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Dimensional analysis and the correspondence between classical and quantum uncertainty

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Abstract. Heisenberg's uncertainty principle is often cited as an example of a "purely quantum" relation with no analogue in the classical limit where $\hbar \rightarrow 0$. However, this formulation of the classical limit is problematic for many reasons, one of which is dimensional analysis. Since \hbar is a dimensionful constant, we may always work in natural units in which $\hbar = 1$. Dimensional analysis teaches us that all physical laws can be expressed purely in terms of dimensionless quantities. This indicates that the existence of a dimensionally consistent constraint on $\Delta x \Delta p$ requires the existence of a dimensionful parameter with units of action, and that any definition of the classical limit must be formulated in terms of dimensionless quantities (such as quantum numbers). Therefore, bounds on classical uncertainty (formulated in terms of statistical ensembles) can only be written in terms of dimensionful scales of the system under consideration, and can be readily compared to their quantum counterparts after being non-dimensionalized. We compare the uncertainty of certain coupled classical systems and their quantum counterparts (such as harmonic oscillators and particles in a box), and show that they converge in the classical limit. We find that since these systems feature additional dimensionful scales, the uncertainty bounds are dependent on multiple dimensionless parameters, in accordance with dimensional considerations.

Keywords: classical limit, dimensional analysis, quantum uncertainty

1. Introduction

Historically, the development of quantum mechanics benefited greatly from the intuition offered by the Lagrangian and Hamiltonian formulations of classical mechanics. Perhaps the most famous example of this was Schrödinger's derivation of his eponymous equation. Realizing that microscopic particles had a fundamentally "undulatory" (wave-like) nature, Schrödinger observed that constant action surfaces[‡], being periodic, would play a major role in developing a robust theory of quantum mechanics [1]. It is at this point in which dimensional analysis played a crucial role in Schrödinger's reasoning: indeed, since the action is a dimensionful quantity, it can only appear as the argument of a periodic function if it is divided by a quantity that

[‡] Here, "action" refers to $W = \int L dt$, where L is the Lagrangian.

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also has units of action. If one cannot “resist the temptation”, as Schrödinger put it,
of assuming that this constant is a fundamental constant, the most obvious candidate
is the Planck constant \hbar , which can be converted into a phase if divided by 2π , leading
to the appearance of the reduced Planck constant $\hbar \equiv \hbar/2\pi$.9
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Schrödinger’s archetypal quantization procedure was later formalized and refined
as the deformation from Poisson brackets to quantum commutators and the promotion
of classical expressions to quantum operators for classical systems. It is therefore
natural to ask whether “purely quantum” phenomena can be formulated in a classical
manner. The question has been approached from multiple angles, and, while it
may be futile to attempt to build a purely classical theory that perfectly replicates
the observations of quantum mechanics, we at least expect quantum mechanics to
reproduce classical predictions corresponding to our everyday experience. This is the
crux of the *correspondence principle* [2–5].18
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The correspondence principle is often expressed as the idea that quantum
mechanics must reduce to classical mechanics in some limit, usually referred to as
the *classical limit*. Stated like this, the correspondence principle is little more than
tautological unless we are very careful to expand on how the classical limit is formally
defined. In most undergraduate textbooks, the correspondence principle is formalized
in terms of Ehrenfest’s theorem, which relates the evolution of the expectation values
of operators to their commutator with the Hamiltonian. However, even Ehrenfest’s
theorem alone does not bridge the conceptual gap between quantum and classical
mechanics [6]. This is only possible if we assign a classical meaning to the expectation
values themselves. In order to do this, consider a quantum system evolving so rapidly
that we can only measure its mean state over many cycles. Hence, the classical limit
may be identified as the limit in which energies (or quantum numbers) are so large that
the law of large numbers is unquestionably in effect and our macroscopic measurements
are, in effect, averages of a very large number of quantum measurements.32
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While Ehrenfest’s theorem provides a formal way of moving from the quantum
realm of uncertainty to the classical realm of certainty, it is helpful to approach the
question from the opposite angle and ask how uncertainty may arise in classical
systems. Of course, for a single system with a fixed, known state, the notion of
an expectation value of a quantity is no different from the quantity itself. However,
since quantum mechanics is an inherently probabilistic theory, it makes more sense
to formulate classical uncertainty by way of an *ensemble* of systems, each with a
different statistical weight. In this framework, the classical probability distribution
is the analogue of the quantum mechanical probability density $\rho = |\psi|^2$. Therefore,
in this way, the statistical average of the ensemble may be identified as the classical
analogue of the quantum average of the system in question.43
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We turn our attention to the well-known Heisenberg uncertainty principle, a
purely quantum mechanical limit on the simultaneous localization of a particle’s
energy and momentum $\Delta x \Delta p \geq \hbar/2$. A naïve treatment of the classical limit
might lead us to expect that no uncertainty exists in the classical case. However, this
would be a mistake: the Heisenberg uncertainty principle presupposes the existence
of a fundamental unit of action (i.e. the Planck constant \hbar), something which does
not exist in the classical realm. Therefore, a meaningful comparison between the
two domains can only be achieved by working with non-dimensionalized quantities,
such as the quantum numbers, recognized by Schrödinger as corresponding to Bohr’s
stationary energy levels of the elliptic orbits in the case of the atom. Schrödinger even
obliquely refers to the correspondence principle at this stage, noting that even though55
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“mathematically”, the atom spreads throughout space as $n \rightarrow \infty$, its quantum nature is restricted to a few angstroms, i.e. for relatively small n [1].

The idea that a comparison of the quantum and classical realm must be made through the use of pure numbers is by no means exclusive to the correspondence principle: it is a result of the Buckingham- π theorem [7], which states that *every* law of physics must be expressible in a non-dimensionalized form. A careful treatment of the harmonic oscillator, infinite square well, and other single-particle systems with simple one-dimensional potentials [8–15] reveals that, once position and momentum are properly non-dimensionalized (with the help of x_0 and p_0 , which are reference scales inherent to the system), then it is possible to arrive at a similar limit for the ensemble uncertainty, given by $\Delta\bar{x}\Delta\bar{p} \geq c$, where c is a pure number.

A natural question to ask at this stage is how the relation between quantum and classical uncertainty is modified for interacting one dimensional systems with additional dimensionful scales (such as coupled systems). Our approach is to define appropriate dimensionless variables, and seek to obtain the uncertainty relations in terms of their dimensionless ratios. In particular, we expect that adding more degrees of freedom to the system will yield more dimensionless ratios in the expression for the uncertainty relations, in accordance with the Buckingham- π theorem.

The outline of the paper is as follows: in Section 2, we briefly review dimensional analysis and consider the implications of the Buckingham- π theorem for the classical and quantum uncertainty principle. In Section 3, we outline how the notion of classical uncertainty may be formalized by providing an overview of the correspondence between classical and quantum uncertainty for one-dimensional harmonic oscillators and the infinite square well. We then go on to examine how the addition of further dimensionful scales modifies the uncertainty relations while still respecting the correspondence principle. In Sections 4 and 5, we study coupled harmonic oscillators, whereas in Section 6, we study two particles in a box with a contact potential. Finally, we discuss our findings in the Conclusions.

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2. Dimensional analysis and the classical limit

Physical dimensions are a concept related to but distinct from that of units. Units are related to *standards* (e.g. the length of a rod, the weight of a bearing, etc.) and correspond to the way in which the same measurements can be expressed: for instance, 1 km and 1000 m both describe the same quantity. Dimensions, however, are inherent qualities of a physical quantity, related to how they are measured within a consistent theoretical and experimental framework [16].

Dimensions of physical quantities can be formalized as vectors living in a “physical dimension” vector space. The dimension of a quantity Q in a system with three “base” dimensions can be expressed as

$$[Q] = M^\alpha L^\beta T^\gamma, \quad (1)$$

where M, L, T correspond to the dimensions of mass, length, and time, respectively. Then, $[Q]$ is a member of a vector space over \mathbb{R} , where vector addition is represented by multiplying $[Q]$ and vector multiplication is represented by raising $[Q]$ to a power:

$$[Q_1^a Q_2^b] = [Q_1]^a [Q_2]^b. \quad (2)$$

The triplet (α, β, γ) is a particular representation of $[Q]$, dependent on the choice of base dimensions. In a system of different fundamental units (and therefore

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6 dimensions), for instance, weight W instead of mass M, the dimension of a quantity
7 will remain the same, even though its coordinates will be different (exactly like vectors
8 are independent of the coordinate system we parametrize them in).9
10 The dimensions of a quantity can be used to determine how it transforms under
11 a change of units. For instance, energy has dimension ML^2T^{-2} and therefore would
12 quadruple if the definition of a meter were to be halved, whereas a dimensionless
13 quantity or “pure number” with dimension 1 would not transform. The usual
14 nomenclature of vector spaces can be applied even further: the quantities Q_i are said
15 to have linearly independent dimensions if there is no non-trivial way of constructing
16 a dimensionless quantity out of them, or, equivalently, if
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$$\left[\prod_i [Q_i]^{\alpha_i} \right] = 1 \quad (3)$$

20 implies $\alpha_i = 0$ for all i .

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22 Dimensional analysis fundamentally rests upon the principle that the laws of
23 nature should not depend on our choice of units. This simple idea turns out to
24 have profound implications: it can be used to actually constrain the form of physical
25 relations. This constraint is often expressed in terms of the *Buckingham- π theorem* [7].
26 The simplest statement of this theorem is that a physically meaningful relation
27 between quantities q_i can always be non-dimensionalized, i.e. written in terms of
28 dimensionless quantities π_j constructed out of q_i . It is not difficult to see why this
29 is the case: since, by definition, only dimensionless quantities are invariant under
30 changes of units, then any invariant law must be constructed from dimensionless
31 quantities only. More specifically, if we have a set of m dimensionful quantities with
32 n linearly independent dimensions, and k dimensionless quantities c_j , then any non-
33 dimensionalized law can be written as

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$$f(\pi_1, \dots, \pi_{m-n}, c_1, \dots, c_k) = 0. \quad (4)$$

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37 It is not difficult to see that this relation encapsulates all possible relations that
38 remain invariant under a change of units. Rayleigh’s method of dimensional analysis
39 (which is what “dimensional analysis” most often refers to in an undergraduate
40 context), a procedure in which a system is analyzed by considering all the dimensionful
41 independent variables that might influence a dependent variable, is a particular
42 application of the Buckingham- π theorem.43 The Buckingham- π theorem, as expressed through (4), gives us a profound insight
44 on what exactly it means to examine a physical relation in a particular limit. We
45 observe that it is meaningless to talk about the limit of a dimensionful quantity: it
46 only makes sense to consider the limit of the dimensionless quantities π_j or c_j . This
47 is an important distinction, because once we have decided to include a dimensionful
48 quantity as part of our dimensional analysis, we are considering a different system.
49 Consider, for example, special relativity. Formally (and naïvely) taking the limit
50 $c \rightarrow \infty$ in hopes of recovering classical mechanics can lead to unexpected results, such
51 as infinite rest mass mc^2 for particles. This is because the very concept of “rest mass”
52 presupposes the existence of a finite speed limit, which does not vanish simply by
53 taking this limit to infinity.54 For the reasons outlined above, studying the correspondence between classical
55 and quantum mechanics is more nuanced than it might originally appear. A common
56 statement of the correspondence principle is due to Dirac [17], who regarded classical
57 mechanics as the limiting case of quantum mechanics when the (reduced) Planck58
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constant $\hbar \rightarrow 0$ (see [18] for a review). This statement is formally valid for a large scope of applications, but as discussed, may lead to misleading results if applied carelessly. For instance, we might be tempted to write $\Delta x \Delta p > 0$ in the classical limit. However, we know that only limits of dimensionless ratios are meaningful. This means that Bohr's statement of the correspondence limit ($n \rightarrow \infty$, where n are the quantum numbers of the system in question) is more precise, and the two statements are not necessarily equivalent [4].

In order to illustrate how the correspondence limit of uncertainty bounds can be viewed through the lens of dimensional analysis, we consider a particle with a one-dimensional trajectory. Immediately, from dimensional analysis, we can see that there is no way to impose a limit on Δx and Δp : there are no dimensionless ratios that can be derived from these quantities. We need to introduce at least one additional scale with dimensions of action (ML^2T^{-1}). In quantum mechanics, such a scale already exists: \hbar denotes the scale at which quantum effects are relevant. In the classical ensemble, however, any meaningful uncertainty limit will necessarily be expressed in terms of some physical scale(s) of the system.

We now consider a particle in the quantum realm with characteristic action A (defined as the integral of momentum over length) along with the quantum unit of action \hbar . We wish to arrive at a relation between Δx and Δp , and so the dimensionless quantities are given by

$$\pi_1 = \frac{\hbar}{A}, \quad \pi_2 = \frac{\hbar}{\Delta x \Delta p}, \quad (5)$$

or, alternatively

$$\pi_1 = \frac{\hbar}{A}, \quad \pi_2 = \frac{A}{\Delta x \Delta p}. \quad (6)$$

Any of these two choices is valid; after all, the Buckingham- π theorem does not tell us which choice of dimensionless ratios is more "physically meaningful". However, it is clear that if the limit $\hbar/A \rightarrow 0$ is taken, the choice (6) contains more information. Alternatively, we may replace π_1 by the quantum number n , since both tell us how prominent quantum effects are. The uncertainty relation is therefore going to satisfy

$$f\left(\frac{\Delta x \Delta p}{A}, \frac{\hbar}{A}\right) = 0. \quad (7)$$

As a result, the uncertainty equation in the classical limit can be recovered by setting $\hbar/A \rightarrow 0$.

If we instead follow a classical treatment without ever introducing \hbar , the uncertainty relation is simply

$$g\left(\frac{\Delta x \Delta p}{A}\right) = 0, \quad (8)$$

and by the correspondence principle, we expect

$$\lim_{\hbar/A \rightarrow 0} f\left(\frac{\Delta x \Delta p}{A}, \frac{\hbar}{A}\right) = g\left(\frac{\Delta x \Delta p}{A}\right). \quad (9)$$

In this limiting case, the minimum uncertainty is necessarily given by $g(\Delta x \Delta p/A) = 0$, and as such, the uncertainty principle is given by

$$\frac{\Delta x \Delta p}{A} > c, \quad (10)$$

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Furthermore, it is possible to straightforwardly calculate the variance of classical operators using the usual definition $\Delta A^2 = \langle A^2 \rangle - \langle A \rangle^2$.

Having put all preliminaries in place, we may turn our attention to the correspondence between the classical and quantum uncertainty in a few select systems of interest. Single-particle systems have been examined thoroughly in the literature, and from a dimensional analysis perspective, two-body systems may be more robust. However, in order to set the scene and allow for a direct comparison with coupled systems, we will cite here the one-particle results for the cases of harmonic oscillator and infinite square well. As discussed in the previous section, it is illuminating to view uncertainty bounds through a non-dimensionalized system of equations, which motivates us to define the dimensionless position and momentum variables as follows

$$\bar{x}_i = \frac{x_i}{A_i}, \quad \bar{p}_i = \frac{p_i}{m_i \omega_i A_i}, \quad (14)$$

where A_i is the initial amplitude of the system, m_i its mass and ω_i is the frequency of oscillation.

With this notation, the uncertainty product for position and momentum in the classical harmonic oscillator case is found to be

$$\Delta \bar{x} \Delta \bar{p} = \frac{1}{2}. \quad (15)$$

Similarly, for a single particle confined to move under a one-dimensional infinite square well potential, the uncertainty product is

$$\Delta \bar{x} \Delta \bar{p} = \frac{1}{\sqrt{12}}. \quad (16)$$

This coincides with the quantum mechanical result in the limit of large principal quantum number n . For a full derivation of these uncertainty results, the reader is referred to [20] and [21].

For single-particle systems such as the harmonic oscillator and the infinite square well, we note that the uncertainty bound reduces to a pure number in the classical limit. This is to be expected, since there are just enough dimensionful parameters in order to define the dimensionless analogues of position and momentum. In the presence of additional dimensionful parameters, however, we expect the uncertainty bounds to depend on residual dimensionless variables even in the limit of large quantum numbers. Coupled systems feature more dimensionful parameters, and for this reason we turn our attention to them in the next section.

In concluding this section, we must stress that recasting quantum mechanics in the classical phase space (effectively deducing its postulates on statistical grounds) is a much more involved procedure than assigning a classical distribution to a particle ensemble as outlined above. This procedure usually involves “geometrizing” quantum mechanics by constructing the phase space of a classical system, which is then endowed with a probability measure. Riccia and Wiener [22], for instance, use stochastic integrals in order to motivate Born’s rule, while Kibble [23] employs symplectic manifolds with a complex structure in order to recover quantum dynamics. Another geometric approach by Heslot [24] reveals \hbar to be the curvature of the space of quantum states, indicating once again that there is a qualitative difference between the classical and quantum realms; the former features a state space which has a natural unit of distance, while the latter does not. The field of quantum information [25, 26] features powerful and sophisticated techniques that make the quantum-classical correspondence much more manifest. These approaches to the

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relation between classical and quantum mechanics are quite involved, and beyond
the scope of the present work. Nonetheless, our simple approach still highlights the
often-overlooked role that dimensional analysis plays in both classical and quantum
uncertainty and the classical limit of quantum mechanics.9
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4. Two coupled harmonic oscillators with equal masses12
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We consider a simple coupled system consisting of two harmonic oscillators with
spring constant k and equal masses m , coupled via a potential of the form
 $V(x_1, x_2) = \frac{1}{2}k'(x_1 - x_2)^2$, where $k \neq k'$ in general, and x_1 and x_2 denote the
displacements of the individual oscillators. For similar treatments of this set-up the
reader is referred to [27–29].17
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*4.1. Classical oscillators*20
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We model the classical system with two blocks constrained to move in a one-
dimensional frictionless surface. Each block is attached to an outer stationary wall by
means of a spring with force constant k . The inner spring has force constant k' . It is
assumed that all springs assume their natural length when the system is at rest.24
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This 1D problem can be easily approached utilizing the Lagrangian formalism
and is part of the repertoire of any undergraduate dynamics course. The Lagrangian
of the system reads

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$$\mathcal{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}k(x_1^2 + x_2^2) - \frac{1}{2}k'(x_1 - x_2)^2. \quad (17)$$

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To decouple (17) and obtain the equations of motions, we perform a change of variables
in favour of the normal mode coordinates $x_{c,r}$ defined as linear combinations of the
position variables $x_{1,2}$, i.e.

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$$x_c = \frac{x_1 + x_2}{\sqrt{2}}, \quad x_r = \frac{x_1 - x_2}{\sqrt{2}}, \quad (18)$$

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which are the centre of mass and relative distance respectively.

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The characteristic frequencies of the two normal modes of oscillation are given
by $\omega_c^2 = k/m$ and $\omega_r^2 = (k + 2k')/m$, corresponding to in phase and out of phase
oscillations respectively.

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The time averages of the normal modes coordinates for position and momentum
are then found through (13) over a period $\tau_i = 2\pi/\omega_i$ with $i = c, r$:

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$$\langle x_i \rangle = 0, \quad \langle x_i^2 \rangle = \frac{A_i^2}{2}, \quad (19)$$

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$$\langle p_i \rangle = 0, \quad \langle p_i^2 \rangle = \frac{1}{2}m^2\omega_i^2 A_i^2. \quad (20)$$

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It is now straightforward to change back to the position variables $x_{1,2}$ in order to
find the uncertainty relation in its conventional formulation as the product of the
uncertainties Δx and Δp .

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The time averages for the position variable x_1 can be readily obtained via of (19)
and (20):

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$$\langle x_1 \rangle = \frac{\sqrt{2}}{2} \langle x_c + x_r \rangle = 0, \quad (21)$$

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$$\langle x_1^2 \rangle = \frac{1}{2} \langle (x_c + x_r)^2 \rangle = \frac{1}{4}(A_c^2 + A_r^2). \quad (22)$$

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This yields the following expression for the variance of x_1 :

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$$\Delta x_1 = \sqrt{\langle x_1^2 \rangle - \langle x_1 \rangle^2} = \frac{\sqrt{A_c^2 + A_r^2}}{2}. \quad (23)$$

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Using the definition, the time averages and variance of the momenta are found to be

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$$\langle p_1 \rangle = 0, \quad \langle p_1^2 \rangle = \frac{1}{4}m^2(A_c^2\omega_c^2 + A_r^2\omega_r^2), \quad (24)$$

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$$\Delta p_1 = m \frac{\sqrt{A_c^2\omega_c^2 + A_r^2\omega_r^2}}{2}. \quad (25)$$

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Repeating the previous steps for the second block yields identical results. The product
of the variances is therefore

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$$\Delta x_1 \Delta p_1 = m \frac{\sqrt{A_c^2 + A_r^2} \sqrt{A_c^2\omega_c^2 + A_r^2\omega_r^2}}{4}, \quad (26)$$

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and similarly for $\Delta x_2 \Delta p_2$. Using the scaled canonical variables defined in (14), (26)
can be written as

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$$\Delta \bar{x}_1 \Delta \bar{p}_1 = \frac{1}{4} \sqrt{1 + \frac{A_c^2}{A_r^2}} \sqrt{1 + \frac{A_c^2\omega_c^2}{A_r^2\omega_r^2}}. \quad (27)$$

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where the variables have been scaled by A_r and $m A_r \omega_r$ respectively. Since the system is
symmetric under particle exchange, the expression for $\Delta x_2 \Delta p_2$ can be easily deduced
from that of $\Delta x_1 \Delta p_1$ by swapping indices 1 and 2. We note that this expression,
which represents the uncertainty relation in terms of dimensionless position and
momentum for a classical ensemble, comprises of two dimensionless ratios involving
the amplitudes $A_{c,r}$ and frequencies $\omega_{c,r}$ respectively, as expected.30
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We may compare the uncertainty product of this system to that of the 1D
uncoupled case of (15) by setting $A_c = A_r$ and $\omega_c = \omega_r$. As such, we observe
that the effect of the coupling is to add another dimensionful degree of freedom to the
system, which results in the possibility for another dimensionless ratio to appear in
the non-dimensionalized expression for the uncertainty product.36
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4.2. *Quantum oscillators*38
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The quantum mechanical analogue of the system we have analyzed so far is that of two
distinguishable coupled one-dimensional harmonic oscillators. We treat the oscillators
as distinguishable to allow for a direct comparison to the uncertainty product of (27)
in the previous section. The system is then described by the following Hamiltonian:

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$$\hat{H} = \frac{\hat{p}_1^2 + \hat{p}_2^2}{2m} + \frac{1}{2}k(x_1^2 + x_2^2) + \frac{1}{2}k'(x_1 - x_2)^2. \quad (28)$$

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Using a variation of the Jacobi coordinates allows to reduce the complexity of the
problem by transforming it to what is essentially a one-dimensional problem in terms
of the centre of mass and relative coordinate of (18) [30].47
After the change of variables, the Hamiltonian becomes separable:

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$$\hat{H} = \frac{\hat{p}_c^2}{2m} + \frac{1}{2}kx_c^2 + \frac{\hat{p}_r^2}{2m} + \frac{1}{2}(k + 2k')x_r^2 \quad (29)$$

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$$= \hat{H}_c + \hat{H}_r. \quad (30)$$

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It is therefore possible to determine the evolution of the system by viewing it as two
single-particle harmonic oscillators with angular frequencies given by $\omega_c^2 = k/m$ and

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Dimensional analysis and quantum-classical correspondence

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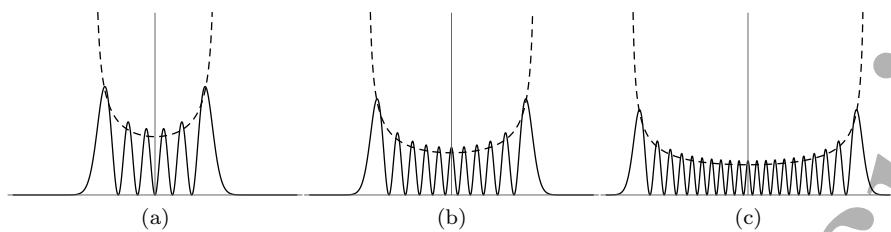


Figure 1. Uncoupled classical (dashed line) and quantum (solid line) probability densities for the harmonic oscillator as functions of the centre of mass coordinate x_c for (a) $n_c = 5$, (b) $n_c = 10$ and (c) $n_c = 20$.

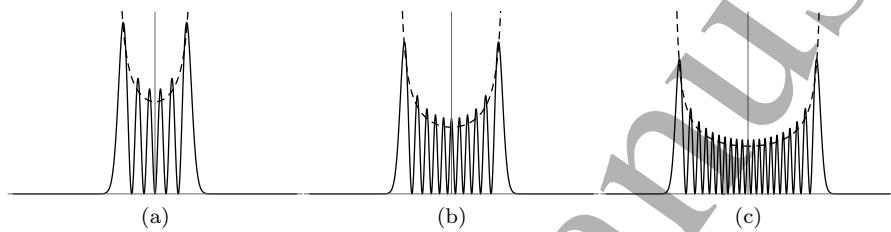


Figure 2. Uncoupled classical (dashed line) and quantum (solid line) probability densities for the harmonic oscillator as functions of the relative coordinate x_r for (a) $n_r = 5$, (b) $n_r = 10$ and (c) $n_r = 20$.

$\omega_r^2 = (k + 2k')/m$. Since the oscillators are treated as distinguishable, the total wavefunction of the system is the product of the single-particle wavefunctions

$$\Psi(x_c, x_r) = \phi_{n_c}(x_c)\phi_{n_r}(x_r), \quad (31)$$

where

$$\phi_n(x) = \left(\frac{\sqrt{\pi}}{x_0 2^n n!} \right)^{1/2} H_n \left(\frac{x}{x_0} \right) \exp \left(-\frac{x^2}{2x_0^2} \right), \quad (32)$$

with $x_0 = \sqrt{\frac{\hbar}{m\omega}}$, and n_r and n_c are the principal quantum numbers associated with the wavefunctions $\phi(x_r)$ and $\phi(x_c)$ respectively.

Using the definition for quantum mechanical expectation values and making use of the orthogonality properties of the Hermite polynomials, one can calculate the expectation values for x_r and x_c and the respective momentum operators for the general wavefunction defined in (31) as

$$\langle x_i \rangle = 0, \quad \langle x_i^2 \rangle = \frac{(2n_i + 1)\hbar}{2m\omega_i}, \quad (33)$$

$$\langle p_i \rangle = 0, \quad \langle p_i^2 \rangle = (2n_i + 1)\frac{m\hbar\omega_i}{2}, \quad (34)$$

where $i = c, r$. We can now transform back to the original spatial variables x_1 and x_2 and obtain the expectation value for positions and momenta using the results (33) and (34) above:

$$\langle x_1 \rangle = 0, \quad \langle x_1^2 \rangle = \frac{1}{2} \left[\frac{(2n_r + 1)\hbar}{2m\omega_r} + \frac{(2n_c + 1)\hbar}{2m\omega_c} \right], \quad (35)$$

Dimensional analysis and quantum-classical correspondence

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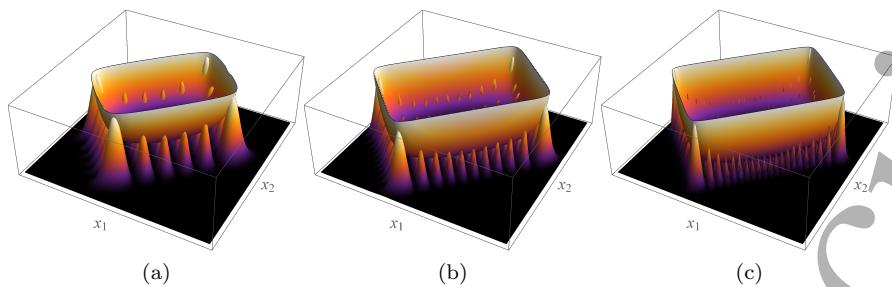


Figure 3. Classical $\rho_{\text{CL}}(x_1, x_2)$ and quantum $\rho_{\text{QM}} = |\psi_n(x_1, x_2)|^2$ probability densities for coupled harmonic oscillators with equal masses vs the spatial variables x_1 and x_2 for (a) $n = 5$, (b) $n = 10$, (c) $n = 20$.

$$\langle p_1 \rangle = 0, \quad \langle p_1^2 \rangle = \frac{1}{2} \left[\frac{m\hbar\omega_r(2n_r + 1)}{2} + \frac{m\hbar\omega_c(2n_c + 1)}{2} \right], \quad (36)$$

and similarly for the second oscillator. Hence, the quantum uncertainties in position and momentum read

$$\Delta x_1 = \frac{1}{2} \sqrt{\frac{(2n_r + 1)\hbar}{m\omega_r} + \frac{(2n_c + 1)\hbar}{m\omega_c}} \quad (37)$$

$$= \frac{1}{2} \sqrt{A_{n_c}^2 + A_{n_r}^2}, \quad (38)$$

$$\Delta p_1 = \frac{1}{2} \sqrt{m\hbar\omega_r(2n_r + 1) + m\hbar\omega_c(2n_c + 1)} \quad (39)$$

$$= \frac{m}{2} \sqrt{\omega_c^2 A_{n_c}^2 + \omega_r^2 A_{n_r}^2}, \quad (40)$$

respectively, where $A_{n_{c,r}}$ are the classical turning points associated with the energy $E_{n_{c,r}}$ of an harmonic oscillator with mass m and frequency $\omega_{c,r}$:

$$A_{n_{c,r}} = \sqrt{\frac{2E_{n_{c,r}}}{m\omega_{c,r}^2}} = \sqrt{\frac{(2n_{c,r} + 1)\hbar}{m\omega_{c,r} n_{c,r}}}. \quad (41)$$

In terms of the dimensionless variables \bar{x}_i and \bar{p}_i of (14) scaled by means of the amplitude $A_{n_{c,r}}$, the uncertainty product can be written as

$$\Delta \bar{x}_1 \Delta \bar{p}_1 = \frac{1}{4} \sqrt{1 + \frac{A_{n_c}^2}{A_{n_r}^2}} \sqrt{1 + \frac{A_{n_c}^2 \omega_c^2}{A_{n_r}^2 \omega_r^2}}, \quad (42)$$

or explicitly in terms of the principal quantum numbers $n_{c,r}$ and harmonic oscillator frequencies $\omega_{c,r}$ as

$$\Delta \bar{x}_1 \Delta \bar{p}_1 = \frac{1}{4} \sqrt{1 + \frac{(2n_c + 1)\omega_r}{(2n_r + 1)\omega_c}} \sqrt{1 + \frac{(2n_c + 1)\omega_c}{(2n_r + 1)\omega_r}}. \quad (43)$$

We note that the last expression only involves dimensionless ratios of the two frequencies of oscillation and the characteristic principal quantum numbers. It is suggestive of the correspondence principle that (42) is in complete agreement with

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the classical uncertainty of (27), and all observations on how the couplings affects the
classical uncertainty product also apply in this case.7
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To offer a graphical comparison between the classical and quantum system,
let $\rho_{\text{CL}}(x_{c,r})$ be the classical probability density defined by

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$$\rho_{\text{CL}}(x) = \frac{1}{\pi} \frac{1}{\sqrt{A_n^2 - x^2}}, \quad (44)$$

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and valid in the range $x \in (-A_n, A_n)$ where A_n is defined in (41) [20, 21]. The
usual representation of 1D quantum and classical probability densities can be found
in the literature (see [21]); as such, we illustrate the coupled case. Figure 1 shows
the quantum probability density, $\rho_{\text{QM}}(x_c)$, along with the classical analogue $\rho_{\text{CL}}(x_c)$
in terms of the centre of mass coordinate for various values of the principal quantum
number n_c . Figure 2 shows the probability densities for the same values of the principal
quantum number n_r as a function of the relative coordinate x_r instead.19
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Finally, let the total classical probability density be given by the product of the
ones associated to the individual modes of oscillation $x_{c,r}$ as

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$$\rho_{\text{CL}}(x_1, x_2) = \rho_{\text{CL}}(x_c)\rho_{\text{CL}}(x_r). \quad (45)$$

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Figure 3 offers a comparison between the quantum probability density, ρ_{QM} and the
classical counterpart $\rho_{\text{CL}}(x_1, x_2)$ which are plotted as functions of the original spatial
variables of the problem for value of n equal to 5, 10 and 20. Noticeably, the shape of
the three dimensional classical probability density is that of a rectangular well with
soft edges. This is the natural and intuitive extension of the 1D distribution shown in
Figures 1 and 2, which can be retrieved by slicing the three dimensional distributions
across lines of $x_1 = 0$ and $x_2 = 0$.30
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5. Two coupled harmonic oscillators with different masses33
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Let us now consider a slight variation of the problem analyzed in the previous section
for which the oscillators have masses m_1 and m_2 with $m_1 \neq m_2$ and the potential
function is $V(x_1, x_2) = \frac{1}{2}k(x_1 - x_2)^2$. We expect that the results in this case
will have a similar form, even though the dimensionless ratios will differ. For similar
treatments of this set-up the reader is referred to [27, 28].38
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5.1. Classical oscillators41
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Similarly to Section 4.1, the classical system consists of two blocks connected to three
springs. The motion of the blocks is described by the following Lagrangian

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$$\mathcal{L} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}\omega^2(m_1x_1^2 + m_2x_2^2) - \frac{1}{2}k(x_1 - x_2)^2, \quad (46)$$

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where we assume the angular frequency ω to be the same for both oscillators.
The equations of motion obtained from Lagrange's equations can be decoupled by
performing a variable transformation to the Jacobi coordinates

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$$x_r = \frac{x_1 - x_2}{\sqrt{2}}, \quad x_c = \frac{m_1x_1 + m_2x_2}{M\sqrt{2}}, \quad (47)$$

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where $M = (m_1 + m_2)/2$. Proceeding with this transformation, (46) can be rewritten
as

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$$\mathcal{L} = \frac{1}{2}M\dot{x}_c^2 + \frac{1}{2}\mu\dot{x}_r^2 - \frac{1}{2}M\omega_c^2x_c^2 - \frac{1}{2}\mu\omega_r^2x_r^2, \quad (48)$$

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where $\mu = (m_1 m_2)/M$ is the reduced mass and $\omega_c^2 = \omega^2$, $\omega_r^2 = \omega^2 + 2k/\mu$ are the two harmonic oscillator frequencies. The time averages of the relative and centre of mass coordinate are given once again by (19). We may then easily transform back to the spatial coordinates x_1 and x_2 by combining the expressions for x_r and x_c from (47), i.e.

$$x_1 = \frac{1}{\sqrt{2}} \left(x_c + \frac{m_2}{M} x_r \right), \quad x_2 = \frac{1}{\sqrt{2}} \left(x_c - \frac{m_1}{M} x_r \right). \quad (49)$$

Thus, using the results of (19), the averages for x_1 are found to be

$$\langle x_1 \rangle = 0, \quad \langle x_1^2 \rangle = \frac{1}{4} \left[A_c^2 + \left(\frac{m_2}{M} \right)^2 A_r^2 \right], \quad (50)$$

with variance given by the following expression

$$\Delta x_1 = \frac{1}{2} \sqrt{A_c^2 + \left(\frac{m_2}{M} \right)^2 A_r^2}. \quad (51)$$

The symmetric properties of the system allow us to easily deduce Δx_2 by simply swapping the indices $1 \rightarrow 2$ in the variables appearing in the expression for Δx_1 .

Repeating the previous steps for the momenta allows to express $p_{1,2}$ in terms of the relative and centre of mass coordinate momenta defined as $p_r = \mu \dot{x}_r$ and $p_c = M \dot{x}_c$ respectively. Hence, the time averages and variances of the momenta are

$$\langle p_1 \rangle = 0, \quad (52)$$

$$\langle p_1^2 \rangle = \frac{1}{4} \left[\mu^2 \omega_r^2 A_r^2 + \left(\frac{m_1}{M} \right)^2 M^2 \omega_c^2 A_c^2 \right], \quad (53)$$

$$\Delta p_1 = \frac{1}{2} \sqrt{\mu^2 \omega_r^2 A_r^2 + \left(\frac{m_1}{M} \right)^2 M^2 \omega_c^2 A_c^2}, \quad (54)$$

for the first oscillator. Once more, the results for the second block are obtained by swapping the subscripts $1 \rightarrow 2$.

Using the scaled variables \bar{x}_i and \bar{p}_i defined in (14), enables us to find the dimensionless uncertainty product, $\Delta \bar{x} \Delta \bar{p}$, as

$$\Delta \bar{x}_1 \Delta \bar{p}_1 = \frac{1}{4} \sqrt{1 + \left(\frac{m_2}{M} \right)^2 \frac{A_r^2}{A_c^2}} \sqrt{\left(\frac{m_1}{M} \right)^2 + \frac{\mu^2 \omega_r^2 A_r^2}{M^2 \omega_c^2 A_c^2}}, \quad (55)$$

and

$$\Delta \bar{x}_2 \Delta \bar{p}_2 = \frac{1}{4} \sqrt{1 + \left(\frac{m_1}{M} \right)^2 \frac{A_r^2}{A_c^2}} \sqrt{\left(\frac{m_2}{M} \right)^2 + \frac{\mu^2 \omega_r^2 A_r^2}{M^2 \omega_c^2 A_c^2}}, \quad (56)$$

for the first and second oscillator respectively.

We note that these expressions are symmetric under particle exchange and comprise of three dimensionless ratios involving the masses, the amplitudes and the frequencies of the oscillators. The extra term involving the masses in (55) is what sets it apart from the uncertainty product of the system of coupled oscillators with identical masses described by (27), where only the other two dimensionless ratios appeared. This may be surprising, since both systems have the same number of fundamental dimensionful variables. The difference in the two expressions can be ascribed to the choice of coordinates of (47) employed to decouple the equations of motion: since the expression for the centre of mass coordinate is weighted by the masses, this “artificially” introduces the extra term, $(m_i/M)^2$ with $i = 1, 2$ in the uncertainty for momentum $\Delta \bar{p}_i$. Such a detail, however, is beyond the capabilities of dimensional analysis to predict.

Dimensional analysis and quantum-classical correspondence

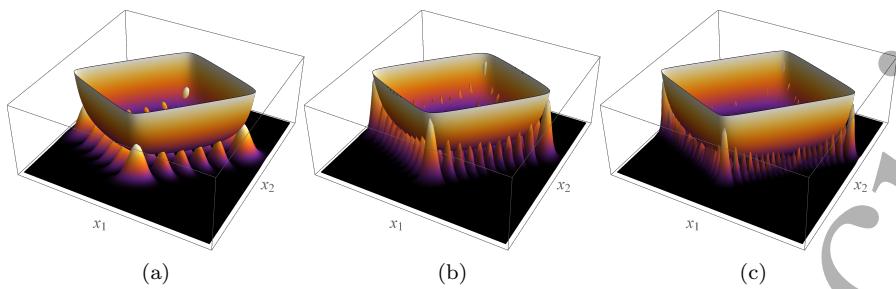


Figure 4. Classical $\rho_{\text{CL}}(x_1, x_2)$ and quantum $\rho_{\text{QM}} = |\psi_n(x_1, x_2)|^2$ probability densities for coupled harmonic oscillators with different masses vs the spatial variables x_1 and x_2 for values of n given by (a) $n = 5$, (b) $n = 10$, (c) $n = 20$.

The graphical representations of the 1D classical and quantum probability densities are analogous to the ones presented in Figures 1 and 2. Figure 4 offers a comparison between the quantum probability density, $\rho_{\text{QM}} = |\Psi_n(x_1, x_2)|^2$ and the classical counterpart $\rho_{\text{CL}}(x_1, x_2)$ which are plotted as functions of the original variables of the problem for various values of n . Notably, the shape of the classical distribution is that of a skewed rectangular well which is not diagonal as the one in Figure 3. It is evident however that the quantum and classical distributions converge in a locally averaged sense for high values of the principal quantum number n in both cases.

6. Two particles in a box coupled via contact potential

The final set-up we consider is a system of two particles in an infinite potential well which are allowed to interact via a contact potential of the form

$$V(x) = \begin{cases} \lambda\delta(x_1 - x_2) & 0 < x_1, x_2 < L \\ \infty & \text{otherwise} \end{cases}, \quad (65)$$

where L is the dimension of the box and the coefficient λ , which has dimensions of ML^2T^{-2} , determines the strength of the interaction. For similar treatments of this set-up without uncertainty considerations, the reader is referred to [31–35] or to [36] for a treatment with uncertainty relations.

6.1. Classical case

Starting with the classical case, the Lagrangian of the system can be written as

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - V, \quad (66)$$

where V is defined in (65) and, in general, $m_1 \neq m_2$. Once again the equations of motion are decoupled via the Jacobi coordinates:

$$x_r = x_1 - x_2, \quad x_c = \frac{m_1x_1 + m_2x_2}{M}, \quad (67)$$

where $M = m_1 + m_2$ is the total mass of the system of particles. Proceeding with this variable transformation allows (66) to be written as

$$\mathcal{L} = \frac{1}{2}M\dot{x}_c^2 + \frac{1}{2}\mu\dot{x}_r^2 - \lambda\delta(x_r), \quad (68)$$

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where $\mu = (m_1 m_2)/M$ is the reduced mass and the expression for \mathcal{L} is valid within
the spatial domain $0 < x_1, x_2 < L$. The equation of motion for the center of mass
coordinate is trivial $M\ddot{x}_c = 0$, as expected for a classical particle confined in a
one-dimensional potential box.9
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11
Let us now evaluate the classical averages relevant to this set-up. Since $0 <$
 $x_1, x_2 < L$ then the spatial domain for the centre of mass coordinate is $0 < x_c < L$.
Hence, the time averages $\langle x_c \rangle$ and $\langle p_c \rangle$ for position and momentum are simply

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$$\langle x_c \rangle = \frac{L}{2}, \quad \langle x_c^2 \rangle = \frac{L^2}{3}, \quad (69)$$

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$$\langle p_c \rangle = 0, \quad \langle p_c^2 \rangle = 2ME_c, \quad (70)$$

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where E_c is the energy of the particle. For the relative coordinate, the equation of
motion involves the derivative of the delta-function which is remarkably challenging to
compute analytically. As a simplification, however, since we are in the classical regime
where no tunnelling can occur, we can assume that the potential acts as a solid wall at
 $x_r = 0$. Essentially, in each half of the box we are dealing with a classical free particle
confined to move in a limited region of space with the condition $-L < x_r < L$. Hence,
the averages will again be those of the one-particle case for a box of length $2L$ with
the condition that $x_r \neq 0$ [20, 21].24
We find the averages to be

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$$\langle x_r \rangle = 0, \quad \langle x_r^2 \rangle = \frac{L^2}{3}, \quad (71)$$

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28
$$\langle p_r \rangle = 0, \quad \langle p_r^2 \rangle = 2\mu E_r. \quad (72)$$

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Switching back to the original variables, the classical averages of the spatial coordinate
 x_1 and the respective momentum p_1 are found to be

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$$\langle x_1 \rangle = \frac{L}{2}, \quad \langle x_1^2 \rangle = \frac{L^2}{3} \left[1 + \left(\frac{m_2}{M} \right)^2 \right], \quad (73)$$

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$$\langle p_1 \rangle = 0, \quad \langle p_1^2 \rangle = 2\mu E_r + \left(\frac{m_1}{M} \right)^2 2ME_c, \quad (74)$$

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and similarly for the second particle:

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$$\langle x_2 \rangle = \frac{L}{2}, \quad \langle x_2^2 \rangle = \frac{L^2}{3} \left[1 + \left(\frac{m_1}{M} \right)^2 \right], \quad (75)$$

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40
$$\langle p_2 \rangle = 0, \quad \langle p_2^2 \rangle = 2\mu E_r + \left(\frac{m_2}{M} \right)^2 2ME_c. \quad (76)$$

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42
To find the dimensionless uncertainty product $\Delta x \Delta p$, we use a similar definition to
that of (14), introducing the following scaled quantities

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44
$$\Delta \bar{x} = \frac{\Delta x}{L}, \quad \Delta \bar{p} = \frac{\Delta p}{\sqrt{2mE}}. \quad (77)$$

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Hence, after finding the variances using the results (73) to (76) and expressing
them in terms of the canonical variables from (77), one finds the dimensionless
uncertainty product of position and momentum to be

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$$\Delta \bar{x}_1 \Delta \bar{p}_1 = \sqrt{\frac{1}{12}} \sqrt{1 + \left(\frac{2m_2}{M} \right)^2} \sqrt{1 + \left(\frac{m_1}{M} \right)^2 \frac{ME_c}{\mu E_r}}. \quad (78)$$

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8 Since the system is symmetric under exchange of particles, the product $\Delta x_2 \Delta p_2$ can
 9 be easily deduced from (78) by exchanging the indices 1 and 2:
 10

$$\Delta \bar{x}_2 \Delta \bar{p}_2 = \sqrt{\frac{1}{12}} \sqrt{1 + \left(\frac{2m_1}{M}\right)^2} \sqrt{1 + \left(\frac{m_2}{M}\right)^2 \frac{ME_c}{\mu E_r}}. \quad (79)$$

11 We can see that (78) (and equivalently (79)) only involves dimensionless ratios of the
 12 masses and energies of the two classical particles. Comparing these results to the
 13 uncoupled case relation, which is given by (16), we see how the two expressions do not
 14 simply coincide when $m_1 = m_2$ and $E_c = E_r$. In fact, in the latter case we obtain
 15 exactly double the uncertainty product of the one-particle case. This can be ascribed
 16 to the fact that this is still a coupled system of particles.

17 Hence, as we have also concluded in the other examples considered so far,
 18 introducing a coupling potential increases the uncertainty product $\Delta \bar{x} \Delta \bar{p}$ from its
 19 minimum value given by (16) through an additional dimensionless ratio of parameters.

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21
6.2. Quantum case

22 The quantum mechanical counterpart of the system under consideration is described
 23 by the following Hamiltonian operator

$$24 \quad 25 \quad \hat{H} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + \lambda \delta(x_1 - x_2). \quad (80)$$

26 It is important to point out that a quantum mechanical system coupled by a contact
 27 interaction of the kind described above can only consist of bosonic states. This follows
 28 from the Pauli exclusion principle: since fermions cannot be at the same point in space
 29 at the same instant in time, a contact potential would not affect fermionic states [29].

30 Proceeding with the analysis, (80) can be decoupled by transforming to the same
 31 Jacobi coordinates defined in (67) for the classical case. The resulting Hamiltonian is
 32 separable in terms of the centre of mass and relative coordinate:

$$34 \quad 35 \quad \hat{H} = -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial x_c^2} - \frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial x_r^2} + \lambda \delta(x_r). \quad (81)$$

36 The solution to the centre of mass part of the Hamiltonian is that of a single particle
 37 harmonic oscillator with mass M in a one-dimensional box of length L , i.e. $0 < x_c < L$.
 38 The expectation values for position and momentum are then given by

$$39 \quad 40 \quad \langle x_c \rangle = \frac{L}{2}, \quad \langle x_c^2 \rangle = L^2 \left(\frac{1}{3} - \frac{1}{2n_c^2 \pi^2} \right), \quad (82)$$

$$41 \quad 42 \quad \langle p_c \rangle = 0, \quad \langle p_c^2 \rangle = \left(\frac{\pi \hbar n_c}{L} \right)^2 = 2ME_{n_c}. \quad (83)$$

43 In addition to the box boundary conditions, the presence of delta-function potential
 44 in the relative coordinate Hamiltonian imposes two sets of boundary conditions on the
 45 wavefunction at the location of the delta-function potential. These are the continuity
 46 of the wavefunction

$$48 \quad \psi|_{x_1=x_2+0} = \psi|_{x_1=x_2-0}, \quad (84)$$

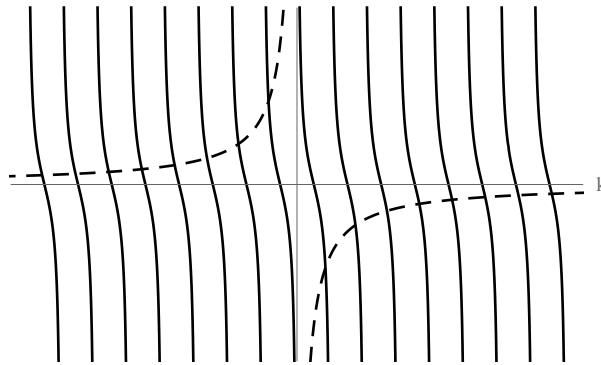
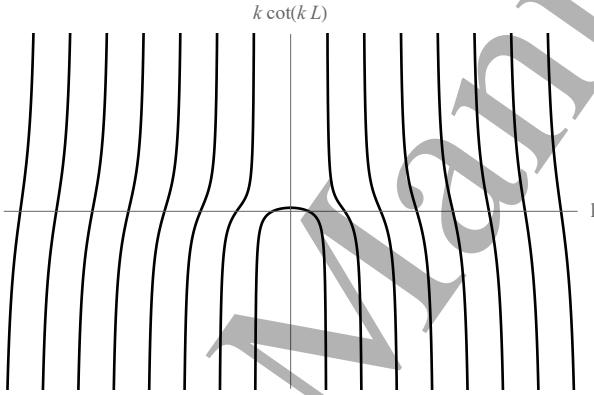
49 and the discontinuity of its first derivative

$$50 \quad 51 \quad \frac{d\psi_{n_r}(x_r)}{dx_r} \Big|_{0+\epsilon} - \frac{d\psi_{n_r}(x_r)}{dx_c} \Big|_{0-\epsilon} = \frac{2\mu}{\hbar^2} \int_{0-\epsilon}^{0+\epsilon} V(x_r) \psi_{n_r}(x_r) dx_r \quad (85)$$

$$52 \quad 53 \quad = \frac{2\mu\lambda}{\hbar^2} \psi_{n_r}(0). \quad (86)$$

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Figure 5. Plot of $\cot(k_{nr})$ (solid line) vs $-1/k_{nr}$ (dashed line) for a box of length $L = 5$ and for $\mu\lambda/\hbar^2 = 1$.33
34
Figure 6. Plot of the self-consistent equation (90) vs k_{nr} for a box of length $L = 5$ within a range $-10 < k_{nr} < 10$.35
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The latter is obtained by integrating the Schrödinger equation with Hamiltonian operator specified by (80) over the small interval $(0 - \epsilon, 0 + \epsilon)$ [37, 38].37
38
From [32, 33], we can see that for a particle confined in a box of length $2L$, i.e. $-L < x_r < L$, with a δ -function potential at position $x_r = pL$, where $-1 < p < 1$, the wavefunction can be written as

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$$\psi_{nr}(x_r) = \begin{cases} A \sin(k_{nr}(x_r + L)) & (-L \leq x_r \leq pL) \\ B \sin[k_{nr}(x_r - L)] & (pL \leq x_r \leq L) \end{cases}, \quad (87)$$

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where $k_{nr} = \sqrt{2\mu E_r}/\hbar$. Moreover, the continuity condition (84) of the wave function at $x_r = pL$ gives

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$$\frac{A}{B} = \frac{\sin[k_{nr} L(p - 1)]}{\sin[k_{nr} L(p + 1)]}. \quad (88)$$

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If $p = 0$, this condition yields $A = -B$. Applying now the discontinuity condition of (86) allows us to obtain a quantization relation for wavenumber k_{nr} , i.e.

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$$k_{nr} \sin(2k_{nr} L) = \frac{2\mu\lambda}{\hbar^2} \sin[k_{nr} L(p - 1)] \sin[k_{nr} L(p + 1)]. \quad (89)$$

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Setting $p = 0$ yields the following expression

$$k_{n_r} \cot(k_{n_r} L) = -\frac{\mu\lambda}{\hbar^2}, \quad (90)$$

which is a self-consistent relation for k_{n_r} and cannot be solved to give a simple analytical answer for the wavenumber. This is shown in Figure 5 where the intersection points between the curves $y = \cot(k_{n_r} L)$ and $y = -1/k_{n_r}$ represent the solutions to (90). Clearly, these solutions are not spaced in a regular manner and a numerical approach becomes necessary. Plotting (90) against k_{n_r} however gives us insight into the way the wavefunction changes over the length of the box. For a box of length $L = 5$ and within a range $-10 < k_{n_r} < 10$, the behaviour of the function is shown in Figure 6.

We note that for large values of the ordinate, the lines become parallel and equally spaced, suggesting that in the high quantum number limit, the k_{n_r} values satisfying the self-consistent equation will repeat periodically. After normalizing the wavefunctions, the expectation values for the relative spatial coordinate, x_r , and momentum are evaluated to be

$$\langle x_r \rangle = 0, \quad \langle x_r^2 \rangle = \frac{2k_{n_r} L^3}{6k_{n_r} L - 3 \sin(2k_{n_r} L)} - \frac{1}{2k_{n_r}^2}, \quad (91)$$

$$\langle p_r \rangle = 0, \quad \langle p_r^2 \rangle = \hbar^2 k_{n_r}^2 = 2\mu E_{n_r}. \quad (92)$$

The expression for $\langle x_r^2 \rangle$ in (91) seems quite complicated at a first glance but reduces to the familiar $L^2/3$ in the limit of large k_{n_r} . This corresponds to the classical regime:

$$\lim_{k_{n_r} \rightarrow \infty} \langle x_r^2 \rangle \rightarrow \frac{L^2}{3}. \quad (93)$$

Hence, the expectation values of x_1 and p_1 are found to be

$$\langle x_1 \rangle = \frac{L}{2}, \quad \langle x_1^2 \rangle = L^2 \left(\frac{1}{3} - \frac{1}{2n_c^2 \pi^2} \right) + \left(\frac{m_2}{M} \right)^2 \langle x_r^2 \rangle, \quad (94)$$

$$\langle p_1 \rangle = 0, \quad \langle p_1^2 \rangle = 2\mu E_r + \left(\frac{m_1}{M} \right)^2 2ME_c, \quad (95)$$

and similarly for the second particle. Using the scaled variables of (77) we can then evaluate the dimensionless uncertainties in position and momentum as

$$\Delta \bar{x}_1 = \sqrt{\frac{1}{12} - \frac{1}{2n_c^2 \pi^2} + \left(\frac{m_2}{M} \right)^2 \frac{\langle x_r^2 \rangle}{L^2}}, \quad (96)$$

$$\Delta \bar{p}_1 = \sqrt{1 + \left(\frac{m_1}{M} \right)^2 \frac{ME_c}{\mu E_r}}. \quad (97)$$

Taking the limit for large principal quantum number n_c and wavenumber k_{n_r} , we step in the classical regime and obtain an equivalent result to (78), i.e.

$$\lim_{n_c, k_{n_r} \rightarrow \infty} \Delta \bar{x}_1 \Delta \bar{p}_1 = \sqrt{\frac{1}{12}} \sqrt{1 + \left(\frac{2m_2}{M} \right)^2} \sqrt{1 + \left(\frac{m_1}{M} \right)^2 \frac{ME_c}{\mu E_r}}. \quad (98)$$

This is in complete agreement with the classical result of (78).

As it is not possible to obtain a value for the wavenumber k_{n_r} in closed form, we will not offer a graphical comparison between quantum and classical probability densities here. The reader is referred to [20] for a comparison of probability density distributions for the simple one-particle infinite square well.

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7. Conclusions
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7 Dimensional analysis is a powerful tool for deriving robust results for a host of
8 situations, and the correspondence between the quantum and classical domains is
9 no exception. The Buckingham- π theorem of dimensional analysis provides us with a
10 fundamental insight regarding the so-called classical limit: its definition cannot involve
11 taking the limit of a dimensionful quantity such as \hbar if it is to be physically meaningful.
12 Even in cases where $\hbar \rightarrow 0$ is a shorthand for some quantum number $n \rightarrow \infty$, it is not
13 possible to recover classical mechanics by naïvely setting $\hbar = 0$.

14 There is a stark divide between a classical world where $\hbar = 0$ and a quantum
15 world viewed by an observer living in the classical limit. The classical probability
16 density is not a limit of the wavefunction in the classical limit: indeed, the classical
17 probability density can be recovered by “smearing out” the quantum probability
18 density (by an amount that decreases as quantum numbers increase). In a classical
19 world, wavefunctions do not exist, but in a quantum world, classical scales are those
20 in which the smearing required to match the two domains is small (i.e. wavenumbers
21 are very large).

22 In order to illustrate how the correspondence between classical and quantum
23 realm fares in the context of dimensional analysis, we focused on quantum uncertainty,
24 popularly regarded as a phenomenon with no classical analogue. For simple systems,
25 the role of \hbar in the classical limit is assumed by a characteristic action scale, as required
26 by the Buckingham- π theorem. For systems with more dimensionful parameters,
27 however, uncertainty limits will in general acquire a more complicated dependence
28 on dimensionless ratios. To demonstrate this, we examined two distinct systems: a
29 coupled harmonic oscillator and a two-body particle-in-a-box set-up, and showed that
30 the uncertainty bounds acquire additional dimensionless terms (and also converge in
31 the limit of large quantum numbers).

32 Dimensional analysis lends support to the idea that any comparison between
33 quantum and classical uncertainties is only possible if made in terms of dimensionless
34 quantities, since the classical realm lacks a fundamental unit of action (much like
35 Newtonian mechanics lacks a fundamental unit of speed). Quantum numbers are not
36 simply the discretized counterparts to the dimensionless ratios of the classical system;
37 they arise precisely because of the existence of the fundamental unit \hbar . This highlights
38 the fundamental conceptual distinction between quantum and classical mechanics: it
39 is only through the tuning of *dimensionless* parameters that a bridge between the two
40 domains can be unambiguously built. This indicates that introducing uncertainty to
41 a system is, in a sense, a one-way street. Uncertainty bounds can only exist alongside
42 a dimensionful parameter that acts as a “universal certainty limit” (just like relativity
43 requires a universal speed limit). Once such a parameter is added to a theory, there
44 is a well-defined limit in which the quantum world appears to be classical, but there
45 is no limit that will return us to a truly classical world.

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