$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Зад. Да се намери обр. матр:

* $\begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}^{-1}$?

Реш: $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{array}\right)^{(3)} \sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 6 & 3 & 1 \end{array}\right) \sim$

$$\sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{6} \end{array}\right)_{(-2)} \sim \left(\begin{array}{cc|cc} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{1}{2} & \frac{1}{6} \end{array}\right)$$

$$\begin{vmatrix} 1 & 2 \\ -3 & 0 \end{vmatrix} = 6 \qquad \begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & -\frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix}$$

* $\begin{pmatrix} 1 & 3 & -7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$?

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -7 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)_{(2)(7)} \sim_{(-3)} \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 7 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to = \begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Матрици от вида $\begin{pmatrix} * & \cdots & * \\ 0 & \ddots & \vdots \\ 0 & \cdots & * \end{pmatrix}$ наричаме

горно-триъгълни

Обратната матрица на горно-триъгълна също е горно-триъгълна

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}^{-1} ?$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) (-1) \sim$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -2 & -1 & 0 & 1 & 0 \\ 0 & -2 & -2 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1/2 & 0 & -1/2 & 0 \\ 0 & 1 & 1 & 0 & 1/2 & 0 & 0 & -1/2 \end{array}\right) (-1) \sim$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 1 & 0 & 1/2 & 0 & 0 & -1/2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1/2 & 1/2 & -1/2 & -1/2 \\ 0 & 1 & 1 & 0 & 1/2 & 0 & 0 & -\frac{1}{2} \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/4 & 1/4 & -1/4 & -1/4 \\ 0 & 1 & 1 & 0 & 1/2 & 0 & 0 & -1/2 \end{array}\right) \begin{array}{c} \\ \\ (-1) \\ \\ \end{array}$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/4 & 1/4 & -1/4 & -1/4 \\ 0 & 0 & 1 & 0 & 1/4 & -\frac{1}{4} & 1/4 & -\frac{1}{4} \end{array}\right) \begin{array}{c} \\ \\ \\ (-1) \end{array}$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/4 & -1/4 & -1/4 & \frac{1}{4} \\ 0 & 1 & 0 & 0 & 1/4 & 1/4 & -1/4 & -1/4 \\ 0 & 0 & 1 & 0 & 1/4 & -1/4 & 1/4 & -1/4 \end{array}\right) \begin{array}{c} \\ (-1) \\ (-1) \\ (-1) \end{array} \sim$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1 & 0 & 0 & 1/4 & 1/4 & -1/4 & -1/4 \\ 0 & 0 & 1 & 0 & 1/4 & -1/4 & 1/4 & -1/4 \\ 0 & 0 & 0 & 1 & 1/4 & -1/4 & -1/4 & 1/4 \end{array}\right)$$

$$\begin{pmatrix} a & 1 & 0 & 0 & \cdots & 0 \\ 0 & a & 1 & 0 & & \vdots \\ 0 & 0 & a & 1 & & \vdots \\ & \cdots & & \ddots & \ddots & \\ & 0 & & & \ddots & \\ & & \cdots & & & a \end{pmatrix}^{-1} \; ?$$

$$\left(\begin{array}{cccc|cccc} a & 1 & & 0 & 1 & & 0 & \\ 0 & a & \ddots & & & 1 & & \\ & & \ddots & 1 & & & \ddots & \\ & 0 & & a & 0 & & & 1 \end{array}\right)\begin{matrix}(-1) \\ \\ \\ (\cdot \frac{1}{a})\end{matrix} \sim$$

$$\sim \left(\begin{array}{ccccc|cccc} a & 1 & & & 0 & 1 & \ddots & & 0 \\ & a & 1 & & & & & & \\ & & \ddots & \ddots & & & 1 & & \\ & 0 & & 1 & & 0 & & 1 & -\frac{1}{a} \\ & & & a & 0 & & & & \\ & & & & a & & & & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccccc|cccccc} a & 1 & & & & 0 & 1 & 0 & & & & 0 \\ & \ddots & \ddots & & & & \vdots & \ddots & 0 & \cdots & 0 & 0 \\ & & \ddots & 1 & & & \vdots & & & 1 & -\frac{1}{a} & \frac{1}{a^2} \\ & & & a & 0 & 0 & & & & 0 & 1 & -\frac{1}{a} \\ & 0 & & & a & 0 & & & & & & \\ & & & & & a & 0 & \cdots & & & 0 & 1 \end{array}\right) \begin{matrix} \\ (\cdot \frac{1}{a}) \end{matrix} \sim$$

$$\sim \cdots \left(\begin{array}{ccc|ccccc} a & & 0 & 1 & -\frac{1}{a} & \frac{1}{a^2} & \cdots & \frac{(-1)^{n-1}}{a^{n-1}} \\ & \ddots & & & 1 & & & \\ & & & & & 1 & & \\ 0 & & a & 0 & & & \ddots & 1 \end{array}\right) \sim$$

$$\Rightarrow \begin{pmatrix} & & \\ & * & \\ & & \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & -\frac{1}{a^2} & -\frac{1}{a^3} & \cdots & \frac{(-1)^{n-1}}{a^n} \\ & \frac{1}{a} & -\frac{1}{a^2} & \cdots & \\ & & \ddots & & \\ & & & \frac{1}{a} & -\frac{1}{a^2} \\ & & & & \frac{1}{a} \end{pmatrix}$$

Зад.
Да се реши матричното уравнение
$AX=B$

$$\begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \\ 1 & 3 & -5 \end{pmatrix} X = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & -1 \\ 3 & 1 & -1 & 2 & 1 \\ 1 & 3 & -5 & 3 & 0 \end{array}\right) \begin{matrix} (-3) & (-1) \end{matrix} \sim \left(\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & -1 \\ 0 & 4 & -7 & -1 & 4 \\ 0 & 4 & -7 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & -1 \\ 0 & 4 & -7 & -1 & 4 \\ 0 & 0 & 0 & 3 & -3 \end{array}\right)$$

Получихме, че първата матрица не е обратима, но последното ни даде, че:

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & -7 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 4 \\ 3 & -3 \end{pmatrix}$$

което ни дава системата:

$$\left|\begin{array}{l} x_1 - x_3 + 2x_5 = 1 \\ x_2 - x_4 + 2x_6 = -1 \\ \qquad 4x_3 - 7x_5 = -1 \\ \qquad 4x_4 - 7x_6 = 4 \\ 0x_1 + 0x_3 + 0x_5 = 0 \\ 0x_2 + 0x_4 + 0x_6 = 0 \end{array}\right.$$

$\Rightarrow$ с-мата няма решение

$\Rightarrow$ матричното уравнение няма решение

Зад. Да се реши матричното у-е:

$$X \begin{pmatrix} 2 & -1 & 3 \\ 3 & 2 & 2 \\ 1 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 4 & 4 \\ 0 & 0 & 0 \\ 18 & 12 & 12 \end{pmatrix} \rightarrow \begin{array}{l} XA = B \\ A^t X^t = B^t \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 6 & 0 & 18 \\ -1 & 2 & 3 & 4 & 0 & 12 \\ 3 & 2 & -1 & 4 & 0 & 12 \end{array}\right) \begin{array}{c} \\ (2)\,(3) \\ \\ \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & 7 & 7 & 14 & 0 & 42 \\ -1 & 2 & 3 & 4 & 0 & 12 \\ 0 & 8 & 8 & 16 & 0 & 48 \end{array}\right) \begin{array}{l} |:7 \\ \\ |:8 \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 2 & 0 & 6 \\ -1 & 2 & 3 & 4 & 0 & 12 \\ 0 & 1 & 1 & 2 & 0 & 6 \end{array}\right) (-1) \sim \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 3 & 4 & 0 & 12 \\ 0 & 1 & 1 & 2 & 0 & 6 \end{array}\right)$$

Получихме, че A^t не е обратима, но все пак сведохме матр. у-е до:

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 0 & 0 \\ 12 & 6 \end{pmatrix}$$

$$\left| \begin{array}{l} -x_1 + 2x_2 + 3x_3 = 4 \\ \vdots \\ \vdots \end{array} \right.$$

9 неизвестни, 6 ур-е
$\Rightarrow$ 3 параметъра

Тази система можем да решим и през матрицата:

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & p & q & s \\ -1 & 2 & 3 & 4 & 0 & 12 \\ 0 & 1 & 1 & 2 & 0 & 6 \end{array}\right) \begin{matrix} (-3) & (-1) \end{matrix} \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & p & q & s \\ -1 & 2 & 0 & 4-3p & -3q & 12-3s \\ 0 & 1 & 0 & 2-p & -q & 6-s \end{array}\right) (-2) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & p & q & s \\ -1 & 0 & 0 & -p & -q & -s \\ 0 & 1 & 0 & 2-p & -q & 6-s \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & p & q & s \\ 0 & 1 & 0 & 2-p & -q & 6-s \\ 0 & 0 & 1 & p & q & s \end{array}\right)$$

$$\Rightarrow \quad X = \begin{pmatrix} p & 2-p & p \\ q & -q & q \\ s & 6-s & s \end{pmatrix}$$

Зад. $AX = A + X$

$$A = \begin{pmatrix} 0 & 4 & -3 \\ 1 & -4 & 5 \\ -1 & 3 & 1 \end{pmatrix}$$

Реш: $(A - E)X = A$

$$\left(\begin{array}{ccc|ccc} -1 & 4 & -3 & 0 & 4 & -3 \\ 1 & -5 & 5 & 1 & -4 & 5 \\ -1 & 3 & 0 & -1 & 3 & 0 \end{array}\right) \begin{array}{c} \\ (1) \\ \\ \end{array} \sim$$

$$\left(\begin{array}{ccc|ccc} 0 & -1 & 2 & 1 & 0 & 2 \\ 1 & -5 & 5 & 1 & -4 & 5 \\ 0 & -2 & 5 & 0 & -1 & 5 \end{array}\right) \begin{array}{c} (-2) \\ \\ \\ \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & -1 & 2 & 1 & 0 & 2 \\ 1 & -5 & 5 & 1 & -4 & 5 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right) \begin{array}{l} \\ (-5) \quad (-2) \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & -1 & 0 & 5 & 2 & 0 \\ 1 & -5 & 0 & 11 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right) \begin{array}{l} (-5) \\ \sim \\ \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & -1 & 0 & 5 & 2 & 0 \\ 1 & 0 & 0 & -14 & -9 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -14 & -9 & 0 \\ 0 & 1 & 0 & -5 & -2 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right)$$

$AXB = C$?

1) $XB := Y \rightarrow AY = C$

2) с нам $Y \rightarrow XB = Y$

3) $(XB)^t = Y^t \rightarrow B^t X^t = Y^t$

4) нам X^t

$$\underbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4 & -5 & 2 \\ 5 & -7 & 3 \end{pmatrix}}_{A} X \underbrace{\begin{pmatrix} 9 & 7 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} 2 & 0 & -2 \\ 18 & 12 & 9 \\ 23 & 15 & 11 \end{pmatrix}}_{C}$$

$Y := XB$

$AY = C$, търсим Y

$$\left(\begin{array}{ccc|ccc} 2 & -3 & 1 & 2 & 0 & -2 \\ 4 & -5 & 2 & 18 & 12 & 9 \\ 5 & -7 & 3 & 23 & 15 & 11 \end{array}\right) \begin{array}{l} (-2)\,(-3) \\ \\ \\ \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & -3 & 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 14 & 12 & 13 \\ -1 & 2 & 0 & 17 & 15 & 17 \end{array}\right) \begin{array}{l} \\ (3) \quad (-2) \\ \\ \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 44 & 36 & 37 \\ 0 & 1 & 0 & 14 & 12 & 13 \\ -1 & 0 & 0 & -11 & -9 & -9 \end{array}\right) \begin{array}{l} \\ \\ (2) \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 22 & 18 & 19 \\ 0 & 1 & 0 & 14 & 12 & 13 \\ 1 & 0 & 0 & 11 & 9 & 9 \end{array}\right)$$

$$\Rightarrow \quad Y = \begin{pmatrix} 11 & 9 & 9 \\ 14 & 12 & 13 \\ 22 & 18 & 19 \end{pmatrix}$$

$$XB = Y \qquad B^t X^t = Y^t$$

$$\begin{array}{r} \\ \\ (-1) \end{array} \left(\begin{array}{c|c} \overbrace{\begin{matrix} 9 & 1 & 1 \\ 7 & 1 & 1 \\ 6 & 2 & 1 \end{matrix}}^{B^t} & \overbrace{\begin{matrix} 11 & 14 & 12 \\ 9 & 12 & 18 \\ 9 & 13 & 19 \end{matrix}}^{y^t} \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 3 & -1 & 0 & 2 & 1 & 3 \\ 1 & -1 & 0 & 0 & -1 & -1 \\ 6 & 2 & 1 & 9 & 13 & 19 \end{array}\right) \begin{array}{l} \\ (-1) \\ (+2) \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 2 & 2 & 4 \\ -1 & 1 & 0 & 0 & 1 & 1 \\ 8 & 0 & 1 & 9 & 11 & 17 \end{array}\right) \begin{array}{l} /:2 \\ \sim \\ \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ -1 & 1 & 0 & 0 & 1 & 1 \\ 8 & 0 & 1 & 9 & 11 & 17 \end{array}\right) \begin{array}{l} (1)\,(-8) \\ \\ \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 3 & 1 \end{array}\right) \quad X^T$$

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Задачи за упражнение

$$\begin{pmatrix} 21 & 1 & 1 \\ 6 & 5 & 4 \\ 13 & 10 & 8 \end{pmatrix} X = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

$$X \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 3 \\ 4 & 3 & 2 \\ 1 & -2 & 5 \end{pmatrix}$$

$AXB = C$, където

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 1 & 5 & 6 \\ -1 & -2 & 7 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 7 & -7 \\ 1 & 6 & 5 \\ 1 & 0 & -6 \end{pmatrix}$$

$$C = \begin{pmatrix} 45 & -167 & 37 \\ 62 & -85 & -216 \\ -10 & 369 & -619 \end{pmatrix}$$

$XA = A + (-4)X$, където

$$A = \begin{pmatrix} -5 & 7 & -5 \\ -1 & 2 & -4 \\ -1 & 1 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ & 1 & & & & \vdots \\ & & \ddots & & & 1 \\ & 0 & & \ddots & & \\ & & & & & 1 \end{pmatrix} X = \begin{pmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & 0 & \\ & 1 & 2 & \ddots & & \\ & & & \ddots & & 1 \\ & 0 & & & 1 & 2 \end{pmatrix}$$

$$X \begin{pmatrix} -35 & 11 & -2 \\ -16 & 5 & -1 \\ 4 & -3 & -5 \end{pmatrix} = \begin{pmatrix} -6 & 1 & -3 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix}$$