Детерминанти 2. Матрични уравнения

Припомняме си детерминантата на Вандермонд:

$$W(x_1, \dots, x_n) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & & x_n \\ x_1^2 & x_2^2 & \cdot & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i)$$

Задачи. Да се пресметне детерминантата

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 3 & & n+1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2^n & 3^n & & (n+1)^n \end{vmatrix} = \prod_{1 \le i < j \le n} (j - i) = n!\,(n-1)! \dots 2!\,1!$$

$$= 1^n 2^{n-1} 3^{n-2} \dots (n-1)^2 n$$

$$\begin{vmatrix} \cos(\alpha_1 - \beta_1) & \cos(\alpha_1 - \beta_2) & \dots & \cos(\alpha_1 - \beta_n) \\ \cos(\alpha_2 - \beta_1) & \cos(\alpha_2 - \beta_2) & \dots & \cos(\alpha_2 - \beta_n) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(\alpha_n - \beta_1) & \cos(\alpha_n - \beta_2) & \dots & \cos(\alpha_n - \beta_n) \end{vmatrix} =$$

$$\cos(\alpha_i - \beta_j) = \cos\alpha_i \cos\beta_j + \sin\alpha_i \sin\beta_j$$

$$
\begin{vmatrix}
\cos\alpha_1\cos\beta_1+\sin\alpha_1\sin\beta_1 & \cos\alpha_1\cos\beta_2+\sin\alpha_1\sin\beta_2 & \cdots & \cos\alpha_1\cos\beta_n+\sin\alpha_1\sin\beta_n \\
\cos\alpha_2\cos\beta_1+\sin\alpha_2\sin\beta_1 & \cos\alpha_2\cos\beta_2+\sin\alpha_2\sin\beta_2 & \cdots & \cos\alpha_2\cos\beta_n+\sin\alpha_2\sin\beta_n \\
\cos\alpha_3\cos\beta_1+\sin\alpha_3\sin\beta_1 & \cos\alpha_3\cos\beta_2+\sin\alpha_3\sin\beta_2 & \cdots & \cos\alpha_3\cos\beta_n+\sin\alpha_3\sin\beta_n \\
\vdots & \vdots & \ddots & \vdots \\
\cos\alpha_n\cos\beta_1+\sin\alpha_n\sin\beta_1 & \cos\alpha_n\cos\beta_2+\sin\alpha_n\sin\beta_2 & \cdots & \cos\alpha_n\cos\beta_n+\sin\alpha_n\sin\beta_n
\end{vmatrix}
$$

$$
= \begin{vmatrix}
\cos\alpha_1 & \sin\alpha_1 & 0 & \cdots & 0 \\
\cos\alpha_2 & \sin\alpha_2 & 0 & \cdots & 0 \\
\vdots & \vdots & & \ddots & \vdots \\
\cos\alpha_n & \sin\alpha_n & 0 & \cdots & 0
\end{vmatrix}
\cdot
\begin{vmatrix}
\cos\beta_1 & \cos\beta_2 & \cdots & \cos\beta_n \\
\sin\beta_1 & \sin\beta_2 & \cdots & \sin\beta_n \\
0 & 0 & & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & & 0
\end{vmatrix}
$$

Сега, при $n>2$ $\Delta_n = 0$

при $n=2$ $\Delta_2 = \sin(\alpha_1-\alpha_2)\sin(\beta_1-\beta_2)$

при $n=1$ $\Delta_1 = \cos(\alpha_1-\beta_1)$

! формулата не работи за $n=1$!

$$
+\begin{vmatrix}
\frac{1-a_1^n b_1^n}{1-a_1b_1} & \frac{1-a_1^n b_2^n}{1-a_1b_2} & \cdots & \frac{1-a_1^n b_n^n}{1-a_1b_n} \\
\frac{1-a_2^n b_1^n}{1-a_2b_1} & \frac{1-a_2^n b_2^n}{1-a_2b_2} & \cdots & \frac{1-a_2^n b_n^n}{1-a_2b_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1-a_n^n b_1^n}{1-a_nb_1} & \frac{1-a_n^n b_2^n}{1-a_nb_2} & \cdots & \frac{1-a_n^n b_n^n}{1-a_nb_n}
\end{vmatrix} =
$$

$$1-x^n = (1-x)(1+x+x^2+\dots+x^{n-1})$$

$$\Rightarrow \frac{1-a_i^n b_j^n}{1-a_i b_j} = 1 + a_i b_j + a_i^2 b_j^2 + \dots + a_i^{n-1} b_j^{n-1}$$

$$= \begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ b_1 & b_2 & \dots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ b_1^{n-1} & b_2^{n-1} & \dots & b_n^{n-1} \end{vmatrix}$$

$$= \prod_{1\le i<j\le n} (a_j - a_i) \prod_{1\le i<j\le n} (b_j - b_i)$$

Рекурентни детерминанти

Такива, в които можем да изразим $n \times n$ детерминантата чрез предходните

Решаване с рекурентни уравнения (вж. предх. файл)

Зад. $\leftarrow$ развиване по първи ред

$$\begin{vmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \ddots & \vdots \\ 0 & 1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 & 2 \end{vmatrix}_{n\times n} = 2 \begin{vmatrix} 2 & 1 & & 0 \\ 1 & 2 & 1 & \\ & 1 & \ddots & 1 \\ 0 & & 1 & 2 \end{vmatrix}_{n-1} - 1 \begin{vmatrix} 1 & 1 & \dots & 0 \\ 0 & 2 & 1 & \\ \vdots & 1 & \ddots & 1 \\ 0 & & 1 & 2 \end{vmatrix}_{n-1} =$$

$\leftarrow$ по първи стълб

$$= 2\Delta_{n-1} - \begin{vmatrix} 2 & 1 & & 0 \\ 1 & 2 & & \\ & & \ddots & 1 \\ 0 & & 1 & 2 \end{vmatrix}_{n-2} = 2\Delta_{n-1} - \Delta_{n-2}$$

Получихме рек у-е:

$$a_n = 2a_{n-1} - a_{n-2}$$

$$x^n - 2x^{n-1} + x^{n-2} = 0 \quad /: x^{n-2}$$

$$x^2 - 2x + 1 = 0 = (x-1)^2$$

$$x_{1,2} = 1$$

$$\Delta_n = A \cdot 1^n + n \cdot B \cdot 1^n = A + nB$$

$$\left.\begin{array}{l} \Delta_1 = |2| = 2 = A + B \\ \Delta_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 = A + 2B \end{array}\right\} \Rightarrow A = B = 1$$

$$\Rightarrow \Delta_n = n + 1$$

$$\rightarrow \begin{vmatrix} 7 & 2 & & & 0 \\ 5 & 7 & 2 & & \\ & 5 & 7 & \ddots & \\ & & \ddots & \ddots & 2 \\ 0 & & & 5 & 7 \end{vmatrix}_{n\times n} = 7 \begin{vmatrix} 7 & 2 & & \\ 5 & 7 & 2 & \\ & 5 & \ddots & \ddots \\ & & \ddots & \ddots & 2 \\ & & & 5 & 7 \end{vmatrix}_{n-1} - 2 \begin{vmatrix} 5 & 2 & & & 0 \\ 0 & 7 & 2 & & \\ & 5 & 7 & 2 & \\ 0 & & \ddots & \ddots & 2 \\ & & & 5 & 7 \end{vmatrix}_{n-1}$$

$$= 7\Delta_{n-1} - 2\cdot 5 \begin{vmatrix} 7 & 2 & & \\ 5 & 7 & \ddots & \\ & \ddots & \ddots & 2 \\ & & 5 & 7 \end{vmatrix} = 7\Delta_{n-1} - 10\Delta_{n-2}$$

$$a_n = 7a_{n-1} - 10a_{n-2}$$

$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$

$$x^2 - 7x + 10 = 0 = (x-2)(x-5) \qquad x_{1,2} = 2, 5$$

$$\Delta_n = A\cdot 2^n + B\cdot 5^n$$

$$\left.\begin{aligned} \Delta_1 &= 2A + 5B = 7 \\ \Delta_2 &= \begin{vmatrix} 7 & 2 \\ 5 & 7 \end{vmatrix} = 39 = 4A + 25B \end{aligned}\right\} \Rightarrow A = -\frac{2}{3},\ B = \frac{5}{3}$$

$$\left(\begin{array}{cc|c} 2 & 5 & 7 \\ 4 & 25 & 39 \end{array}\right) (-2) \sim \left(\begin{array}{cc|c} 2 & 5 & 7 \\ 0 & 15 & 25 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 5 & 7 \\ 0 & 1 & \frac{5}{3} \end{array}\right) (-5)$$

$$\sim \left(\begin{array}{cc|c} 2 & 0 & -\frac{4}{3} \\ 0 & 1 & \frac{5}{3} \end{array}\right)$$

$$\Rightarrow \Delta_n = \frac{1}{3}\left(5^{n+1} - 2^{n+1}\right)$$

$$* \begin{vmatrix} \alpha+\beta & \alpha & & & 0 \\ \beta & \alpha+\beta & \alpha & & \\ & \beta & \alpha+\beta & \ddots & \\ & & \ddots & \ddots & \alpha \\ 0 & & & \beta & \alpha+\beta \end{vmatrix} = (\alpha+\beta)\Delta_{n-1} - \alpha \begin{vmatrix} \beta & \alpha & & 0 \\ & \alpha+\beta & \alpha & \\ & \beta & \ddots & \ddots \\ & & \ddots & \alpha \\ 0 & & \beta & \alpha+\beta \end{vmatrix}$$

$$= (\alpha+\beta)\Delta_{n-1} - \alpha\beta\,\Delta_{n-2}$$

$$a_n = (\alpha+\beta)a_{n-1} - \alpha\beta\, a_{n-2}$$

$$x^n - (\alpha+\beta)x^{n-1} + \alpha\beta\, x^{n-2} = 0$$

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

Формули на Виет: $x_1 + x_2 = \alpha + \beta$, $x_1 x_2 = \alpha\beta$ $\Rightarrow$ $x_1 = \alpha$, $x_2 = \beta$

1сл. $\alpha \neq \beta$

$$\Delta_n = A\cdot\alpha^n + B\beta^n$$

$$\Delta_1 = \alpha+\beta = A\alpha + B\cdot\beta$$

$$\Delta_2 = A\alpha^2 + B\beta^2 = \alpha^2 + \alpha\beta + \beta^2$$

$$\begin{array}{c} (-\alpha) \end{array}\left(\begin{array}{cc|c} \alpha & \beta & \alpha+\beta \\ \alpha^2 & \beta^2 & \alpha^2+\alpha\beta+\beta^2 \end{array}\right)$$

$$\left(\begin{array}{cc|c} \alpha & \beta & \alpha+\beta \\ 0 & \beta(\beta-\alpha) & \beta^2 \end{array}\right) \Rightarrow B = \frac{\beta}{\beta-\alpha},\ A = \frac{-\alpha}{\beta-\alpha}$$

$$\Rightarrow \Delta_n = \frac{1}{\alpha-\beta}\left(\alpha^{n+1} - \beta^{n+1}\right)$$

2сл. $\alpha = \beta$

$$\Delta_n = A\cdot\alpha^n + n\cdot B\cdot\alpha^n,\quad \Delta_1 = 2\alpha = \alpha A + \alpha B \Rightarrow A + B = 2$$

$$\Delta_2 = 3\alpha^2 = 2\alpha^2 A + \alpha^2 B \Rightarrow 2A + B = 3 \Rightarrow A = B = 1$$

$$\Delta_n = \alpha^n + n\cdot\alpha^n = (n+1)\alpha^n$$

Задачи за упражнение:

Зад. Да се пресметне детерминантата

* $$\begin{vmatrix} x_1y_1 & x_1y_2 & x_1y_3 & \dots & x_1y_n \\ x_1y_2 & x_2y_2 & x_2y_3 & \dots & x_2y_n \\ x_1y_3 & x_2y_3 & x_3y_3 & \dots & x_3y_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1y_n & x_2y_n & \dots & \dots & x_ny_n \end{vmatrix}$$

(Hint: изнесете x_1 от първия ред, и резултатът извадете от i-тия ред, умножен по x_i, после развийте по последния стълб)

* $$\begin{vmatrix} 1 & 1 & \dots & \dots & \dots & 1 \\ 1 & 1 & \dots & 1 & x_1 & 1 \\ 1 & \dots & \dots & x_2 & 1 & 1 \\ \vdots & & \cdot^{\cdot^{\cdot}} & 1 & 1 & 1 \\ 1 & \cdot^{\cdot^{\cdot}} & & & & \vdots \\ x_n & 1 & \dots & \dots & \dots & 1 \end{vmatrix}$$

(Hint: $\begin{vmatrix} 0 & & a_n \\ & \cdot^{\cdot^{\cdot}} & \\ a_1 & & 0 \end{vmatrix} = (-1)^n a_1 \dots a_n$ различно от $\begin{vmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{vmatrix}$)

* $$\begin{vmatrix} 1 & 1 & \dots & \dots & 1 \\ 1 & 0 & 1 & & \vdots \\ \vdots & 1 & 0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 1 & \dots & \dots & 1 & 0 \end{vmatrix}$$

* $$\begin{vmatrix} 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & x & \dots & x \\ \vdots & x & 0 & x & x \\ \vdots & \vdots & & \ddots & x \\ 1 & x & \dots & x & 0 \end{vmatrix}$$

* $$\begin{vmatrix} a_0 & a_1 & \dots & a_{n-1} & a_n \\ -1 & x & 0 & \dots & 0 \\ 0 & -1 & x & \ddots & \vdots \\ \vdots & \ddots & \ddots & x & 0 \\ 0 & \dots & 0 & -1 & x \end{vmatrix}$$

(Hint: развивайте по последния стълб)

$$*\ \begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & \cdots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \cdots & 1+x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ 1+x_ny_1 & 1+x_ny_2 & \cdots & 1+x_ny_n \end{vmatrix}$$

(Hint $\Delta_n = 0$ за $n > 2$, като в последната задача от предното упражнение)

$$*\ \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \sin\alpha_1 & \sin\alpha_2 & \cdots & \sin\alpha_n \\ \sin^2\alpha_1 & \sin^2\alpha_2 & \cdots & \sin^2\alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \sin^{n-1}\alpha_1 & \sin^{n-1}\alpha_2 & \cdots & \sin^{n-1}\alpha_n \end{vmatrix}$$

$$*\ \begin{vmatrix} 3 & 2 & & & 0 \\ 1 & 3 & 2 & & \\ & 1 & 3 & 2 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & & 2 \\ & & & 1 & 3 \end{vmatrix} \qquad *\ \begin{vmatrix} 3 & 3 & 1 & & & 0 \\ 1 & 3 & 3 & 1 & & \\ & 1 & 3 & 3 & 1 & \\ & & \ddots & \ddots & \ddots & 1 \\ 0 & & & 1 & 3 & 3 \\ & & & & 1 & 3 \end{vmatrix}$$

$$*\ \begin{vmatrix} \alpha+\beta & \alpha\beta & & & 0 \\ 1 & \alpha+\beta & \alpha\beta & & \\ & 1 & \alpha+\beta & \alpha\beta & \\ 0 & & \ddots & \ddots & \alpha\beta \\ & & & 1 & \alpha+\beta \end{vmatrix}$$

Обратни матрици. Матрични уравнения

Една кв. матрица $B \in M_n(\mathbb{R})$ наричаме обратна на $A \in M_n(\mathbb{R})$, ако

$AB = BA = E$, B бележим с A^{-1}

Не всяка матрица е обратима:

$\Theta_{n \times n} B = \Theta_{n \times n} \neq E_n$

нулевата матр.

Намиране на обратна матрица:

$$(A \mid E) \sim \ldots \sim (E \mid A^{-1})$$

елементарни преобр. по редове

Зад. Да се намери обр. матр:

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right) \overset{(-3)}{\sim} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right) \overset{(2)}{\sim}$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & -1 & -3 & 1 \end{array}\right) \underset{\cdot(-1)}{\sim} \left(\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} \\ /:5 \\ /:6 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/6 \end{array}\right) \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/6 \end{pmatrix}$$

$$\begin{array}{c} \\ (\cdot a) \\ \\ (\cdot a) \\ (\cdot a) \end{array} \left(\begin{array}{ccccc|ccc} 1 & a & a^2 & \dots & a^n & 1 & & \\ & 1 & a & \dots & a^{n-1} & & 1 & 0 \\ & & \ddots & & & & \ddots & \\ & 0 & & 1 & a & 0 & & \ddots \\ & & & & 1 & & & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|ccccc} 1 & & & & 1 & -a & & 0 & \\ & 1 & & & & 1 & -a & & \\ & & \ddots & & & & \ddots & \ddots & \\ & & & \ddots & 0 & & & 1 & -a \\ & & & 1 & & & & & 1 \end{array}\right)$$

Матрични уравнения

Уравнение от вида $AX = B$

Ако $AX = E$, то X - обр. $\to X = A^{-1}$

Тогава

$(A \mid E) \sim \dots \sim (E \mid A^{-1})$

при $AX = B$:

$$(A \mid B) \sim \ldots \sim (E \mid X)$$

елементарни преобр. по редове

Ако $A \in M_n(\mathbb{R})$ - обратима, то $X = A^{-1}B$

Зад. $\underset{2\times 2}{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}} \underset{2\times 3}{X} = \underset{2\times 3}{\begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}}$

$$\begin{array}{c} \\ (-2) \end{array}\left(\begin{array}{cc|ccc} 2 & 1 & 3 & 0 & 3 \\ 1 & 2 & 0 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{cc|ccc} 0 & -3 & 3 & -6 & 3 \\ 1 & 2 & 0 & 3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|ccc} 0 & 1 & -1 & 2 & -1 \\ 1 & 2 & 0 & 3 & 0 \end{array}\right)(-2) \sim \left(\begin{array}{cc|ccc} 0 & 1 & -1 & 2 & -1 \\ 1 & 0 & 2 & -1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|ccc} 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & -1 & 2 & -1 \end{array}\right)$$

$*$ $\underset{3\times 3}{\begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \\ 1 & 3 & -5 \end{pmatrix}} \underset{3\times 2}{Y} = \underset{3\times 2}{\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 0 & 3 \end{pmatrix}}$

$$\left(\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & -1 \\ 3 & 1 & -1 & 2 & 1 \\ 1 & 3 & -5 & 0 & 3 \end{array}\right)(-3)\,(-1) \sim \left(\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & -1 \\ 0 & 4 & -7 & -1 & 4 \\ 0 & 4 & -7 & -1 & 4 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & -1 \\ 0 & 4 & -7 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \begin{array}{c} \\ (+1) \\ \end{array}\left(\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & -1 \\ 0 & 4 & -7 & -1 & 4 \\ 0 & 0 & 1 & p & q \end{array}\right)$$

! Когато имаме изцяло нулев ред, добавяме параметри!

! Когато имаме нули отляво, но ненулев елемент отдясно - няма решение!

$$\sim \left(\begin{array}{ccc|cc} 1 & 0 & 1/4 & 3/4 & 0 \\ 0 & 1 & -7/4 & -1/4 & 1 \\ 0 & 0 & 1 & p & q \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 3/4 - \frac{1}{4}p & -\frac{1}{4}q \\ 0 & 1 & 0 & -1/4 + \frac{7}{4}p & 1 + \frac{7}{4}q \\ 0 & 0 & 1 & p & q \end{array}\right)$$

$$\underbrace{\begin{pmatrix} 3/4 - \frac{1}{4}p & -\frac{1}{4}q \\ -1/4 + \frac{7}{4}p & 1 + \frac{7}{4}q \\ p & q \end{pmatrix}}_{Y}$$

(p, q - произв.)

Транспонирана на произв.

$$(XA)^t = B^t \Rightarrow A^t X^t = B^t$$

Зад.

$$X \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 3 \\ 4 & 3 & 2 \\ 1 & -2 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} X^t = \begin{pmatrix} 1 & 4 & 1 \\ -1 & 3 & -2 \\ 3 & 2 & 5 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 4 & 1 \\ 1 & 1 & 0 & -1 & 3 & -2 \\ -1 & 0 & 1 & 3 & 2 & 5 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & 2 & 0 & 4 & 6 & 6 \\ 1 & 1 & 0 & -1 & 3 & -2 \\ -1 & 0 & 1 & 3 & 2 & 5 \end{array}\right) \begin{array}{l} :2 \\ \\ \\ \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 2 & 3 & 3 \\ 1 & 1 & 0 & -1 & 3 & -2 \\ -1 & 0 & 1 & 3 & 2 & 5 \end{array}\right) \begin{array}{l} (-1) \\ \\ \\ \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 2 & 3 & 3 \\ 1 & 0 & 0 & -3 & 0 & -5 \\ -1 & 0 & 1 & 3 & 2 & 5 \end{array}\right) \begin{array}{l} \\ (1) \\ \\ \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 0 & -5 \\ 0 & 1 & 0 & 2 & 3 & 3 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{array}\right)$$

$$AXB = C \ ?$$

1) $XB := Y \rightarrow AY = C$

2) с нам $Y \rightarrow XB = Y$

3) $(XB)^t = Y^t \rightarrow B^t X^t = Y^t$

4) нам X^t

$$\underbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4 & -5 & 2 \\ 5 & -7 & 3 \end{pmatrix}}_{A} \underbrace{X \underbrace{\begin{pmatrix} 9 & 7 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}}_{B}}_{Y := XB} = \underbrace{\begin{pmatrix} 2 & 0 & -2 \\ 18 & 12 & 9 \\ 23 & 15 & 11 \end{pmatrix}}_{C}$$

$AY = C$, търсим Y

$$\left(\begin{array}{ccc|ccc} 2 & -3 & 1 & 2 & 0 & -2 \\ 4 & -5 & 2 & 18 & 12 & 9 \\ 5 & -7 & 3 & 23 & 15 & 11 \end{array}\right) \begin{array}{l} (-2)\ (-3) \\ \sim \\ \\ \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & -3 & 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 14 & 12 & 13 \\ -1 & 2 & 0 & 17 & 15 & 17 \end{array}\right) \begin{array}{l} \\ (3) \quad (-2) \\ \\ \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 44 & 36 & 37 \\ 0 & 1 & 0 & 14 & 12 & 13 \\ -1 & 0 & 0 & -11 & -9 & -9 \end{array}\right) \begin{array}{l} \\ \\ (2) \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 22 & 18 & 19 \\ 0 & 1 & 0 & 14 & 12 & 13 \\ 1 & 0 & 0 & 11 & 9 & 9 \end{array}\right)$$

$$\Rightarrow \quad Y = \begin{pmatrix} 11 & 9 & 9 \\ 14 & 12 & 13 \\ 22 & 18 & 19 \end{pmatrix}$$

$$XB = Y \qquad B^t X^t = Y^t$$

$$\begin{array}{c} \\ \\ (-1) \end{array} \underbrace{\begin{pmatrix} 9 & 1 & 1 \\ 7 & 1 & 1 \\ 6 & 2 & 1 \end{pmatrix}}_{B^t} \underbrace{\begin{pmatrix} 11 & 14 & 12 \\ 9 & 12 & 18 \\ 9 & 13 & 19 \end{pmatrix}}_{Y^t} \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 3 & -1 & 0 & 2 & 1 & 3 \\ 1 & -1 & 0 & 0 & -1 & -1 \\ 6 & 2 & 1 & 9 & 13 & 19 \end{array}\right) \begin{array}{l} \\ (-1) \\ (+2) \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 2 & 2 & 4 \\ -1 & 1 & 0 & 0 & 1 & 1 \\ 8 & 0 & 1 & 9 & 11 & 17 \end{array}\right) \begin{array}{l} /:2 \\ \sim \\ \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ -1 & 1 & 0 & 0 & 1 & 1 \\ 8 & 0 & 1 & 9 & 11 & 17 \end{array}\right) \begin{array}{l} (1)\,(-8) \\ \\ \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 3 & 1 \end{array}\right) \quad X^T$$

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Задачи за упражнение

Зад. Да се намерят обратните матрици на матриците:

* $\begin{pmatrix} 1 & 3 & -7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

* $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix}$

* $\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}$

* $\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$

Симетрична матрица ($A = A^t$). Симетрична ли е обратната ѝ?

* $\begin{pmatrix} 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 0 \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}$

* $\begin{pmatrix} 1+a & 1 & \dots & \dots & \dots & 1 \\ 1 & 1+a & 1 & \dots & \dots & 1 \\ 1 & 1 & 1+a & 1 & \dots & 1 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 1 \\ 1 & \dots & \dots & \dots & 1 & 1+a \end{pmatrix}$

Зад. Да се решат матричните уравнения:

* $\begin{pmatrix} 2 & -1 & -1 \\ 3 & 4 & -2 \\ 3 & -2 & 4 \end{pmatrix} X = \begin{pmatrix} 4 \\ 11 \\ 11 \end{pmatrix}$

$$* \quad X\begin{pmatrix} 2 & -1 & 3 \\ 3 & 2 & 2 \\ 1 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 4 & 4 \\ 0 & 0 & 0 \\ 18 & 12 & 12 \end{pmatrix}$$

(K2, KH 2022)

$$\begin{cases} \begin{pmatrix} 1 & -2 & -2 \\ -1 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix}(X+Y) = \begin{pmatrix} 5 & 10 & 12 \\ -6 & -9 & -10 \\ -5 & -12 & -15 \end{pmatrix} \\ (X-Y)\begin{pmatrix} 1 & -1 & -1 \\ -2 & 1 & 3 \\ -2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 6 & 5 \\ 10 & -15 & -8 \\ 12 & -10 & -15 \end{pmatrix} \end{cases}$$

(K4, ИНФО 2023)

$A = \begin{pmatrix} 6 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 2 \\ 1 & -2 \end{pmatrix}$

?X

$$AX(D+BX)^{-1} = C$$

$$* \quad \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \\ 1 & 3 & -5 \end{pmatrix} Z = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 0 \end{pmatrix}$$