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ON PRESSURE BOUNDARY CONDITIONS FOR THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS*

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SUMMARY

The pressure is a somewhat mysterious quantity in incompressible flows. It is not a thermodynamic variable as there is no 'equation of state' for an incompressible fluid. It is in one sense a mathematical artefact—a Lagrange multiplier that constrains the velocity field to remain divergence-free; i.e., incompressible—yet its gradient is a relevant physical quantity: a force per unit volume. It propagates at infinite speed in order to keep the flow always and everywhere incompressible; i.e., it is always *in equilibrium* with a time-varying divergence-free velocity field. It is also often difficult and/or expensive to compute. While the pressure is perfectly well-defined (at least up to an arbitrary additive constant) by the governing equations describing the conservation of mass and momentum, it is (ironically) less so when more directly expressed in terms of a Poisson equation that is both derivable from the original conservation equations and used (or misused) to replace the mass conservation equation. This is because in this latter form it is also necessary to address directly the subject of pressure boundary conditions, whose proper specification is crucial (in many ways) and forms the basis of this work. Herein we show that the same principles of mass and momentum conservation, combined with a continuity argument, lead to the correct boundary conditions for the pressure Poisson equation: viz., a Neumann condition that is derived simply by applying the normal component of the momentum equation at the boundary. It usually follows, but is not so crucial, that the tangential momentum equation is also satisfied at the boundary.

KEY WORDS Boundary conditions Incompressible flow Pressure Poisson equation Navier–Stokes equations

INTRODUCTION

In some of the recent literature dealing with the numerical (approximate) solution of the incompressible Navier–Stokes (NS) equations, the issue of boundary conditions (BCs) for the associated (and derived) Poisson equation for the pressure is addressed in ways that to us appear to be, to say the least, unclear. While the majority of researchers seem to be content with what we shall attempt to demonstrate is the proper BC, there are some who are, in one sense or another, not quite satisfied with this approach; these ostensibly minority opinions are addressed in this paper. Hence, to those many (if not most) readers who already know what we are about to say, we apologize; but

* Based on an invited lecture.

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to those who don't, we believe the effort is worthwhile (it has been for us) and perhaps even important.

In the references to be cited (merely a *selective* sample of relevant recent works, to be sure) and in this work, attention is restricted to the case wherein Dirichlet BCs apply to the velocities; this includes, for example, the common case of rigid (impenetrable) no-slip boundaries at which the velocity vector is prescribed (specified). It is important to point out at the onset that the NS equations require no *a priori* BCs on the pressure; velocity (or traction, not discussed herein) BCs applied to the momentum equations are sufficient to allow the determination of both velocity *and* pressure.

We begin by discussing the review paper by Orszag and Israeli,[1] who seem to confound the issues of the Weyl (additive) decomposition of a given vector (into a solenoidal part and an irrotational part) and solid wall BCs for *viscous* flow; while in the former case it is sufficient to specify *either* the normal *or* the tangential component(s) of the solenoidal part in order that both components be uniquely determined, the NS equations require that *all* components of the velocity vector be prescribed on the boundary. They go on to conclude that either the normal or the tangential component(s) of the (vector) momentum (NS) equation is permissible as a BC for the pressure Poisson equation and then seem to imply that this is a serious dilemma. (The former leads directly to a Neumann BC and the latter leads indirectly to a Dirichlet BC.) They then express the (valid) concern that a finite difference solution to the discrete pressure Poisson equation with the Neumann BC could be different than that using the Dirichlet BC. They conclude this discussion by mentioning (almost in passing) the following key point, upon which we shall elaborate owing to its vital importance: 'An alternative, physically well-motivated procedure is to derive boundary conditions on the pressure by directly enforcing incompressibility on the fluid in the grid cells adjacent to the walls'. This, as they state, is 'particularly natural in staggered-mesh schemes...'; it is also, and this is somewhat important, the procedure that is *built in* to a Galerkin finite element method (GFEM) approximation to the primitive variables form of the NS equations, for both staggered and non-staggered meshes. We believe and assert that this procedure is also *mathematically* 'well-motivated'. Finally, the very important point, not mentioned by Orszag and Israeli and to be demonstrated in this paper, is the fact that the above procedure leads directly to the Neumann BC for the pressure Poisson equation; i.e., it is the BC 'selected' by a well-motivated procedure and hence (although not solely for this reason) is the BC of choice. This BC 'ambiquity' was largely resolved in a later publication, however.[2] We present further resolution herein.

Moin and Kim[3] picked up on the 'problem' of Neumann versus Dirichlet BCs and made the following true but vague statement: 'In general, however, the Neumann and Dirichlet problems for pressure may not have the same solution'. A more relevant and particular statement, which we shall attempt to prove in this paper, is: 'The Neumann and Dirichlet problems for the pressure, if properly derived from a well-posed NS problem, will have the same solution, at least for $t > 0$'.

In fact, Dierieck[4] has already shown, for the restricted but important case of steady 2D Stokes flow, that the pressure does indeed satisfy both the Neumann and Dirichlet BCs.

The paper by Donea *et al.*[5] is interesting in that they first derive (via the GFEM) and explicitly present the discrete pressure equations, both at an interior 'cell' and at the boundary, and then proceed to *reject* the latter equation as being 'incorrect' because it does not look like a discrete approximation to the Laplacian. What it *does* look like, and this was pointed out by Gresho[6] but apparently not appreciated or not believed by Donea *et al.*, is a Neumann BC for the (built-in) pressure Poisson equation; i.e., the normal component of the NS equation on the boundary.

In a well-titled paper, 'On the divergence-free condition in computational fluid dynamics: how important is it?', Gustafson and Halasi[7] present what we believe is the right philosophy, but then seem to go astray. They state that the BCs for the pressure Poisson equation are obtained by applying the momentum equation (normal component) and the continuity equation at the walls, but they then present the wrong equation for said BC. In a more recent and extended version of this work, however,[8] the BC descriptions are corrected—and, according to Gustafson (personal communication), were correct, although wrongly stated, in the first reference. Herein we shall show that the answer to the above question is: 'The divergence-free condition is of the utmost importance, in both theoretical and computational fluid dynamics'.

The paper by Deville *et al.*[9] focuses (in part) on the desirability of the initial velocity field satisfying certain 'compatibility conditions' so that the evolving pressure solution will satisfy *both* Dirichlet and Neumann BCs, initially and for all $t > 0$. However, as pointed out by Heywood and Rannacher,[10] who refer to it as an overdetermined Neumann problem, and Temam,[11] these compatibility conditions are generally *not* satisfied and one must accept and contend with this fact. (These compatibility conditions are also exceptional in that the incompressibility constraint causes them to be global rather than confined to the boundary as, for example, in the case of the Dirichlet or Neumann problem for the transient heat equation.) We agree with the latter philosophy and take the position that compatibility conditions on the data are not generally satisfied (i.e., they are usually 'too difficult' to satisfy) and thus, while an otherwise well-posed problem will satisfy both BCs for $t > 0$, we permit less regularity and even singularities as $t \to 0$. Heywood and Rannacher also show that *only* the Neumann BC is proper and applicable at $t = 0$.

A recent paper by Morino[12] addresses the issue of compatibility conditions in a rather different, but interesting and probably quite useful, way. His conclusions, if slightly 'reinterpreted', agree with those expressed by Heywood and by us. In what is called the 'key issue of this work', 'the core of the present work', Morino clearly recognizes the loss of 'regularity' as $t \to 0$, but he states it differently: after properly diagnosing that the tangential momentum equation is generally waived at a solid boundary at $t = 0$ in favour of the normal momentum equation, he interprets this behaviour as a loss of the no-slip BC. While the end result is the same (the equations and their solution are indifferent to the manner of interpretation), we prefer to 'retain' the no-slip BC but permit discontinuities in, for example, the tangential acceleration (at the wall at $t = 0$).

Last, in what is probably the most controversial of the references that we have seen, Strikwerda[13] actually claims that the Neumann BC is 'not satisfactory' and that it leaves the system 'underdetermined'. He then uses this as one of his reasons for not solving the pressure Poisson equation—he solves instead the discretized momentum and continuity equations. (There are some good reasons for following this route and we shall state them in due course; but Strikwerda's is not one of them.) It is possible that he was misled by his apparent belief that the normal component of the momentum equation on the boundary has already been employed so that its repeated use in the pressure equation (as a BC) is necessarily redundant. The fact is that the momentum equations *per se* are *not* employed on boundaries for which Dirichlet BCs on the velocity are invoked; they are thus indeed 'available', and properly so, as BCs for the pressure Poisson equation. See also Roache[14] for further discussion of these issues.

In this paper we attempt to eliminate the confusion using a three-pronged approach: (1) via the continuum partial differential equations (PDEs), (2) via the analysis of several consistent discretized approximations to the PDEs and (3) by numerical examples using one of the above approximations. By this route, we plan to demonstrate the following:

(i) To solve the continuum Poisson equation for the pressure, only the Neumann BC is always appropriate; i.e., it provides a unique solution for $t \geqslant 0$. The Dirichlet BC is generally only appropriate for $t > 0$; it often does not apply at $t = 0$. The unique solution obtained using either BC will, for $t > 0$, satisfy the other BC provided the Neumann BC is applied at $t = 0$.

(ii) Any consistent discrete approximation of the original (primitive) equations contains, as an automatic and built-in BC for the (implied) discrete pressure Poisson equation, the Neumann BC; for $t \geqslant 0$. It does not obviously satisfy the Dirichlet BC.

(iii) The converged numerical solution from (ii) will, however, also satisfy the Dirichlet BC owing to (i); but in general only for $t > 0$.

Finally we will discuss another and different approach to a Dirichlet BC—one that is essentially equivalent to the Neumann BC but is not obtained by applying the tangential momentum equation at the boundary. Rather, it is (or can be) obtained by asking: 'What boundary pressure corresponds to the normal derivative of pressure associated with the Neumann BC?'. It turns out that the answer necessarily involves Green's functions and, in the discrete case, leads to terms like influence matrix, capacitance matrix.

Although the time-dependent equations are employed throughout, the primary results and conclusions apply equally well to the steady-state equations—subject only to the reasonable and important proviso that *every steady solution attained is considered to be the $t \to \infty$ limit of a transient solution.*

ANALYSIS OF THE CONTINUUM EQUATIONS

The continuum PDE's of interest are the time-dependent incompressible Navier–Stokes equations for $\mathbf{u}(\mathbf{x}, t)$ and $P(\mathbf{x}, t)$ (velocity and kinematic pressure i.e., pressure divided by density),

$$(\partial \mathbf{u}/\partial t) + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \nu \nabla^2 \mathbf{u}, \tag{1a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1b}$$

for $t > 0$ in a bounded domain Ω subjected to Dirichlet BCs

$$\mathbf{u} = \mathbf{w}(\mathbf{x}, t) \quad \text{on} \quad \Gamma \equiv \partial\Omega, \tag{2a}$$

with

$$\int_\Gamma \mathbf{n}\cdot\mathbf{w} = 0, \tag{2b}$$

where $\mathbf{w}(\mathbf{x}, t)$ is a given function on Γ, and prescribed initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in} \quad \bar{\Omega} \equiv \Omega \oplus \Gamma, \tag{3a}$$

with

$$\nabla\cdot\mathbf{u}_0 = 0 \quad \text{in} \quad \Omega \tag{3b}$$

and

$$\mathbf{u}_0\cdot\mathbf{n} = \mathbf{w}(\mathbf{x}, 0)\cdot\mathbf{n} \quad \text{on} \quad \Gamma. \tag{3c}$$

The kinematic viscosity ν is a constant and $\geqslant 0$. A useful and important alternative interpretation of (1) that will recur in this paper is: Given an appropriate solenoidal velocity field, (1) can be used to determine the concomitant pressure field, which pressure ensures that the acceleration is also solenoidal. This is perhaps easier seen by taking the time derivative of (1b) and rewriting (1) as $\mathbf{a} + \nabla P = \mathbf{f}(\mathbf{u})$, $\nabla\cdot\mathbf{a} = 0$, where $\mathbf{a} \equiv \partial\mathbf{u}/\partial t$ and $f(\mathbf{u}) \equiv \nu\nabla^2\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u}$; given $\mathbf{u}$ such that $\nabla\cdot\mathbf{u} = 0$, these equations can yield $\mathbf{a}$ and P. Of course (2) and (3) must also be invoked to complete the problem specification.

Remarks

(i) A common and important special case of (2) is 'simple contained flow', for which $\mathbf{n}\cdot\mathbf{w} = 0$ on Γ.

(ii) There are no (*a priori*) BCs on the pressure. Yet the solution of (1)–(3) will yield P, albeit only up to an arbitrary additive constant (the hydrostatic pressure level).

(iii) The boundary condition solvability constraint, (2b), is of course a statement of global mass conservation and is derived by integrating (1b) over Ω and using the divergence theorem and (2a).

(iv) Constraints (3b) and (3c) are required in order for the problem to be well-posed (see, e.g., Tenam[15]); and the former causes (1b) to apply for *all* $t \geqslant 0$. $\mathbf{u}_0(\mathbf{x})$ is *not* required to satisfy the tangential component(s) of BC (2a); but if it does, the resulting solution may be smoother. Furthermore, by applying the divergence theorem to a small subdomain attached to Γ (a Gaussian pillbox) and letting its thickness approach zero, it follows that the normal component of (2a) is actually the manifestation of (1b) *on* Γ; i.e., *an alternate and useful statement of* $\nabla\cdot\mathbf{u}=0$ on Γ is, for the only case of interest wherein $\mathbf{n}\cdot\mathbf{u}$ is a continuous function of $\mathbf{x}$ as $\mathbf{x}\to\Gamma$ where the boundary unit normal vector is imagined to be translated (in the $-\mathbf{n}$ direction) into Ω to form $\mathbf{n}\cdot\mathbf{u}$ near Γ, $\mathbf{n}\cdot\mathbf{u}=\mathbf{n}\cdot\mathbf{w}$ on Γ. So we see, from (2a) and (3c), that (1b) applies 'everywhere and all the time': $\nabla\cdot\mathbf{u}=0$ in $\bar{\Omega}$ for $t\geqslant 0$. [Note that in this interpretation, a problem such as a square cavity containing a fluid initially at rest and shear-driven by a 'rubber sheet' at $y=1$ (e.g., $\mathbf{w}\cdot\boldsymbol{\tau}=1-|x|$ for $-1\leqslant x\leqslant 1$) is well-posed even though $\nabla\cdot\mathbf{u}$, when interpreted strictly as $(\partial u/\partial x)+(\partial v/\partial y)$ on Γ, is non-zero (it is ± 1) on a portion of Γ at $t=0$; i.e., $\mathbf{n}\cdot\mathbf{u}=\mathbf{n}\cdot\mathbf{w}=0$ on Γ and $\nabla\cdot\mathbf{u}=0$ in Ω for $t\geqslant 0$.]

(v) These are referred to as the primitive form or primitive variables form of the NS equations.

(vi) In a common dimensionless form of the equations, the kinematic viscosity ν is replaced by $1/Re$, where Re is the Reynolds number.

(vii) The addition of a non-conservative 'body force' to the right side of (1a), such as the buoyancy term in the Boussinesq equations, causes no additional difficulties.

Having stated a well-posed problem in the primitive variables, we now turn to the major focus of this work: the derived equation for the pressure and its associated BCs. *Assuming 'appropriate' differentiability*, the pressure Poisson equation is obtained by first applying the divergence operator to (1a) to give

$$\nabla\cdot((\partial\mathbf{u}/\partial t)+\mathbf{u}\cdot\nabla\mathbf{u})+\nabla^2 P=\nu\nabla\cdot(\nabla^2\mathbf{u}).$$

Assuming next that div and $\partial/\partial t$ can be commuted, and using the identity

$$\nabla^2\mathbf{u}=\nabla(\nabla\cdot\mathbf{u})-\nabla\times\nabla\times\mathbf{u}$$

and the fact that div curl of any (differentiable) vector field is identically zero, we obtain

$$\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})+\nabla^2 P=(\nu\nabla^2\Theta-\partial\Theta/\partial t)\quad\text{in}\quad\Omega,$$

where $\Theta\equiv\nabla\cdot\mathbf{u}$ is the velocity divergence. But according to Remark (iv) above, we have

$$\nabla\cdot\mathbf{u}=0\quad\text{in}\quad\bar{\Omega}\quad\text{for}\quad t\geqslant 0, \tag{1c}$$

which leads to the pressure Poisson equation (PPE)

$$\nabla^2 P=-\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})\quad\text{in}\quad\Omega\quad\text{for}\quad t\geqslant 0. \tag{4a}$$

Having easily arrived at the statement of the conventional PPE, we next pose the logical, important and seemingly innocuous question: 'Can we go backwards?; i.e., given that (1a) and (1c) imply (4a), do (1a) and (4a) imply (1c)?'. The (surprising?) answer is: 'No—at least not always'. Subtracting (4a) from the divergence of (1a) gives $\nabla\cdot((\partial\mathbf{u}/\partial t)-\nu\nabla^2\mathbf{u})=0$ in Ω, which is, after commuting the operators, the transient heat equation for Θ. Since Θ is initially zero in $\bar{\Omega}$, *à la* (3b), it will remain zero—thus satisfying (1c)—*if and only if* Θ (or $\partial\Theta/\partial n$) is held at zero on Γ. On the other hand, consider the following, ostensibly equivalent PPE:

$$\nabla^2 P = \nabla\cdot(\nu\,\nabla^2\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u}). \tag{4b}$$

A similar analysis now leads to $\partial(\nabla\cdot\mathbf{u})/\partial t = 0$ in Ω, *regardless* of the value of Θ on Γ. Integration in time then gives $\nabla\cdot\mathbf{u} = g(\mathbf{x})$; i.e., $\nabla\cdot\mathbf{u}$ is independent of t. But (3b) states that $\nabla\cdot\mathbf{u}_0 = 0$, and since $\nabla\cdot\mathbf{u} = g(\mathbf{x})$ for *all* t, we must have $g(\mathbf{x}) = 0$. So we see that (1a) and (4b) *do* imply (1c) for a well-posed problem, but that (1a) and (4a) might need additional 'help' to imply (1c).

We shall return to these somewhat subtle, delicate and important issues later. For now, we just make the following

Remarks:

(i) We will soon address the issue of applying (4) at $t = 0$.

(ii) If (3b) is *not* satisfied, (1a) and (4b) imply that $\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{u}_0 \neq 0$; the initial divergence is preserved for all time. If, however, (4a) rather than (4b) is used to get P under these same conditions, the initial divergence will diffuse out to the boundaries *if* $\Theta = 0$ there, but will diffuse toward a constant value in Ω if $\partial\Theta/\partial n = 0$ on Γ. (If neither Θ nor $\partial\Theta/\partial n$ is held at zero on Γ, additional and unknown 'divergence' can be diffused into or out of Ω.) Thus the Poisson equation approach can ostensibly solve an otherwise ill-posed problem—and has probably often although inadvertently done so.

(iii) We have also just proven the following important facts (modulo pressure BCs, which come later and can have a profound effect on these 'facts'): Any divergence-free velocity field *induces* a (computable) pressure field. This pressure field *ensures* a divergence-free acceleration, which is then also computable; i.e., the induced pressure assures that the velocity field *remains* divergence-free—at least up to the imposition of pressure BCs, which we discuss later.

(iv) The pressure is thus seen to (also) play the role of a Lagrange multiplier acting to enforce satisfaction of the solenoidal constraint.

(v) If the solution $(\mathbf{u}, P)$ is not sufficiently smooth, owing to some combination of rough data—$\mathbf{u}_0(\mathbf{x})$, $\mathbf{w}(\mathbf{x}, t)$ and Γ— it is possible that the initial pressure field will *not* satisfy (4). In such cases, the pressure (and concomitant acceleration) can only be obtained by solving the *coupled* system of (1). This is a consequence of the fact that (1a) and (1c) encompass a larger class of solutions than do (1a) and (4).

Focusing now on 'sufficiently smooth' 2D domains for simplicity, we assume that a Cartesian coordinate system with $\mathbf{u} = (\mathbf{u}_x, u_y) = (u, v)$ will be utilized and that a *local* Cartesian system can be erected at each point on Γ such that the local normal to Γ coincides with one of its axes. (An ostensibly analogous analysis could then be applied to 3D domains of sufficient regularity.)

Denoting by $\mathbf{n}$ the outward-pointing unit normal vector and by τ the unit tangent vector on Γ according to Figure 1, we have the following identities on Γ:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial n^2} + \frac{\partial^2}{\partial \tau^2} \tag{5a}$$

and

$$\mathbf{u}\cdot\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} = u_n\frac{\partial}{\partial n} + u_\tau\frac{\partial}{\partial \tau}, \tag{5b}$$

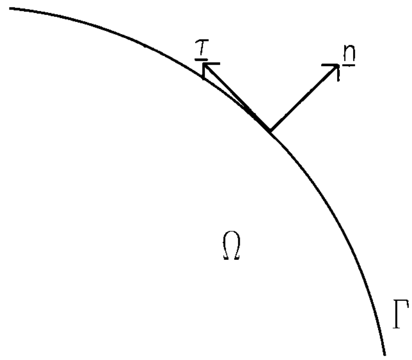

Figure 1.

where of course derivatives with respect to n are 'one-sided'.

Now to complete the specification of the problem for the pressure, we must set BCs on Γ. Since (4) is a *derived* equation, it is not surprising that the concomitant BCs must also be derived. And the obvious way of doing so is simply to apply (1a) on the boundary itself. But since (1a) is a vector equation and a scalar BC is required, we seem to have a choice: either the normal or tangential projection of (1a) onto Γ is ostensibly sufficient to supply a BC for (4). We first choose the former to get

$$\mathbf{n}\cdot\nabla P \equiv \partial P/\partial n = \nu\nabla^2 u_n - ((\partial u_n/\partial t) + \mathbf{u}\cdot\nabla u_n) \quad \text{on} \quad \Gamma \quad \text{for} \quad t \geqslant 0, \tag{6}$$

so that (4) and (6) provide a Neumann problem for the pressure; i.e., given a velocity field satisfying (1)–(3), the (induced) pressure field can be obtained from (4) and (6), at least up to an arbitrary additive constant.

Remarks

(i) As with (4), we will soon address the use of (6) at $t = 0$.

(ii) It is well known that a solvability constraint is associated with a Neumann problem. The solvability constraint associated with (4b) and (6), which is just a restatement of global mass conservation, is automatically satisfied when (1)–(3) are satisfied; i.e., when the original NS problem is well-posed, so is the associated Poisson/Neumann problem. The proof is simple: first restate the original problem as

$$(\partial\mathbf{u}/\partial t) + \nabla P = \nu\nabla^2\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} \equiv \mathbf{f}$$

and

$$\nabla\cdot(\partial\mathbf{u}/\partial t) = 0 \quad \text{in} \quad \Omega;$$

these imply

$$\nabla^2 P = \nabla\cdot\mathbf{f} \quad \text{in} \quad \Omega;$$

also

$$\partial P/\partial n = \mathbf{n}\cdot(\mathbf{f} - (\partial\mathbf{u}/\partial t)) \quad \text{on} \quad \Gamma.$$

Solvability then requires

$$\int_\Omega \nabla^2 P = \int_\Omega \nabla\cdot\mathbf{f} \Rightarrow$$

$$\int_\Gamma \partial P/\partial n = \int_\Gamma \mathbf{n}\cdot\mathbf{f} \Rightarrow$$

$$\int_\Gamma \mathbf{n}\cdot(\mathbf{f}-(\partial\mathbf{u}/\partial t)) = \int_\Gamma \mathbf{n}\cdot\mathbf{f} \Rightarrow$$

$$0 = \int_\Gamma \mathbf{n}\cdot(\partial\mathbf{u}/\partial t) = \int_\Gamma \mathbf{n}\cdot\dot{\mathbf{w}}(\mathbf{x},t) = (\mathrm{d}/\mathrm{d}t)\int_\Gamma \mathbf{n}\cdot\mathbf{w}(\mathbf{x},t),$$

which *is* satisfied since (2b) is true for all $t \geqslant 0$. Q.E.D.

(iii) We will show, in the Discussion section, that (4a) and (6) is also well-posed.

On the other hand, the tangential component of (1a) on Γ gives the so-called Dirichlet condition,

$$\boldsymbol{\tau}\cdot\nabla P = \partial P/\partial\tau = \nu\nabla^2 u_\tau - ((\partial u_\tau/\partial t) + \mathbf{u}\cdot\nabla u_\tau), \tag{7}$$

where the value of P on Γ (i.e., Dirichlet data) is provided (in principle, at least) by integration of (7) along τ.

Next we address the question: 'Does a solution of (4) and (6) satisfy (7)?'. To (partially) answer this, we first *assume* that (4a) applies on Γ to give, using (5),

$$\nabla^2 P = \frac{\partial^2 P}{\partial n^2} + \frac{\partial^2 P}{\partial \tau^2} = -\nabla\cdot(\mathbf{u}\cdot\nabla)\mathbf{u} = -\left(\frac{\partial}{\partial n}(\mathbf{u}\cdot\nabla u_n) + \frac{\partial}{\partial\tau}(\mathbf{u}\cdot\nabla u_\tau)\right). \tag{8}$$

Next, differentiate (6) in the $\mathbf{n}$ direction,

$$\frac{\partial^2 P}{\partial n^2} = \frac{\partial}{\partial n}\left[\nu\nabla^2 u_n - \left(\frac{\partial u_n}{\partial t} + \mathbf{u}\cdot\nabla u_n\right)\right],$$

and insert the result into (8) to give

$$\frac{\partial^2 P}{\partial\tau^2} = -\frac{\partial}{\partial\tau}(\mathbf{u}\cdot\nabla u_\tau) - \nu\nabla^2\frac{\partial u_n}{\partial n} + \frac{\partial}{\partial t}\frac{\partial u_n}{\partial n}, \tag{9}$$

where $\partial/\partial n$ has been commuted with ∇^2 and $\partial/\partial t$. Now use the continuity equation, (1b), on Γ in the form $(\partial u_n/\partial n) + (\partial u_\tau/\partial\tau) = 0$ in (9) to give, commuting the operators again,

$$\frac{\partial^2 P}{\partial\tau^2} = -\frac{\partial}{\partial\tau}(\mathbf{u}\cdot\nabla u_\tau) + \frac{\partial}{\partial\tau}\left(\cdot\nu\nabla^2 u_\tau - \frac{\partial u_\tau}{\partial t}\right)$$

or

$$\frac{\partial}{\partial\tau}\left(\frac{\partial u_\tau}{\partial t} + \mathbf{u}\cdot\nabla u_\tau + \frac{\partial P}{\partial\tau} - \nu\nabla^2 u_\tau\right) = 0, \tag{10}$$

which implies that the term in parentheses is independent of τ (on Γ). But since the equation of motion (and its tangential component) must be satisfied in the neighbourhood of Γ (if not actually on Γ), this term must vanish identically; i.e., we obtain (7): (4a) and (6) *do* imply (7)—at least for sufficiently smooth solutions. [A similar 'proof' follows easily beginning with (4b) and (6).]

Hence, if the Neumann BC is applied to the pressure Poisson equation, the solution (if sufficiently smooth) will also satisfy the Dirichlet BC. This result can also be interpreted as another way of saying that both normal and tangential components of the momentum equation, (1a), apply on (or at least very close to) the boundary. *But*, as shown by Heywood[16] and Heywood and Rannacher,[10] only the *normal* component of the momentum equation applies on Γ at $t = 0$ for the general case. They also show that the *initial* pressure field is obtained from the Poisson/Neumann problem: (4)

and (6) at $t=0$; i.e., these equations apply for $t \geqslant 0$, a result that coincides with (1c) but is not coincidental in the common sense of the word. Hence, as we will demonstrate and further discuss later, the tangential momentum equation on the boundary (and concomitant Dirichlet BC for the PPE) only applies for $t>0$ in the general case.

Thus, in general, the *initial* solution is *not* sufficiently smooth to validate the above argument; for $t>0$, however, the solution *is* generally smooth enough.

Remarks

(i) Just as (4) and (6) imply (7), it can be shown (in the same way) that (4) and (7) imply (6).
(ii) For the simpler but more common case of $\mathbf{u}=\mathbf{0}$ on Γ, (6) and (7) become $\partial P/\partial n = \nu\, \partial^2 u_n/\partial n^2$ and $\partial P/\partial \tau = \nu\, \partial^2 u_\tau/\partial n^2$ respectively.
(iii) For $Re \gg 1$ and $\mathbf{u}=\mathbf{0}$ on Γ (for simplicity), (6) can ostensibly [but see (iv) below] be approximated by $\partial P/\partial n=0$ and (7) by $\partial P/\partial \tau = 0$; indeed, the first of these approximations is frequently and successfully made in practice (e.g., in boundary layer theory).
(iv) For the inviscid (Euler) equations ($\nu=0$, $Re=\infty$), however, the results are *different*, owing to the loss of the no-slip BC: while (1), (6) and (7) still apply (with $\nu=0$), (2) is changed to $\mathbf{n}\cdot\mathbf{u}=w(\mathbf{x},t)$ on Γ and $\int_\Gamma w(\mathbf{x},t)=0$; the tangential velocity on Γ is unconstrained. If, further, $w=0$, (6) becomes $\partial P/\partial n=0$ (consistent with the above remark), but (7) becomes [using (5b)]

$$\partial u_\tau/\partial t = -((\partial P/\partial \tau) + (u_\tau \partial u_\tau/\partial \tau)) = -(\partial/\partial \tau)(P + \tfrac{1}{2}u_\tau^2),$$

consistent with a streamline on Γ but *not* consistent with the high-*Re* approximation above. This is a manifestation of the singular limit as $\nu \to 0$ and provides a hint, at least, of the difficulty associated with large *Re*. It also helps to reinforce the assertion that the Neumann BC is inherently preferable and suggests, in complete harmony with boundary layer theory, that (in dimensionless form)

$$\partial^2 u_\tau/\partial n^2 = O(Re) \quad \text{for} \quad Re \gg 1 \text{ (and } t>0),$$

so that $\partial P/\partial \tau$ remains bounded. It also suggests that the Dirichlet BC would then be extremely difficult to approximate numerically, since $\partial^2 u_\tau/\partial n^2$ is both large and rapidly varying.
(v) Finally, for Stokes flow ($Re=0$), (4a) becomes $\nabla^2 P=0$ [i.e., $\mathbf{u}\cdot\nabla\mathbf{u}$ is omitted from (1) and thus from (4)] and the entire pressure field is thus 'driven' by the BCs [which are (6) or (7) with the (non-linear) advection term omitted]. Clearly the sometimes-used BC $\partial P/\partial n=0$ on Γ is (usually) a very bad approximation to (6) for this limiting case, since it would yield $P=$ constant, which is (usually) wrong.
(vi) If the solution of (1)–(3) is smooth enough that (4) is valid, the (same) velocity and pressure fields will be obtained by solving (1)–(3) or (1a), (2)–(4) and (6). [Provided (6) is used to compute the *initial* ($t=0$) pressure field, then BC (6) can be replaced by BC (7) for $t>0$ and the results will still be the same.]

After presenting some discrete equations and some numerical examples, we will return to the subject of Dirichlet BCs for pressure, but viewed from a different perspective.

DISCRETE APPROXIMATIONS TO THE CONTINUUM EQUATIONS

We, like Chorin,[17] Strikwerda[13] and many others, advocate the 'direct attack' on the original primitive variables form of the equations without the introduction of the continuum-derived Poisson equation; i.e., we work with (1a) and (1c) rather than (1a) and (4). The main reasons for this, which were also discussed by Gresho *et al.*,[18] are:

(i) The Poisson equation approach, being higher-order in the spatial derivatives, induces the requirement of more-than-originally-necessary smoothness (differentiability) in both pressure and velocity.
(ii) The BCs, which must also be (carefully) derived, are difficult to implement owing (at least) to the necessity of approximating second-order derivatives of velocity at the boundary.
(iii) There is generally no discrete divergence-free condition that will be satisfied by the computed velocity field.
(iv) The associated solvability constraint is often (usually) difficult to satisfy. This point, which is considered by Pat Roache (personal communication) to be of special importance, will be discussed further below.

A final reason is the result to be demonstrated below; viz.,

(v) The primitive variable approach simultaneously removes the ambiguity regarding pressure BCs (there are *none*), yet automatically, implicitly, and perhaps ironically, leads to the 'proper' choice; i.e., the analyst then need not and cannot make the choice.

Retaining time as a continuous variable, the semi-discrete version of (1) and (2a) can be written as the following differential-algebraic system (Petzold and Lötstedt[19]):

$$M\dot{u} + A(u)u + GP = Ku + f(t), \tag{11a}$$

$$Du = g(t), \tag{11b}$$

where u is now a vector containing all *internal* nodal degrees of freedom associated with the velocity (u and v in 2D, or u, v and w in 3D) and P is a vector containing the discrete pressures (all—internal and boundary). M is the 'mass matrix' which, when not diagonal (e.g., GFEM), couples the accelerations, and $A(u)$, K, G, and D are the matrices representing advection, diffusion, gradient, and divergence, respectively. Finally, the given vectors f and g represent the effects of the Dirichlet boundary conditions on velocity. (g is obtained by transposing all terms involving specified boundary velocities in the discretized continuity equations at or near Γ.) The initial conditions are $u(0) = u_0$ where $Du_0 = g(0) \equiv g_0$ is required in order to have a well-posed problem, *à la* (3b).

To complete the specification of the semi-discrete problem, we must address the BC solvability constraint, the analogue of (2b). But since this issue is fraught with difficulty in the finite difference method (see, e.g., Briley,[20] Ghia *et al.*,[21,22] Alfrink[23] and Strikwerda[13]), we take what for us is the easy way out, using the GFEM. In this case we have the important result that $D = G^{\mathrm{T}}$: the gradient and divergence matrices are the transposes of each other—and this result (see also Strang[24]) holds for *all other* BCs as well as Dirichlet. [Actually, here (11b) represents the negative of the divergence-free constraint, $-\nabla\cdot\mathbf{u} = 0$; i.e., the convergence-free constraint.] Letting $b \equiv Ku + f(t) - A(u)u$ and replacing G by C to obtain notational consistency with our previously published works on NS (whence $M^{-1}C$ is the gradient operator, or, more precisely, the weak gradient operator), we rewrite the differential-algebraic system as

$$M\dot{u} + CP = b(u, t), \quad u(0) = u_0, \tag{12a}$$

$$C^{\mathrm{T}}u = g(t), \quad C^{\mathrm{T}}u_0 = g_0. \tag{12b}$$

Remarks

(i) The actual weak divergence operator is $M_p^{-1}C^{\mathrm{T}}$, where M_p is the pressure mass matrix; i.e., the Gram matrix associated with the pressure basis functions—see, e.g., Sani *et al.*[25] or Engleman *et al.*[26]

(ii) Denoting $G_{\mathrm{w}} \equiv M^{-1}C$ as the weak gradient operator, $D_{\mathrm{w}} \equiv M_p^{-1}C^{\mathrm{T}}$ as the weak divergence operator and $(u,v)_A \equiv u^{\mathrm{T}}Av$, we have the FEM analogue of the identity discussed by Chorin;[27] viz., $(p, D_{\mathrm{w}}u)_{M_p} = (u, G_{\mathrm{w}}p)_M$ for all p and u; and the entire equation $\rightarrow 0 = 0$ as $h \rightarrow 0$, where h is the generic mesh size.

(iii) Rewriting (12), after differentiating (12b) with respect to time, as

$$\begin{bmatrix} M & C \\ C^{\mathrm{T}} & O \end{bmatrix}\begin{bmatrix} \dot{u} \\ P \end{bmatrix} = \begin{bmatrix} b \\ \dot{g} \end{bmatrix},$$

which can be solved for $\dot{u}$ and P, emphasizes the (*important*) point that: 'Given a divergence-free velocity field, the NS equations generate the corresponding pressure and (divergence-free) acceleration'.

We now consider the BC solvability constraint related to global mass conservation *à la* (2b). Since it is known (e.g., Sani *et al.*[25]) that the C matrix possesses a non-trivial null space in the presence of Dirichlet BCs, comprising at least what is called the hydrostatic pressure mode, $P_{\mathrm{H}}^{\mathrm{T}} = (1 \rightarrow 1)^{\mathrm{T}}$ and $CP_{\mathrm{H}} = 0$, the system (12) has a solution *if and only if* $g(t)$ is orthogonal to this null space; in particular, we have

$$P_{\mathrm{H}}^{\mathrm{T}}C^{\mathrm{T}}u = u^{\mathrm{T}}CP_{\mathrm{H}} = 0 = P_{\mathrm{H}}^{\mathrm{T}}g(t) \tag{13}$$

as a constraint on the boundary data, $g(t)$. This is the proper analogue of (2b) and provides the important assurance that the applied BCs represent a global mass balance for the semi-discrete system. It is noteworthy that constraint (13) is 'more important' than its continuum counterpart, (2b), in that a numerical solution exists if and only if (13) is satisfied, a situation which often precludes the simple use of interpolations of continuum fields satisfying (2b) as initial conditions for the discrete problem. (See also Engelman *et al.*[28] and Glowinski.[29]) There also sometimes exist 'spurious pressure modes', *à la* Sani *et al.*, which also induce their own associated (and spurious) solvability constraints, but we shall not delve further into this matter except to state that we henceforth assume that these constraints, too, are satisfied. Returning briefly now to 'reason (iv)' above, if we were to discretize (4) rather than (1b), (12b) would be replaced by $\tilde{K}P = k$, where $k(u)$ is known and $\tilde{K}$ corresponds to the Laplacian matrix with Neumann BCs and is thus singular. Since $\tilde{K}P_{\mathrm{H}} = 0$ and $\tilde{K}$ is symmetric, the solvability condition then would be $P_{\mathrm{H}}^{\mathrm{T}}k = 0$, which is not often satisfied if k is obtained in the seemingly appropriate manner—via a discretization of (4) and (6). Again, for further elucidation, see, e.g., Ghia *et al.*[22] and Alfrink.[23] Finally, to see the type of difficulty that can arise if $D \neq G^{\mathrm{T}}$ when the primitive variables, (11), are employed, see Strikwerda.[13]

Given that the constraints of (12b) and (13) are satisfied by the initial and boundary data, we have a well-posed differential-algebraic system and can proceed to derive the discrete analogue of (4), the pressure Poisson equation: since $C^{\mathrm{T}}u = g(t)$ for all time, we have $C^{\mathrm{T}}\dot{u} = \dot{g}$ which is used in (12a) to obtain

$$(C^{\mathrm{T}}M^{-1}C)P = C^{\mathrm{T}}M^{-1}b(u,t) - \dot{g}, \tag{14}$$

which we call the *consistent* discretized pressure Poisson equation {for reasons mentioned in Gresho *et al.*,[30] the most important of which for our purposes here is that the discrete approximations to the pressure BCs are [still, as in (12)] built in—*automatically*}.

Remarks

(i) When employing (14) in practice (see Gresho *et al.*[30]), it is expedient to employ 'mass lumping' which converts M (and, more importantly, M^{-1}) to a simple diagonal matrix. (In most finite difference methods, M is inherently diagonal.)

(ii) The 'Laplacian' matrix, $C^{\mathrm{T}}M^{-1}C$, is singular; it annihilates P_{H} as well as any spurious pressure modes. But, because (11)–(13) are satisfied, the solvability constraint associated with (14) is automatically satisfied—i.e., as in the continuum, a well-posed system in the primitive equations will always generate a solvable Neumann problem:

$$P_{\mathrm{H}}^{\mathrm{T}}C^{\mathrm{T}}M^{-1}b - P_{\mathrm{H}}^{\mathrm{T}}\dot{g} = b^{\mathrm{T}}M^{-1}CP_{\mathrm{H}} - P_{\mathrm{H}}^{\mathrm{T}}\dot{g} = 0.$$

Also, since (by definition, see Sani *et al.*[25]) the C matrix annihilates all pressure modes, the pressure *gradient* is always meaningful—even in those cases where the pressure itself is not. [Here, though, it is noteworthy that, if $\dot{g}(t) = 0$ so that $g = g_0$,(14) is solvable for *any* value of g_0, well-posed or *not*; i.e., (13) needn't then be satisfied for (14) to possess a solution. The solution, however, will be devoid of physical meaning if (13) is violated. It will satisfy $C^{\mathrm{T}}\dot{u} = 0$, as discussed below, but not $C^{\mathrm{T}}u = g$ uniformly in time.]

(iii) It may be important to note that the above Laplacian matrix is actually a 'generalized' Laplacian in the sense that it has access to the larger class of (less smooth) solutions referred to in Remark (v) below equation (4).

(iv) In the term $C^{\mathrm{T}}M^{-1}b(u,t)$ is $C^{\mathrm{T}}M^{-1}Ku$, which approximates $\nu\nabla\cdot(\nabla^2\mathbf{u})$ in Ω. Away from boundaries, this term is small but generally non-zero since the matrices don't commute. It is identically zero, however, on certain simple meshes (e.g., equal rectangles) and it always vanishes as $h \to 0$; see also Gresho and Chan.[31] At (or near) boundaries, however, this is the *very* (and non-zero) term that provides the (crucial!) viscous contribution to the (Neumann) BC given by (6); i.e., it no longer approximates $\nu\nabla\cdot(\nabla^2\mathbf{u})$, just as $(C^{\mathrm{T}}M^{-1}C)P$ no longer approximates $\nabla^2 P$ there. The omission of Ku on the right-hand side of (14), which would *appear* to mimic (4a) more closely, would actually lead to large errors in any *viscous* flow simulation as it would approximate the *inviscid* BC; e.g., $\partial P/\partial n = 0$ from (6) with $\nu = u_n = 0$—an especially bad BC for low Re; it would mimic (4a) but not (6). As a final curse on this idea, the resulting solution would violate (12b)—especially near boundaries. It thus seems reasonable to suggest that (4b) is a better starting point than (4a) for generating approximate solutions via the PPE approach.

(v) Interpolated initial discrete velocities derived from the continuum equations that satisfy (3) will not generally satisfy (13); yet it is the latter that is required for well-posedness of the discrete problem. (See Example 1 below.)

(vi) Consistent PPEs have been utilized in several early and important finite difference papers: Harlow and Welch,[32] Chorin[27] and Williams.[33]

(vii) It may be worth pointing out that many finite element codes but few finite difference (or finite volume) codes bypass the PPE route of (14) by solving the coupled system of (12) directly. The principal advantage of this approach is that the consistent mass matrix can easily be retained. The disadvantage is that the fully coupled system is more expensive to solve—implicit time integration is then virtually mandatory.

(viii) Finally, similar again to the continuum, just as (12a) and (12b) imply (14), so too do (12a) and (14) imply (12b). [Either set of (differential-algebraic) equations can be used to generate the same approximate solutions to the NS equations.]

Proof. Inserting (14) into (12a) gives (formally)

$$M\dot{u} + C(C^{T}M^{-1}C)^{-1}[C^{T}M^{-1}b(u,t) - \dot{g}] = b(u,t)$$

and thus

$$C^{T}\dot{u} = C^{T}M^{-1}b(u,t) - (C^{T}M^{-1}C)(C^{T}M^{-1}C)^{-1}[C^{T}M^{-1}b(u,t) - \dot{g}] = \dot{g},$$

which integrates to $C^{T}u = g(t) + k$, where k is a constant vector. Since this result is true for all time, we have $C^{T}u_0 = g_0 + k$ and thus $C^{T}u = g(t) + C^{T}u_0 - g_0$. Finally, imposing the solvability condition from the original primitive equations, $C^{T}u_0 = g_0$, gives $C^{T}u = g(t)$. Q.E.D.

[If the BCs are independent of time, $g(t) = g_0$ and the general result from (12a) and (14) is $C^{T}\dot{u} = 0$ or $C^{T}u = C^{T}u_0$ and, as stated above, $C^{T}u_0 = g_0$ is ostensibly not required in order that a solution exist; but then any *initial* violation of mass conservation is carried on forevermore in time, again in complete analogy with the continuum—see Remark (ii) following (4).]

Returning now to the main issue, we focus on the meaning of (14) on or near Γ, since the BCs for (4) are *built in* to the linear algebraic system (14) that approximates (4). We shall thus construct portions of the equations represented by (14); and we shall do so directly—i.e., by inserting the appropriate accelerations of (11a) into the time-differentiated form of (11b). The equations of interest are those wherein the discrete continuity equation involves velocities on Γ (i.e., those that are specified). We shall explicitly construct the relevant and typical equation at a simple but representative boundary given by $x = L$; other cases then follow easily. The demonstration will be performed for three different discretization schemes; two finite difference and one finite element. The chosen approach also allows us to avoid the issue of whether $D = G^{T}$.

We begin with the staggered mesh (MAC) scheme (e.g., Harlow and Welch,[32] Roache[34]) shown in Figure 2 and write the continuity equation (11b) for the cell containing P_0:

$$\frac{\dot{u}_e - \dot{u}_w}{l} + \frac{\dot{v}_n - \dot{v}_s}{h} = 0, \tag{15}$$

where $\dot{u}_e$ (on Γ) is given and the three other accelerations must be obtained from the semi-discretized momentum equations, (11a); viz.,

(i) *Node w*:

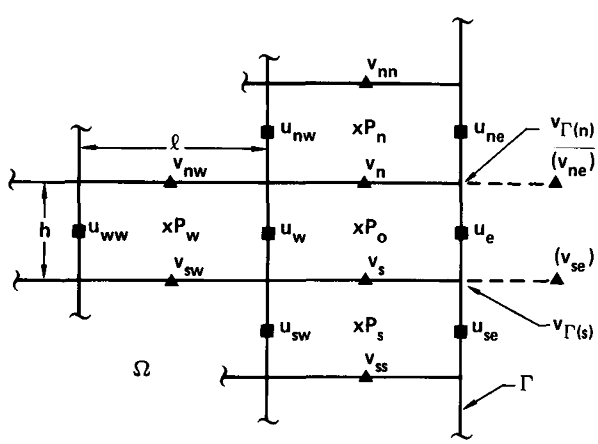

Figure 2. MAC grid near the boundary $x = L$

$$\dot{u}_{\mathrm{w}}+A(u)u_{\mathrm{w}}+\frac{P_0-P_{\mathrm{w}}}{l}=\nu\left(\frac{u_{\mathrm{ww}}-2u_{\mathrm{w}}+u_{\mathrm{e}}}{l^2}+\frac{u_{\mathrm{nw}}-2u_{\mathrm{w}}+u_{\mathrm{sw}}}{h^2}\right),$$

where $A(u)u_{\mathrm{w}}=(u(\partial u/\partial x)+v(\partial u/\partial y))_{\mathrm{w}}+O(l,h)$ or $O(l^2,h^2)$ depending on whether advection is approximated via upwind or centred differencing respectively. We do not write the $A(u)u$ terms in detail since all that matters here is that they represent a consistent approximation to $\mathbf{u}\cdot\nabla u$.

(ii) *Node n:*

$$\dot{v}_{\mathrm{n}}+A(u)v_{\mathrm{n}}+\frac{P_{\mathrm{n}}-P_0}{h}=\nu\left(\frac{v_{\mathrm{ne}}-2v_{\mathrm{n}}+v_{\mathrm{nw}}}{l^2}+\frac{v_{\mathrm{nn}}-2v_{\mathrm{n}}+v_{\mathrm{s}}}{h^2}\right),$$

where v_{ne} is the value of v at a 'phantom' node, or fictitious node, outside of Ω. It seems that in most 'MAC-type' references, u_{ne} is eliminated by linear extrapolation from v_{n} through the known value on Γ, $v_{\Gamma(\mathrm{n})}$, via $v_{\mathrm{ne}}=2v_{\Gamma(\mathrm{n})}-v_{\mathrm{n}}$. But since this appears to be a slightly inconsistent approximation to $\partial^2v/\partial x^2|_{\mathrm{n}}$ [it leads, via Taylor series, to $(2v_{\Gamma(\mathrm{n})}-3v_{\mathrm{n}}+v_{\mathrm{nw}})/l^2=\frac{3}{4}(\partial^2v/\partial x^2)|_{\mathrm{w}}+O(l)$], we will use the minimum higher-order extrapolation given by $v_{\mathrm{ne}}=\frac{1}{3}(8v_{\Gamma(\mathrm{n})}+v_{\mathrm{nw}}-6v_{\mathrm{n}})$, giving $\frac{4}{3}(2v_{\Gamma(\mathrm{n})}-3v_{\mathrm{n}}+v_{\mathrm{nw}})/l^2=\partial^2v/\partial x^2|_{\mathrm{n}}+O(l)$. For further discussion of this and related issues, see Peyret and Taylor.[35]

(iii) *Node s:* In a similar way, we write, given $v_{\Gamma(\mathrm{s})}$,

$$\dot{v}_{\mathrm{s}}+A(u)v_{\mathrm{s}}+\frac{P_0-P_{\mathrm{s}}}{h}=\nu\left(\frac{4}{3}\frac{2v_{\Gamma(\mathrm{s})}-3v_{\mathrm{s}}+v_{\mathrm{sw}}}{l^2}+\frac{v_{\mathrm{n}}-2v_{\mathrm{s}}+v_{\mathrm{ss}}}{h^2}\right).$$

Inserting the above accelerations into (15) yields

$$\begin{aligned}&\frac{1}{l}\left[\dot{u}_{\mathrm{e}}+A(u)u_{\mathrm{w}}+\frac{P_0-P_{\mathrm{w}}}{l}-\nu\left(\frac{u_{\mathrm{ww}}-2u_{\mathrm{w}}+u_{\mathrm{e}}}{l^2}+\frac{u_{\mathrm{nw}}-2u_{\mathrm{w}}+u_{\mathrm{sw}}}{h^2}\right)\right]\\&+\frac{1}{h}\left[-A(u)v_{\mathrm{n}}-\frac{P_{\mathrm{n}}-P_0}{h}+\nu\left(\frac{4}{3}\frac{2v_{\Gamma(\mathrm{n})}-3v_{\mathrm{n}}+v_{\mathrm{nw}}}{l^2}+\frac{v_{\mathrm{nn}}-2v_{\mathrm{n}}+v_{\mathrm{s}}}{h^2}\right)\right.\\&\left.+A(u)v_{\mathrm{s}}+\frac{P_0-P_{\mathrm{s}}}{h}-\nu\left(\frac{4}{3}\frac{2v_{\Gamma(\mathrm{s})}-3v_{\mathrm{s}}+v_{\mathrm{sw}}}{l^2}+\frac{v_{\mathrm{n}}-2v_{\mathrm{s}}+v_{\mathrm{ss}}}{h^2}\right)\right]=0,\end{aligned}$$

which we multiply by l and rearrange to

$$\begin{aligned}\frac{P_0-P_{\mathrm{w}}}{l}=&\,\nu\left(\frac{u_{\mathrm{ww}}-2u_{\mathrm{w}}+u_{\mathrm{e}}}{l^2}+\frac{u_{\mathrm{nw}}-2u_{\mathrm{w}}+u_{\mathrm{sw}}}{h^2}\right)-[\dot{u}_{\mathrm{e}}+A(u)u_{\mathrm{w}}]\\&+l\left[\frac{A(u)v_{\mathrm{n}}-A(u)v_{\mathrm{s}}}{h}+\frac{P_{\mathrm{n}}-2P_0+P_{\mathrm{s}}}{h^2}\right.\\&+\frac{4}{3}\frac{\nu}{h}\left(\frac{2v_{\Gamma(\mathrm{s})}-3v_{\mathrm{s}}+v_{\mathrm{sw}}}{l^2}-\frac{2v_{\Gamma(\mathrm{n})}-3v_{\mathrm{n}}+v_{\mathrm{nw}}}{l^2}\right)\\&\left.+\frac{\nu}{h}\left(\frac{v_{\mathrm{n}}-2v_{\mathrm{s}}+v_{\mathrm{ss}}}{h^2}-\frac{v_{\mathrm{nn}}-2v_{\mathrm{n}}+v_{\mathrm{s}}}{h^2}\right)\right].\end{aligned}\tag{16}$$

Taylor series expansion of all quantities about the point P_0 then yields (almost 'by inspection'), as $l,h\rightarrow 0$,

$$\frac{\partial P}{\partial x} = \nu \nabla^2 u - \left(\frac{\partial u}{\partial t} + \mathbf{u}\cdot\nabla u\right) + l\frac{\partial}{\partial y}\left(\mathbf{u}\cdot\nabla v + \frac{\partial P}{\partial y} - \nu\nabla^2 v\right) + O(l, h)$$

or

$$\frac{\partial P}{\partial n} \simeq \nu \nabla^2 u_{\mathrm{n}} - \left(\frac{\partial u_{\mathrm{n}}}{\partial t} + \mathbf{u}\cdot\nabla u_{\mathrm{n}}\right),$$

which is just (6) applied at $x = L$.

Next, consider the non-staggered mesh of Figure 3, in which each node carries a u, v and P. The continuity equation corresponding to P_0, at node 0, is

$$\frac{1}{2l}(3\dot{u}_0 - 4\dot{u}_{\mathrm{w}} + \dot{u}_{\mathrm{ww}}) + \frac{1}{2h}(\dot{v}_{\mathrm{n}} - \dot{v}_{\mathrm{s}}) = 0, \tag{17}$$

where the first term represents a second-order accurate approximation to $\partial u/\partial x$ at node 0, following Chorin.[17] Since u_0, v_{n} and v_{s} are given by the boundary conditions, we need in this case only the u-momentum equations at nodes w and ww.

(i) *Node w:*

$$\dot{u}_{\mathrm{w}} + A(u)u_{\mathrm{w}} + \frac{P_0 - P_{\mathrm{ww}}}{2l} = \nu\left(\frac{u_{\mathrm{ww}} - 2u_{\mathrm{w}} + u_0}{l^2} + \frac{u_{\mathrm{nw}} - 2u_{\mathrm{w}} + u_{\mathrm{sw}}}{h^2}\right).$$

(ii) *Node ww:*

$$\dot{u}_{\mathrm{ww}} + A(u)u_{\mathrm{ww}} + \frac{P_{\mathrm{ww}} - P_{\mathrm{www}}}{2l} = \nu\left(\frac{u_{\mathrm{w}} - 2u_{\mathrm{ww}} + u_{\mathrm{www}}}{l^2} + \frac{u_{\mathrm{nww}} - 2u_{\mathrm{ww}} + u_{\mathrm{sww}}}{h^2}\right).$$

Inserting these accelerations into (17) yields

$$\frac{1}{2l}\Bigg[3\dot{u}_0 - 4\nu\left(\frac{u_{\mathrm{ww}} - 2u_{\mathrm{w}} + u_0}{l^2} + \frac{u_{\mathrm{nw}} - 2u_{\mathrm{w}} + u_{\mathrm{sw}}}{h^2}\right) + 4A(u)u_{\mathrm{w}} + 4\frac{P_0 - P_{\mathrm{ww}}}{2l}$$
$$+ \nu\left(\frac{u_{\mathrm{w}} - 2u_{\mathrm{ww}} + u_{\mathrm{www}}}{l^2} + \frac{u_{\mathrm{nww}} - 2u_{\mathrm{ww}} + u_{\mathrm{sww}}}{h^2}\right) - A(u)u_{\mathrm{ww}} - \frac{P_{\mathrm{w}} - P_{\mathrm{www}}}{2l}\Bigg]$$
$$+ \frac{1}{2h}(\dot{v}_{\mathrm{n}} - \dot{v}_{\mathrm{s}}) = 0$$

or, after multiplying by $2l/3$ and rearranging,

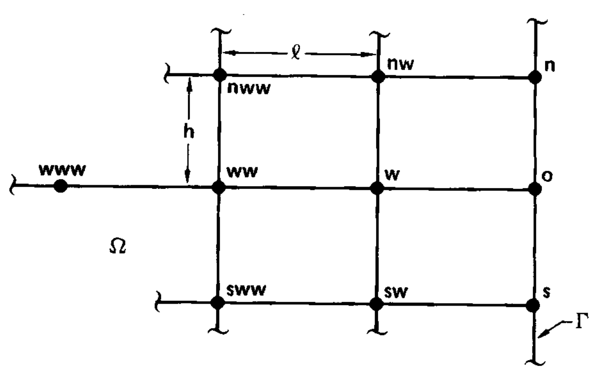

Figure 3. Non-staggered grid near $x = L$

$$\frac{4(P_0 - P_{ww}) - (P_w - P_{www})}{6l} = \nu\left[\frac{4}{3}\left(\frac{u_{ww} - 2u_w + u_0}{l^2} + \frac{u_{nw} - 2u_w + u_{sw}}{h^2}\right)\right.$$
$$\left.-\frac{1}{3}\left(\frac{u_w - 2u_{ww} + u_{www}}{l^2} + \frac{u_{nww} - 2u_{ww} + u_{sww}}{h^2}\right)\right]$$
$$-\left(\dot{u}_0 + \frac{4}{3}A(u)u_w - \frac{1}{3}A(u)u_{ww}\right) - \frac{l}{3}\left(\frac{\dot{v}_n - \dot{v}_s}{h}\right), \tag{18}$$

which clearly consistently approximates

$$\frac{\partial P}{\partial x} = \nu\nabla^2 u - \left(\frac{\partial u}{\partial t} + \mathbf{u}\cdot\nabla u\right) - \frac{l}{3}\frac{\partial}{\partial y}\frac{\partial v}{\partial t}$$

to $O(h, l)$ at node 0 or, as h, $l \to 0$, we again recover (6), applied at $x = L$.

Remarks

(i) If we had used a first-order approximation to $\partial u/\partial x$ at node 0, viz., $(u_0 - u_w)/2l$, the essential final result would be the same; i.e., the Neumann BC on Γ.

(ii) If we modified ('regularized') the scheme *a là* Strikwerda,[13] we would still recover the Neumann BC since his scheme is just an $O(h^2)$ perturbation from the non-staggered scheme analysed above.

Last, for a finite element derivation of the same final result (except that $l = h$), using bilinear approximation for u and v and piecewise constant aproximation for P (an FEM version of a staggered grid), see Gresho.[6] (Note here that the pressure for this element. like that in the MAC finite difference scheme, belongs to the 'generalized' Laplacian referred to earlier, since the pressure is discontinuous from one element to the next.)

These sample results, while certainly not representing a proof, are sufficiently suggestive to permit the assertion that any consistent discretized approximation of the primitive variables will 'automatically include' (i.e., imply) a consistent pressure Poisson equation in Ω *and* the appropriate Neumann BCs on Γ.

Finally, careful analysis of these discrete equations leads to a more succinct and perhaps better way to express all of the above results: given $\mathbf{f}$, to find $\mathbf{a}$ and ∇P from (i)$\mathbf{a} + \nabla P = \mathbf{f}$ and (ii)$\nabla\cdot\mathbf{a} = 0$, it is sufficient to solve $\nabla^2 P = \nabla\cdot\mathbf{f}$ in Ω and $\partial P/\partial n = \mathbf{n}\cdot(\mathbf{f} - \mathbf{a})$ on Γ, where the Neumann BC is also the realization of (ii) on Γ. This argument also makes clear the need to know $\mathbf{n}\cdot\mathbf{a}$ on Γ in order to have a completely-posed problem.

NUMERICAL EXAMPLES

Example 1

This simple but powerful example was inspired by Heywood and Rannacher: Consider (1)–(3) with $\nu = 1$, $\mathbf{w}(t) = \mathbf{0}$ and $\mathbf{u}_0(\mathbf{x})$ with compact support; viz., $\mathbf{u}_0(\mathbf{x})$ is only non-zero in a subdomain (Ω_s) of Ω that does not include any portion of Γ. Figure 4 shows the (25 × 25) domain and the initial velocity field on a uniform 50 × 50 element mesh, which is given by the stream function

$$\psi(x, y) = [\sin^2 \pi(x - x_0)/l][\sin^2 \pi(y - y_0)/h], \tag{19}$$

where $l \times h$ is the inner 11 × 7 rectangle (Ω_s) centred at $(x_0, y_0) = (16{\cdot}5, 7{\cdot}5)$.

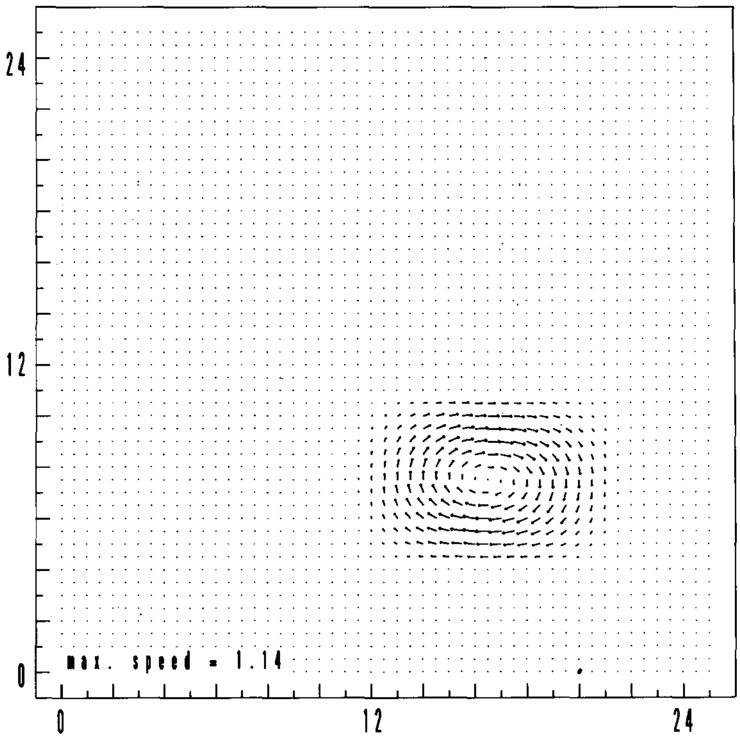

Figure 4. An initial velocity field of compact support that is discretely divergence-free

The initial velocity field is obtained from $u_0 = \partial\psi/\partial y$ and $v_0 = -\partial\psi/\partial x$ evaluated at each node point within Ω_s, with $u_0, v_0 = 0$ in $\Omega - \Omega_s$. While the continuous velocity field is solenoidal, the discrete interpolated one is not; i.e., if $\tilde{v}_0$ represents the nodal interpolant, $C^T\tilde{v}_0 \neq g_0 = 0$ and the problem, as posed, is ill-posed. This is a minor issue, however, if one has the 'mass consistency adjustment mechanism', as described by Gresho *et al.*[30] in his code; viz., $\tilde{v}_0$ is projected onto the discretely divergence-free subspace by first solving for the Lagrange multiplier λ from

$$(C^TM^{-1}C)\lambda = C^T\tilde{v}_0 \tag{20a}$$

and then computing the mass consistent velocity field $\tilde{u}_0$ from

$$\tilde{u}_0 = \tilde{v}_0 - M^{-1}C\lambda, \tag{20b}$$

which obviously satisfies $C^T\tilde{u}_0 = 0$ and is thus a legitimate initial condition for (12); this is the vector field shown in Figure 4, in which $\tilde{u}_0 = 0$ outside of Ω_s. The corresponding initial pressure field P_0 from (14) with $t = 0$, $\dot{g} = 0$ and $b(\tilde{u}_0, t) = K\tilde{u}_0 - A(\tilde{u}_0)\tilde{u}_0$ is shown in Figure 5. It is noteworthy that P_0 is independent of the viscosity for this case—both in the continuum and the discrete case.

At this point we recall the possible BCs for the pressure Poisson equation, (4). For this case the Neumann BC, (6), gives $\partial P_0/\partial n = 0$ and the Dirichlet BC, (7), gives $\partial P_0/\partial\tau = 0$, and we see an obvious contradiction—while either BC is mathematically legitimate for the Poisson equation, the solutions from each would be very different. Clearly only the former is satisfied in Figure 5 and this is in complete agreement with Heywood and Rannacher; the 'overdetermined Neumann problem' is *not* satisfied at $t = 0$. And it is just this violation of $\partial P_0/\partial\tau = 0$ that causes 'regularity problems'—the solution is not as smooth as it would be if (7) were also satisfied at $t = 0$—an

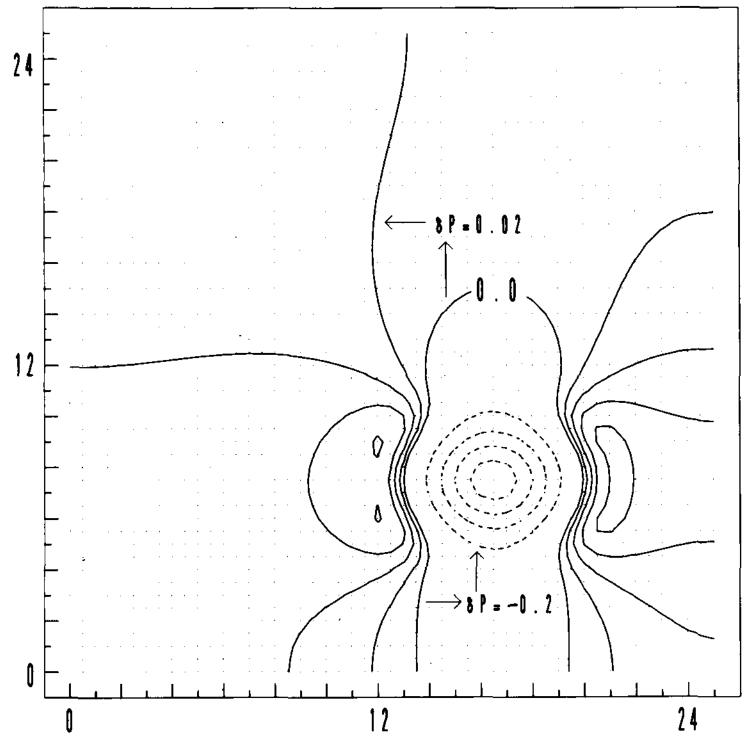

Figure 5. Initial pressure field corresponding to the velocity field of Figure 4. Note the 10-fold increase in contour interval for $P < 0$

apparent impossibility for the problem posed. It is also noteworthy that wall vorticity production is caused by (proportional to) $\partial P/\partial \tau$ *and* the no-slip BC (see, e.g., Panton[36]). For inviscid flow, although $\partial P_0/\partial \tau$ would be the *same*, there would be slip and no vorticity generation—as well as a more regular solution, since then we would have $\partial u_\tau/\partial t = -\partial P/\partial \tau$ on Γ at $t = 0$; the flow would accelerate along Γ instead of generating vorticity there.

Consider, for example, the point at $x = 20$, $y = \varepsilon$, where $0 < \varepsilon \ll 1$; i.e., a point just in from the boundary. At $t = 0$, the x-momentum equation gives $\partial u_0/\partial t = -\partial P_0/\partial x$ for the acceleration in the tangential direction. But at $t = 0^+$, the same equation reads $\partial u/\partial t = -\partial P/\partial x + \nu \nabla^2 u - \mathbf{u}\cdot\nabla u$, wherein the last term (advection) is a quadratic quantity and is negligibly small (i.e., we effectively have a developing Stokes flow at early time), as is $\partial^2 u/\partial x^2$ compared to $\partial^2 u/\partial y^2$. Since $u = 0$ on Γ for all time and $\partial P/\partial x$ is finite (in Ω and on Γ) and (to first approximation) slowly varying both in space and time, the tangential acceleration and $\nu\, \partial^2 u/\partial y^2$ must suffer a jump as $t \to 0$ and $y \to 0$. (The data do not satisfy the global compatibility conditions discussed by Heywood and Rannacher[10] and Temam.[11]) Here too we have an example that would tend to vitiate Morino's[12] interpretation of the behaviour as $t \to 0$, since we indeed see no slip—and the tangential equation of motion is violated at the walls.

For further elucidation of this important point, we show in Figure 6 the significant terms comprising the two pressure BCs (normal and tangential) at $x = 20$, $y = 0$ as a function of time for two uniform mehses: $50 \times 50 = 2500$ elements and $100 \times 100 = 10000$ elements. Realizing that on any given mesh in which some sort of boundary layer phenomenon is being simulated there is a time below which the numerical results are not worth much (the 'minimum time of believability'

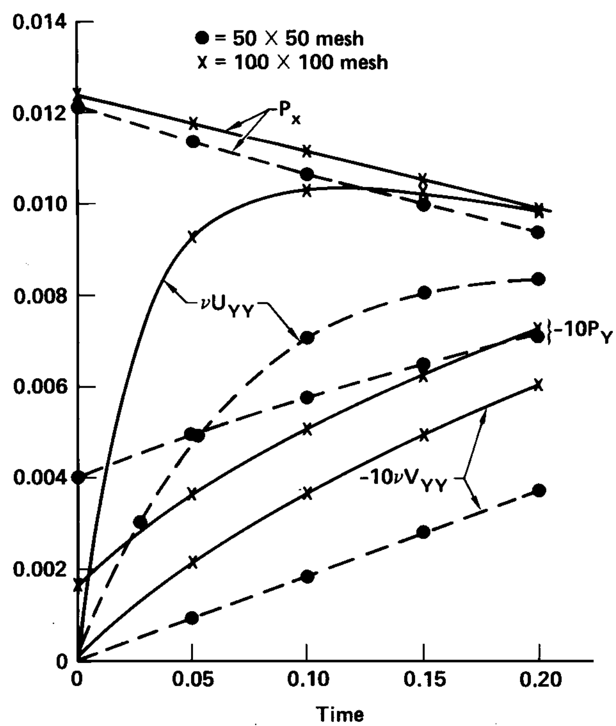

Figure 6. Test of the normal ($\partial P/\partial y = \nu\partial^2 v/\partial y^2$) and tangential ($\partial P/\partial x = \nu\partial^2 u/\partial y^2$) boundary conditions

discussed by Gresho and Lee[37] and Gresho,[38] which time is $O(\Delta y^2/\nu)$, or $O(0{\cdot}25)$ on the 50×50 and $O(0{\cdot}0625)$ on the 100×100 mesh for this problem, the numerical results are quite consistent with the theory: (1) the difference between $\partial P/\partial y$ and $\nu\,\partial^2 v/\partial y^2$ is small, nearly independent of time, and is a measure of the spatial truncation error (it decreases by a factor of $\sim 2{\cdot}8$ from the first to the second mesh); i.e., the Neumann BC is (closely) satisfied—for *all* time; (2) $\nu\,\partial^2 u/\partial y^2$ is trying very hard (for $t > 0$) to equal $\partial P/\partial x$ and is apparently only restricted from doing so by the diffusional time constant of the mesh and then by spatial truncation errors; i.e., the Dirichlet BC is (closely) satisfied for $t > 0$, but is clearly violated at $t = 0$.

(To generate Figure 6, a four-point, second-order accurate finite difference formula, described in the next example, was used to estimate $\partial^2 u/\partial n^2$ and $\partial^2 v/\partial n^2$ at the wall. To get $\partial P/\partial n$ and $\partial P/\partial \tau$, the presence of a checker-board pressure mode—see Sani *et al.*[25]—required first the strict use of $M^{-1}CP$ to remove this mode; after computing $\partial P/\partial y$ and $\partial P/\partial x$ in this way at the first two nodes up from the boundary at $x = 20$, simple second-order extrapolation was employed to estimate the gradient *at* the wall.)

Two other points are worthy of mention: (1) the simplest finite difference solution to $\partial u/\partial t = \nu\,\partial^2 u/\partial y^2 - \partial P/\partial x$, where $\partial P/\partial x$ as a function of time is taken from Figure 6, gives almost the same curve for $\nu\,\partial^2 u/\partial y^2$ at $y = 0$ as is labelled νu_{yy} in Figure 6; (2) the numerical results are also trying to satisfy $\partial u/\partial t = (-\partial P/\partial x)\,\mathrm{erf}[y/2\sqrt{(\nu t)}]$ near $y = 0$ at small t, which is the time derivative of the analytic solution of this 1D 'heat' equation with $\partial P/\partial x$ held constant at its initial value (see, e.g., pp. 78, 130 of Garslaw and Jaeger[39]). This analytic solution,

$$u = \frac{1}{\nu}\left(-\frac{\partial P}{\partial x}\right)\left[\frac{2y^2 + \delta^2}{4}\,\mathrm{erf}\left(\frac{y}{\delta}\right) + \frac{\delta y}{\sqrt{(4\pi)}}\exp\left(-\frac{y^2}{\delta^2}\right) - \frac{y^2}{2}\right],$$

where $\delta \equiv \sqrt{(4\nu t)}$, exhibits a discontinuity in $\partial u/\partial t$ and $\partial^2 u/\partial y^2$ at $y=0$, $t=0$. This close approximation also suggests a more general and probably very important result: the tangential velocity component(s) near a no-slip wall and for small time will respond to the (parabolic) 1D heat equation in which the tangential pressure gradient plays the role of a heat source and is obtained from (i.e., is a consequence of) the initial ($t=0$) Neumann problem for the pressure.

Example 2

Here we revisit the ubiquitous lid-driven cavity and examine some steady-state results for both Stokes flow and high-*Re* flows (5000 and 10000). The solutions (vectors and isobars in the upper right quadrant of the cavity) shown in Figures 7–9 on a 50 × 50 graded mesh of bilinear velocity/piecewise-constant pressure elements were obtained using the modified finite element technique [applied to (12)] described by Gresho *et al.*[30] Figure 10 shows the tangential velocities at two locations near the solid boundaries for the two high-*Re* cases, $v(x)$ versus x at $y=0.75$ and $u(y)$ versus y at $x=0.75$, corresponding respectively to the points marked A and B in Figures 7–9. The points marked on these curves show that the boundary layer thickness follows fairly well the equation $\delta = 4/\sqrt{Re}$; i.e., the solution in the neighbourhood of these points is probably well-described by boundary layer theory.

To test these results with respect to BC satisfaction, we estimated $\partial P/\partial n$, $\partial P/\partial \tau$, $\partial^2 u_n/\partial n^2$ and $\partial^2 u_\tau/\partial n^2$ at points A and B of Figures 7–9. Since the numerical solution for velocity and pressure is probably about second-order accurate, we use second-order approximations to the derivatives. To

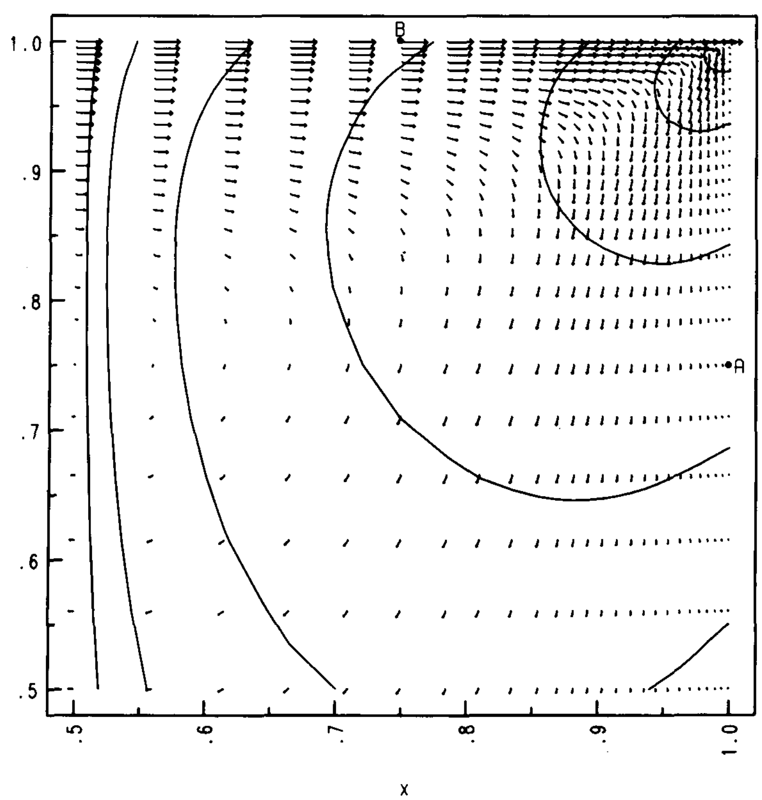

Figure 7. Vectors and isobars in the upper right quadrant of the lid-driven cavity; $Re=0$. Contours shown are: 10^{-4}, 3×10^{-4}, 10^{-3}, 3×10^{-3}, 10^{-2}, 3×10^{-2}, 10^{-1}

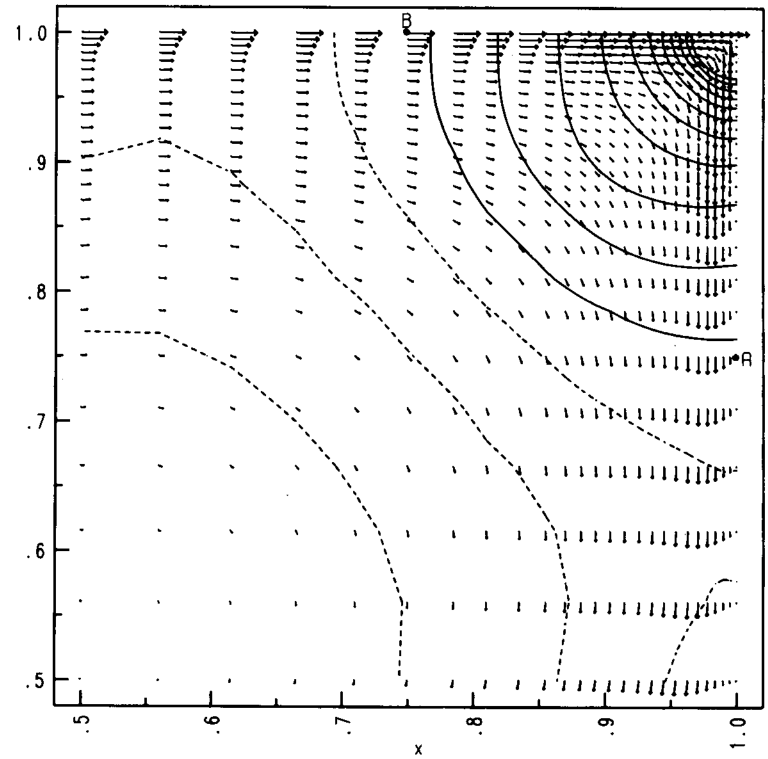

Figure 8. Same as Figure 7 except $Re = 5000$ and the contours range from $-0{\cdot}06(0{\cdot}02)$ to $+0{\cdot}20$

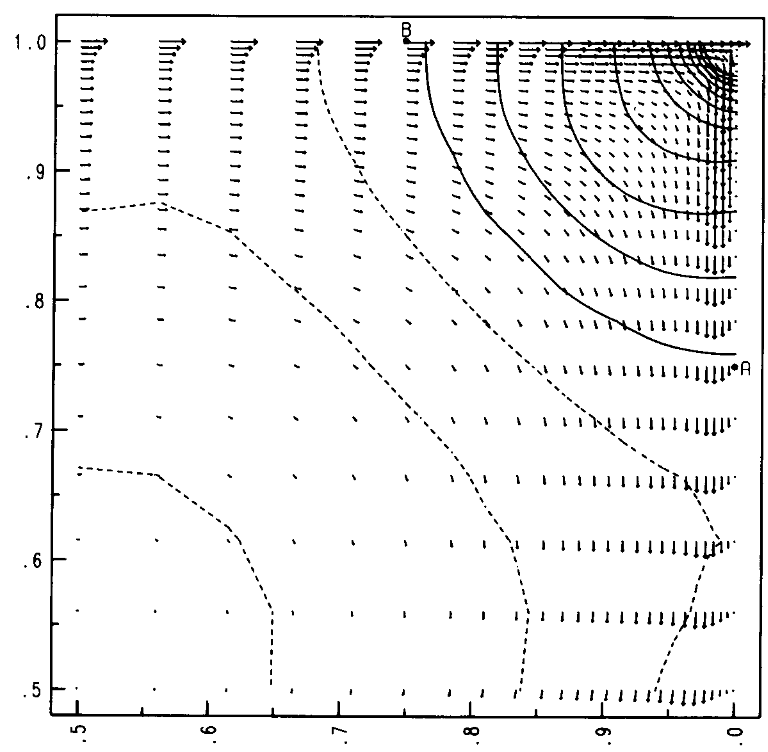

Figure 9. Same as Figure 8 except $Re = 10000$

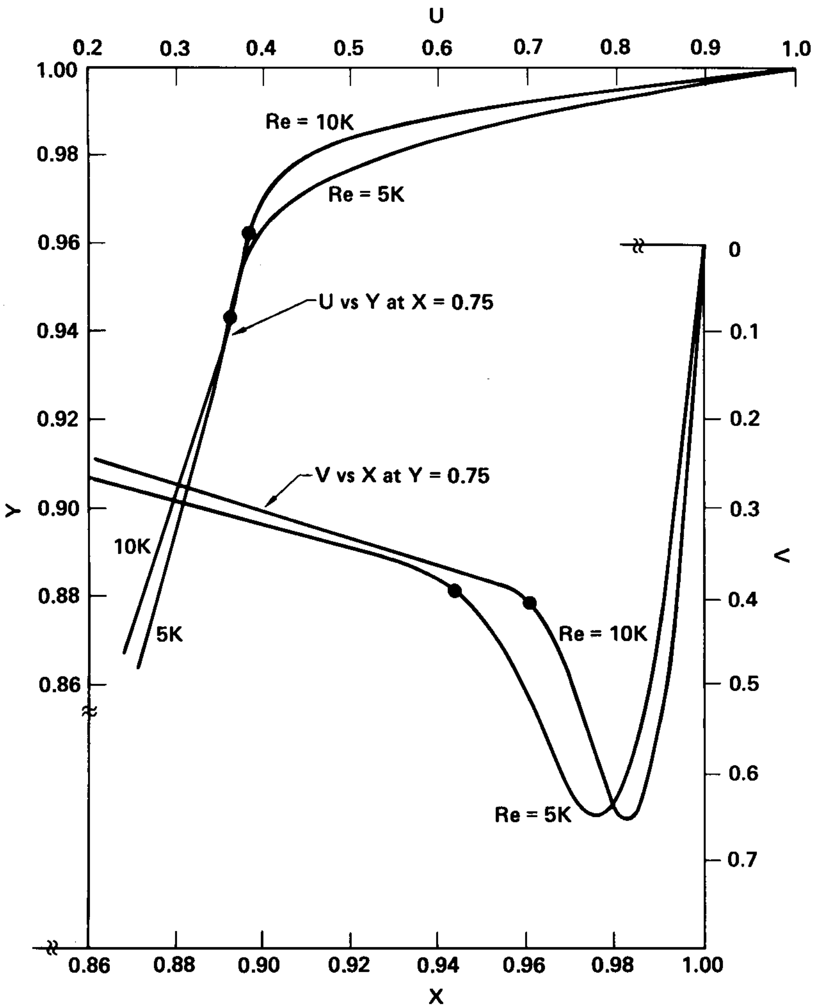

Figure 10. Tangential velocities near the walls for two Reynolds numbers. The full circles are placed at a distance $4/\sqrt{Re}$ from the walls

compute $\partial P/\partial x$ at the wall, for example, we first compute $M_x^{-1} C_x P$ from the raw data (M is a lumped mass matrix), which converts element centroid pressures to nodal pressure gradients, at the first two nodes in from the wall (which involves six elements). Then a linear extrapolation from these nodes to the wall is utilized to obtain $\partial P/\partial n$ and $\partial P/\partial \tau$ at the walls. For the velocity derivatives we used the following one-sided finite difference formula,

$$\begin{aligned}\partial^2\phi/\partial n^2 = {} & 2(2l_1 + l_2 + l_3)(\phi_2 - \phi_0)/l_2 l_3(l_1 + l_2) \\ & - 2(2l_1 + 2l_2 + l_3)(\phi_1 - \phi_0)/l_1 l_2(l_2 + l_3) \\ & - 2(2l_1 + l_2)(\phi_3 - \phi_0)/l_3(l_2 + l_3)(l_1 + l_2 + l_3),\end{aligned}$$

where node 0 is at the wall, node 1 is at l_1 (normal distance from the wall), node 2 is at $(l_1 + l_2)$ and node 3 is at $(l_1 + l_2 + l_3)$; and ϕ is u or v.

The results are shown in Table I, where we selected $\nu = 10^{-3}$ for Stokes flow simply to put the results on a similar scale to the other two cases (the pressure scales with ν for Stokes flows).

Table I. Driven cavity test results ($\nu = 0{\cdot}001$ for $Re = 0$, $\nu = 1/Re$ for $Re > 0$)

Re	Neumann BC		Dirichlet BC	
	$\frac{\partial P}{\partial n}$	$\nu\frac{\partial^2 u_n}{\partial n^2}$	$\frac{\partial P}{\partial \tau}$	$\nu\frac{\partial^2 u_\tau}{\partial n^2}$
	Right sidewall ($x = 1, y = 0{\cdot}75$; point A)			
0	$-0{\cdot}014$	$-0{\cdot}016$	0·036	0·035
5000	$-0{\cdot}029$	$-0{\cdot}025$	0·27	0·23
10000	$-0{\cdot}021$	$-0{\cdot}016$	0·27	0·25
	Top wall ($x = 0{\cdot}75, y = 1$; point B)			
0	$-0{\cdot}0079$	$-0{\cdot}0078$	0·023	0·027
5000	$-0{\cdot}0073$	$-0{\cdot}0086$	0·32	0·27
10000	$-0{\cdot}0036$	$-0{\cdot}0036$	0·30	0·25

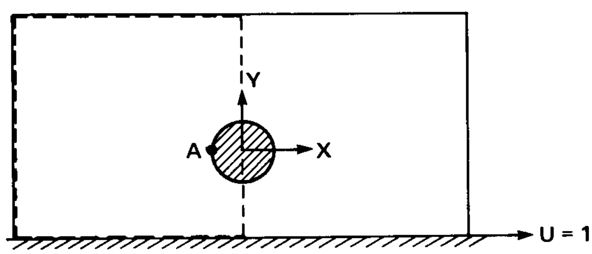

Figure 11. Geometry and computational domain (broken line) for Stokes flow about a stationary cylinder near a moving wall

While it is apparent that both BCs are reasonably well-satisfied, we must admit that similar calculations at other wall locations (e.g., close to the singularity at $x = y = 1$ or close to one of the lower corners where all terms are quite small) were not so encouraging. On the other hand, the BCs on pressure were usually satisfied to about the same accuracy as were the momentum equations themselves (using all terms) away from the wall—again using difference quotients to approximate derivatives.

Finally, Figures 7–9 show that, while $\partial P/\partial n = 0$ on the walls is nearly true for the large-Re cases, except of course very close to the corner singularities (not shown), it is far from true for Stokes flow.

Example 3

The final example is steady Stokes flow (for $\nu = 1$) about a stationary cylinder located close to a moving wall, as illustrated in Figure 11. Both an exact solution and detailed numerical results generated via a primitive variable GFEM formulation are available (Maslanik *et al.*[40]), the streamlines for which are shown in Figure 12. The exact and GFEM nodal velocities and elemental pressures for that portion of the mesh illustrated in Figure 13 are presented in Tables II and III. Using these results and the second-order accurate finite difference formulae discussed in the previous example, the relative error in the Neumann and Dirichlet boundary conditions at, for example, point A (node 26) in Figures 11 and 13 can be estimated, i.e.,

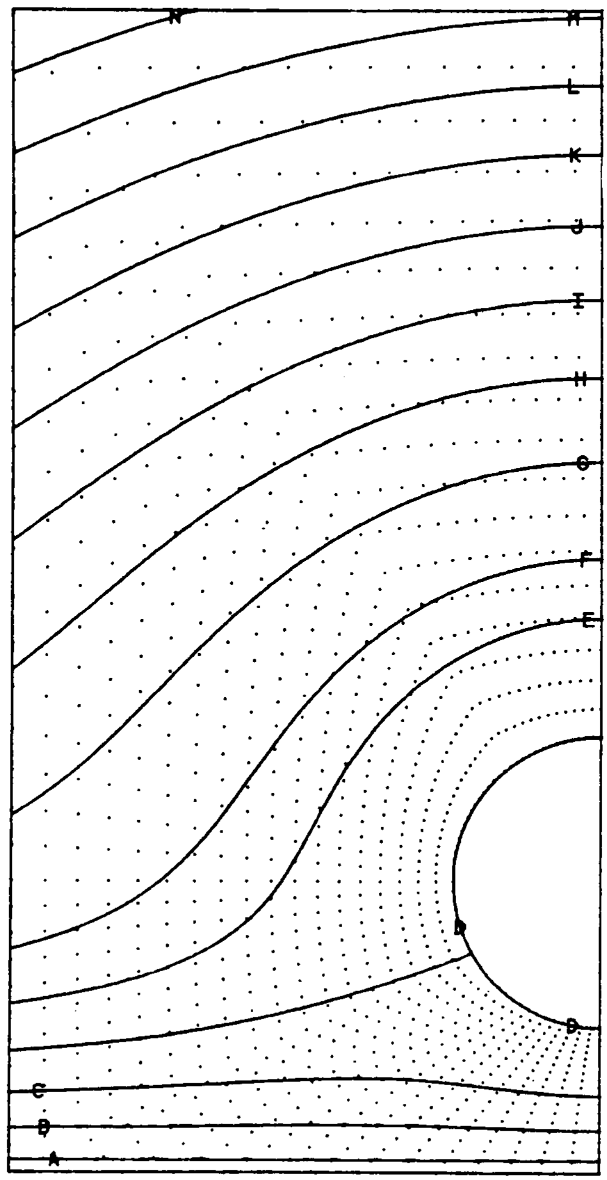

Figure 12. Typical mesh and streamlines for flow around the cylinder of Figure 11

$$\mathrm{EN} \equiv \left|\frac{\partial^2 u}{\partial x^2} - \frac{\partial p}{\partial x}\right| \Big/ \left|\frac{\partial^2 u}{\partial x^2}\right| \quad \text{and} \quad \mathrm{ED} \equiv \left|\frac{\partial^2 v}{\partial x^2} - \frac{\partial p}{\partial y}\right| \Big/ \left|\frac{\partial^2 v}{\partial x^2}\right|.$$

Since both exact as well as GFEM results are available, these errors can be evaluated for both cases in order to also obtain some assessment of the truncation error of the difference scheme on this mesh. The errors from the GFEM solution are EN = 0·07 and ED = 0·25, and the corresponding errors for the exact values are EN = 0·003 and ED = 0·32. Again here the primitive variable GFEM method is trying to satisfy both the Neumann- and Dirichlet-type boundary

Table II. FEM and exact nodal velocities for mesh of Figure 13

Node	x	y	u	v	u_{ex}	v_{ex}
24	−0·2480	−0·03133	0·0	0·0	$-0{\cdot}9027 \times 10^{-7}$	$0{\cdot}7146 \times 10^{-6}$
25	−0·2495	−0·01570	0·0	0·0	$-0{\cdot}4855 \times 10^{-7}$	$0{\cdot}7717 \times 10^{-6}$
26	−0·2500	0·0	0·0	0·0	0·0	$0{\cdot}2665 \times 10^{-14}$
27	−0·2497	0·01266	0·0	0·0	$-0{\cdot}1762 \times 10^{-7}$	$-0{\cdot}3474 \times 10^{-6}$
28	−0·2487	0·02529	0·0	0·0	$-0{\cdot}9484 \times 10^{-7}$	$-0{\cdot}4327 \times 10^{-6}$
81	−0·2769	−0·03442	$-0{\cdot}1746 \times 10^{-2}$	$0{\cdot}5479 \times 10^{-1}$	$-0{\cdot}1492 \times 10^{-2}$	$0{\cdot}5446 \times 10^{-1}$
82	−0·2783	−0·01724	$0{\cdot}5995 \times 10^{-3}$	$0{\cdot}6005 \times 10^{-1}$	$0{\cdot}8075 \times 10^{-3}$	$0{\cdot}5977 \times 10^{-1}$
83	−0·2788	0·0	$0{\cdot}3716 \times 10^{-2}$	$0{\cdot}6468 \times 10^{-1}$	$0{\cdot}3884 \times 10^{-2}$	$0{\cdot}6444 \times 10^{-1}$
84	−0·2785	0·01513	$0{\cdot}7007 \times 10^{-2}$	$0{\cdot}6826 \times 10^{-1}$	$0{\cdot}7147 \times 10^{-2}$	$0{\cdot}6818 \times 10^{-1}$
85	−0·2776	0·03025	$0{\cdot}1072 \times 10^{-1}$	$0{\cdot}7176 \times 10^{-1}$	$0{\cdot}1091 \times 10^{-1}$	$0{\cdot}7176 \times 10^{-1}$
139	−0·3084	−0·01881	$0{\cdot}9588 \times 10^{-2}$	0·1052	$0{\cdot}1007 \times 10^{-1}$	0·1047
140	−0·3088	0·0	$0{\cdot}1388 \times 10^{-1}$	0·1137	$0{\cdot}1432 \times 10^{-1}$	0·1133
141	−0·3085	0·01765	$0{\cdot}1916 \times 10^{-1}$	0·1209	$0{\cdot}1960 \times 10^{-1}$	0·1206
196	−0·3397	−0·02041	$0{\cdot}2472 \times 10^{-1}$	0·1397	$0{\cdot}2543 \times 10^{-1}$	0·1392
197	−0·3401	0·0	$0{\cdot}2898 \times 10^{-1}$	0·1514	$0{\cdot}2967 \times 10^{-1}$	0·1509
198	−0·3398	0·0202	$0{\cdot}3529 \times 10^{-1}$	0·1619	$0{\cdot}3593 \times 10^{-1}$	0·1614

Table III. FEM and exact elemental pressures for mesh of Figure 13

Element	x_c	y_c	p	p_{ex}
24	−0·2632	0·4753	5·5793	5·5759
25	−0·2642	0·4918	5·3080	5·2645
26	−0·2643	0·5069	4·9761	4·9821
27	−0·2636	0·5208	4·7662	4·7293
80	−0·2927	0·4730	5·2518	5·2051
81	−0·2936	0·4910	4·9494	4·9425
82	−0·2937	0·5082	4·7359	4·6913
83	−0·2931	0·5246	4·4599	4·4534
137	−0·3242	0·4902	4·6777	4·6325
138	−0·3243	0·5095	4·4181	4·4087

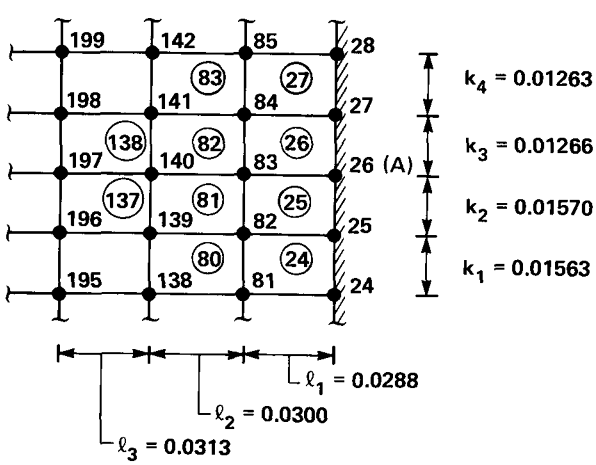

Figure 13. Schematic of the finite element mesh (bilinear velocity, piecewise constant pressure) near point A of Figure 11

conditions (as does the analytic solution); moreover, this case is an example of one in which the sometimes-used homogeneous Neumann boundary condition would be inappropriate.

DISCUSSION

First we return to reconsider, amplify and re-emphasize an important point made earlier. The incompressible NS equations can be written as

$$\mathbf{a} + \nabla P = \mathbf{f}(\mathbf{u}) \quad \text{in} \quad \Omega, \tag{21a}$$

$$\nabla \cdot \mathbf{a} = 0 \quad \text{in} \quad \bar{\Omega}, \tag{21b}$$

where $\mathbf{a} = \partial \mathbf{u}/\partial t$ is the acceleration and $f(\mathbf{u}) = \nu \nabla^2 \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u}$. Given a (sufficiently smooth) solenoidal velocity field that satisfies the BC on the normal component ($\mathbf{n} \cdot \mathbf{u} = \mathbf{n} \cdot \mathbf{w}$), $\mathbf{f}(\mathbf{u})$ is known and (21) can be used to compute $\mathbf{a}$ and P. This can be done as follows:

(i) Solve $\nabla^2 P = \nabla \cdot \mathbf{f}(\mathbf{u})$ in Ω, which can be used in lieu of (and is equivalent to) (21b) in the domain.
(ii) Use $\partial P/\partial n = \mathbf{n} \cdot [\mathbf{f}(\mathbf{u}) - \mathbf{a}]$ on Γ, which is the equivalent manifestation of (21a) *and* (21b) on the boundary.
(iii) Set $\mathbf{n} \cdot \mathbf{a} = \mathbf{n} \cdot \mathrm{d}\mathbf{w}/\mathrm{d}t = \dot{w}_n$ [see (2a)] on Γ, which sets the value of the BC in (ii) and completes the specification of a well-posed problem for P; it is also *another* boundary realization of $\nabla \cdot \mathbf{a} = 0$ and assures continuity in $\mathbf{n} \cdot \mathbf{a}$ as $\mathbf{x} \to \Gamma$.
(iv) Compute $\mathbf{a}$ in Ω from (21a). If also $\boldsymbol{\tau} \cdot \mathbf{u} = \boldsymbol{\tau} \cdot \mathbf{w}$ on Γ via (no-slip) BCs, which requires $\nu > 0$ and generally requires $t > 0$, then (21a) will also apply on Γ and yields $\mathbf{a} = \dot{\mathbf{w}}$ there. (At $t = 0$, it is not necessary to have $\boldsymbol{\tau} \cdot \mathbf{u} = \boldsymbol{\tau} \cdot \mathbf{w}$ on Γ from which it follows that the tangential acceleration, $\boldsymbol{\tau} \cdot \mathbf{a}$, is generally singular for $\mathbf{x} \to \Gamma$ and $\nu > 0$—although it is always true that $\mathbf{n} \cdot \dot{\mathbf{a}} = \mathbf{n} \cdot \dot{\mathbf{w}}$ on Γ at $t = 0$.)
(v) A simple (in principle!) time integration of $\mathbf{a}$ then gives the velocity field.

Having now accomplished some of our major goals, it is also of interest to compare the correct approach to the PPE and BCs to others, some of which have been used with varying degrees of success to generate numerical algorithms.

Given the ICs and BCs of (2) and (3), we consider a sequence of problems, defined below, each of which involves the selection of one PPE and one associated BC:

A. *Consistent PPE*:

$$\nabla^2 P = \nabla \cdot (\nu \nabla^2 \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u}). \tag{CPPE}$$

B. *Simplified PPE*:

$$\nabla^2 P = -\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}). \tag{SPPE}$$

C. *Consistent Neumann BC*:

$$\partial P/\partial n = \mathbf{n} \cdot (\nu \nabla^2 \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} - \partial \mathbf{u}/\partial t). \tag{CNBC}$$

D. *Arbitrary Neumann BC*:

$$\partial P/\partial n = N(\mathbf{x}). \tag{ANBC}$$

E. *Arbitrary Dirichlet BC*:

$$P = D(\mathbf{x}). \tag{ADBC}$$

We now examine the consequences of solving the CPPE or the SPPE with one of the above three BCs, along with the momentum equation (1a), and discuss the nature of the resulting solutions (when they exist). To do so, we shall, in each case, perform two principal steps:

(1) Form an equation for $\nabla\cdot\mathbf{u}$ by subtracting the PPE from the divergence of (1a); i.e., from

$$\nabla^2 P = \nabla\cdot(\nu\,\nabla^2\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} - \partial\mathbf{u}/\partial t).$$

(2) Apply the divergence theorem to the PPE and its BC; this will test for consistency and BC 'compatibility' (to be defined below).

The results from step (1) are easy to state:

(i) The CPPE approach implies

$$\nabla\cdot(\partial\mathbf{u}/\partial t) \equiv \nabla\cdot\mathbf{a} = 0 \quad \text{in} \quad \Omega;$$

the acceleration remains divergence-free and thus so does the velocity.

(ii) The SPPE approach implies

$$\nabla\cdot(\partial\mathbf{u}/\partial t - \nu\,\nabla^2\mathbf{u}) = 0$$

or, what is presumably equivalent,

$$\partial(\nabla\cdot\mathbf{u})/\partial t = \nu\,\nabla^2(\nabla\cdot\mathbf{u}),$$

a transient 'heat' equation for the divergence $\Theta \equiv \nabla\cdot\mathbf{u}$. Here, while Θ starts at zero, it will only assuredly remain zero if some (additional) mechanism exists for holding $\nabla\cdot\mathbf{u}$ (or its normal derivative) at zero in the boundary. For solutions which are at least C^0, this is equivalent to enforcing $\mathbf{n}\cdot\mathbf{u} = \mathbf{n}\cdot\mathbf{w}$ on Γ, as discussed earlier. Since no *a priori* 'mechanism' is obviously operative in general, it must be assumed that in such a case the velocity would wander away from the divergence-free subspace. We will return to this issue below.

It is interesting and perhaps even paradoxical that the SPPE was generated by assuming $\nabla\cdot\mathbf{u} = 0$ yet its use cannot guarantee same. But if we do *not* assume $\nabla\cdot\mathbf{u} = 0$ (in the viscous term), the CPPE is obtained and it *assures* that $\mathbf{u}$ remains divergence-free: If you include it, you don't need it; if you don't include it, you need it. Perhaps this paradox, which we shall resolve in due course, is related to some numerical methods of the past that behaved 'strangely'.

Now we move to the BCs and apply step (2) above to the six possible cases:

(1) *CPPE + CNBC*. Application of the divergence theorem to CPPE, using CNBC, gives simply $\int_\Gamma \mathbf{n}\cdot\mathbf{a} = 0$ as a solvability/consistency constraint. But, as noted earlier,

$$\int_\Gamma \mathbf{n}\cdot\mathbf{a} = \int_\Gamma \mathbf{n}\cdot\left(\frac{\partial\mathbf{u}}{\partial t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \mathbf{n}\cdot\mathbf{u} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \mathbf{n}\cdot\mathbf{w},$$

which is zero because of (2b) and (3c). This is the 'clean' case.

(2) *CPPE + ANBC*. Here we obtain the constraint

$$\int_\Gamma N = \int_\Gamma \mathbf{n}\cdot(\nu\,\nabla^2\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u}),$$

which will not generally be satisfied; i.e., no solution exists for general $N(x)$. For a special and (too?) common case, however, a solution *will* exist; viz., if $\mathbf{u} = \mathbf{0}$ on Γ and N is taken as zero (i.e., $\partial P/\partial n = 0$ on Γ), the solvability condition is trivially satisfied—all terms are zero. (To show that the viscous term does indeed vanish, use

$$\int_\Gamma \mathbf{n}\cdot\nabla^2\mathbf{u} = \int_\Omega \nabla\cdot(\nabla^2\mathbf{u}) = \int_\Omega \nabla\cdot[\nabla(\nabla\cdot\mathbf{u}) - \nabla\times\nabla\times\mathbf{u}]$$

In this case, however, as discussed earlier, the solution will generally not be the desired one; i.e., it will not 'smoothly' satisfy (1)–(3) because the normal momentum equation does not want to see $\partial P/\partial n = 0$ on Γ. Rather, it will lead to a BC 'incompatibility', discussed below.

(3) *CPPE + ADBC*. This case is especially subtle since a solution *always* exists (there is no solvability constraint), $\nabla\cdot\mathbf{u}$ remains zero in Ω and $\mathbf{u} = \mathbf{w}$ on Γ, yet the solutions will generally not be 'right'. The reason is simply that the acceleration $\mathbf{a}$, and thus $\mathbf{u}$, will be wrong near Γ because P, and thus ∇P, is. $\mathbf{a}$ will disagree with $\dot{\mathbf{w}}$ both in magnitude and direction; i.e., the solution would actually *tend* to (try to) violate the imposed BC($\mathbf{u} = \mathbf{w}$ on Γ)—although it will not in fact do so. To see this, write the NS equations (again) as $\mathbf{a} = \mathbf{f}(\mathbf{u}) - \nabla P$, where $\nabla^2 P = \nabla\cdot\mathbf{f}$, so that $\nabla\cdot\mathbf{a} = 0$ in Ω; but for a given $\mathbf{f}$ there is only *one* vector field, ∇P, that will give $\mathbf{a}\to\dot{\mathbf{w}}$ as $\mathbf{x}\to\Gamma$. Since ∇P is a function of $D(\mathbf{x})$, there is clearly only one correct $D(\mathbf{x})$. Thus, for *arbitrary* $D(\mathbf{x})$, $\mathbf{u}$ will not *smoothly* approach $\mathbf{w}$ as $\mathbf{x}\to\Gamma$, leading to a loss of regularity at Γ. Perhaps the consequence of this behaviour is most easily seen if $\mathbf{w} = \mathbf{0}$; here the solution (very) near Γ would generally tend to display 'spurious' inflows and outflows—but since $\nabla\cdot\mathbf{u} = 0$ in Ω, the net inflow 'tendency' will balance the net outflow 'tendency'. (Note that $\nabla\cdot\mathbf{u} = 0$ in $\Omega \Rightarrow \int_\Gamma \mathbf{u}\cdot\mathbf{n} = 0$, but *not* $\mathbf{u}\cdot\mathbf{n} = 0$ on Γ; there are many ways to satisfy the *global* constraint, but only one, viz., $\mathbf{u}\cdot\mathbf{n} = \mathbf{w}\cdot\mathbf{n}$ on Γ, that will also give a *continuous* normal velocity field in $\bar{\Omega}$—and thus be correct.) We will refer to such behaviour as a 'boundary condition incompatibility'.

(4) *SPPE + CNBC*. The solvability condition here is

$$\int_\Gamma \mathbf{n}\cdot(\nu\nabla^2\mathbf{u} - (\partial\mathbf{u}/\partial t)) = 0$$

which is the compatible BC for the transient diffusion equation associated with the SPPE; i.e., this combination is solvable. Also, the CNBC provides just that 'mechanism' to ensure $\mathbf{n}\cdot\mathbf{a}\to\mathbf{n}\cdot\dot{\mathbf{w}}$ as $\mathbf{x}\to\Gamma$, so that $\Theta = 0$ is the (implied) BC for the (implied) divergence 'heat' equation; i.e., ∇P is correct, $\Theta = 0$ in $\bar{\Omega}$ and the correct solution is realized.

Considering also case (1), *this BC makes (4a) and (4b) equivalent*, and thus (4a) and (6) are also well-posed.

(5) *SPPE + ANBC*. As for the CPPE case, it is clear that no solution exists in general. Again as in that case, the special combination of $\partial P/\partial n = 0$ and $\mathbf{u} = \mathbf{0}$ on Γ *will* yield a solution—but here it will be exceptionally poor; it will generally not conserve mass ($\nabla\cdot\mathbf{u} \neq 0$) and it will display the BC incompatibility.

(6) *SPPE + ADBC*. In this last case, we obtain another bad result in general: while the solution will always exist, it will generally violate $\nabla\cdot\mathbf{u} = 0$ and be incompatible at the boundary. As in case (3), however, there is a *special* $D(\mathbf{x})$ which gives the right answer. Before pursuing this important point, we present in Table IV a summary of the six cases considered thus far.

It is interesting and relevant to study further the case CPPE + ADBC and (especially) the last

Table IV. Behaviour of various 'pressure schemes'

Scheme	Solves the NS equations and BCs, (1)–(3)	Satisfies $\nabla\cdot\mathbf{u}=0$ but displays BC incompatibility	Violates $\nabla\cdot\mathbf{u}=0$ and displays BC incompatibility	No solution exists
CPPE + CNBC	●			
CPPE + ANBC		●†		●*
CPPE + ADBC		●‡		
SPPE + CNBC	●			
SPPE + ANBC			●†	●*
SPPE + ADBC			●‡	

Notes:
* The general case.
† Special cases (e.g., $\partial P/\partial n=0, \mathbf{u}=\mathbf{0}$ on Γ).
‡ If the Dirichlet data are appropriate (and special, particular, as discussed below) rather than arbitrary, then the BC incompatibility vanishes and (1)–(3) are satisfied.

case, SPPE + ADBC, for reasons that will soon become clear. Consider the Poisson/Dirichlet problem

$$\nabla^2 P=S \quad \text{in} \quad \Omega, \quad P=D \quad \text{on} \quad \Gamma, \tag{22}$$

and the related Green's function problem

$$\nabla^2 G=-\delta(\mathbf{x}-\boldsymbol{\xi}) \quad \text{in} \quad \Omega, \quad G=0 \quad \text{on} \quad \Gamma. \tag{23}$$

If $G(\mathbf{x};\boldsymbol{\xi})$ were known, which we assume to be the case, the solution to (22) could be obtained by integration; viz.,

$$P(\mathbf{x})=-\int_\Omega S(\boldsymbol{\xi})G(\mathbf{x};\boldsymbol{\xi})\,\mathrm{d}\Omega_\xi-\int_\Gamma D(\boldsymbol{\xi})\frac{\partial G}{\partial n_\xi}(\mathbf{x};\boldsymbol{\xi})\,\mathrm{d}\Gamma_\xi \quad \text{for} \quad \mathbf{x}\in\bar{\Omega}, \tag{24}$$

and the pressure gradient is the vector field

$$\nabla P(\mathbf{x})=-\int_\Omega S(\boldsymbol{\xi})\nabla_x G(\mathbf{x};\boldsymbol{\xi})\,\mathrm{d}\Omega_\xi-\int_\Gamma D(\boldsymbol{\xi})\frac{\partial}{\partial n_\xi}\nabla_x G(\mathbf{x};\boldsymbol{\xi})\,\mathrm{d}\Gamma_\xi. \tag{25}$$

Inserting, e.g., $S=-\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})$ for the SPPE into (25) and (25) into the linear momentum equation, (1a), produces an aceleration field

$$\mathbf{a}=\nu\nabla^2\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{u}-\int_\Omega \nabla_\xi\cdot(\mathbf{u}\cdot\nabla_\xi\mathbf{u})\nabla_x G(\mathbf{x};\boldsymbol{\xi})\,\mathrm{d}\Omega_\xi+\int_\Gamma D(\boldsymbol{\xi})\frac{\partial}{\partial n_\xi}\nabla_x G(\mathbf{x};\boldsymbol{\xi})\,\mathrm{d}\Gamma_\xi. \tag{26}$$

But, as mentioned earlier, this acceleration will not agree with $\mathrm{d}\mathbf{w}/\mathrm{d}t$ on Γ in general and the solution would therefore display discontinuities at $\mathbf{x}\in\Gamma$ and, if the SPPE were used, become non-divergence-free as well. If, however, $D(\mathbf{x})$ were special, rather than arbitrary, the correct solution *could* be obtained. So we will now use (26) to determine the *proper* boundary pressure, $D(\mathbf{x})$, by setting $\mathbf{n}\cdot\mathbf{a}=\mathbf{n}\cdot\mathrm{d}\mathbf{w}/\mathrm{d}t\equiv\dot{w}_n$ on Γ to smoothly enforce mass conservation on and near the boundary:

$$\int_{\Gamma} D(\xi)\frac{\partial}{\partial n_\xi}\frac{\partial}{\partial n_x}G(\mathbf{x};\xi)\,\mathrm{d}\Gamma_\xi = \dot{w}_n + \mathbf{n}\cdot(\mathbf{u}\cdot\nabla\mathbf{u} - \nu\nabla^2\mathbf{u}) + \int_{\Omega}\nabla_\xi\cdot(\mathbf{u}\cdot\nabla_\xi\mathbf{u})\frac{\partial}{\partial n_x}G(\mathbf{x};\xi)\,\mathrm{d}\Omega_\xi \quad \text{for} \quad x\in\Gamma, \tag{27}$$

which is a boundary integral equation for $P_\Gamma \equiv D(\xi)$; i.e., the right-hand side of (27) is known, as is $G(\mathbf{x};\xi)$, so (27) yields P—at least in principle.

Another and simpler approach to the *consistent* Dirichlet boundary condition (CDBC) begins with the following (generalized/modified) Green's function problem:

$$\nabla^2\tilde{G} = -\delta(\mathbf{x}-\xi) + K \quad \text{in} \quad \Omega, \quad \partial\tilde{G}/\partial n = 0 \quad \text{in} \quad \Gamma, \tag{28}$$

where the addition of the constant K, with $K^{-1} \equiv \int_\Omega \mathrm{d}\Omega$, is required in order to have a well-posed problem. Again assuming that $\tilde{G}(\mathbf{x};\xi)$ is available, the pressure is (more) easily obtained from

$$P(\mathbf{x}) = \int_\Gamma N(\xi)\tilde{G}(\mathbf{x};\xi)\,\mathrm{d}\Gamma_\xi - \int_\Omega S(\xi)\tilde{G}(\mathbf{x};\xi)\,\mathrm{d}\Omega_\xi \tag{29}$$

for $\mathbf{x}, \xi\in\bar{\Omega}$, where the arbitrary constant has been set via $\int_\Omega P(\mathbf{x})\,\mathrm{d}\Omega = 0$ and $N(\xi)$ is the consistent Neumann data for $\partial P/\partial n$, obtained from (6). In particular, the boundary pressure is obtained by restricting $\mathbf{x}$ to $\mathbf{x}\in\Gamma$ and performing a simple quadrature rather than solving a boundary integral equation. Thus, to determine P_Γ, it is actually only necessary to know $\tilde{G}(\mathbf{x};\xi)$ for $\mathbf{x}\in\Gamma$, even though we need $\xi\in\bar{\Omega}$.

The resulting boundary pressure, from (24) or (29) applied on Γ, is the *unique* (up to an additive constant) pressure that simultaneously removes the BC incompatibility (there is no more jump in the normal velocity) and assures that $\nabla\cdot\mathbf{u}$ remains zero in $\bar{\Omega}$; i.e., for the SPPE, this Dirichlet BC on pressure (the CDBC) provides just that 'mechanism' required to prevent Θ from being generated on Γ and diffusing into Ω. *If* this BC is employed, it is clear that (4a) and (4b) are again equivalent—as for the CNBC.

Since (29) looks somewhat more 'attractive' than (27) for computing P_Γ, it may be interesting to at least speculate upon the construction of an approximation to $\tilde{G}$. To do this, let

$$\tilde{G}(\mathbf{x};\xi) \equiv F(\mathbf{x};\xi) + \Psi(\mathbf{x};\xi), \tag{30}$$

where $F(\mathbf{x};\xi)$ is a 'fundamental solution' to Laplace's equation—i.e., a 'free space' Green's function (see, e.g., Greenberg,[41] Stakgold[42])—and $\Psi(\mathbf{x};\xi)$ is the regular part of $\tilde{G}$ in Ω. Thus, since F satisfies $\nabla^2 F = -\delta(\mathbf{x}-\xi)$ in Ω and is *known*, the Ψ-problem is

$$\nabla^2\Psi(\mathbf{x};\xi) = K \quad \text{in} \quad \Omega, \tag{31a}$$

$$\frac{\partial\Psi}{\partial n_x}(\mathbf{x};\xi) = -\frac{\partial F}{\partial n_x}(\mathbf{x};\xi) \quad \text{on} \quad \Gamma \tag{31b}$$

where, since $\tilde{G}$, F and Ψ are symmetric in $\mathbf{x}$ and ξ, we need only solve (31) for $\xi\in\Gamma$, although $\mathbf{x}\in\bar{\Omega}$. Thus, on a discrete mesh, (31) need be solved once for each boundary 'pressure node' (and only once per NS *problem*) in order that (29) can be used to compute P_Γ at these same nodes at any $t \geqslant 0$.

Having shown how to relate the Neumann data for the PPE to Dirichlet data, we remark that ideas similar to these are sometimes used when discrete time integration is invoked in which at least *some semi*-implicitness is employed—most typically in the viscous terms. These methods also are typically applied to the SPPE rather than the CPPE and are exemplified in, e.g., in Glowinski and

Pironneau[43] (see also Thomasset[44] and Glowinski[29]), Kleiser and Schumann,[45] Marcus,[46] le Quere and de Roquefort[47] and Quartapelle and Napolitano.[48] While the first three and the last employ finite elements to generate a numerical algorithm, the other three use spectral methods. One difference in our derivation/interpretation is the identification of $\mathbf{u}\cdot\mathbf{n}=\mathbf{w}\cdot\mathbf{n}$ on Γ as the *boundary realization* of $\nabla\cdot\mathbf{u}=0$. [See Gresho and Chan[31] and Bullister[52] for an alternate approach to semi-implicit time integration.]

Upon comparing the above references to, e.g., Harlow and Welch[32] or Gresho *et al.*,[30] it is noteworthy that *explicit* time integration is much more amenable to the 'uncoupling' of equations (P then $\mathbf{u}$) that is one of the reputed advantages of replacing $\nabla\cdot\mathbf{u}=0$ by the PPE. That is, only when the velocity field is *known*, both in Ω and on Γ, can the right-hand side of the PPE and the BCs be easily and explicitly formed and the appropriate pressure thus directly computed. And this advantage would also accrue with higher-order explicit methods, such as the Adams–Bashforth or Runge–Kutta families. The flip-side of this observation, however, is the well-known stability limits of explicit techniques.

To conclude, we observe that the efforts in this paper and in many before it have evolved to the following

Equivalence theorem of incompressible flow:

Given $\mathbf{u}_0(\mathbf{x})$ in $\bar{\Omega}\equiv\Omega\oplus\Gamma$ and $\mathbf{w}(\mathbf{x},t)$ on Γ with $\nabla\cdot\mathbf{u}_0=0$ in $\Omega, \mathbf{u}_0\cdot\mathbf{n}=\mathbf{w}(\mathbf{x},0)\cdot\mathbf{n}$ on Γ, and $\int_\Gamma \mathbf{n}\cdot\mathbf{w}(\mathbf{x},t)=0$, then there exists a velocity $\mathbf{u}(\mathbf{x},t)$ with $\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x})$ in Ω and $\mathbf{u}(\mathbf{x},t)=\mathbf{w}(\mathbf{x},t)$ on Γ for $t>0$ and a pressure $P(\mathbf{x},t)$ that satisfy the following nearly *equivalent* systems:

(1) $$\partial\mathbf{u}/\partial t+\nabla P=\nu\nabla^2\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{u} \quad \text{in} \quad \bar{\Omega} \quad \text{for} \quad t>0$$

and

$$\nabla\cdot\mathbf{u}=0 \quad \text{in} \quad \bar{\Omega} \quad \text{for} \quad t\geqslant 0,$$

(2) $$\partial\mathbf{u}/\partial t+\nabla P=\nu\nabla^2\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{u} \quad \text{in} \quad \bar{\Omega} \quad \text{for} \quad t>0$$

with in addition

$$\mathbf{n}\cdot(\partial\mathbf{u}/\partial t+\nabla P-\nu\nabla^2\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u})=0 \quad \text{on} \quad \Gamma \quad \text{at} \quad t=0$$

and

$$\nabla^2 P=-\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) \quad \text{in} \quad \Omega \quad \text{for} \quad t\geqslant 0.$$

Remarks

(i) They are only 'nearly' equivalent because system (1) can actually admit a larger class of solutions owing to the additional smoothness requirements implied by system (2).

(ii) The initial pressure is completely determined by the (initial) Neumann problem and the fact that the tangential components of the momentum equation are generally not satisfied at $t=0$ is caused by the no-slip (vorticity-producing) BC ($\nu>0$) and leads to a loss of regularity as $t\to 0$. (For example, the tangential acceleration near a no-slip wall may suffer a jump at $t=0$ and the short-term behaviour of the tangential velocity will often closely obey the transient heat equation with a source term.) For inviscid flow ($\nu=0$), the (slippery) solution is more regular at $t=0$—because the BC on $\mathbf{u}$ is relaxed to $\mathbf{u}\cdot\mathbf{n}=\mathbf{w}\cdot\mathbf{n}$ on Γ.

(iii) If $\nabla\cdot\mathbf{u}_0\neq 0$, system (2) can still have a solution. The solution will have $\nabla\cdot\mathbf{u}=\nabla\cdot\mathbf{u}_0$ for all time and is not a solution of system (1) which is ill-posed and has none.

(iv) A well-posed modified problem can, however, be easily derived from an *arbitrary* field, $\tilde{\mathbf{u}}(\mathbf{x})$ in $\bar{\Omega}$, as follows:

(*a*) Solve $\nabla^2\lambda = \nabla\cdot\tilde{\mathbf{u}}$ in Ω with $\partial\lambda/\partial n = \mathbf{n}\cdot(\tilde{\mathbf{u}} - \mathbf{w})$ on Γ.
(*b*) Set $\mathbf{u}_0 = \tilde{\mathbf{u}} - \nabla\lambda$ in $\bar{\Omega}$.
(*c*) Use $\mathbf{u}_0(\mathbf{x})$ as the initial velocity for the Navier–Stokes equations.

The modified field, $\mathbf{u}_0(\mathbf{x})$, is the appropriate projection of $\tilde{\mathbf{u}}(\mathbf{x})$ onto the solenoidal subspace, and is therefore divergence-free in $\bar{\Omega}$; i.e., it also satisfies $\mathbf{n}\cdot\mathbf{u}_0 = \mathbf{w}\cdot\mathbf{n}$ on Γ. It is also the 'closest' divergence-free field to $\tilde{\mathbf{u}}$ in a least-squares sense; it possesses the same vorticity as $\tilde{\mathbf{u}}(\mathbf{x})$, but it generally does *not* satisfy $\boldsymbol{\tau}\cdot\mathbf{u}_0 = \boldsymbol{\tau}\cdot\mathbf{w}$ on Γ. See also Chorin and Marsden[49] and Gresho *et al.*[30]

(v) The 'theorem' is actually only an assertion. Unfortunately, we must leave it to others to elevate it to a theorem.

Towards the end of this research, we obtained a draft manuscript by S. Orszag, M. Israeli and M. Deville, entitled 'Boundary conditions for incompressible flows'; it has since recently appeared in *J. Sci. Comput.*, **1**, 75. It appears to be complementary to our own, since it emphasizes discrete-time, continuous-space methods. Another contemporary paper with similar goals is that by A. Ku and T. Taylor, entitled 'Pseudospectral methods for solution of the incompressible Navier–Stokes equations', submitted to *Comput. Fluids*. Finally, a recent interesting pair of papers by Pironneau[50,51] addresses the subject of pressure boundary conditions of a more general type.

SUMMARY AND CONCLUSIONS

The condition $\nabla\cdot\mathbf{u} = 0$ in $\bar{\Omega}$ for $t \geqslant 0$ is very powerful and pervasive; it profoundly affects all aspects of incompressible flows, from theoretical understanding through algorithm design and numerical simulation. The appropriate realization of this incompressibility condition on Γ is $\mathbf{u}\cdot\mathbf{n} = \mathbf{w}\cdot\mathbf{n}$ for all $t \geqslant 0$.

The pressure in incompressible flows is always *in equilibrium* with a given (solenoidal) and generally time-varying velocity field, a fact that has caused much confusion; i.e., a divergence-free velocity field *induces* a pressure field, which pressure field is special (Lagrange multiplier) in that it ensures that the resulting acceleration field is divergence-free and thus that the velocity remains divergence-free. Proper BCs are *essential* in order to find this 'special' pressure if the PPE is utilized.

The Neumann BC (normal momentum equation on Γ) is always appropriate for the PPE. It is another manifestation of $\nabla\cdot\mathbf{u} = 0$ on Γ, just as the PPE ensures $\nabla\cdot\mathbf{u} = 0$ in Ω. It is also 'stable' in the sense that it is valid for all $Re, 0 \leqslant Re < \infty$ (and for $\nu = 0$), and all $t \geqslant 0$. This result is related to the fact that $\partial P/\partial n = \mathbf{n}\cdot\mathbf{f}$ is the natural BC for the (scalar) Poisson equation, $\nabla^2 P = \nabla\cdot\mathbf{f}$, that is derived from the (vector) equation, $\nabla P = \mathbf{f}$, where of course $\mathbf{f}$ is curl-free. For the NS equations, simply replace $\mathbf{f}$ by $\mathbf{f} - \mathbf{a}$ and enforce $\nabla\cdot\mathbf{a} = 0$ in $\bar{\Omega}$; it then follows that it is necessary to specify $\mathbf{n}\cdot\mathbf{a}$ on Γ to determine P and that the normal velocity BC, $\mathbf{a}\cdot\mathbf{n} = \mathbf{n}\cdot\dot{\mathbf{w}}$, is appropriate (necessary and sufficient). When $\mathbf{u}\cdot\mathbf{n}$ is specified on Γ, the boundary realization of the continuity equation is simply the normal component of the momentum equation.

The appropriate and associated Dirichlet BC for the PPE is that derived from the Green's function for the Poisson/Dirichlet or Poisson/Neumann problem. This BC is equivalent to the above Neumann BC. It is also equivalent to the Dirichlet BC derived by applying the tangential momentum equation on the boundary, but only for $t > 0$ since the latter does generally not apply at $t = 0$. Any scheme that purports to invoke a Dirichlet BC for the pressure will only be correct if the resulting pressure also satisfies the proper Neumann BC.

For $t \to 0$ and $\mathbf{x} \to \Gamma$, the tangential velocity will often (e.g., if $\mathbf{u} = 0$ on Γ and $\nu > 0$) be well-described by the transient heat equation in which the tangential component of the pressure gradient, 'supplied' by the Neumann problem, acts like a prescribed 'heat source' term—and generates vorticity along no-slip walls.

We have tried to clarify some of the issues involved with Dirichlet boundary data for velocity for both the continuum equations (PDEs) and the semi-discrete equations. For this simple but important case, the results presented herein should help to provide a proper basis and starting point for all ensuing numerical approximation methods—in time and/or space—finite difference, finite element, spectral, etc. In the future, we hope to extend this work in some obvious (and some not so obvious) directions, For example, we and others often have difficulty with flow-through domains and many have gotten into trouble when numerical time integration is attempted. Free surface flows are yet another area of interest.

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