Automata and Space-Filling Curves
Final Project Report

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1 Project Goals

An interesting question related to both automata theory and real analysis is the following: given a real function f and its domain D, does there exist a Büchi automaton that recognizes the set $\{(x, f(x)) \mid x \in D\}$? If the answer is yes, then the related function is called **regular**.
In this project, we are interested in the properties of regular continuous functions and regular space-filling curves. In particular, we find some examples of regular space-filling curves and explore the relation between regularity and differentiability for continuous function.

2 Background

In order to recognize real functions, we first code real number into infinite array of digits $\{n \in \mathbb{Z}_{\geq 0} : n < \beta\}$, where β is the base of the number. For example, $1/3$ is coded into $[0, 3, 3, \cdots]$ in base-10 and $[0, 1, 1, 1, \cdots]$ in base-3. Then we can define the input alphabet:

$$\Sigma_{\beta_1,\beta_2} = \{(n_1, n_2) : n_1, n_2 \in \mathbb{Z}_{\geq 0}, n_1 < \beta_1, n_2 < \beta_2\}$$

In short, one input letter $\sigma \in \Sigma$ is one digit of $(x, f(x))$ where x is coded in base β_1 and $f(x)$ is coded in base β_2.

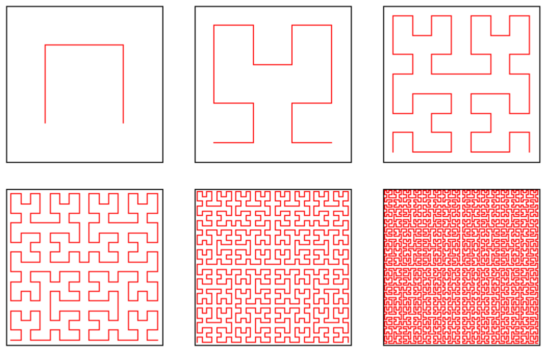
Definition. *A real function $f : D \subset \mathbb{R} \to \mathbb{R}$ is called* ***regular with input base*** β_1 ***and output base*** β_2*, or shortly* ***regular****, if there exists a Büchi automaton with input alphabet Σ_{β_1,β_2} which accepts runs $(x, f(x))$ for all $x \in \mathbb{D}$*

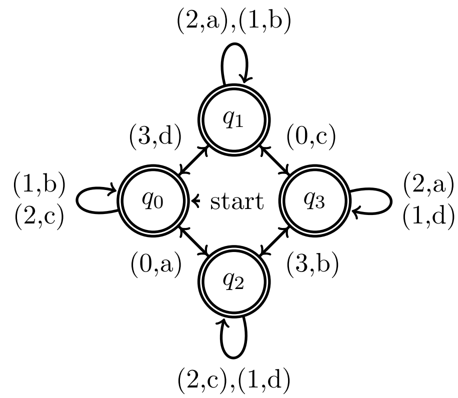
3 Progress and Results

We have two main results in this project.

3.1 Automata of space-filling curves

Space-filling curve is a special fractal whose range covers the entire 2-dimensional square. Hilbert curve is an example of it. The graph on the left is the first to 6th iteration of Hilbert curve. We prove that Hilbert curve is regular by finding an automaton which recognizes it (the picture on the right). Here the input base is 4 and output base is 2×2, and $a = (0, 0), b = (1, 0), c = (1, 1), d = (0, 1)$.





This example gives us the intuition that the input base and output base has important impact on the continuity of the function. If we use only the x-coordinate of the Hilbert curve as the output and input stay the same, this function should be continuous and nowhere differentiable and is recognized by an automaton from base 4 to base 2. This inspired us to get the second result.

3.2 Property of regular continuous function

Chaudhuri, Sankaranarayanan and Vardi (2013) proved that a continuous regular function from base 2 to base 2 is Lipschitz continuous, but mistakenly claimed this could be generalized to any base. We correct this result for arbitrary input and output bases.

Theorem. *Given a real function $f : \mathbb{R} \to \mathbb{R}$, if f is continuous and regular, with x coded in base β_1 and $f(x)$ coded in base β_2, then*

- *if $\beta_1 < \beta_2$, then f is constant.*
- *if $\beta_1 = \beta_2$, then f is Lipschitz continuous.*
- *if $\beta_1 > \beta_2$, then f is Hölder continuous with exponent $\alpha = log_{\beta_1}(\beta_2)$.*

4 Future Directions

Our faculty advisors Hieronymi and Walsberg (2017) proved that:

Theorem. *If a function f is regular and continuously differentiable on a bounded open interval, and coded in the same base for input and output, then f is affine.*

Following this theorem, a natural question is whether in the same setting, do regular continuous functions also have this property? Or **does there exist a regular continuous function that is nowhere affine?**

We were unable to find such a function yet. But from the theorem we proved, for any regular continuous function f, if x and $f(x)$ are in the same base, f is Lipschitz continuous, which implies f is differentiable almost everywhere. So if such a non-affine regular continuous function exists, it must be nondifferentiable only on a measure zero dense set. Other possible topics include:

- Build more examples of regular space filling curves. In particular, build automata that don't go from base n to base $(\sqrt{n}, \sqrt{n})$.
- Find other properties of regular functions.

5 References

[1] Khoussainov, Bakhadyr.Nerode, Anil. (2001) *Automata Theory and its ApplicationsBoston*, MA : Birkhäuser Boston

[2] Swarat Chaudhuri, Sriram Sankaranarayanan, and Moshe Y. Vardi. (2013) *Regular Real Analysis* In Proceedings of the 2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS '13). IEEE Computer Society, Washington, DC, USA, 509-518.

[3] Hieronymi Philipp, Walsberg Erik. (2017) *On continuous functions definable in expansions of the ordered real additive group*