Numerical methods for incompressible flows (i.e. $\nabla\cdot\underset{\sim}{v}=0$)

We consider single phase flows: $\rho=\rho_0$. The equations are then:

$$\nabla\cdot\underset{\sim}{v}=0 \quad (\equiv \text{constraint!}) \qquad \text{Continuity}$$

$$\rho_0 \frac{D\underset{\sim}{v}}{Dt} = -\nabla p + \rho_0 \underset{\sim}{g} + \nabla\cdot\underset{\approx}{\tau} \qquad \text{Momentum}$$

$$\rho_0 c \frac{DT}{Dt} = \Phi - \nabla\cdot\underset{\sim}{q} \qquad \text{Energy}$$

where $\underset{\approx}{\tau} = 2\mu\,\underset{\approx}{d}$ is the shear stress tensor with $\underset{\approx}{d}=\frac{1}{2}\left((\nabla\underset{\sim}{v})+(\nabla\underset{\sim}{v})^T\right)$ the strain rate tensor, μ the dynamic viscosity

$\underset{\sim}{q}=-k\nabla T$ is the heat flux with k the thermal conductivity. c is the specific heat.

$\Phi \triangleq \underset{\approx}{\tau}\cdot\underset{\approx}{d} = 2\mu\,\underset{\approx}{d}\cdot\underset{\approx}{d} \geq 0$ the viscous dissipation

Moreover, as $\underset{\sim}{g}=-g\,\hat{\underset{\sim}{e}}_z=-\nabla W$ with $W=gz$, we can lump the hydrostatic pressure contribution $\rho_0 W$ and p together. We hence obtain:

$$\rho_0\frac{D\underset{\sim}{v}}{Dt} = -\nabla(p+\rho_0 W) + \nabla\cdot(2\mu\,\underset{\approx}{d})$$

$$\rho_0 c\frac{DT}{Dt} = 2\mu\,\underset{\approx}{d}\cdot\underset{\approx}{d} + \nabla\cdot(k\nabla T)$$

In general, we have that $\mu=\mu(T)$, $k=k(T)$ and $c=c(T)$ and the fluids mechanics are coupled to the temperature $\Rightarrow$ We need to solve them together.

If we consider problems where the variations of μ are negligible, and take μ constant, we can solve the fluid mechanics without having to solve the temperature. We consider this case. We then obtain:

$$\begin{cases} \nabla\cdot \underset{\sim}{v} = 0 \\ \rho_0 \dfrac{D\underset{\sim}{v}}{Dt} = -\nabla(p+\rho_0 W) + \mu \overbrace{\nabla^2 \underset{\sim}{v}}^{\text{Notation for } \nabla\cdot(\nabla v)} \end{cases}$$

We define $P \triangleq \dfrac{(p+\rho_0 W)}{\rho_0}$ the kinematic pressure

$\nu \triangleq \dfrac{\mu}{\rho}$ the kinematic viscosity

Dividing by ρ_0, we finally write:

$$\boxed{\begin{aligned} &\nabla\cdot \underset{\sim}{v} = 0 \\ &\frac{D\underset{\sim}{v}}{Dt} = -\nabla P + \nu \nabla^2 \underset{\sim}{v} \end{aligned}}$$

Often also noted: $\dfrac{\partial \underset{\sim}{v}}{\partial t} + (\underset{\sim}{v}\cdot\nabla)\underset{\sim}{v}$

$\Uparrow$

with $\dfrac{D\underset{\sim}{v}}{Dt} = \dfrac{\partial \underset{\sim}{v}}{\partial t} + (\nabla\underset{\sim}{v})\cdot\underset{\sim}{v}$ $\equiv$ Advective form

Alternatively, since $\nabla\cdot(\underset{\sim}{v}\otimes\underset{\sim}{v}) = (\nabla\underset{\sim}{v})\cdot\underset{\sim}{v} + \underbrace{(\nabla\cdot\underset{\sim}{v})}_{=0}\underset{\sim}{v}$, we can also use:

$$\frac{\partial v}{\partial t} + \nabla\cdot(\underset{\sim}{v}\otimes\underset{\sim}{v}) \equiv \text{Divergence form}$$

we hence write:

$$\frac{\partial v}{\partial t} + \underset{\sim}{A} = -\nabla P + \nu\nabla^2\underset{\sim}{v} \quad \text{with} \quad \begin{aligned} &\underset{\sim}{A} = (\nabla\underset{\sim}{v})\cdot\underset{\sim}{v} \triangleq (\underset{\sim}{v}\cdot\nabla)\underset{\sim}{v} \\ \text{or } &\underset{\sim}{A} = \nabla\cdot(\underset{\sim}{v}\otimes\underset{\sim}{v}) \\ \text{or } &\underset{\sim}{A} = \text{any combination} \end{aligned}$$

Of course, we can also solve the temperature equation if we want to. When k and c are supposed constant, it reduces to:

$$\rho_0 c \frac{DT}{Dt} = \underbrace{2\mu\, \underline{\underline{d}}:\underline{\underline{d}}}_{=\Phi} + k\nabla^2 T$$

this term is often negligible!

Considering cases when Φ is negligible, and dividing by ρ_0, we finally write

$$\boxed{\frac{DT}{Dt} = \alpha\nabla^2 T}$$ with $\alpha \triangleq \dfrac{k}{\rho_0 c}$ the thermal diffusivity

Note on Boussinesq Approximation

The flow is still incompressible, but buoyancy effects due to thermal variations are taken into account:

$$\rho = \rho_0 + \left(\frac{\partial \rho}{\partial T}\right)_p (T - T_0) =$$

$$\rho = \rho_0 \left(1 - \beta(T - T_0)\right) \quad \text{with } \beta \triangleq -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial T}\right)_p > 0$$

volumetric thermal expansion coefficient

$$\rho_0 \frac{D\underline{v}}{Dt} = -\nabla p + \rho_0\left(1 - \beta(T - T_0)\right)\underline{g} + \mu \nabla^2 \underline{v}$$

$$= -\nabla p + \rho_0 \underline{g} - \rho_0 \beta (T - T_0)\underline{g} + \mu \nabla^2 \underline{v}$$

$$= -\nabla(p + \rho_0 W) - \rho_0 \beta (T - T_0)\underline{g} + \mu \nabla^2 \underline{v}$$

$$\hookrightarrow \frac{D\underline{v}}{Dt} = -\nabla \underbrace{\left(\frac{p + \rho_0 W}{\rho_0}\right)}_{P''} - \beta(T - T_0)\underline{g} + \nu \nabla^2 \underline{v}$$

Finally: considering cases where Φ is negligible

$$\boxed{\begin{aligned} &\nabla \cdot \underline{v} = 0 \\ &\frac{D\underline{v}}{Dt} = -\nabla P - \beta(T - T_0)\underline{g} + \nu \nabla^2 \underline{v} \\ &\frac{DT}{Dt} = \alpha \nabla^2 T \end{aligned}}$$

2D Flows

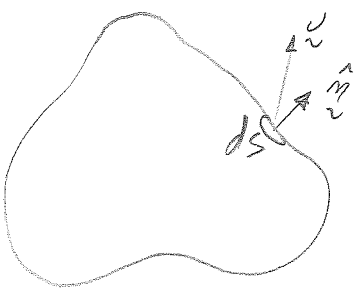
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + a + \frac{\partial P}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + b + \frac{\partial P}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

	Advective Form	Divergence Form
a	$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$	$\frac{\partial}{\partial x}(uu) + \frac{\partial}{\partial y}(uv)$
b	$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$	$\frac{\partial}{\partial x}(vu) + \frac{\partial}{\partial y}(vv)$

$$\nabla\cdot\underset{\sim}{u} = 0 \Rightarrow \int_V \nabla\cdot\underset{\sim}{u}\, dv = 0$$



$$\Leftrightarrow \int_S \underset{\sim}{u}\cdot\hat{\underset{\sim}{n}}\, ds = 0$$

↳ The initial condition and the boundary conditions must be consistent with that

Note

$\nabla\cdot\underset{\sim}{u} = 0$ is not an evolution equation. It is a constraint that must be satisfied everywhere.

This makes the problem "elliptic" as all points are connected to all points

↳ We will need to solve a Poisson equation

Iterative methods for the case of steady flows

We wish to obtain the solution of:

(Of course, the steady state of the computed flow must exist)

$$\boxed{\begin{array}{l} \mathbf{A} + \nabla P = \nu \nabla^2 \mathbf{v} \\ \text{with } \nabla \cdot \mathbf{v} = 0 \end{array}}$$

Method of "artificial evolution"

Vladimirova et al. (1965) (Yanenko 1971)

Chorin (1967)

Chorin:

$$\boxed{\begin{array}{l} \dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{A} + \nabla P = \nu \nabla^2 \mathbf{v} \\ \dfrac{\partial P}{\partial t} + c_0^2 \, \nabla \cdot \mathbf{v} = 0 \end{array}}$$

⚠ Here, t is not a physical time. It is a pseudo-time solely used for "iterating in time".

We use use an artificial evolution equation for P such that, at convergence, we obtain the steady state solution of the flow of interest:

At convergence

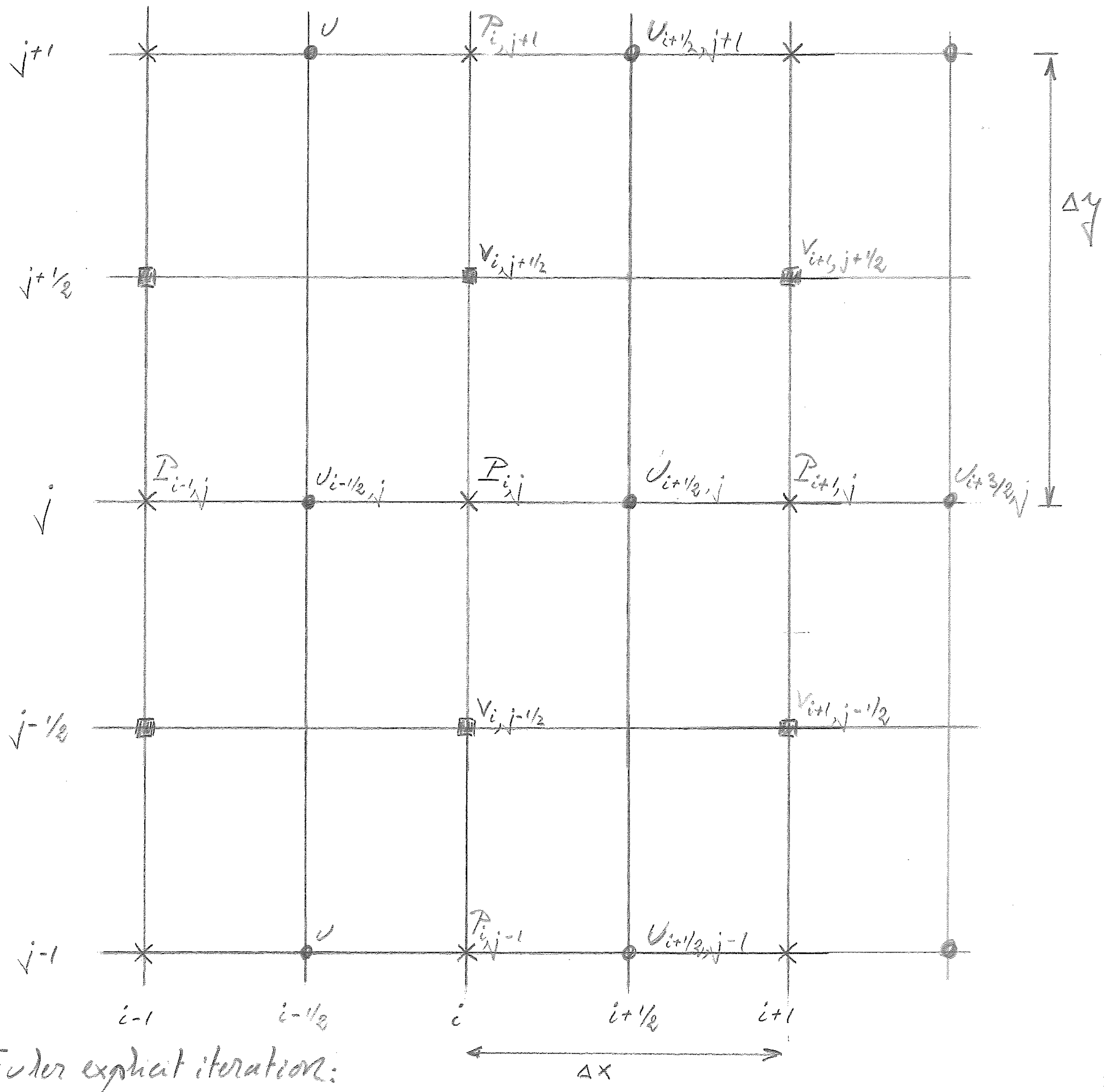
$\dfrac{\partial \mathbf{v}}{\partial t} = 0$ and $\dfrac{\partial P}{\partial t} = 0$ $\Rightarrow$ O.K.!

The parameter c_0^2 has dimensions of $\left[\dfrac{L^2}{T^2}\right]$

↳ c_0 is a velocity. Its choice should hence be made in relation to the characteristic velocity of the flow. [Note that one could even use a c_0 that varies in space and/or time: e.g. $c_0^2 \propto (u^2 + v^2)$]

Discretization

Staggered "marker-and-cell" (MAC) mesh (Harlow and Welsh 1965)



Euler explicit iteration:

$$\frac{\left(u^{n+1}_{i+1/2,j} - u^{n}_{i+1/2,j}\right)}{\Delta t} + a^{n}_{i+1/2,j} + \frac{\left(P^{n}_{i+1,j} - P^{n}_{i,j}\right)}{\Delta x}$$

$$= \nu\left(\frac{\left(u^{n}_{i+3/2,j} - 2\,u^{n}_{i+1/2,j} + u^{n}_{i-1/2,j}\right)}{(\Delta x)^2} + \frac{\left(u^{n}_{i+1/2,j+1} - 2\,u^{n}_{i+1/2,j} + u^{n}_{i+1/2,j-1}\right)}{(\Delta y)^2}\right)$$

$$\frac{\left(v^{n+1}_{i,j+1/2} - v^{n}_{i,j+1/2}\right)}{\Delta t} + b^{n}_{i,j+1/2} + \frac{\left(P^{n}_{i,j+1} - P^{n}_{i,j}\right)}{\Delta y}$$

$$= \nu\left(\frac{\left(v^{n}_{i+1,j+1/2} - 2v^{n}_{i,j+1/2} + v^{n}_{i-1,j+1/2}\right)}{(\Delta x)^2} + \frac{\left(v^{n}_{i,j+3/2} - 2v^{n}_{i,j+1/2} + v^{n}_{i,j-1/2}\right)}{(\Delta y)^2}\right)$$

Then

$$\frac{\left(P^{n+1}_{i,j} - P^{n}_{i,j}\right)}{\Delta t} + c^2\left(\frac{\left(u^{n+1}_{i+1/2,j} - u^{n+1}_{i-1/2,j}\right)}{(\Delta x)} + \frac{\left(v^{n+1}_{i,j+1/2} - v^{n+1}_{i,j-1/2}\right)}{(\Delta y)}\right) = 0$$

Note: we use u^{n+1} and v^{n+1} as we have them!

Discretization of the advective form:

$$a_{i+1/2,j} = \frac{1}{2}\left(u_{i+1,j}\frac{\left(u_{i+3/2,j} - u_{i+1/2,j}\right)}{\Delta x} + u_{i,j}\frac{\left(u_{i+1/2,j} - u_{i-1/2,j}\right)}{\Delta x}\right)$$

$$+ \frac{1}{2}\left(v_{i+1/2,j+1/2}\frac{\left(u_{i+1/2,j+1} - u_{i+1/2,j}\right)}{\Delta y} + v_{i+1/2,j-1/2}\frac{\left(u_{i+1/2,j} - u_{i+1/2,j-1}\right)}{\Delta y}\right)$$

with $u_{i+1,j} = \frac{\left(u_{i+3/2,j} + u_{i+1/2,j}\right)}{2}$ and $v_{i+1/2,j+1/2} = \frac{\left(v_{i+1,j+1/2} + v_{i,j+1/2}\right)}{2}$

$b_{i,j+1/2} = \ldots$

Discretization of the divergence form:

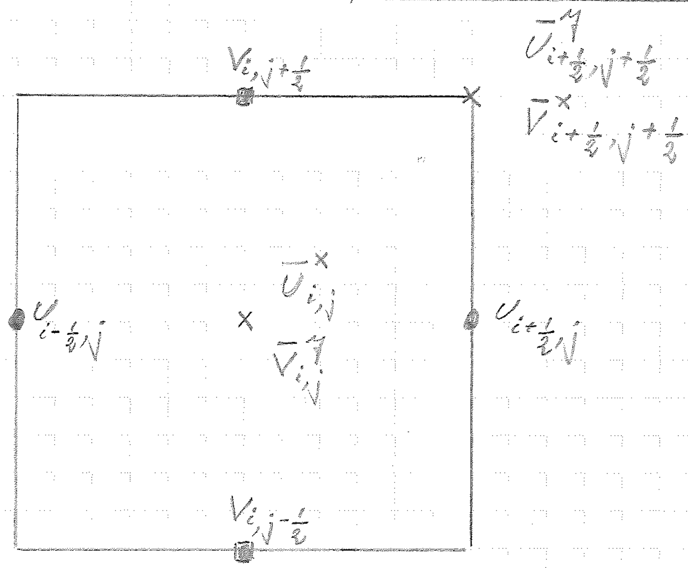
$$a_{i+1/2,j} = \frac{\left((uu)_{i+1,j} - (uu)_{i,j}\right)}{\Delta x} + \frac{\left((uv)_{i+1/2,j+1/2} - (uv)_{i+1/2,j-1/2}\right)}{\Delta y}$$

with $(uu)_{i+1,j} = u_{i+1,j}\,u_{i+1,j} = \left(\frac{\left(u_{i+3/2,j} + u_{i+1/2,j}\right)}{2}\right)^2$

and $(uv)_{i+1/2,j+1/2} = u_{i+1/2,j+1/2}\,v_{i+1/2,j+1/2} = \frac{\left(u_{i+1/2,j+1} + u_{i+1/2,j}\right)}{2}\,\frac{\left(v_{i+1,j+1/2} + v_{i,j+1/2}\right)}{2}$

$b_{i,j+1/2} = \ldots$

Complement on discretization



Notation: $\bar{u}^x_{i,j} \triangleq \frac{1}{2}\left(u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}\right)$

$$\bar{v}^y_{i,j} \triangleq \frac{1}{2}\left(v_{i,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}}\right)$$

$$\bar{u}^y_{i+\frac{1}{2},j+\frac{1}{2}} \triangleq \frac{1}{2}\left(u_{i+\frac{1}{2},j+1} + u_{i+\frac{1}{2},j}\right)$$

$$\bar{v}^x_{i+\frac{1}{2},j+\frac{1}{2}} \triangleq \frac{1}{2}\left(v_{i+1,j+\frac{1}{2}} + v_{i,j+\frac{1}{2}}\right)$$

Divergence Form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uu) + \frac{\partial}{\partial y}(vu) = \dots$$

$$\hookrightarrow \frac{\partial u}{\partial t} + \frac{\delta}{\delta x}\left(\bar{u}^x \bar{u}^x\right) + \frac{\delta}{\delta y}\left(\bar{v}^x \bar{u}^y\right) = \dots$$

$$\rightarrow \frac{d}{dt} u_{i+\frac{1}{2},j} + \frac{1}{\Delta x}\left(\bar{u}^x_{i+1,j}\,\bar{u}^x_{i+1,j} - \bar{u}^x_{i,j}\,\bar{u}^x_{i,j}\right) + \frac{1}{\Delta y}\left(\bar{v}^x_{i+\frac{1}{2},j+\frac{1}{2}}\,\bar{u}^y_{i+\frac{1}{2},j+\frac{1}{2}} - \bar{v}^x_{i+\frac{1}{2},j-\frac{1}{2}}\,\bar{u}^y_{i+\frac{1}{2},j-\frac{1}{2}}\right) = \dots$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(vv) = \dots$$

$$\hookrightarrow \frac{\partial v}{\partial t} + \frac{\delta}{\delta x}\left(\bar{u}^y \bar{v}^x\right) + \frac{\delta}{\delta y}\left(\bar{v}^y \bar{v}^y\right) = \dots$$

Généralization $(x_1 = x,\ x_2 = y,\ u_1 = u,\ u_2 = v)$

$$\frac{\partial}{\partial t} u_1 + \frac{\delta}{\delta x_1}\left(\overline{u_1}^{x_1}\ \overline{u_1}^{x_1}\right) + \frac{\delta}{\delta x_2}\left(\overline{u_2}^{x_1}\ \overline{u_1}^{x_2}\right) = \dots$$

$$\frac{\partial}{\partial t} u_2 + \frac{\delta}{\delta x_1}\left(\overline{u_1}^{x_2}\ \overline{u_2}^{x_1}\right) + \frac{\delta}{\delta x_2}\left(\overline{u_2}^{x_2}\ \overline{u_2}^{x_2}\right) = \dots$$

↳ 3D: $\frac{\partial}{\partial t} u_i + \frac{\delta}{\delta x_j}\left(\overline{u_j}^{x_i}\ \overline{u_i}^{x_j}\right) = \dots$

Advective form of KAJISHIMA

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \dots$$

$$\hookrightarrow \frac{\partial u}{\partial t} + \overline{\overline{u}^x \frac{\delta u}{\delta x}}^x + \overline{\overline{v}^x \frac{\delta u}{\delta y}}^y = \dots$$

$$\hookrightarrow \frac{d}{dt} u_{i+\frac{1}{2},j} + \frac{1}{2}\left(\left(\overline{u}^x \frac{\delta u}{\delta x}\right)_{i+1,j} + \left(\overline{u}^x \frac{\delta u}{\delta x}\right)_{i,j}\right) + \frac{1}{2}\left(\left(\overline{v}^x \frac{\delta u}{\delta y}\right)_{i+\frac{1}{2},j+\frac{1}{2}} + \left(\overline{v}^x \frac{\delta u}{\delta y}\right)_{i+\frac{1}{2},j-\frac{1}{2}}\right) = \dots$$

$$\begin{aligned}\frac{d}{dt} u_{i+\frac{1}{2},j} &+ \frac{1}{2}\Bigg(\frac{1}{2}\left(u_{i+\frac{3}{2},j} + u_{i+\frac{1}{2},j}\right)\frac{1}{\Delta x}\left(u_{i+\frac{3}{2},j} - u_{i+\frac{1}{2},j}\right) \\ &\qquad + \frac{1}{2}\left(u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}\right)\frac{1}{\Delta x}\left(u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}\right)\Bigg) \\ &+ \frac{1}{2}\Bigg(\frac{1}{2}\left(v_{i+1,j+\frac{1}{2}} + v_{i,j+\frac{1}{2}}\right)\frac{1}{\Delta y}\left(u_{i+\frac{1}{2},j+1} - u_{i+\frac{1}{2},j}\right) \\ &\qquad + \frac{1}{2}\left(v_{i+1,j-\frac{1}{2}} + v_{i,j-\frac{1}{2}}\right)\frac{1}{\Delta y}\left(u_{i+\frac{1}{2},j} - u_{i+\frac{1}{2},j-1}\right)\Bigg) = \dots\end{aligned}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \dots$$

$$\hookrightarrow \frac{\partial v}{\partial t} + \overline{\overline{u}^y \frac{\delta v}{\delta x}}^x + \overline{\overline{v}^y \frac{\delta v}{\delta y}}^y = \dots$$

Generalization

$$\frac{\partial}{\partial t} v_1 + \overline{\bar{v}_1^{x_1} \frac{\delta v_1}{\delta x_1}}^{x_1} + \overline{\bar{v}_2^{x_1} \frac{\delta v_1}{\delta x_2}}^{x_2} = \ldots$$

$$\frac{\partial}{\partial t} v_2 + \overline{\bar{v}_1^{x_2} \frac{\delta v_2}{\delta x_1}}^{x_1} + \overline{\bar{v}_2^{x_2} \frac{\delta v_2}{\delta x_2}}^{x_2} = \ldots$$

↳ 3D:

$$\frac{\partial}{\partial t} v_i + \overline{\bar{v}_j^{x_i} \frac{\delta v_i}{\delta x_j}}^{x_j} = \ldots$$

Skew-Symmetric Form:

$$\frac{\partial v_i}{\partial t} + \frac{1}{2}\left(\frac{\delta}{\delta x_j}\left(\bar{v}_j^{x_i}\, \bar{v}_i^{x_j}\right) + \overline{\bar{v}_j^{x_i} \frac{\delta v_i}{\delta x_j}}^{x_j} \right) = \ldots$$

↳ Conserves (1) the kinetic energy when $\nu = 0$.

(2) the momentum, when the continuity is satisfied (i.e when $\frac{\delta v_i}{\delta x_i} = 0$).

- Stability ⇒ linearized equations:

$$
① \begin{cases} \dfrac{\partial u}{\partial t} + u_0 \dfrac{\partial u}{\partial x} + v_0 \dfrac{\partial u}{\partial y} + \dfrac{\partial P}{\partial x} = \nu \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right) \\[2ex] \dfrac{\partial v}{\partial t} + u_0 \dfrac{\partial v}{\partial x} + v_0 \dfrac{\partial v}{\partial y} + \dfrac{\partial P}{\partial y} = \nu \left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} \right) \end{cases}
$$

$$
② \quad \frac{\partial P}{\partial t} + c_0^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0
$$

As we used the Euler explicit scheme for the pseudo-time integration (= iteration), we use the modal analysis with discretized space and time

(a) Analysis of the convection-diffusion equations without pressure gradient:

Case $\Delta x = \Delta y = h$, then $R = \dfrac{(|u_0| + |v_0|)\, h}{\nu}$ ≡ mesh Reynolds number

if $R \le 4$, the scheme is stable if $\boxed{r \le \frac{1}{4}}$

if $R > 4$, the scheme is stable if $\boxed{r R^2 \le 4}$

(b) Analysis of step ① followed by step ②, neglecting the convection:

Case $\Delta x = \Delta y = h$

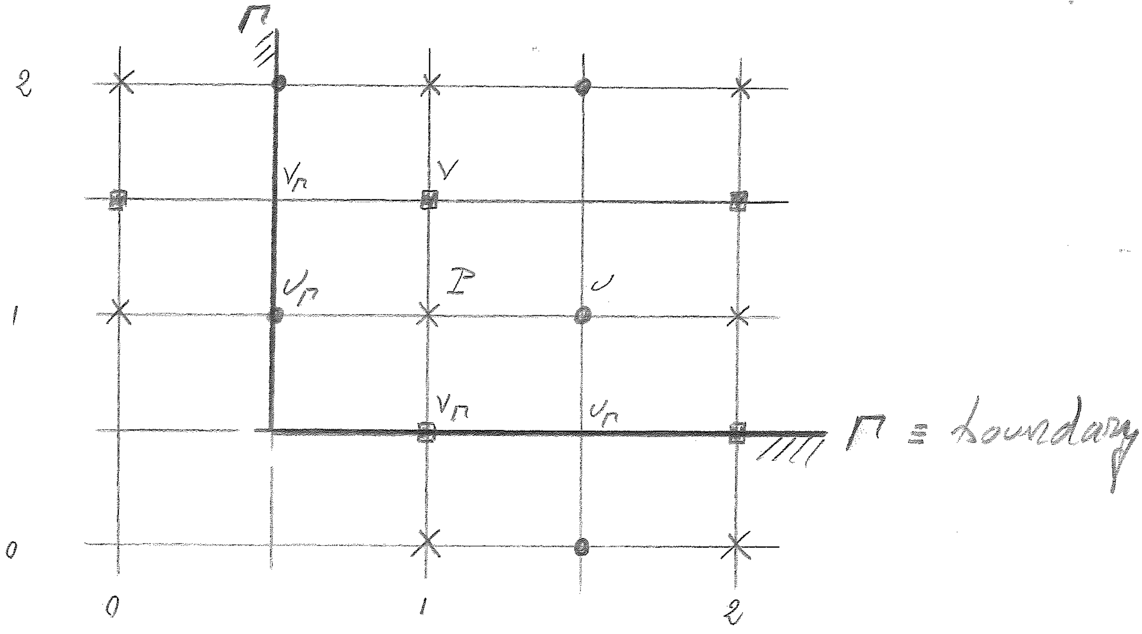
The scheme is stable if $\boxed{r + \frac{1}{2} \left(\frac{c_0 \Delta t}{h} \right)^2 \le \frac{1}{4}}$

Example: c_0 taken as $c_0 = |u_0| + |v_0|$

$$
\hookrightarrow \frac{c_0 \Delta t}{h} = \frac{(|u_0| + |v_0|)\Delta t}{h} = \beta = r R \quad \Rightarrow \quad r + \frac{1}{2}(rR)^2 \le \frac{1}{4}
$$

$$
r\left(1 + \frac{1}{2} r R^2\right) \le \frac{1}{4}
$$

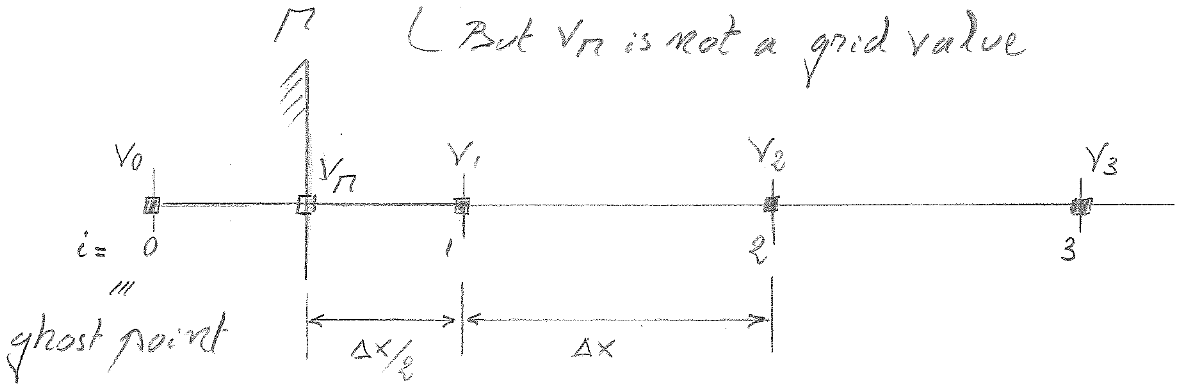
- <u>Imposition of boundary conditions on a boundary Γ:</u>



- P is not on Γ $\Rightarrow$ OK as we do not have a B.C. on Γ.
  ↳ the MAC mesh is also well adapted for that.

- <u>v_Γ imposed on horizontal Γ or V_Γ imposed on vertical Γ:</u>

We here consider V_Γ imposed on vertical Γ:



To compute $\dfrac{(V_1^{n+1} - V_1^n)}{\Delta t}$, we need to evaluate $\left.\dfrac{\partial V}{\partial x}\right|_1$ and $\left.\dfrac{\partial^2 V}{\partial x^2}\right|_1$ with $O((\Delta x)^2)$ accuracy.

We can use a "ghost point" 0, of value V_0, and program as for the the interior points: $\left.\dfrac{\partial V}{\partial x}\right|_1 = \dfrac{(V_2 - V_0)}{2\Delta x}$ and $\left.\dfrac{\partial^2 V}{\partial x^2}\right|_1 = \dfrac{(V_2 - 2V_1 + V_0)}{(\Delta x)^2}$.

Of course, one must put a proper value in V_0!

Use Taylor series

$$V_{\Gamma} = V_1 - \left(\frac{\Delta x}{2}\right)\frac{\partial V}{\partial x}\Big|_1 + \left(\frac{\Delta x}{2}\right)^2\frac{1}{2!}\frac{\partial^2 V}{\partial x^2}\Big|_1 - \left(\frac{\Delta x}{2}\right)^3\frac{1}{3!}\frac{\partial^3 V}{\partial x^3}\Big|_1 + \left(\frac{\Delta x}{2}\right)^4\frac{1}{4!}\frac{\partial^4 V}{\partial x^4}\Big|_1 - \dots$$

$$V_2 = V_1 + (\Delta x)\frac{\partial V}{\partial x}\Big|_1 + (\Delta x)^2\frac{1}{2!}\frac{\partial^2 V}{\partial x^2}\Big|_1 + (\Delta x)^3\frac{1}{3!}\frac{\partial^3 V}{\partial x^3}\Big|_1 + (\Delta x)^4\frac{1}{4!}\frac{\partial^4 V}{\partial x^4}\Big|_1 + \dots$$

$$V_3 = V_1 + (2\Delta x)\frac{\partial V}{\partial x}\Big|_1 + (2\Delta x)^2\frac{1}{2!}\frac{\partial^2 V}{\partial x^2}\Big|_1 + (2\Delta x)^3\frac{1}{3!}\frac{\partial^3 V}{\partial x^3}\Big|_1 + (2\Delta x)^4\frac{1}{4!}\frac{\partial^4 V}{\partial x^4}\Big|_1 + \dots$$

$\Downarrow$

$V_0 =$	$\frac{\partial V}{\partial x}\Big\vert_1 = \frac{(V_2 - V_0)}{2\Delta x}$	$\frac{\partial^2 V}{\partial x^2}\Big\vert_1 = \frac{(V_2 - 2V_1 + V_0)}{(\Delta x)^2}$
$-(V_1 - 2V_{\Gamma})$	$O(\Delta x)$	Incorrect
$\frac{1}{3}(V_2 - 6V_1 + 8V_{\Gamma})$	$O((\Delta x)^2)$	$O(\Delta x)$
$-\frac{1}{5}(V_3 - 5V_2 + 15V_1 - 16V_{\Gamma})$	$O((\Delta x)^2)$	$O((\Delta x)^2)$

← to verify as exercise

The Poisson equation for the pressure

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{A} + \nabla P = \nu\, \nabla\cdot(\nabla \underset{\sim}{v}) \qquad \text{as } \nabla\times(\nabla\times \underset{\sim}{a}) = -\nabla\cdot(\nabla \underset{\sim}{a}) + \nabla(\nabla\cdot \underset{\sim}{a})$$

$$= \nu\Big[-\nabla\times\underbrace{(\nabla\times \underset{\sim}{v})}_{\underset{\sim}{\omega}} + \nabla\underbrace{(\nabla\cdot \underset{\sim}{v})}_{=0}\Big]$$

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{A} + \nabla P = -\nu\, \nabla\times \underset{\sim}{\omega}$$

$$\hookrightarrow \nabla\cdot\left[\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{A} + \nabla P\right] = -\nu\, \nabla\cdot(\nabla\times \underset{\sim}{\omega})$$

$$\frac{\partial}{\partial t}\underbrace{(\nabla\cdot \underset{\sim}{v})}_{=0} + \nabla\cdot \underset{\sim}{A} + \underbrace{\nabla\cdot(\nabla P)}_{=\nabla^2 P} = -\nu\, \underbrace{\nabla\cdot(\nabla\times \underset{\sim}{\omega})}_{=0 \text{ cas } \nabla\cdot(\nabla\times \underset{\sim}{a}) = 0}$$

Verification

$$V_\Gamma = V_1 - \left(\frac{\Delta x}{2}\right)\frac{\partial V}{\partial x}\Big|_1 + \left(\frac{\Delta x}{2}\right)^2\frac{1}{2}\frac{\partial^2 V}{\partial x^2}\Big|_1 - \left(\frac{\Delta x}{2}\right)^3\frac{1}{6}\frac{\partial^3 V}{\partial x^3}\Big|_1 + \left(\frac{\Delta x}{2}\right)^4\frac{1}{24}\frac{\partial^4 V}{\partial x^4}\Big|_1 - \dots$$

$$V_2 = V_1 + (\Delta x)\frac{\partial V}{\partial x}\Big|_1 + (\Delta x)^2\frac{1}{2}\frac{\partial^2 V}{\partial x^2}\Big|_1 + (\Delta x)^3\frac{1}{6}\frac{\partial^3 V}{\partial x^3}\Big|_1 + (\Delta x)^4\frac{1}{24}\frac{\partial^4 V}{\partial x^4}\Big|_1 + \dots$$

$$V_3 = V_1 + (2\Delta x)\frac{\partial V}{\partial x}\Big|_1 + (2\Delta x)^2\frac{1}{2}\frac{\partial^2 V}{\partial x^2}\Big|_1 + (2\Delta x)^3\frac{1}{6}\frac{\partial^3 V}{\partial x^3}\Big|_1 + (2\Delta x)^4\frac{1}{24}\frac{\partial^4 V}{\partial x^4}\Big|_1 + \dots$$

$$\boxed{\text{Case } V_\Gamma = \frac{(V_1+V_0)}{2} \;\Downarrow\; V_0 = 2V_\Gamma - V_1}$$

$$2V_\Gamma - V_2 = V_1 - 2(\Delta x)\frac{\partial V}{\partial x}\Big|_1 - \frac{1}{2}(\Delta x)^2\frac{1}{2}\frac{\partial^2 V}{\partial x^2}\Big|_1 - \dots$$

$$\frac{\partial V}{\partial x}\Big|_1 = \underset{\displaystyle\frac{(V_2 - V_0)}{2(\Delta x)}}{\frac{(V_2 + V_1 - 2V_\Gamma)}{2(\Delta x)}} \underbrace{- \frac{1}{8}(\Delta x)\frac{\partial^2 V}{\partial x^2}\Big|_1 + \dots}_{\text{truncation error is } O(\Delta x)}$$

Then $\dfrac{(V_2 - 2V_1 + V_0)}{(\Delta x)^2} = \dfrac{(V_2 - 3V_1 + 2V_\Gamma)}{(\Delta x)^2} = \dots = \dfrac{3}{4}\dfrac{\partial^2 V}{\partial x^2}\Big|_1 + (\Delta x)\dfrac{1}{8}\dfrac{\partial^3 V}{\partial x^3}\Big|_1 + \dots$

This is incorrect! It should be 1.

$$2V_\Gamma + V_2 = 3V_1 + 0 + \frac{3}{2}(\Delta x)^2\frac{1}{2}\frac{\partial^2 V}{\partial x^2}\Big|_1 + \frac{3}{4}(\Delta x)^3\frac{1}{6}\frac{\partial^3 V}{\partial x^3}\Big|_1 + \dots$$

$$\frac{\partial^2 V}{\partial x^2}\Big|_1 = \frac{4}{3}\frac{(V_2 - 3V_1 + 2V_\Gamma)}{(\Delta x)^2} \underbrace{- (\Delta x)\frac{1}{6}\frac{\partial^3 V}{\partial x^3}\Big|_1 + \dots}_{\text{truncation error is } O(\Delta x)}$$

$$= \frac{(V_2 - 2V_1 + V_0)}{(\Delta x)^2} \iff V_2 - 2V_1 + V_0 = \frac{4}{3}(V_2 - 3V_1 + 2V_\Gamma)$$

$$3V_2 - 6V_1 + 3V_0 = 4V_2 - 12V_1 + 8V_\Gamma$$

$$3V_0 = V_2 - 6V_1 + 8V_\Gamma$$

$$\boxed{V_0 = \frac{1}{3}(V_2 - 6V_1 + 8V_\Gamma)}$$

Then: $\dfrac{(V_2 - V_0)}{2(\Delta x)} = \dfrac{(3V_2 - 3V_0)}{6(\Delta x)} = \dfrac{(3V_2 - (V_2 - 6V_1 + 8V_\Gamma))}{6(\Delta x)}$

$$= \frac{(2V_2 + 6V_1 - 8V_\Gamma)}{6(\Delta x)} = \frac{(V_2 + 3V_1 - 4V_\Gamma)}{3(\Delta x)}$$

but: $4V_\Gamma - V_2 = 3V_1 - 3(\Delta x)\dfrac{\partial V}{\partial x}\Big|_1 + 0 - \dfrac{3}{2}(\Delta x)^3\dfrac{1}{6}\dfrac{\partial^3 V}{\partial x^3}\Big|_1 + \dots$

hence $\frac{(V_2 - V_0)}{2(\Delta x)} = \frac{(V_2 + 3V_1 - 4V_{\bar{1}})}{3(\Delta x)} = \frac{\partial V}{\partial x}\Big|_1 + (\Delta x)^2 \frac{1}{12} \frac{\partial^3 V}{\partial x^3}\Big|_1 + \dots$

$$\hookrightarrow \quad \frac{\partial V}{\partial x}\Big|_1 = \frac{(V_2 + 3V_1 - 4V_{\bar{1}})}{3(\Delta x)} \underbrace{- (\Delta x)^2 \frac{1}{12} \frac{\partial^3 V}{\partial x^3}\Big|_1}_{\text{truncation error is } O((\Delta x)^2)} + \dots$$

$$a V_{\bar{1}} + b V_2 + V_3 = (a+b+1) V_1 + \left(-\frac{a}{2} + b + 2\right)(\Delta x) \frac{\partial V}{\partial x}\Big|_1$$
$$+ \left(\frac{a}{4} + b + 4\right)(\Delta x)^2 \frac{1}{2} \frac{\partial^2 V}{\partial x^2}\Big|_1 + \left(-\frac{a}{8} + b + 8\right)(\Delta x)^3 \frac{1}{6} \frac{\partial^3 V}{\partial x^3}\Big|_1$$
$$+ \left(\frac{a}{16} + b + 16\right)(\Delta x)^4 \frac{1}{24} \frac{\partial^4 V}{\partial x^4}\Big|_1 + \dots$$

$$\begin{cases} -a + 2b + 4 = 0 \\ -a + 8b + 64 = 0 \end{cases} \qquad \begin{cases} a - 2b = 4 \\ a - 8b = 64 \end{cases} \qquad 6b = -60 \rightarrow b = -10$$
$$\hookrightarrow a = 64 - 80 = -16$$

$$\hookrightarrow -16 V_{\bar{1}} - 10 V_2 + V_3 = -25 V_1 + 0 + (-4 - 10 + 4)(\Delta x)^2 \frac{1}{2} \frac{\partial^2 V}{\partial x^2}\Big|_1 + 0$$
$$+ (-1 - 10 + 16)(\Delta x)^4 \frac{1}{24} \frac{\partial^4 V}{\partial x^4}\Big|_1 + \dots$$
$$= -25 V_1 - 10 (\Delta x)^2 \frac{1}{2} \frac{\partial^2 V}{\partial x^2}\Big|_1 + 5(\Delta x)^4 \frac{1}{24} \frac{\partial^4 V}{\partial x^4}\Big|_1 + \dots$$

$$\hookrightarrow \frac{\partial^2 V}{\partial x^2}\Big|_1 = \frac{(-V_3 + 10 V_2 - 25 V_1 + 16 V_{\bar{1}})}{5(\Delta x)^2} + \underbrace{(\Delta x)^2 \frac{1}{24} \frac{\partial^4 V}{\partial x^4}\Big|_1}_{\text{truncation error is } O((\Delta x)^2)} + \dots$$

$$= \frac{(V_2 - 2V_1 + V_0)}{(\Delta x)^2} \Longleftrightarrow V_2 - 2V_1 + V_0 = \frac{1}{5}(-V_3 + 10 V_2 - 25 V_1 + 16 V_{\bar{1}})$$

$$5V_2 - 10 V_1 + 5 V_0 = -V_3 + 10 V_2 - 25 V_1 + 16 V_{\bar{1}}$$

$$5 V_0 = -V_3 + 5 V_2 - 15 V_1 + 16 V_{\bar{1}}$$

$$\boxed{V_0 = -\frac{1}{5}(V_3 - 5V_2 + 15 V_1 - 16 V_{\bar{1}})}$$

$\hookrightarrow$ then $\frac{(V_2 - V_0)}{2(\Delta x)} = \frac{(5V_2 - 5V_0)}{10(\Delta x)} = \frac{(5V_2 + (V_3 - 5V_2 + 15 V_1 - 16 V_{\bar{1}}))}{10(\Delta x)}$

(Note: no V_2!) $= \frac{(V_3 + 15 V_1 - 16 V_{\bar{1}})}{10(\Delta x)} = \frac{\partial V}{\partial x}\Big|_1 + \underbrace{(\Delta x)^2 \frac{1}{6} \frac{\partial^3 V}{\partial x^3}\Big|_1}_{\text{truncation error is } O((\Delta x)^2)}$

as $16 V_{\bar{1}} - V_3 = 15 V_1 - 10(\Delta x) \frac{\partial V}{\partial x}\Big|_1 + 0 - 10(\Delta x)^3 \frac{1}{6} \frac{\partial^3 V}{\partial x^3}\Big|_1 + \dots$

$$\hookrightarrow \quad \boxed{\nabla^2 P = -\nabla \cdot \underset{\sim}{A}}$$

This is the Poisson equation for the pressure in incompressible flows

B.C. on P?

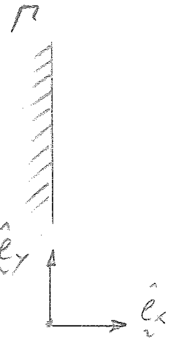
It is obtained by taking the normal component of the momentum equation

$$\underset{\sim}{\hat{n}} \cdot \left[\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{A} + \nabla P \right]_\Gamma = \underset{\sim}{\hat{n}} \cdot \left[\nu \nabla^2 \underset{\sim}{u} \right]_\Gamma$$

Example: vertical Γ with $u_\Gamma = 0$ and $v_\Gamma(y)$ given

$u_\Gamma = 0$

v_Γ given



$$\underbrace{\frac{\partial}{\partial t} u_\Gamma}_{=0} + \underbrace{u_\Gamma}_{=0} \cdot \frac{\partial u}{\partial x}\Big|_\Gamma + v_\Gamma \cdot \underbrace{\frac{\partial u}{\partial y}\Big|_\Gamma}_{=0} + \frac{\partial P}{\partial x}\Big|_\Gamma = \nu \left(\frac{\partial^2 u}{\partial x^2}\Big|_\Gamma + \underbrace{\frac{\partial^2 u}{\partial y^2}\Big|_\Gamma}_{=0} \right)$$

$$\hookrightarrow \boxed{\frac{\partial P}{\partial x}\Big|_\Gamma = \nu \frac{\partial^2 u}{\partial x^2}\Big|_\Gamma} \equiv \text{Neumann B.C.}$$

In fact, the "artificial evolution" method presented before basically amounts to an iterative method that also solves the Poisson equation for the pressure. Indeed:

- $u_{i+1/2,j}^{n+1} = u_{i+1/2,j}^{n} - \Delta t \left\{ a_{i+1/2,j}^{n} + \frac{\delta_x}{\Delta x} P_{i+1/2,j}^{n} - \nu \left(\frac{\delta_x^2}{(\Delta x)^2} + \frac{\delta_y^2}{(\Delta y)^2} \right) u_{i+1/2,j}^{n} \right\}$
- $u_{i-1/2,j}^{n+1} = \dots$
- $v_{i,j+1/2}^{n+1} = v_{i,j+1/2}^{n} - \Delta t \left\{ b_{i,j+1/2}^{n} + \frac{\delta_y}{\Delta y} P_{i,j+1/2}^{n} - \nu \left(\frac{\delta_x^2}{(\Delta x)^2} + \frac{\delta_y^2}{(\Delta y)^2} \right) v_{i,j+1/2}^{n} \right\}$
- $v_{i,j-1/2}^{n+1} = \dots$

$$P_{i,j}^{n+1} = P_{i,j}^{n} - c_0^2 \Delta t \left\{ \frac{\left(u_{i+1/2,j}^{n+1} - u_{i-1/2,j}^{n+1} \right)}{\Delta x} + \frac{\left(v_{i,j+1/2}^{n+1} - v_{i,j-1/2}^{n+1} \right)}{\Delta y} \right\}$$

$\Updownarrow$ is equivalent to doing this:

$$P_{ij}^{m+1} = P_{ij}^{m} - c_0^2 \Delta t \, \nabla_h \cdot \underset{\sim}{u}_{ij}^{m} + c_0^2 (\Delta t)^2 \left\{ \nabla_h^2 P_{ij}^{m} + \nabla_h \cdot \left(\underset{\sim}{A}_{ij}^{m} - \nu \nabla_h^2 \underset{\sim}{u}_{ij}^{m} \right) \right\}$$

At convergence: $P_{ij}^{m+1} = P_{ij}^{m}$

$\hookrightarrow \nabla_h \cdot \underset{\sim}{u}_{ij} = 0$

$\hookrightarrow \nabla_h^2 P_{ij} = \nu \nabla_h^2 \underbrace{(\nabla_h \cdot \underset{\sim}{u}_{ij})}_{=0} - \nabla_h \cdot \underset{\sim}{A}_{ij}$

$\hookrightarrow \boxed{\nabla_h^2 P_{ij} = - \nabla_h \cdot \underset{\sim}{A}_{ij}}$

$\hookrightarrow$ O.K. P_{ij} satisfies the discretized Poisson equation

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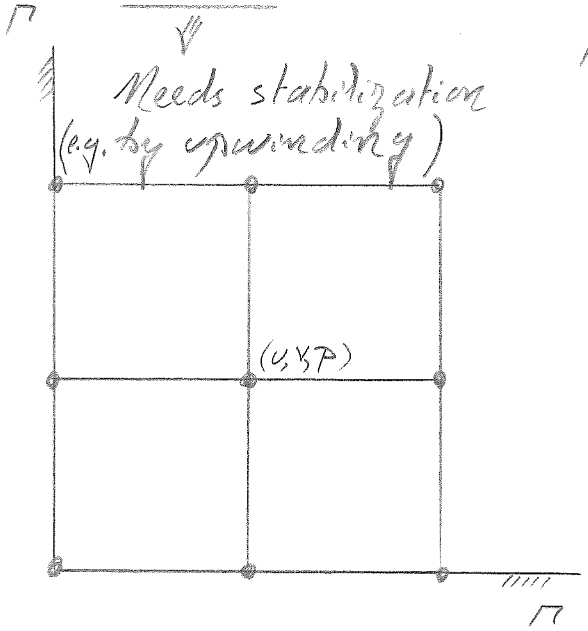
Other possible meshes? Yes, but not usual.

collocated

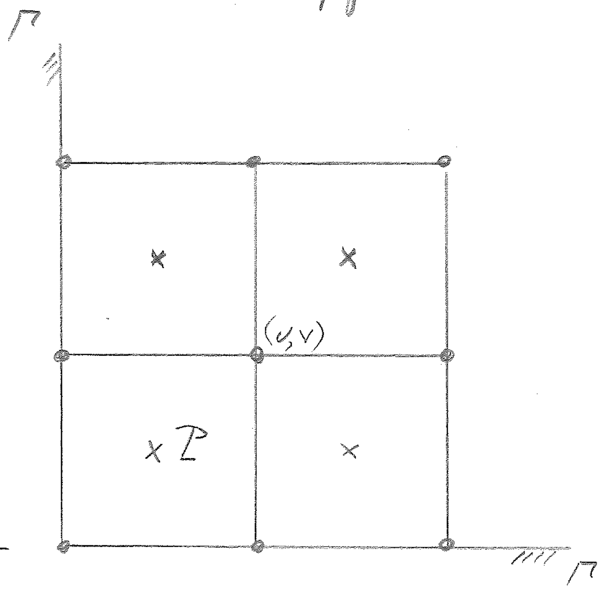
$\Downarrow$

Needs stabilization (e.g. by upwinding)



Chorin (1967)

still staggered



Kuznetsov (1968)

Fortin (1971)

Use of ADI on convection and diffusion terms

(1) $$\frac{u^*_{i+1/2,j} - u^n_{i+1/2,j}}{\Delta t/2} + u^n_{i+1/2,j}\left(\frac{u^*_{i+3/2,j} - u^*_{i-1/2,j}}{2\Delta x}\right) + v^n_{i+1/2,j}\left(\frac{u^n_{i+1/2,j+1} - u^n_{i+1/2,j-1}}{2\Delta y}\right)$$

$$+ \left(\frac{P^n_{i+1,j} - P^n_{i,j}}{\Delta x}\right) = \nu\left(\left(\frac{u^*_{i+3/2,j} - 2u^*_{i+1/2,j} + u^*_{i-1/2,j}}{(\Delta x)^2}\right) + \left(\frac{u^n_{i+1/2,j+1} - 2u^n_{i+1/2,j} + u^n_{i+1/2,j-1}}{(\Delta y)^2}\right)\right)$$

$$\left(\frac{v^*_{i,j+1/2} - v^n_{i,j+1/2}}{\Delta t/2}\right) = \dots$$

(2) $$\left(\frac{u^{n+1}_{i+1/2,j} - u^*_{i+1/2,j}}{\Delta t/2}\right) = \dots$$

→ No stability constraints for the convection-diffusion part

$$\left(\frac{v^{n+1}_{i,j+1/2} - v^*_{i,j+1/2}}{\Delta t/2}\right) = \dots$$

(3) $$\left(\frac{P^{n+1}_{i,j} - P^n_{i,j}}{\Delta t}\right) + c_0^2\left(\left(\frac{u^{n+1}_{i+1/2,j} - u^{n+1}_{i-1/2,j}}{\Delta x}\right) + \left(\frac{v^{n+1}_{i,j+1/2} - v^{n+1}_{i,j-1/2}}{\Delta y}\right)\right) = 0$$

→ Stable if $\boxed{\frac{c_0\Delta t}{\Delta x} \le 2}$

For the steps (1) and (2):

- Need to solve tridiagonal systems.
  - diagonal dominance if $\frac{|u|\Delta x}{\nu} \le 2$ and $\frac{|v|\Delta y}{\nu} \le 2$
  - otherwise, need $\frac{\nu\Delta t}{(\Delta x)^2} \le \frac{2}{\left(\frac{|u|\Delta x}{\nu} - 2\right)}$ and $\frac{\nu\Delta t}{(\Delta y)^2} \le \frac{2}{\left(\frac{|v|\Delta y}{\nu} - 2\right)}$

  to ensure the diagonal dominance.

Methods for unsteady flows

Harlow and Welsh, 1965 (MAC)

Chorin 1968, Temam 1969, Fortin 1971

Fortin 1971 : explicit , $\mathcal{O}(\Delta t)$

(1) $$\frac{(\underset{\sim}{u}^* - \underset{\sim}{u}^n)}{\Delta t} + \underset{\sim}{A}^n = \nu \nabla^2 \underset{\sim}{u}^n$$

(3) $$\frac{(\underset{\sim}{u}^{n+1} - \underset{\sim}{u}^*)}{\Delta t} + \nabla P = 0$$

with the constraint: $\nabla \cdot \underset{\sim}{u}^{n+1} = 0$

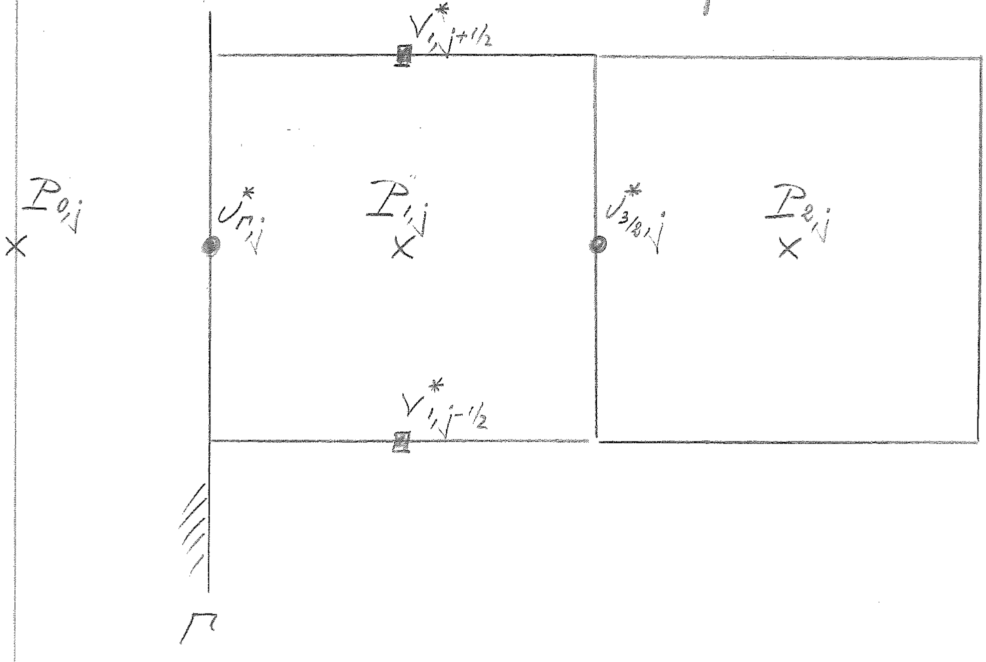
$$\Rightarrow \boxed{\nabla^2 P = \frac{\nabla \cdot \underset{\sim}{u}^*}{\Delta t}} \quad (2)$$

Poisson equation

B.C.

$$\left.\frac{\partial P}{\partial n}\right|_{\Gamma} = -\frac{1}{\Delta t}\left(\underbrace{\underset{\sim}{u}_{\Gamma}^{n+1}}_{\text{known}} - \underbrace{\underset{\sim}{u}_{\Gamma}^{*}}_{\text{not known}}\right) \cdot \hat{\underset{\sim}{n}}$$

$\underset{\sim}{u}_\Gamma^* \cdot \hat{\underset{\sim}{n}}$ is not known but this is not a problem with the MAC mesh. Consider for instance the discretization of the Poisson equation and boundary condition on a vertical Γ:



$$\left(\frac{P_{2,j} - 2P_{1,j} + P_{0,j}}{(\Delta x)^2}\right) + \left(\frac{P_{1,j+1} - 2P_{1,j} + P_{1,j-1}}{(\Delta y)^2}\right)$$

$$= \frac{1}{\Delta t}\left(\left(\frac{u^*_{3/2,j} - u^*_{\Gamma,j}}{\Delta x}\right) + \left(\frac{v^*_{1,j+1/2} - v^*_{1,j-1/2}}{\Delta y}\right)\right)$$

with $\left(\frac{P_{1,j} - P_{0,j}}{\Delta x}\right) = -\frac{1}{\Delta t}\left(u^{n+1}_{\Gamma,j} - u^*_{\Gamma,j}\right)$

$$\Downarrow$$

$$\left(\frac{P_{2,j} - P_{1,j}}{(\Delta x)^2}\right) + \frac{1}{\Delta x}\cdot\frac{1}{\Delta t}\left(u^{n+1}_{\Gamma,j} - \cancel{u^*_{\Gamma,j}}\right) + \left(\frac{P_{1,j+1} - 2P_{1,j} + P_{1,j-1}}{(\Delta y)^2}\right)$$

$$= \frac{1}{\Delta t}\left(\left(\frac{u^*_{3/2,j} - \cancel{u^*_{\Gamma,j}}}{\Delta x}\right) + \left(\frac{v^*_{1,j+1/2} - v^*_{1,j-1/2}}{\Delta y}\right)\right)$$

We see that the term $-\frac{u^*_{\Gamma,j}}{\Delta t\,\Delta x}$ is on both sides: it simplifies!

Hence, the solution does not depend on the choice we make for $\underset{\sim}{u}^*_\Gamma$.

<u>For instance</u>, we can use $\underset{\sim}{u}^*_\Gamma = \underset{\sim}{u}^{n+1}_\Gamma \Rightarrow \left.\frac{\partial P}{\partial n}\right|_\Gamma = 0$

Globally, the scheme is equivalent to:

$$\frac{\left(\underset{\sim}{u}^{n+1} - \underset{\sim}{u}^n\right)}{\Delta t} + \underset{\sim}{A}^n + \nabla P = \nu \nabla^2 \underset{\sim}{u}^n \Rightarrow O(\Delta t)$$

Semi-implicit scheme equivalent to

$$\frac{(\underset{\sim}{u}^{n+1}-\underset{\sim}{u}^{n})}{\Delta t}+\underbrace{\left(\frac{3}{2}\underset{\sim}{A}^{n}-\frac{1}{2}\underset{\sim}{A}^{n-1}\right)}_{\text{Adams-Bashforth (explicit)}}+\nabla P=\nu\underbrace{\left(\frac{1}{2}\nabla^2\underset{\sim}{u}^{n+1}+\frac{1}{2}\nabla^2\underset{\sim}{u}^{n}\right)}_{\text{Crank-Nicolson (implicit)}}$$

with $\nabla\cdot\underset{\sim}{u}^{n+1}=0$

Crank-Nicolson (implicit) $\Downarrow$ Needs ADI

↳ Scheme is $\mathcal{O}((\Delta t)^2)$.

(1) $$\frac{(\underset{\sim}{u}^{*}-\underset{\sim}{u}^{n})}{\Delta t}+\left(\frac{3}{2}\underset{\sim}{A}^{n}-\frac{1}{2}\underset{\sim}{A}^{n-1}\right)=\frac{\nu}{2}\nabla^2\underset{\sim}{u}^{n}$$

ADI here

(3) $$\frac{(\underset{\sim}{u}^{n+1}-\underset{\sim}{u}^{*})}{\Delta t}+\nabla P=\frac{\nu}{2}\nabla^2\underset{\sim}{u}^{n+1}$$

with the constraint: $\nabla\cdot\underset{\sim}{u}^{n+1}=0$

$$\Longrightarrow\ \boxed{\nabla^2 P=\frac{\nabla\cdot\underset{\sim}{u}^{*}}{\Delta t}}\quad(2)$$

This P is basically $P^{n+1/2}$

→ BC:

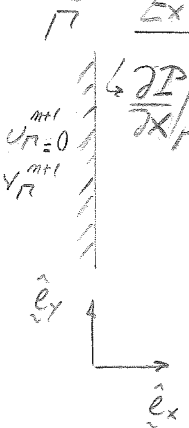
$$\left.\frac{\partial P}{\partial n}\right|_{\Gamma}=-\left(\frac{1}{\Delta t}\left(\underset{\sim}{u}_{\Gamma}^{n+1}-\underset{\sim}{u}_{\Gamma}^{*}\right)-\frac{\nu}{2}\nabla^2\underset{\sim}{u}_{\Gamma}^{n+1}\right)\cdot\underset{\sim}{\hat{n}}$$

Again, the solution does not depend on the choice of $\underset{\sim}{u}_{\Gamma}^{*}$.

↳ If we take $\underset{\sim}{u}_{\Gamma}^{*}=\underset{\sim}{u}_{\Gamma}^{n+1}$, we have $\left.\frac{\partial P}{\partial n}\right|_{\Gamma}=\frac{\nu}{2}\underbrace{\nabla^2\underset{\sim}{u}_{\Gamma}^{n+1}\cdot\hat{n}}$

Still needs to evaluate that!

Example: vertical Γ with solid wall



$$\left.\frac{\partial P}{\partial x}\right|_{\Gamma}=\frac{\nu}{2}\left(\left.\frac{\partial^2 u}{\partial x^2}\right|_{\Gamma}^{n+1}+\underbrace{\left.\frac{\partial^2 u}{\partial y^2}\right|_{\Gamma}^{n+1}}_{=0}\right)$$

$$\to=-\left.\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)\right|_{\Gamma}^{n+1}\quad\text{as}\quad\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

evaluate this using forward differences with $\left.\frac{\partial v}{\partial y}\right|_{\Gamma}^{n+1}$ known

↳ not easy! Not very practical.

Kim and Moin, Journal of Computational Physics 59 (1985)

- $\mathcal{O}((\Delta t)^2)$
- Implicit on viscous term $\Rightarrow$ needs ADI

$$(1)\quad \frac{(\underset{\sim}{u}^* - \underset{\sim}{u}^n)}{\Delta t} + \left(\frac{3}{2}\underset{\sim}{A}^n - \frac{1}{2}\underset{\sim}{A}^{n-1}\right) = \nu\left(\frac{1}{2}\nabla^2\underset{\sim}{u}^* + \frac{1}{2}\nabla^2\underset{\sim}{u}^n\right)$$

$$(3)\quad \frac{(\underset{\sim}{u}^{n+1} - \underset{\sim}{u}^*)}{\Delta t} + \nabla\varphi = 0$$

with the constraint: $\nabla\cdot\underset{\sim}{u}^{n+1} = 0$

$$\Rightarrow \boxed{\nabla^2\varphi = \frac{1}{\Delta t}\nabla\cdot\underset{\sim}{u}^*}\quad (2)$$

BC: $\left.\frac{\partial\varphi}{\partial n}\right|_\Gamma = -\frac{1}{\Delta t}\left(\underset{\sim}{u}_\Gamma^{n+1} - \underset{\sim}{u}_\Gamma^*\right)\cdot\underset{\sim}{\hat{n}}$

$\hookrightarrow = 0$ if $\underset{\sim}{u}_\Gamma^*$ is set to $\underset{\sim}{u}_\Gamma^{n+1}$

$\Downarrow$

The scheme is equivalent to:

$$\frac{(\underset{\sim}{u}^{n+1} - \underset{\sim}{u}^n)}{\Delta t} + \left(\frac{3}{2}\underset{\sim}{A}^n - \frac{1}{2}\underset{\sim}{A}^{n-1}\right) + \nabla\varphi = \nu\left(\frac{1}{2}\nabla^2\underset{\sim}{u}^* + \frac{1}{2}\nabla^2\underset{\sim}{u}^n\right)$$

$$= \nu\left(\frac{1}{2}\nabla^2\left(\underset{\sim}{u}^{n+1} + \Delta t.\nabla\varphi\right) + \frac{1}{2}\nabla^2\underset{\sim}{u}^n\right)$$

$$= \nu\left(\frac{1}{2}\nabla^2\underset{\sim}{u}^{n+1} + \frac{1}{2}\nabla^2\underset{\sim}{u}^n\right) + \frac{\nu\Delta t}{2}\underbrace{\nabla^2(\nabla\varphi)}_{\nabla(\nabla^2\varphi)}$$

$$\hookrightarrow \frac{(\underset{\sim}{u}^{n+1} - \underset{\sim}{u}^n)}{\Delta t} + \left(\frac{3}{2}\underset{\sim}{A}^n - \frac{1}{2}\underset{\sim}{A}^{n-1}\right) + \nabla\left(\varphi - \frac{\nu\Delta t}{2}\nabla^2\varphi\right)$$

$$= \nu\left(\frac{1}{2}\nabla^2\underset{\sim}{u}^{n+1} + \frac{1}{2}\nabla^2\underset{\sim}{u}^n\right)$$

$\hookrightarrow$ The pressure is $\boxed{P^{n+\frac{1}{2}} = \varphi - \frac{\nu\Delta t}{2}\nabla^2\varphi}$

Lee and Kim J. Computational Physics 168 (2001)

- $\mathcal{O}((\Delta t)^2)$
- Implicit on viscous term $\Rightarrow$ needs ADI

$$(1)\quad \frac{(\underset{\sim}{v}^* - \underset{\sim}{v}^n)}{\Delta t} + \left(\frac{3}{2}\underset{\sim}{A}^n - \frac{1}{2}\underset{\sim}{A}^{n-1}\right) + \nabla P^n = \nu\left(\frac{1}{2}\nabla^2 \underset{\sim}{v}^* + \frac{1}{2}\nabla^2 \underset{\sim}{v}^n\right)$$

$$(3)\quad \frac{(\underset{\sim}{v}^{n+1} - \underset{\sim}{v}^*)}{\Delta t} + \nabla\varphi = 0$$

with the constraint $\nabla\cdot\underset{\sim}{v}^{n+1} = 0$ $\Rightarrow$ $\boxed{\nabla^2\varphi = \frac{1}{\Delta t}\nabla\cdot\underset{\sim}{v}^*}$ (2)

$$\text{BC:}\quad \left.\frac{\partial\varphi}{\partial n}\right|_\Gamma = -\frac{1}{\Delta t}\left(\underset{\sim}{v}_\Gamma^{n+1} - \underset{\sim}{v}_\Gamma^*\right)\cdot\hat{\underset{\sim}{n}}$$

$\hookrightarrow = 0$ if $\underset{\sim}{v}_\Gamma^*$ is set to $\underset{\sim}{v}_\Gamma^{n+1}$

$$(4)\quad P^{n+1} = P^n + \varphi$$

$\Downarrow$ The scheme is equivalent to:

$$\frac{(\underset{\sim}{v}^{n+1} - \underset{\sim}{v}^n)}{\Delta t} + \left(\frac{3}{2}\underset{\sim}{A}^n - \frac{1}{2}\underset{\sim}{A}^{n-1}\right) + \nabla(P^n + \varphi)$$

$$= \nu\left(\frac{1}{2}\nabla^2\underset{\sim}{v}^* + \frac{1}{2}\nabla^2\underset{\sim}{v}^n\right)$$

$$= \nu\left(\frac{1}{2}\nabla^2\left(\underset{\sim}{v}^{n+1} + \Delta t\,\nabla\varphi\right) + \frac{1}{2}\nabla^2\underset{\sim}{v}^n\right)$$

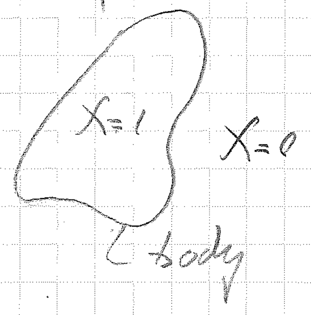
$$= \nu\left(\frac{1}{2}\nabla^2\underset{\sim}{v}^{n+1} + \frac{1}{2}\nabla^2\underset{\sim}{v}^n\right) + \frac{\nu\Delta t}{2}\underbrace{\nabla^2(\nabla\varphi)}_{\begin{array}{c}\parallel\\ \nabla(\nabla^2\varphi)\end{array}}$$

$$\hookrightarrow \frac{(\underset{\sim}{v}^{n+1} - \underset{\sim}{v}^n)}{\Delta t} + \left(\frac{3}{2}\underset{\sim}{A}^n - \frac{1}{2}\underset{\sim}{A}^{n-1}\right) + \nabla\left(P^n + \varphi - \frac{\nu\Delta t}{2}\nabla^2\varphi\right)$$

$$= \nu\left(\frac{1}{2}\nabla^2\underset{\sim}{v}^{n+1} + \frac{1}{2}\nabla^2\underset{\sim}{v}^n\right)$$

$$\hookrightarrow \boxed{P^{n+\frac{1}{2}} = P^n + \varphi - \frac{\nu\Delta t}{2}\nabla^2\varphi = P^{n+1} - \frac{\nu\Delta t}{2}\nabla^2\varphi}$$

Penalization method to take into account the presence of a body intersecting the grid in an arbitrary way



$$(1)\quad \frac{(\underset{\sim}{v}^* - \underset{\sim}{v}^n)}{\Delta t} = \overbrace{\left(\underset{\sim}{NS}\right)}^{\text{as before}} - \overbrace{\chi\left(\frac{\underset{\sim}{v}^* - \underset{\sim}{v}_b^{n+1}}{\Delta\tau}\right)}^{\text{penalization term}}$$

where

- χ is a mask: $\begin{cases} \chi = 1 & \text{inside the body} \\ \chi = 0 & \text{outside the body} \end{cases}$

- $\lambda_p = -\frac{1}{\Delta\tau}$ is a parameter that has dimension of $1/\text{time}$

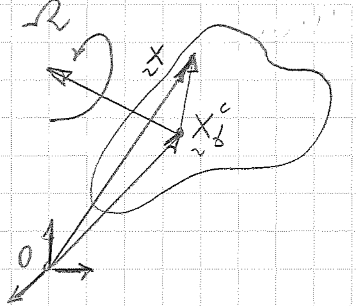
  ↳ hence, it must be chosen relatively to the time step Δt $\quad \boxed{-\lambda_p \Delta t = \frac{\Delta t}{\Delta\tau}}$

- $\underset{\sim}{v}_b$ is the velocity field inside the body

  e.g.: fixed: $\underset{\sim}{v}_b(\underset{\sim}{x}, t) = 0$

  solid body rotation: $\underset{\sim}{v}_b(\underset{\sim}{x}, t) = \underset{\sim}{\Omega}(t) \times (\underset{\sim}{x} - \underset{\sim}{x}_b^c)$ ← center of rotation

  etc.



$\underset{\sim}{\Omega}(t) = \Omega(t)\, \hat{\underset{\sim}{e}}_\Omega$ ← rotation rate

First: We consider a body without change of volume: $\nabla \cdot \underset{\sim}{v}_b = 0$

↳ we want $\nabla \cdot \underset{\sim}{v}^{n+1} = 0$ outside and inside the body

$$(3)\quad \frac{(\underset{\sim}{v}^{n+1} - \underset{\sim}{v}^*)}{\Delta t} + \nabla q = 0 \quad \text{with} \quad \boxed{\nabla^2 q = \frac{1}{\Delta t} \nabla \cdot \underset{\sim}{v}^*} \quad (2)$$

The larger the value of $(-\lambda_p \Delta t) = \frac{\Delta t}{\Delta\tau}$, the better the condition that $\underset{\sim}{v} = \underset{\sim}{v}_b$ inside the body will be satisfied.

Stability analysis of the penalization term inside the body:

$$\underset{\sim}{v}^* = \Delta t(\underset{\sim}{NS}) + \underset{\sim}{v}^n - \frac{\Delta t}{\Delta\tau}\left(\underset{\sim}{v}^* - \underset{\sim}{v}_b^{n+1}\right)$$

$$\left(1+\frac{\Delta t}{\Delta \tau}\right)\underset{\sim}{v}^* = \Delta t\,(\underset{\sim}{NS}) + \underset{\sim}{v}^n + \frac{\Delta t}{\Delta \tau}\,\underset{\sim}{v}_b^{n+1}$$

$$\underset{\sim}{v}^* = \underbrace{\frac{1}{\left(1+\frac{\Delta t}{\Delta \tau}\right)}}_{\rho}\,\underset{\sim}{v}^n + \frac{\Delta t}{\left(1+\frac{\Delta t}{\Delta \tau}\right)}\,(\underset{\sim}{NS}) + \frac{\frac{\Delta t}{\Delta \tau}}{\left(1+\frac{\Delta t}{\Delta \tau}\right)}\,\underset{\sim}{v}_b^{n+1}$$

ρ ≡ amplication factor of implicit Euler scheme

We have $|\rho| \leq 1$ for any $\frac{\Delta t}{\Delta \tau}$ ⇒ unconditionally stable!

<u>For large $\frac{\Delta t}{\Delta \tau}$</u> :

$$\left.\begin{aligned} \rho &\to 0 \\ \frac{\Delta t}{\left(1+\frac{\Delta t}{\Delta \tau}\right)} &\to 0 \\ \frac{\frac{\Delta t}{\Delta \tau}}{\left(1+\frac{\Delta t}{\Delta \tau}\right)} &\to 1 \end{aligned}\right\}\ \text{good!}$$

For instance, we can use $\frac{\Delta t}{\Delta \tau} = 10^3, 10^4$

Then, whatever the value of the time step Δt, we obtain that $\underset{\sim}{v}^* \to \underset{\sim}{v}_b^{n+1}$ ≡ good!

<u>Note</u> <u>If we use an explicit Euler scheme on the penalization term:</u>

$$\underset{\sim}{v}^* = \Delta t\,(\underset{\sim}{NS}) + \underset{\sim}{v}^n - \frac{\Delta t}{\Delta \tau}\left(\underset{\sim}{v}^n - \underset{\sim}{v}_b^{n+1}\right)$$

$$\underset{\sim}{v}^* = \underbrace{\left(1-\frac{\Delta t}{\Delta \tau}\right)}_{\rho}\underset{\sim}{v}^n + \Delta t\,(\underset{\sim}{NS}) + \frac{\Delta t}{\Delta \tau}\,\underset{\sim}{v}_b^{n+1}$$

ρ ≡ amplication factor of explicit Euler scheme

<u>Need</u> $|\rho| \leq 1$ for stability ⇔ $\frac{\Delta t}{\Delta \tau} \leq 2$ ≡ conditionally stable!

<u>Optimal choice</u> is then $\frac{\Delta t}{\Delta \tau} = 1$ as $\rho = 0$

Then, we obtain that $\underset{\sim}{v}^* = \underbrace{\Delta t\,(\underset{\sim}{NS})}_{\text{bad!}} + \underset{\sim}{v}_b^{n+1}$

This "Navier-Stokes residual" inside the body only goes to zero when Δt goes to zero.

Note: the use of the implicit Euler scheme for the penalization term does not require any implicit solver as this term is local. We simply do:

$$\frac{(\underset{\sim}{u}^* - \underset{\sim}{u}^n)}{\Delta t} = (\underset{\sim}{NS}) - \chi \frac{(\underset{\sim}{u}^* - \underset{\sim}{u}_b^{n+1})}{\Delta \tau}$$

$$\hookrightarrow \underset{\sim}{u}^* = \underset{\sim}{u}^n + \Delta t (\underset{\sim}{NS}) - \chi \frac{\Delta t}{\Delta \tau} (\underset{\sim}{u}^* - \underset{\sim}{u}_b^{n+1})$$

$$\hookrightarrow \left(1 + \chi \frac{\Delta t}{\Delta \tau}\right) \underset{\sim}{u}^* = \underset{\sim}{u}^n + \Delta t (\underset{\sim}{NS}) + \chi \frac{\Delta t}{\Delta \tau} \underset{\sim}{u}_b^{n+1}$$

Note If we use a Crank-Nicolson scheme on the penalization term:

$$\frac{(\underset{\sim}{u}^* - \underset{\sim}{u}^n)}{\Delta t} = (\underset{\sim}{NS}) - \chi \frac{\left(\frac{\underset{\sim}{u}^* + \underset{\sim}{u}^n}{2}\right) - \underset{\sim}{u}_b^{n+1}}{\Delta \tau}$$

$$\hookrightarrow \underset{\sim}{u}^* = \underset{\sim}{u}^n + \Delta t (\underset{\sim}{NS}) - \chi \frac{\Delta t}{\Delta \tau} \left(\frac{(\underset{\sim}{u}^* + \underset{\sim}{u}^n)}{2} - \underset{\sim}{u}_b^{n+1}\right)$$

$$\hookrightarrow \left(1 + \frac{\chi}{2} \frac{\Delta t}{\Delta \tau}\right) \underset{\sim}{u}^* = \left(1 - \frac{\chi}{2} \frac{\Delta t}{\Delta \tau}\right) \underset{\sim}{u}^n + \Delta t (\underset{\sim}{NS}) + \frac{\chi}{2} \frac{\Delta t}{\Delta \tau} \underset{\sim}{u}_b^{n+1}$$

$$\hookrightarrow \rho = \frac{\left(1 - \frac{1}{2} \frac{\Delta t}{\Delta \tau}\right)}{\left(1 + \frac{1}{2} \frac{\Delta t}{\Delta \tau}\right)} \Rightarrow |\rho| \leq 1 \text{ for any } \frac{\Delta t}{\Delta \tau}$$

$\equiv$ unconditionally stable $=$ good

Optimal choice is $\frac{\Delta t}{\Delta \tau} = 2$ as $\rho = 0$

Then, we obtain that $\underset{\sim}{u}^* = \underbrace{\frac{\Delta t}{2} (\underset{\sim}{NS}) + \underset{\sim}{u}_b^{n+1}}_{\equiv \text{ bad (as for explicit Euler)}}$

$\rho < 0$ for $\frac{\Delta t}{\Delta \tau} > 2$ $\Rightarrow$ non monotonic behavior $\equiv$ bad!

$\rho \to -1$ for large $\frac{\Delta t}{\Delta \tau}$ $\equiv$ very bad!

What about a body with change of volume?

$\underset{\sim}{v_b}$ is then such that $\nabla \cdot \underset{\sim}{v_b} \neq 0$

↳ After the step (1), we reproject in such a way that

$$\nabla \cdot \underset{\sim}{v}^{n+1} = \begin{cases} 0 & \text{outside the body} \\ \nabla \cdot \underset{\sim}{v_b}^{n+1} & \text{inside the body} \end{cases}$$

↳ The constraint is $\boxed{\nabla \cdot \underset{\sim}{v}^{n+1} = \chi \ \nabla \cdot \underset{\sim}{v_b}^{n+1}}$

↳ (3) $\dfrac{(\underset{\sim}{v}^{n+1} - \underset{\sim}{v}^{*})}{\Delta t} + \nabla \varphi = 0$ with $\boxed{\nabla^2 \varphi = \dfrac{1}{\Delta t}\left(\nabla \cdot \underset{\sim}{v}^{*} - \chi \ \nabla \cdot \underset{\sim}{v_b}^{n+1}\right)}$

(2)

Vorticity - velocity formulation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{A} + \nabla P = \nu \nabla^2 \mathbf{v} \quad \text{and} \quad \overset{\text{vorticity}}{\overset{\equiv}{\boldsymbol{\omega}}} \triangleq \nabla \times \mathbf{v} \Rightarrow \nabla \cdot \boldsymbol{\omega} = 0$$

$$\hookrightarrow \nabla \times \boldsymbol{\omega} = \nabla \times (\nabla \times \mathbf{v}) = -\nabla \cdot (\nabla \mathbf{v}) + \nabla (\overset{\equiv 0}{\nabla \cdot \mathbf{v}})$$

$$= -\nabla^2 \mathbf{v}$$

$$\text{with } \mathbf{A} = \begin{cases} (\nabla \mathbf{v}) \cdot \mathbf{v} \\ \nabla \cdot (\mathbf{v} \circ \mathbf{v}) \\ \boldsymbol{\omega} \times \mathbf{v} + \nabla \left(\dfrac{\mathbf{v} \cdot \mathbf{v}}{2} \right) \equiv \text{yet another form} \end{cases}$$

Consider the form: ($\equiv$ Bernoulli form)

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\omega} \times \mathbf{v} + \nabla \underbrace{\left(P + \frac{\mathbf{v} \cdot \mathbf{v}}{2} \right)}_{\equiv B} = -\nu \nabla \times \boldsymbol{\omega}$$

And take the curl: $\nabla \times [\ldots]$. We obtain:

$$\frac{\partial}{\partial t} \boldsymbol{\omega} + \underbrace{\nabla \times (\boldsymbol{\omega} \times \mathbf{v})}_{\equiv} + 0 = -\nu \underbrace{\nabla \times (\nabla \times \boldsymbol{\omega})}_{\equiv}$$

$$\nabla \times (\nabla \times \boldsymbol{\omega}) \equiv -\nabla \cdot (\nabla \boldsymbol{\omega}) + \nabla (\underset{= 0}{\nabla \cdot \boldsymbol{\omega}}) = -\nabla^2 \boldsymbol{\omega}$$

$$\nabla \times (\boldsymbol{\omega} \times \mathbf{v}) \equiv (\nabla \boldsymbol{\omega}) \cdot \mathbf{v} - (\nabla \mathbf{v}) \cdot \boldsymbol{\omega} + \boldsymbol{\omega} (\underset{= 0}{\nabla \cdot \mathbf{v}}) - \mathbf{v} (\underset{= 0}{\nabla \cdot \boldsymbol{\omega}})$$

$$\hookrightarrow \boxed{\underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t} + (\nabla \boldsymbol{\omega}) \cdot \mathbf{v}}_{\equiv \frac{D}{Dt} \boldsymbol{\omega}} = \underbrace{(\nabla \mathbf{v}) \cdot \boldsymbol{\omega}}_{\equiv \text{vortex stretching term}} + \nu \nabla^2 \boldsymbol{\omega}}$$

$\equiv$ Advective form

Other form $\equiv$ Divergence form

$$\boxed{\frac{\partial}{\partial t}\underset{\sim}{\omega} + \underbrace{\nabla\cdot(\underset{\sim}{\omega}\circ\underset{\sim}{v})}_{\text{as } \nabla\cdot\underset{\sim}{v}=0} = \underbrace{\nabla\cdot(\underset{\sim}{v}\circ\underset{\sim}{\omega})}_{\text{as } \nabla\cdot\underset{\sim}{\omega}=0} + \nu\nabla^2\underset{\sim}{\omega}}$$

In indicial notation:

$$\underbrace{\frac{\partial}{\partial t}\omega_i + \frac{\partial\omega_i}{\partial x_j}v_j}_{\equiv\ \frac{D}{Dt}\omega_i} = \frac{\partial v_i}{\partial x_j}\cdot\omega_j + \nu\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}\omega_i$$

$$\frac{\partial}{\partial t}\omega_i + \underbrace{\frac{\partial}{\partial x_j}(\omega_i v_j)}_{\text{as } \frac{\partial v_j}{\partial x_j}=0} = \underbrace{\frac{\partial}{\partial x_j}(v_i\omega_j)}_{\text{as } \frac{\partial\omega_j}{\partial x_j}=0} + \nu\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}\omega_i$$

$$\frac{\partial}{\partial t}\omega_i + \frac{\partial}{\partial x_j}(\omega_i v_j - \omega_j v_i) = \nu\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}\omega_i$$

Also $\nabla\cdot\underset{\sim}{v}=0 \iff \underset{\sim}{v}=\nabla\times\underset{\sim}{\psi}$ with $\underset{\sim}{\psi}$ the streamfunction

$$\underset{\sim}{\omega}=\nabla\times\underset{\sim}{v}=\nabla\times(\nabla\times\underset{\sim}{\psi})=-\nabla\cdot(\nabla\underset{\sim}{\psi})+\nabla(\underbrace{\nabla\cdot\underset{\sim}{\psi}}_{=0})$$

We choose $\underset{\sim}{\psi}$ with $\nabla\cdot\underset{\sim}{\psi}=0$ ≡ Lorentz gauge

$\Downarrow$

$$\boxed{\begin{aligned}\nabla\cdot(\nabla\underset{\sim}{\psi})&=-\underset{\sim}{\omega}\\ &\equiv\\ \nabla^2\underset{\sim}{\psi}&=-\underset{\sim}{\omega}\end{aligned}}$$ ≡ Poisson equation for $\underset{\sim}{\psi}$

2-D: $\Downarrow$ No vortex stretching

$$\begin{cases}\underset{\sim}{\omega}=\omega\,\hat{\underset{\sim}{e}}_z & \text{with } \omega=\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\\ \underset{\sim}{\psi}=\psi\,\hat{\underset{\sim}{e}}_z & \text{and } u=\dfrac{\partial\psi}{\partial y},\quad v=-\dfrac{\partial\psi}{\partial x}\end{cases}$$

$$\boxed{\nabla^2\psi=-\omega}$$

$$\boxed{\begin{aligned}&\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=\nu\nabla^2\omega\\ &\frac{\partial\omega}{\partial t}+\underbrace{\frac{\partial}{\partial x}(\omega u)+\frac{\partial}{\partial y}(\omega v)}_{\equiv}=\nu\nabla^2\omega\end{aligned}}$$

≡ Advective form

≡ Divergence form

∇ notation for the two forms

BC: u and v given on Γ $\iff$ $\dfrac{\partial\psi}{\partial x}$ and $\dfrac{\partial\psi}{\partial y}$ given

$\iff$ ψ and $\dfrac{\partial\psi}{\partial n}$ given $\Uparrow$ see later

Problem! Mathematically, we need one BC on ψ and one BC on ω...

Formulation "ψ only" for the case of steady flows

$$\omega = -\nabla^2 \psi \;\to\; \frac{\partial \omega}{\partial x} = -\frac{\partial}{\partial x}\left(\nabla^2 \psi\right), \quad \frac{\partial \omega}{\partial y} = -\frac{\partial}{\partial y}\left(\nabla^2 \psi\right)$$

bi-harmonic operator

$$\nabla^2 \omega = -\nabla^4 \psi \quad \text{with } \nabla^4 \psi \triangleq \nabla^2\left(\nabla^2 \psi\right)$$

↳
$$\boxed{J = \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\nabla^2 \psi\right) - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial y}\left(\nabla^2 \psi\right) = \nu \nabla^4 \psi}$$

or

$$\boxed{J = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\nabla^2 \psi\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\nabla^2 \psi\right) = \nu \nabla^4 \psi}$$

Need 2 BC: ψ and $\frac{\partial \psi}{\partial n}$ ⇒ ok, we have them.

↳ The problem is well-posed.

Of course, one still needs to develop a numerical method to solve this equation (e.g., an iterative method...)

↳ This is not easy nor classical. Yet this has been done.

Iterative method for the case of steady flows

and using an "artificial evolution"

We wish to obtain the solution of

$$\boxed{\begin{aligned} J &= \nu \nabla^2 \omega \\ \nabla^2 \psi &= -\omega \end{aligned}}$$

We use "artificial evolution" equations:

$$\boxed{\begin{aligned} \frac{\partial \omega}{\partial t} + J &= \nu \nabla^2 \omega \\ \frac{\partial \psi}{\partial t} &= \alpha \left(\nabla^2 \psi + \omega \right) \end{aligned}}$$

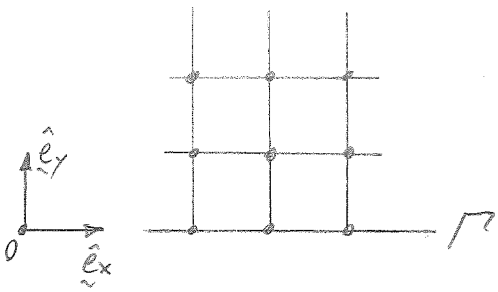
⚠ Here, t is not a physical time.

↳ α is a diffusivity, to be chosen in relation to ν.

Problem of the BC on ω

We here consider a horizontal Γ

e.g.: Γ is a solid wall
↳ $v(x,0) = -\frac{\partial \psi}{\partial x}(x,0) = 0$
↳ $\psi(x,0) = f_0$



On Γ: $\left.\begin{aligned} &\psi(x,0) = f(x) \\ &\text{and} \\ &u(x,0) = \frac{\partial \psi}{\partial y}(x,0) = g(x) \end{aligned}\right\}$ given

$$\hookrightarrow \omega(x,0) = -\frac{\partial^2 \psi}{\partial x^2}(x,0) - \frac{\partial^2 \psi}{\partial y^2}(x,0) = -f''(x) - \frac{\partial^2 \psi}{\partial y^2}(x,0)$$

Taylor series

$$\psi(x,\Delta y) = \psi(x,0) + \Delta y \frac{\partial \psi}{\partial y}(x,0) + \frac{(\Delta y)^2}{2} \frac{\partial^2 \psi}{\partial y^2}(x,0) + O\left((\Delta y)^3\right)$$

$$= f(x) + \Delta y \, g(x) + \frac{(\Delta y)^2}{2} \frac{\partial^2 \psi}{\partial y^2}(x,0) + O\left((\Delta y)^3\right)$$

$\Downarrow$

$$\frac{\partial^2 \psi}{\partial y^2}(x,0) = \frac{2}{(\Delta y)^2}\Big(\psi(x,\Delta y) - f(x)\Big) - \frac{2}{\Delta y}\, g(x) + O(\Delta y)$$

$\Downarrow$

$$\boxed{\omega(x,0) = -f''(x) - \frac{2}{(\Delta y)^2}\Big(\psi(x,\Delta y) - f(x)\Big) + \frac{2}{\Delta y}\, g(x) + O(\Delta y)}$$

$\equiv$ BC with error of first order.

One can do better by using ψ from two levels:

$$\psi(x,\Delta y) = \psi(x,0) + \Delta y\,\frac{\partial \psi}{\partial y}(x,0) + \frac{(\Delta y)^2}{2}\frac{\partial^2 \psi}{\partial y^2}(x,0) + \frac{(\Delta y)^3}{6}\frac{\partial^3 \psi}{\partial y^3}(x,0) + O\big((\Delta y)^4\big)$$

$$= f(x) + \Delta y\, g(x) + \frac{(\Delta y)^2}{2}\frac{\partial^2 \psi}{\partial y^2}(x,0) + \frac{(\Delta y)^3}{6}\frac{\partial^3 \psi}{\partial y^3}(x,0) + O\big((\Delta y)^4\big)$$

$$\psi(x,2\Delta y) = \psi(x,0) + (2\Delta y)\frac{\partial \psi}{\partial y}(x,0) + \frac{(2\Delta y)^2}{2}\frac{\partial^2 \psi}{\partial y^2}(x,0) + \frac{(2\Delta y)^3}{6}\frac{\partial^3 \psi}{\partial y^3}(x,0) + O\big((\Delta y)^4\big)$$

$$= f(x) + 2\Delta y\, g(x) + \frac{4}{2}(\Delta y)^2\frac{\partial^2 \psi}{\partial y^2}(x,0) + \frac{8}{6}(\Delta y)^3\frac{\partial^3 \psi}{\partial y^3}(x,0) + O\big((\Delta y)^4\big)$$

$\Downarrow$ Eliminate $\frac{\partial^3 \psi}{\partial y^3}(x,0)$

$$8\psi(x,\Delta y) - \psi(x,2\Delta y) = 7f(x) + 6\Delta y\, g(x) + 2(\Delta y)^2\frac{\partial^2 \psi}{\partial y^2}(x,0) + O\big((\Delta y)^4\big)$$

$$\rightarrow \frac{\partial^2 \psi}{\partial y^2}(x,0) = \frac{1}{2(\Delta y)^2}\Big(8\psi(x,\Delta y) - \psi(x,2\Delta y) - 7f(x)\Big) - \frac{3}{\Delta y}\, g(x) + O\big((\Delta y)^2\big)$$

$\Downarrow$

$$\boxed{\omega(x,0) = -f''(x) - \frac{1}{2(\Delta y)^2}\Big(8\psi(x,\Delta y) - \psi(x,2\Delta y) - 7f(x)\Big) + \frac{3}{\Delta y}\, g(x) + O\big((\Delta y)^2\big)}$$

$\equiv$ BC with error of second order.

Another way (Woods 1954): use nearest ψ and ω information

$$\psi(x,\Delta y) = \psi(x,0) + \Delta y \frac{\partial \psi}{\partial y}(x,0) + \frac{(\Delta y)^2}{2} \frac{\partial^2 \psi}{\partial y^2}(x,0) + \frac{(\Delta y)^3}{6} \frac{\partial^3 \psi}{\partial y^3}(x,0) + O((\Delta y)^4)$$

$$= f(x) + \Delta y\, g(x) + \frac{(\Delta y)^2}{2} \frac{\partial^2 \psi}{\partial y^2}(x,0) + \frac{(\Delta y)^3}{6} \frac{\partial^3 \psi}{\partial y^3}(x,0) + O((\Delta y)^4)$$

Also:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega$$

$$\Downarrow$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial^3 \psi}{\partial y^3} = -\frac{\partial \omega}{\partial y}$$

$$\hookrightarrow \frac{\partial^3 \psi}{\partial y^3} = -\frac{\partial \omega}{\partial y} - \frac{\partial^2}{\partial x^2}\left(\frac{\partial \psi}{\partial y}\right)$$

$$\frac{\partial^3 \psi}{\partial y^3}(x,0) = -\frac{\partial \omega}{\partial y}(x,0) - g''(x) = -\frac{\left(\omega(x,\Delta y) - \omega(x,0)\right)}{\Delta y} - g''(x) + O(\Delta y)$$

Finally:

$$\frac{\partial^2 \psi}{\partial y^2}(x,0) = \frac{2}{(\Delta y)^2}\left(\psi(x,\Delta y) - f(x)\right) - \frac{2}{\Delta y} g(x) - \frac{1}{3}\Delta y \frac{\partial^3 \psi}{\partial y^3}(x,0) + O((\Delta y)^2)$$

$$= \frac{2}{(\Delta y)^2}\left(\psi(x,\Delta y) - f(x)\right) - \frac{2}{\Delta y} g(x) + \frac{1}{3}\left(\left(\omega(x,\Delta y) - \omega(x,0)\right) + \Delta y\, g''(x)\right) + O((\Delta y)^2)$$

Introduce that into: $\omega(x,0) = -f''(x) - \frac{\partial^2 \psi}{\partial y^2}(x,0)$

We obtain:

$$\boxed{\omega(x,0) = -\frac{3}{2} f''(x) - \frac{3}{(\Delta y)^2}\left(\psi(x,\Delta y) - f(x)\right) + \frac{3}{\Delta y} g(x) - \frac{1}{2}\omega(x,\Delta y) - \frac{\Delta y}{2} g''(x) + O((\Delta y)^2)}$$

$\equiv$ BC with error of second order and more compact than the other one

ADI Method for

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + J = \nu \nabla^2 \omega \\ \dfrac{\partial \psi}{\partial t} = \alpha \left(\nabla^2 \psi + \omega \right) \end{cases}$$

$$(1) \quad \left(\frac{\omega^*_{i,j} - \omega^n_{i,j}}{\Delta t/2} \right) + u^n_{i,j} \left(\frac{\omega^*_{i+1,j} - \omega^*_{i-1,j}}{2\Delta x} \right) + v^n_{i,j} \left(\frac{\omega^n_{i,j+1} - \omega^n_{i,j-1}}{2 \Delta y} \right)$$

$$= \nu \left(\left(\frac{\omega^*_{i+1,j} - 2\omega^*_{i,j} + \omega^*_{i-1,j}}{(\Delta x)^2} \right) + \left(\frac{\omega^n_{i,j+1} - 2\omega^n_{i,j} + \omega^n_{i,j-1}}{(\Delta y)^2} \right) \right)$$

$$(2) \quad \left(\frac{\omega^{n+1}_{i,j} - \omega^*_{i,j}}{\Delta t/2} \right) + u^n_{i,j} \left(\frac{\omega^*_{i+1,j} - \omega^*_{i-1,j}}{2\Delta x} \right) + v^n_{i,j} \left(\frac{\omega^{n+1}_{i,j+1} - \omega^{n+1}_{i,j-1}}{2 \Delta y} \right)$$

$$= \nu \left(\left(\frac{\omega^*_{i+1,j} - 2\omega^*_{i,j} + \omega^*_{i-1,j}}{(\Delta x)^2} \right) + \left(\frac{\omega^{n+1}_{i,j+1} - 2\omega^{n+1}_{i,j} + \omega^{n+1}_{i,j-1}}{(\Delta y)^2} \right) \right)$$

<u>with</u> $u^n_{i,j} = \left(\dfrac{\psi^n_{i,j+1} - \psi^n_{i,j-1}}{2\Delta y} \right)$ and $v^n_{i,j} = - \left(\dfrac{\psi^n_{i+1,j} - \psi^n_{i-1,j}}{2\Delta x} \right)$

$$= \frac{1}{2} \left[\left(\frac{\psi^n_{i,j+1} - \psi^n_{i,j}}{\Delta y} \right) + \left(\frac{\psi^n_{i,j} - \psi^n_{i,j-1}}{\Delta y} \right) \right] = \frac{1}{2} \left[u^n_{i,j+\frac{1}{2}} + u^n_{i,j-\frac{1}{2}} \right]$$

$$(3) \quad \left(\frac{\psi^*_{i,j} - \psi^n_{i,j}}{\Delta t/2} \right) = \alpha \left(\omega^{n+1}_{i,j} + \left(\frac{\psi^*_{i+1,j} - 2\psi^*_{i,j} + \psi^*_{i-1,j}}{(\Delta x)^2} \right) + \left(\frac{\psi^n_{i,j+1} - 2\psi^n_{i,j} + \psi^n_{i,j-1}}{(\Delta y)^2} \right) \right)$$

$$(4) \quad \left(\frac{\psi^{n+1}_{i,j} - \psi^*_{i,j}}{\Delta t/2} \right) = \alpha \left(\omega^{n+1}_{i,j} + \left(\frac{\psi^*_{i+1,j} - 2\psi^*_{i,j} + \psi^*_{i-1,j}}{(\Delta x)^2} \right) + \left(\frac{\psi^{n+1}_{i,j+1} - 2\psi^{n+1}_{i,j} + \psi^{n+1}_{i,j-1}}{(\Delta y)^2} \right) \right)$$

- Unconditionally stable but need to solve tridiagonal systems.
  - diagonal dominance if $\dfrac{|u|\Delta x}{\nu} \leq 2$ and $\dfrac{|v|\Delta y}{\nu} \leq 2$
  - otherwise, need $\dfrac{\nu \Delta t}{(\Delta x)^2} \leq \dfrac{2}{\left(\frac{|u|\Delta x}{\nu} - 2 \right)}$ and $\dfrac{\nu \Delta t}{(\Delta y)^2} \leq \dfrac{2}{\left(\frac{|v|\Delta y}{\nu} - 2 \right)}$

    to ensure the diagonal dominance

Simulation of unsteady flows

$$\frac{\partial \omega}{\partial t} + J = \nu \nabla^2 \omega$$

with $\nabla^2 \psi = -\omega$

Example

$$\frac{(\omega^{n+1} - \omega^n)}{\Delta t} + \underbrace{\left(\frac{3}{2} J^n - \frac{1}{2} J^{n-1}\right)}_{\text{Adams-Bashforth}} = \nu \underbrace{\left(\frac{1}{2} \nabla^2 \omega^{n+1} + \frac{1}{2} \nabla^2 \omega^n\right)}_{\text{Crank-Nicolson (implicit)}}$$

↳ needs ADI

or

$$\frac{(\omega^{n+1} - \omega^n)}{\Delta t} + \underbrace{\left(\frac{3}{2} J^n - \frac{1}{2} J^{n-1}\right)}_{\text{Adams-Bashforth}} = \underbrace{\nu \nabla^2 \omega^n}_{\text{Euler explicit}}$$

Then solve $\nabla^2 \psi^{n+1} = -\omega^{n+1}$

to obtain ψ^{n+1} and thus also $\underset{\sim}{u}^{n+1}$