then we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = kf$$

Note that for $f$ homogeneous of degree $k$, $g = 0, g' = 0$. Therefore

$$x(\partial_x f)(h(t)) + y(\partial_y f)(h(t)) + z(\partial_z f)(h(t))) = kt^{k-1} f(x, y, z)$$

Note that this equation is valid for all $t > 0$, in particular for $t = 1$, which gives us our desired result:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = kf$$

---

Suppose that $f$ satisfies

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = kf \quad (1)$$

(c) Show that $g$ (as defined above) satisfies $g(1) = 0$ and is a solution of the ODE

$$tg' - kg = 0$$

We have

$$\begin{aligned} tg' - kg =&(tx)(\partial_x f)(tx,ty,tz) + (ty)(\partial_y f)(tx,ty,tz) \\ &+ (tz)(\partial_z f)(tx,ty,tz) \\ &- kt^k f(x,y,z) - kf(tx,ty,tz) + kt^k f(x,y,z) \\ =&(tx)(\partial_x f)(tx,ty,tz) + (ty)(\partial_y f)(tx,ty,tz) \\ &+ (tz)(\partial_z f)(tx,ty,tz) - kf(tx,ty,tz) \\ =&kf(tx,ty,tz) - kf(tx,ty,tz) \quad (1) \\ =&0 \end{aligned}$$

For the second part we have $g(1) = f(x,y,z) - f(x,y,z) = 0$.

(d) Deduce that $f$ is homogeneous of degree $k$.
When considering the ODE $tg' - kg$, by seperation of variables(section 1.5), we can deduce that $g(t) = Ct^{k-1}$.
Since we've already proven, that $g(1) = C1^k = 0$, we follow that $C = 0$ and therefore $g(t) = 0, \forall t > 0$.
We conclude that since $g(t) = f(tx,ty,tz) - t^k f(x,y,z) = 0, \forall t > 0$, it must hold, that

$$f(tx,ty,tz) = t^k f(x,y,z), \forall t > 0.$$

---

Let $f$ be any differentiable function of one variable. Show that all tangent planes of the surface

$$z = y \cdot f(\frac{x}{y})$$

pass through the point $(0,0,0)$. Let $G(x,y) := y \cdot f(\frac{x}{y})$. Then the surface $z$ is equal to the graph of $G$. It holds that

$$\partial_x G = G_x(x,y) = f'(\frac{x}{y}), \partial_y G = G_y(x,y) = f(\frac{x}{y}) - \frac{x}{y} f'(\frac{x}{y}).$$

The tangent planes in the points $(x_0, y_0, G(x_0,y_0))$ (with $y \neq 0$) is

$$\begin{aligned} z =&G(x_0,y_0) + G_x(x_0,y_0)(x - x_0) + G_y(x_0,y_0)(y - y_0) \\ =&xf'\left(\frac{x_0}{y_0}\right) + y\left[f\left(\frac{x_0}{y_0}\right) - \frac{x_0}{y_0} f'\left(\frac{x_0}{y_0}\right)\right] \end{aligned}$$

The point $(0,0,0)$ satisfies this equation, which means that $(0,0,0)$ lies on this tangent plane. In other words, all tangent planes pass through the origin.

---

Where does the tangent plane at $(1,1,1)$ to the surface $S = \{(x,y,e^{x-y}) : (x,y \in \mathbb{R}^2)\} \subset \mathbb{R}^3$ intersect with the $z$-axis?
The surface is parameterized by $f(x,y) = (x,y,e^{x-y})$. Notice that $f(1,1) = (1,1,1)$ and $Df(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ e^{x-y} & -e^{x-y} \end{pmatrix}$. The tangent plane $g(x,y)$ is given by:

$$\begin{aligned} g(x,y) =&f(x_0,y_0) + Df(x_0,y_0)\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \\ =&\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} \\ =&\begin{pmatrix} x \\ y \\ 1 + x - y \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \\ &\implies x = 0, y = 0 \\ &\implies z = 1 \end{aligned}$$

---

The equation $z = 2y^2 + x^2$ describes a surface $S$ in $\mathbb{R}^3$, which contains the point $P = (1,1,3)$. Find the coordinates of the other point of $S$ that lies on the normal to $S$ at $P$. Let $f(x,y,z) = x^2 + y^2 - z$. Then $S = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0\}$ is a level set.
The normal at a point $(x,y,z)$ is given by $\nabla f = (2x, 4y, -1)$.
For the point $P = (1,1,3)$ the normal is $\nabla f(1,1,3) = (2,4,-1)$.
Therefore

$$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

is the equation for the normal at $P$.
We now insert this equation into $f$ to find another point on this normal contained in the level set.

$$\begin{aligned} f(1+2t, 1+4t, 3-t) &= 0 \\ (1+2t)^2 + 2(1+4t)^2 - (3-t) &= 0 \\ 36t^2 + 21t &= 0 \quad | : t, t_1 = 0 \implies P \\ 36t + 21 &= 0 \quad | -21, : 36 \\ t = -\frac{21}{36} &= -\frac{7}{12} \end{aligned}$$

$$\implies P_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} - \frac{7}{12}\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} \\ -\frac{4}{3} \\ \frac{43}{12} \end{pmatrix}$$

### 7.3 Linear ODE with constant factors

Find all **real** solutions of the differential equation

$$u'' - 6u' + 13u = (3x+2)e^x$$

1. Homogeneous:
   $\lambda^2 - 6\lambda + 13 = 0 \quad \implies \quad \lambda_1 = 3 + 2i$ and $\lambda_2 = 3 - 2i$.

   Complex Solution: $K_1 \cdot e^{(3+2i)x} + K_2 \cdot e^{(3-2i)x}$ for $K_{1,2} \in \mathbb{C}$
   $\iff e^{3x}\left(K_1 \cdot e^{2ix} + K_2 \cdot e^{-2ix}\right)$
   $\iff e^{3x}\left(K_1 \cdot (\cos(2x) + i\sin(2x)) + K_2 \cdot (\cos(2x) - i\sin(2x))\right)$
   $\iff e^{3x}\left((K_1 + K_2) \cdot \cos(2x) + (K_1 - K_2)i \cdot sin(2x)\right)$
   Real Solution: Let $c_1 = K_1 + K_2$ and $c_2 = (K_1 - K_2)i$ such that $c_{1,2} \in \mathbb{R}$
   $\iff e^{3x}\left(c_1 \cdot \cos(2x) + c_2 \cdot sin(2x)\right)$ for all $c_{1,2} \in \mathbb{R}$
2. Particular Solution:
   Ansatz (from educated guess):

$$\begin{aligned} U_p(x) &= (Ax+B)e^x \\ U_p'(x) &= Ae^x + (Ax+B)e^x \\ U_p''(x) &= 2Ae^x + (Ax+B)e^x \end{aligned}$$

   Then fill in Ansatz into the equation and set it equal to inhomogenous Part.

## 8 Sources

This cheatsheet is forked from Nicolas Wehrli's one. I adapted a few things that it fits to the 2024 course.