Introductory Notes to UA-MATH 325

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The purpose of these notes is to cover the material of Analysis 325 corresponding to Chapters 0-1, since it will differ a bit from Lebl´s book and we will cover some parts more lightly. We also cover properties about open and closed sets, which is included in this course's syllabus but not on Lebl book (except on Chapter 7, but in a much more abstract setting).

Starting in Chapter 2, we will follow Lebl's book much more closely, and lecture notes will not be provided.

1 Sets

Definition 1.1.

- A set is a collection of *elements*. If x is an element and A is a set, we write $x \in A$ to denote that x is an element of A, and $x \notin A$ to denote that x is not an element of A
- Two sets A and B are equal ($A = B$) if they have exactly the same elements.
- The set with no elements is called the empty set and denoted by $\varnothing$.
- A set A is contained in a set B (we denote it by $A \subset B$) when every element of A is also an element of B. We have that $\varnothing \subset A$ for any set A

Example 1.2.

- We denote sets with $\{\quad\}$. For example $A = \{0, 1, 2\}$ or $B = \{\text{banana, apple, orange}\}$ are two different sets, each with three elements. We will generally work with sets of numbers.
- The set $\{2, 0, 1\}$ is equal to A, since it has the same elements (the different ordering does not matter).
- Note that the definition also allows for sets with infinitely many elements. We denote by $\mathbb{N} = \{1, 2, 3, \ldots\}$ the set of natural numbers and $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$ the set of integers. We trivially have $\mathbb{N} \subset \mathbb{Z}$.
- We may also write properties inside the set notation. It is standard to use the notation : as a short-hand for "where" or "such that". For example, the set of rational numbers $\mathbb{Q}$ is defined as

  $$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N} \right\} = \left\{ \frac{a}{b} \text{ where } a \in \mathbb{Z}, b \in \mathbb{N} \right\}.$$

  The elements of the set are placed before the : symbol and the properties that need to be enforced are placed after the : symbol. In some textbooks the symbol | is used, instead of :
- The set of real numbers is denoted by $\mathbb{R}$

Remark 1.3. When we want to prove that $A \subset B$ we will take a generic element $x \in A$ and show that $x \in B$. This shows that every $x \in A$ belongs to B, and therefore $A \subset B$

Remark 1.4. When we want to prove that $A = B$, we will need to show that elements from A are in B and viceversa. Thus, we will divide the proof in two parts: $A \subset B$ and $B \subset A$. Once those parts are proved, it must happen that A and B have the same elements, and therefore we can claim $A = B$.

Definition 1.5.

- The union of two sets A and B is denoted by $A \cup B$ and defined as

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

- The intersection of two sets A and B is denoted by $A \cap B$ and defined as:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

  We say that the sets are **disjoint** if $A \cap B = \varnothing$. That is, if A and B have no common elements

- The difference of two sets A and B is denoted by $A \backslash B$ and defined as

$$A \backslash B = \{x : x \in A \text{ and } x \notin B\}$$

We can also consider families of sets, which we denote by $\{A_\lambda\}_{\lambda \in \Lambda}$. Here, λ is an **index**, Λ is and **indexing set** and each A_λ are sets (different for every λ. For example, taking $\Lambda = \mathbb{N}$, we can define $A_n = \{0, -n, n\}$. Each A_n is a set with three elements, and the set A_n depends on the value of the index n, which ranges over all the possible natural numbers.

Definition 1.6. We can also define unions and intersections over families of sets

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for some } \lambda \in \Lambda\},$$
$$\bigcap_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for all } \lambda \in \Lambda\}.$$

Example 1.7. With the $A_n = \{0, -n, n\}$ from above, one can check that

$$\bigcup_{n \in \mathbb{N}} A_n = \mathbb{Z} \qquad \bigcap_{n \in \mathbb{N}} A_n = \{0\}$$

Theorem 1.8 (De Morgan's laws)**.** *Suppose that $A, B \subset X$ for some 'ambient' space X (e.g. the real numbers). In such case we simply denote $A^c = X \backslash A$ for the complementary set. We have that:*

$$(A \cup B)^c = A^c \cap B^c \qquad \textit{and} \qquad (A \cap B)^c = A^c \cup B^c$$

Proof. We prove the first one, the second one being analogous. As noted above, we split the proof in $(A \cup B)^c \subset A^c \cap B^c$ and $(A \cup B)^c \supset A^c \cap B^c$.

$\boxed{(A \cup B)^c \subset A^c \cap B^c}$. We take an element $x \in (A \cup B)^c$. That means $x \in X$ and $x \notin A \cup B$. Since the elements of $A \cup B$ are the ones either in A or in B, we see that x is neither an element of A nor an element of B. Thus, we have that $x \in A^c$ and $x \in B^c$. By the definition of intersection, $x \in A^c \cap B^c$.

$\boxed{(A \cup B)^c \supset A^c \cap B^c}$. Take any $x \in A^c \cap B^c$. By the definition of intersection, we see $x \in A^c$ and $x \in B^c$. Therefore, we have $x \in X$, $x \notin A$, and $x \notin B$. Since $A \cup B$ is formed by elements either in A or in B, we have $x \notin A \cup B$. Thus, $x \in (A \cup B)^c$. ☐

2 Induction

Suppose that we want to prove some statement that depends on n, which we call $P(n)$. Suppose also that the statement is true for $n = 1$ and that every case n implies the next one. Then, the case $n = 1$ implies the $n = 2$, which itself implies $n = 3$, and so on and so forth. Thus, $P(n)$ must be true for all n. This is rigorously stated in the principle of induction

Theorem 2.1. *Suppose that $P(n)$ is a statement depending on $n \in \mathbb{N}$. Assume:*

- *Base Case: $P(1)$ holds.*
- *Induction Step: If $P(n)$ is true, then $P(n+1)$ is true.*

Under such assumptions, $P(n)$ holds for all $n \in \mathbb{N}$

Proof. We prove it by contradiction. This technique of proof consists on assuming the conclusion of the theorem fails and reaching contradiction. Thus, it must happens that the conclusion holds.

Suppose that $P(n)$ does not hold for all $n \in \mathbb{N}$, and let m to be the first natural number such that $P(m)$ fails. Since $P(1)$ is true (base case), $m > 1$, so $m - 1 \in \mathbb{N}$. Moreover, since m is the minimum number for which P fails, we must have that $P(m-1)$ holds. The induction step then says that $P(m-1+1) = P(m)$ must hold, which is a contradiction with the fact that $P(m)$ fails. Thus, we conclude the theorem. ☐

Example 2.2. We will show by induction that the sum of the first n odd numbers is n^2. That is:

$$1 + 3 + 5 + \ldots (2n-1) = n^2. \tag{2.1}$$

First, we check the base case, $n = 1$. Indeed, for $n = 1$ both the left-hand side and the right-hand side are just 1, so equation (2.1) holds.

Now, we do the induction step. We assume (2.1) for some fixed n, and we try to show it for $n + 1$. That is, we assume (2.1), and try to show

$$1 + 3 + 5 + \ldots (2n-1) + (2n+1) = (n+1)^2. \tag{2.2}$$

Using (2.1) on the first n terms, we have that

$$\underbrace{1 + 3 + 5 + \ldots (2n-1)}_{n^2} + (2n+1) = n^2 + (2n+1) = (n+1)^2$$

so we conclude the proof of (2.2). This concludes the proof by induction, and we can claim that (2.1) holds for every n.

3 Cartesian product and functions

Given two sets A and B we can define their cartesian product as the pairs formed by one element of A and another one of B.

Definition 3.1. Let A and B be sets. We define

$$A \times B := \{(x, y) : x \in A, y \in B\}$$

Example 3.2. $\mathbb{R} \times \mathbb{R}$ is the plane, since it is formed by the points (x, y) with $x, y \in \mathbb{R}$. We can also apply the product more times, for example $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is the three-dimensional space formed by points (x, y, z) with $x, y, z \in \mathbb{R}$.

Once we have that definition we can give the definition of function

Definition 3.3. Let A, B be sets. A function $f : A \to B$ is a subset of $A \times B$ such that for every $x \in A$ there exists exactly one $y \in B$ such that $(x, y) \in f$. We denote $y = f(x)$.

The set A is called the **domain** of f and the set B is called **codomain**

The **range** of f (denoted as $f(A)$) is the subset of B given by the values that f achieves, that is

$$f(A) = \{f(x) : x \in A\}$$

This definition may seem confusing at first, but it is more clear with an example. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. If we plot this function, we plot it in the plane. Concretely, the plot consists on the points (x, y) such that $y = x^2$. We simply have defined f as the set of those points, that is $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} \text{ such that } y = x^2\}$. The important property here is that for every x there exists a unique value of y such that $(x, y) \in f$. For example, $f(x) = \pm\sqrt{x}$ is not a valid function, since it asigns two different values of y. $f(x) = 1/x$ is also not a valid function from $\mathbb{R}$ to $\mathbb{R}$ (unless specified at 0) since it does not asign a real value to $f(0)$.

Definition 3.4. Consider a function $f : A \to B$. We define the direct image of any $C \subset A$ as

$$f(C) = \{f(x) : x \in C\}$$

and the inverse image of any $D \subset B$ as

$$f^{-1}(D) = \{x \in A : f(x) \in D\}$$

Notice that this definition is consistent with our previous notation for range. Indeed, when $C = A$ we simply recover the definition of range, and that is why we denote the range as $f(A)$. Notice that $f^{-1}(B) = A$ but it is not necessarily true that $f(A) = B$ (the function may not achieve all values in B).

Proposition 3.5. *We have the following properties regarding how unions and intersections interact with direct and inverse images. Suppose $f : A \to B$, $C, D \subset A$ and $E, F \subset B$. Note the difference on the second one*

- $f(C \cup D) = f(C) \cup f(D)$.
- $f(C \cap D) \subset f(C) \cap f(D)$.
- $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$
- $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$

Proof. We prove the last one, the rest are exercises. As usual, we show $=$ by showing both $\subset$ and $\supset$.

$\boxed{\subset}$. We take $x \in f^{-1}(E \cap F)$. By definition of inverse image, this means exactly that $f(x) \in E \cap F$. By definition of intersection, $f(x) \in E$ and $f(x) \in F$. Now, $y = f(x)$ with $y \in E$, so the definition of inverse image yields $x \in f^{-1}(E)$. Similarly, $x \in f^{-1}(F)$. Once we have $x \in f^{-1}(E)$ and $x \in f^{-1}(F)$ the definition of intersection yields $x \in f^{-1}(E) \cap f^{-1}(F)$.

$\boxed{\supset}$. We take any $x \in f^{-1}(E) \cap f^{-1}(F)$ and show that it is also in $f^{-1}(E \cap F)$. Since $x \in f^{-1}(E) \cap f^{-1}(F)$, we have that $x \in f^{-1}(E)$ and $x \in f^{-1}(F)$. Thus, by definition of inverse image, $f(x) \in E$ and $f(x) \in F$. Therefore, $f(x) \in E \cap F$, which yields that $x \in f^{-1}(E \cap F)$. $\square$

Example 3.6. To see how the item example does not need to be an equality, take $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$. Take $C = \{-4, 3\}$ and $D = \{3, 4\}$.

We have $f(C) = \{9, 16\}$ and $f(D) = \{9, 16\}$, so that $f(C) \cap f(D) = \{9, 16\}$.

However, $C \cap D = \{3\}$, so we have $f(C \cap D) = \{9\}$.

The issue here is that 16 is being obtained as $f(-4)$ when looking at $f(C)$ and $f(4)$ when looking at $f(D)$, so there is no common $x \in C \cap D$ that outputs $f(x) = 16$.

Definition 3.7. We say that $f : A \to B$ is **injective** (or one-to-one) if $f(x_1) = f(x_2)$ only when $x_1 = x_2$. In other words, for each $y \in B$ there is **at most** one $x \in A$ with $f(x) = y$.

We say that f is **surjective** (or onto) if $f(A) = B$. That is, for every $y \in B$ there exists **at least** one element $x \in A$ with $f(x) = y$.

If f is both surjective and injective, we say it is a **bijection** (or bijective). In that case, every element $y \in B$ has exactly one $x \in A$ such that $f(x) = y$. Thus, in that case we can define the inverse function $f^{-1} : B \to A$ that for every $y \in B$ assigns the unique $x \in A$ with $f(x) = y$.

Exercise 3.8. Why do we need f to be bijective in order to define the inverse function f^{-1}? Go back to the definition of function and think about it.

Definition 3.9. Given a function $f : A \to B$ and $g : B \to C$ we can simply define their composition $g \circ f : A \to C$ as

$$(g \circ f)(x) = g(f(x))$$

For example, if $f, g : \mathbb{R} \to \mathbb{R}$ are given by $f(x) = x + 1$ and $g(x) = x^2$, we have that $(g \circ f)(x) = (x+1)^2$, while $(f \circ g)(x) = x^2 + 1$. Clearly, those two functions are different, so the order of the composition matters.

4 Ordering

Now, we want to define an ordering. The examples we will talk about are the usual ones (for example, the order $\leqslant$ on the real numbers), but we want to give the rigorous definition on any set. For an abstract set A, we will say that an ordering $\leqslant$ is just a subset of $A \times A$ that outputs the pairs $(x, y) \in A \times A$ for which $x \leqslant y$. Moreover, we ask for some standard properties of ordering

Definition 4.1. An ordering relation $\leqslant$ on a set A is a subset of the cartesian product $A \times A$. Instead of writting $(x, y) \in \leqslant$, we normally write $x \leqslant y$ to indicate that a pair is in the ordering. Moreover, we ask for the following properties to be satisfied so that $\leqslant$ is an ordering.

1. (Reflexive). For every $x \in A$ we have that $x \leqslant x$.
2. (Antisymmetric) If we have that $x \leqslant y$ and $y \leqslant x$, it must happen that $x = y$.
3. (Transitive). If we have that $x \leqslant y$ and $y \leqslant z$, it must happen that $x \leqslant z$.

Moreover, when every two elements are comparable (that is, it is always the case that either $x \leqslant y$ or $y \leqslant x$) we say that the order is **total**. Caution: Lebl's book confuses order and total order. In this course we will just deal with total orders so this is not a big issue, but rigorously speaking an order does not need to be total

Example 4.2. If we consider A the set of people in the world, the relation of being an ancestor (that is, parent, grandparent, great-grandparent, etc) is an ordering relation. That is, we say that $x \leqslant y$ whenever y is an ancestor of x (including the case $x = y$).

With that definition, $x \leqslant x$, because x is an ancestor of itself. It is also clear that if x and y are both ancestors of each other, then $x = y$, since you cannot be the parent of your parent, or the parent of your grandparent, etc (we are assuming no time-travelling here). Lastly, if $x \leqslant y$ and $y \leqslant z$ we have that $x \leqslant z$. For example, the grandparent of your grandparent is also your ancestor, etc.

This is an example of an ordering that is non-total. For two given people x and y, they may be siblings, or they may be unrelated, and in those cases neither of them is an ancestor of the other one.

From now on, we will work only with total orders, which is the usual case for numbers. For the rest of the section, we assume that S is a set equipped with a total order $\leqslant$. We also use the symbol $y \geqslant x$ to denote $x \leqslant y$, as usual.

Definition 4.3. We say that $E \subset S$ is bounded from above if there exists $b \in S$ such that $x \leqslant b$ for every $x \in E$. In that case, we say that b is an upper bound.

We say that E has a maximum if there exists an upper bound $b \in E$. In that case, we denote $b = \max E$.

We define analogously lower bounded, lower bound and $\min E$ replacing $x \leqslant b$ by $x \geqslant b$

Remark 4.4. If E has a maximum element, it is unique. Indeed, suppose b and b' are both maximums of E, which means that $b, b' \in E$ and b, b' are both upper bounds of E.

Since b is upper bound of E and $b' \in E$, we have that $b' \leqslant b$. Similarly, since b' is upper bound of E and $b \in E$, we have that $b \leqslant b'$. Thus, using the antisymmetric property, we obtain $b = b'$.

Definition 4.5. Let us assume $E \subset S$ is bounded from above and define $U \subset S$ the set of possible upper bounds. In particular, $U \neq \varnothing$. Whenever U has a minimum, we define the supremum of E as

$$\sup(E) = \min(U)$$

In other words, the supremum of E is the least possible upper bound. We define analogously the infimum of a set bounded from below as the maximum of the set of lower bounds.

One should understand the concept of supremum and infimum as relaxations of the concepts of maximum and minimum. The idea will be that not all sets bounded from above have maximum, but all of them have supremum (if we are working on $\mathbb{R}$).

First, let us see that a maximum is a supremum.

Proposition 4.6. *Assume that E has a maximum $m = \max E$. Then, m is the supremum of E.*

Proof. Let U be the set of upper bounds of E. By the definition of maximum, we know that m is an upper bound of E, that is, $m \in U$. We defined $\sup(E) = \min(U)$ (the supremum is the least possible upper bound). Therefore, we just need to consider any other upper bound $b \in U$ and show that $m \leqslant b$.

Since $m \in E$ and b is an upper bound of E, it is clear that $m \leqslant b$. Therefore $m = \sup(E)$ $\square$

Example 4.7. There are sets that have supremum but not maximum. For example, let us consider $E = (0, 1)$, which is the set of numbers $x \in \mathbb{R}$ with $0 < x < 1$.

It is clear that E does not have a maximum since for every $x \in E$ we will have that $x < 1$, so we can find some other $x' \in E$ with $x' > x$ (for example, $x' = x + \frac{1-x}{2}$).

The set of upper bounds U is just given by the numbers $x \in \mathbb{R}$ with $x \geqslant 1$. Indeed, 1 is an upper bound of E, and any number above 1 is also an upper bound. Therefore, U does have a minimum, which is just 1. We conclude that $\sup(E) = 1$.

This is the typical example of a supremum that is not maximum. When we look at the set E (numbers strictly between 0 and 1) we would like to say that 1 is the maximum. However, $1 \notin E$, so it cannot be the maximum. The definition of supremum allows us to say that $1 = \sup(E)$.

The importance of the real numbers when doing analysis emanates from the following fundamental property.

Theorem 4.8 (Least upper bound property)**.** *Suppose that $S = \mathbb{R}$ (with the usual ordering) and consider some $E \subset \mathbb{R}$ which is upper bounded (and $E \neq \emptyset$).*

Then, E has a supremum. Analogously, if E is lower bounded, it has an infimum.

Proof. Outside the scope of this course. It actually has to do with how one defines rigorously the real numbers, which we have not done. □

Example 4.9. Probably the best way to understand the least upper bound property is with an example of a set that does not satisfy it. That is the case of $\mathbb{Q}$, the rational numbers.

Indeed, consider $S = \mathbb{Q}$ with the usual ordering, and $E = \{x \in \mathbb{Q} \text{ such that } x^2 \leqslant 2,\ x \geqslant 0\}$. That is, E consists on the rational numbers between 0 and $\sqrt{2}$.

The maximum is not $\sqrt{2}$ because $\sqrt{2} \notin E$, given that $\sqrt{2}$ is not a rational number. The set of upper bounds U is formed by the **rational numbers** which are bigger or equal than $\sqrt{2}$. That is

$$U = \{x \in \mathbb{Q} : x \geqslant \sqrt{2}\}$$

However, such set U has no minimum. The non-rigorous reason is that $\sqrt{2} \notin U$, so it cannot be the minimum. Therefore, we see that $E \subset \mathbb{Q}$ has no supremum in $\mathbb{Q}$. This means that the rational numbers do not satisfy the least upper bound property.

The following Proposition regarding the supremum will be very useful

Proposition 4.10. *Assume $E \subset \mathbb{R}$ is bounded from above. Then, for every $\varepsilon > 0$ as small as we want, we have that there is an element $x \in E$ with $\sup S - \varepsilon < x \leqslant \sup S$.*

Proof. Fix any $\varepsilon > 0$. Since $\sup S$ is the least upper bound of E, we know that $\sup S - \varepsilon$ is not an upper bound of E. Therefore, there exists $x \in E$ such that $x > \sup S - \varepsilon$.

Moreover, since $\sup S$ is an upper bound of E and $x \in E$, we have that $x \leqslant \sup S$.

Combining those two things, we conclude $\sup S - \varepsilon < x \leqslant \sup S$.

□

Example 4.11. Let us do one computation of supremum / infimum. Consider the set $A = \{1/n : n \in \mathbb{N}\}$. We will show that $\inf(A) = 0$. We need to show that 0 is a lower bound of A and that moreover is the greatest possible lower bound

It is clear that 0 is a lower bound of A, since $0 < 1/n$ for every $n \in \mathbb{N}$.

Now we need to show that any $x > 0$ is not a lower bound of A. We simply take some natural number n which is bigger than $1/x$. Therefore, since $n < 1/x$, we have that $1/n < x$. Since $1/n \in A$, we conclude that x is not an upper bound of A.

5 Archimedean Property of $\mathbb{R}$

Theorem 5.1 (Archimedean Property)**.** *Between two different real numbers, we can always find a rational number. That is, for any $x, y \in \mathbb{R}$ with $x < y$, there exists $r \in \mathbb{Q}$ such that*

$$x < r < y$$

Proof. Since we want to find $r \in \mathbb{Q}$, we will express $r = p/q$, with $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and reduce to find p and q. First, notice that the rational numbers of denominator q are separated by $1/q$. Thus, intuitively, if we ask for $1/q < y - x$, we should be able to find one rational between x and y. That intuition leads us to fix any $q \in \mathbb{N}$ such that $q > \frac{1}{y-x}$ so that we have

$$\frac{1}{q} < y - x$$

Our task now is to find $p \in \mathbb{Z}$ such that $x < \frac{p}{q} < y$. Now, we define p to be the smallest integer strictly greater than qx. In particular, we must have that

$$qx < p \qquad \text{and} \qquad qx \geqslant p - 1$$

given that p is greater than qx and (since it is the minimum) $p - 1$ is not. From the second inequality, we obtain that

$$p \leqslant qx + 1 = q\left(x + \frac{1}{q}\right) < q(x + y - x) = qy$$

Therefore, we have that $qx < p < qy$, which implies that $x < \frac{p}{q} < y$. This concludes the proof, since we have found a rational number between x and y. $\square$

6 Triangle Inequality and Intervals

We first recall the notation that you are probably familiar with regarding intervals. Open intervals and closed intervals are defined as the subsets of $\mathbb{R}$ defined by:

$$(a, b) = \{x : a < x < b\}, \qquad \text{and} \qquad [a, b] = \{x : a \leqslant x \leqslant b\}. \tag{6.1}$$

One can also mix the extremum of the intervals, for example $(a, b]$ is the set of numbers $x \in \mathbb{R}$ with $a < x \leqslant b$. When the extremum of the interval is open, we can also put $\pm\infty$, for example, $(-\infty, b]$ is the set of numbers $x \in \mathbb{R}$ with $x \leqslant b$.

Definition 6.1. The absolute value of x, denoted by $|x|$ is given by:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$$

Dividing into cases it is trivial to deduce properties like $|x|^2 \geqslant 0$, with $|x| = 0$ if and only if $x = 0$. We also have that $-|x| \leqslant x \leqslant |x|$ or that $|xy| = |x| \cdot |y|$.

A more interesting property is the triangle inequality. Its name comes from the two-dimensional interpretation, where it says that the longest side of a triangle has a smaller length than the sum of the other two. However, here we will state it in the one dimensional case.

Lemma 6.2 (Triangle Inequality)**.** *For any $x, y \in \mathbb{R}$, we have that*

$$|x + y| \leqslant |x| + |y|. \tag{6.2}$$

We also have the similar version

$$|x - y| \leqslant |x| + |y| \tag{6.3}$$

and the reverse triangle inequality

$$||x| - |y|| \leqslant |x - y| \tag{6.4}$$

Proof. Regarding the proof of (6.2), we note that $-|x| \leqslant x \leqslant |x|$ and $-|y| \leqslant y \leqslant |y|$. Adding those inequalities, we deduce

$$-(|x| + |y|) \leqslant x + y \leqslant (|x| + |y|)$$

Now, we divide in cases. If $x + y$ is positive, we clearly have that $|x + y| = x + y \leqslant |x| + |y|$. If $x + y$ is negative, the inequality $-(|x| + |y|) \leqslant x + y = -|x + y|$ yields that $|x + y| \leqslant |x| + |y|$ (remember that changing signs in both sides of the equation also swaps the inequality). That concludes the proof of (6.3).

Regarding (6.3), simply take $z = -y$ and apply the original triangle inequality. Indeed

$$|x - y| = |x + z| \leqslant |x| + |z| = |x| + |-y| = |x| + |y|$$

which concludes the proof.

Lastly, we show (6.4). First of all, notice that without loss of generality, we can assume $|x| \geqslant |y|$. If we have $|y| > |x|$, simply swap the names of x and y and observe that both sides of the equation remain unchanged, given that $|x - y| = |y - x|$ and $||x| - |y|| = ||y| - |x||$. Thus, we can assume $|x| \geqslant |y|$. In that case, the inequality reduces to show

$$|x| - |y| \leqslant |x - y| \Leftrightarrow |x| \leqslant |y| + |x - y|$$

However, notice that taking $w = x - y$, we can reexpress this inequality as $|w + y| \leqslant |y| + |w|$, which is the original triangle inequality. Therefore, we conclude the proof of (6.4). $\square$

7 Open and closed sets of $\mathbb{R}$

Definition 7.1. We say that $U \subset \mathbb{R}$ is an **open set** if for every $x \in U$, there exists some $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset U$.
We say that some set $F \subset \mathbb{R}$ is **closed** if $\mathbb{R} \backslash F$ (which we will simply write as F^c) is open.

Proposition 7.2. *The open intervals are open and the closed intervals are closed.*

Proof. We have six differen cases, namely intervals of the types: (a, b), $[a, b]$, (a, ∞), $(-\infty, b)$, $[a, \infty)$, $(-\infty, b]$. Let us do one of them, for example $[a, \infty)$ is closed.
First, we look at its complement, and since $[a, \infty) = \{x \in \mathbb{R} \quad x \geqslant a\}$, the complement is the set $\{x \in \mathbb{R}, \quad x < a\}$, that is, $(-\infty, a)$. Thus, we need to show $(-\infty, a)$ is open. Let $x \in (-\infty, a)$, so that $x < a$, and take $\varepsilon < a - x$. We have that $(x - \varepsilon, x + \varepsilon)$ only contains elements smaller than $x + \varepsilon < x + (a - x) = a$, so it is clear that $(x - \varepsilon, x + \varepsilon)$ is a subset of $(-\infty, a)$.

Proposition 7.3. *We have the following properties for subsets of* $\mathbb{R}$

- $\varnothing$ *and* $\mathbb{R}$ *are open.*
- *Suppose* Λ *is some (possibly infinite) set and let* A_λ *be open sets for all* $\lambda \in \Lambda$. *Then*

$$\bigcup_{\lambda \in \Lambda} U_\lambda = \{x \in \mathbb{R} \quad \textit{such that } x \in U_\lambda \textit{ quad for some } \lambda \in \Lambda\}$$

  is also open.
- *If* U_k *are open sets for* $k = 0, 1, 2, \ldots n$ *then their* ***finite intersection*** *is also open:*

$$\bigcap_{k=0}^{n} U_k = \{x \in \mathbb{R} \quad \textit{such that } x \in U_k \textit{ for all } k \in \{0, 1, 2, \ldots n\} quad \forall k \in \mathbb{N}, k \leqslant n\}$$

  is also open

Proof. $\boxed{\varnothing}$ $\varnothing$ is open since the hypothesis of the definition never holds. For all $x \in \varnothing$ there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset \varnothing$ because there are no $x \in \varnothing$ to begin with (thus, such implication is always true).

$\boxed{\mathbb{R}}$ $\mathbb{R}$ is open because for every $x \in \mathbb{R}$, we simply take $\varepsilon = 1$ and $(x - 1, x + 1) \subset \mathbb{R}$.

$\boxed{\text{Arbitrary union}}$ Now, let U_λ be open sets and let us show that $\bigcup_{\lambda \in \Lambda} U_\lambda$ is open as well. Take any $x \in \bigcap_{\lambda \in \Lambda} U_\lambda$. We know that $x \in U_{\lambda_0}$ for at least one $\lambda_0 \in \Lambda$. On the one hand, U_{λ_0} is open, so there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset U_{\lambda_0}$. On the other hand, $U_{\lambda_0} \subset \bigcup_{\lambda \in \Lambda} U_\lambda$, by the definition of the union. Therefore, we conclude that $(x - \varepsilon, x + \varepsilon) \subset \bigcup_{\lambda \in \Lambda} U_\lambda$. As a recapitulation, we have shown that for any $x \in \bigcup_{\lambda \in \Lambda} U_\lambda$, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset \bigcup_{\lambda \in \Lambda} U_\lambda$. Thus, we have shown that $\bigcup_{\lambda \in \Lambda} U_\lambda$ is open.

$\boxed{\text{Finite intersection}}$ First, we show it for the intersection of two open sets. Let $U_0, U_1 \subset \mathbb{R}$ open, we will show $U_0 \cap U_1$ is open. Let $x \in U_0 \cap U_1$, so that $x \in U_0, x \in U_1$. Since U_0, U_1 are open, we have that there exist $\varepsilon_0, \varepsilon_1 > 0$ such that

$$(x - \varepsilon_0, x + \varepsilon_0) \subset U_0, \quad \text{and} \quad (x - \varepsilon_1, x + \varepsilon_1) \subset U_1.$$

Taking $\varepsilon = \min\{\varepsilon_0, \varepsilon_1\}$, we have that $(x-\varepsilon, x+\varepsilon) \subset (x-\varepsilon_0, x+\varepsilon_0) \subset U_0$ and similarly $(x-\varepsilon, x+\varepsilon) \subset (x-\varepsilon_1, x+\varepsilon_1) \subset U_1$. Thus, we see $(x-\varepsilon, x+\varepsilon) \subset U_0 \cap U_1$ and we are done with the intersection of two sets.

Now, we show by induction that $\bigcap_{k=0}^n U_k$ is open when U_k are open. The base case $n = 1$ coresponds to the intersection of two sets, which is already proved. Regarding the induction step, we assume that the intersection of $n+1$ open sets (from $k = 0$ until $k = n$) is open and show it for $n+2$ sets (from $k = 0$ until $k = n + 1$). We then notice that

$$\bigcap_{k=0}^{n+1} U_k = \left(\bigcap_{k=1}^{n} U_k \right) \cap U_{n+1}$$

The set in parenthesis is open by induction hypothesis and U_{n+1} is open by assumption. Therefore, the intersection of those two sets is also open by the case of intersecting two sets. □

Proposition 7.4. *We also have that*

- *$\varnothing$ and $\mathbb{R}$ are closed.*
- *The intersection of arbitrarily many closed sets is closed. That is if F_λ are closed for all $\lambda \in \Lambda$, then*

$$\bigcap_{\lambda \in \Lambda} F_\lambda := \{x \text{ such that } x \in F_\lambda \text{ for all } \lambda \in \Lambda\}$$

  is closed.
- *The finite union of closed sets is closed. That is, if F_k is closed for $k = 0, 1, 2, \ldots n$, then*

$$\bigcup_{k=0}^{n} F_k := \{x \text{ such that } x \in F_k \text{ for some } k \in \{0, 1, 2, \ldots n\}\}$$

  is closed

□

Proof. Exercise □

8 Closure, interior, boundary, cluster points

Definition 8.1. Let $A \subset \mathbb{R}$. The closure of A is defined as

$$\overline{A} := \{x \in \mathbb{R} \quad \text{such that for every } \varepsilon > 0 \text{ we have} \quad (x-\varepsilon, x+\varepsilon) \cap A \neq \varnothing\}.$$

The interior of A is defined as

$$A^\circ := \{x \in \mathbb{R} \quad \text{such that there exists } \varepsilon > 0 \text{ with} \quad (x-\varepsilon, x+\varepsilon) \subset A\}.$$

Remark 8.2. We have that $A^\circ \subset A \subset \overline{A}$. For the first inclusion, note that if $x \in A^\circ$, then $(x-\varepsilon, x+\varepsilon) \subset A$ for some $\varepsilon > 0$. In particular, $x \in A$. Regarding the second inclusion, if $x \in A$, we have that $x \in (x-\varepsilon, x+\varepsilon) \cap A$ for all $\varepsilon > 0$. In particular $(x-\varepsilon, x+\varepsilon) \cap A \neq \varnothing$, so $x \in \overline{A}$.

Definition 8.3. The boundary of A is denoted by ∂A and defined as $\partial A = \overline{A} \backslash A^\circ$.

The cluster points of A is a set denoted by A' and defined as $A' := \{x \in \mathbb{R} : \text{ for every } \varepsilon > 0 \text{ we have } (x-\varepsilon, x+\varepsilon) \cap A \backslash \{x\} \neq \varnothing\}$

The definition of cluster point is very similar to the definition of closure, with the difference that we exclude the point x. That is, when $A \cap (x-\varepsilon, x+\varepsilon) = \{x\}$, we still have that x is a point in the closure (even though there are no other points of A close by, the fact that $x \in A$ is enough to claim $x \in \overline{A}$). However, if $A \cap (x-\varepsilon, x+\varepsilon) = \{x\}$, the point x would not be a cluster point, since the definition asks that there should be elements in that intersection appart from x. Intuitively, every point of the closure is a cluster point except those points which are 'isolated'

Example 8.4. Consider $A = [0,1) \cup \{2\}$.

Let us show that its interior is $A^\circ = (0,1)$**.**

First of all, let us recall from Proposition 7.2 that $(0,1)$ is open, so for all $x \in (0,1)$ we have some $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset (0,1) \subset A$.

Since $A^\circ \subset A$, the only two possible elements that can be in A° apart from $(0,1)$ are 0 and 2. Regarding 0, we see that $0 \notin A^\circ$, since for any ε, we will have that $-\varepsilon/2 \in (-\varepsilon, \varepsilon)$ but $-\varepsilon/2 \notin A$. Similarly, $2 \notin A^\circ$, because for any $\varepsilon > 0$, we will have that $2 + \varepsilon/2 \in (2-\varepsilon, 2+\varepsilon)$ but $2 + \varepsilon/2 \notin A$.

All in all, $A^\circ = (0,1)$.

One could similarly show that $\overline{A} = [0,1] \cup \{2\}$ or that $A' = [0,1]$. From the definition of boundary, it follows that $\partial A = \{0,1,2\}$.