This homework is about systems of hyperbolic equations. The problem to solve here originates from electromagnetics and is slightly more complicated than the wave equation that was seen in class. We call $E(x,t)$ and $H(x,t)$ the electric and magnetic field, respectively and $\varepsilon(x)$ and $\mu(x)$ the electric permittivity and the magnetic permeability of the medium at position x, respectively. Consider the one-dimensional Maxwell's equations:

$$\varepsilon \frac{\partial E}{\partial t} + \frac{\partial H}{\partial x} = 0 \tag{1}$$

$$\mu \frac{\partial H}{\partial t} + \frac{\partial E}{\partial x} = 0. \tag{2}$$

We want to find $E(x,t)$ and $H(x,t)$ solutions of this system of equations.

1. For ε and μ constant, transform the system into two decoupled wave equations, similar to what you have studied in class. What is the characteristic velocity of the waves? To what famous physical constant does it correspond?

2. With ε and μ still constant, find the analytical solutions for $E(x,t)$ and $H(x,t)$ in infinite domain given the initial conditions

$$E(x,0) = 0, \quad H(x,0) = e^{-(10x/L)^2}.$$

3. Obtain a discontinuous Galerkin formulation for the system of equations. To do so, multiply the equations on each element D^k by each test function $\mathcal{P}_i^k(x)$ and integrate, use integration by parts and plug the expression of the following discrete approximations on each element k, similarly to what was done in class:

$$E(x,t) \simeq E_h^k(x,t) = \sum_{j=0}^{p} \hat{E}_j^k(t)\,\mathcal{P}_j^k(x)$$

$$H(x,t) \simeq H_h^k(x,t) = \sum_{j=0}^{p} \hat{H}_j^k(t)\,\mathcal{P}_j^k(x)$$

You can assume that the element size is small enough such that ε and μ can be taken as constant on each element:

$$\varepsilon(x) = \varepsilon^k, \quad \mu(x) = \mu^k, \quad x \in D^k.$$

To compute the numerical fluxes $\left(E^k\right)^*$ and $\left(H^k\right)^*$, follow the Riemann approach (see Example 2.6 from the reference book).

Express your final discontinuous Galerkin formulation, in the weak form, in terms of the vectorized solutions $\hat{\mathbf{E}}^k(t), \hat{\mathbf{H}}^k(t) \in \mathbb{R}^{p+1}$. Be extremely rigorous in the notations!

4. Adapt your code from homework 1 to solve Maxwell's equations using the formulation written in question 3. We choose the following finite domain, without loss of generality: $x \in [-L, L]$. Our domain $x \in [-L, L]$ **is thus not periodic anymore.** Impose a zero numerical flux equal at both extremities of the domain: this is equivalent to imposing **a perfect reflection of the electromagnetic wave at the boundaries of the domain**. You should thus write a `python` code that solves the problem. At the end, you must end up with a function

```
def maxwell1d(L, E0, H0, n, eps, mu, dt, m, p, rktype, bctype):
```

Input parameters are described below:

- We assume that the mesh is uniform both in space and time. Thus, the 1D domain $[-L, L]$ is divided into `2n` elements of size L/n (we thus have $2n+1$ nodes) and we assume a constant time step `dt`. There are m time steps in the simulation so the final time is $m \times dt$.
- On each element $D^k := \left[\frac{(-n+k-1)L}{n}, \frac{(-n+k)L}{n}\right]$ $(k \in 1, \ldots, 2\,n)$, we choose to represent the solution locally as locally as a polynomial of arbitrary order p as

$$E_h^k(x(r), t) = \sum_{j=0}^{p} \hat{E}_j^k(t)\,\mathcal{P}_j(r)$$

$$H_h^k(x(r), t) = \sum_{j=0}^{p} \hat{H}_j^k(t)\,\mathcal{P}_j(r)$$

  where $\mathcal{P}_j$ is the normalized Legendre polynomial at order j and $x(r)$ is an affine mapping from the standard element $r \in [-1, 1]$ to element D^k.
- Parameters `E0` and `H0`. For example, the initial conditions from question 2 would be defined by

```
def E0(x, L):
    return 0
```

```
def H0(x, L):
    return np.exp(-(10*x/L)**2)
```

- `eps` and `mu` are NumPy arrays of size $2n$ containing the permittivity and permeability of each element.
- ~~Parameter a controls the upwindness of the scheme. Choosing $a = 1$ leads to a full upwind scheme, $a = 0$. leads to a centered scheme and $a = -1$ leads to a downwind (unstable) scheme.~~
- Parameter `rktype` is a string that defines the time stepping scheme that is used during the simulation. You must implement the following schemes: `"ForwardEuler"`, `"RK22"` and `"RK44"`.
- ~~When parameter `anim` is set to `True`, the solution should be plotted as an animation.~~
- Parameter `bctype` is a string that defines the boundary condition to apply. You must implement the following boundary conditions: `"periodic"`, `"reflective"`, `"non-reflective"`.

The output parameter of `sol` is a `numpy` array of size $2 \times (p+1) \times 2n \times (m+1)$:

$$\text{sol}(q, i, j, k) = \hat{u}^j_{q,i}(k \times dt).$$

$q = 0$ corresponds to E and $q = 1$ corresponds to H.

5. Compare the numerical solution found in item 4 with the analytical solution of item 2. The numerical and analytical solutions should compare well in early times but will diverge when the wave will reach the boundaries (because the true solution is not reflected at $x = \pm L$).

6. The biggest challenge in this situation is to apply an appropriate non-reflective boundary conditions at $x = \pm L$. Find a numerical flux that actually models a non-reflective boundary condition. Compare your solution with the analytical solution and comment.

7. Solve the same problem (same initial conditions), yet with glass for $x \in [-L/2, L/2]$ and air elsewhere. Comment.

Practical information:

Groups: *The groups are the ones indicated on Moodle (the same as for homework 1).*

Collaboration: *You are allowed, and even encouraged, to exchange ideas on how to address this assignment with students from other groups. However, you must do all the writing (report and codes) only with your own group; it is strictly forbidden to share the production of your group. Plagiarism will be checked.*

Writing: *The report should contain your answers to the questions as well as your figures. The report should have* ***maximum 3 pages (not including the header page, the figures and the bibliography)****. Please, do not include your code in the report. Take care of the writing of your report and the structure of your codes. Concision and quality of the writing, clarity of explanations and rigor will be taken into account.*

Language: *All reports and communications are equally accepted in French and English.*

Deliverables: *Each group is asked to submit on Moodle*

- *A report named* `report.pdf`*,*
- *A Python module named* `maxwell1d.py` *containing the function* `maxwell1d` *as specified above; this module should not execute anything so that it can be tested.*
- *A Python script named* `run.py` *that reproduces all the figures in your report.*
- *Optionally, animations can be provided as* `.gif` *files.*

Deadline: *The homework is due Thursday 10 March at 18:15.*

Questions: *You can address questions either at the Q&A session on Thursday the 3rd March at 2 p.m. (****be careful, there will be only one Q&A session for this homework !****), or by sending an email to the teaching assistants. Direct messages on Teams will not be considered.*