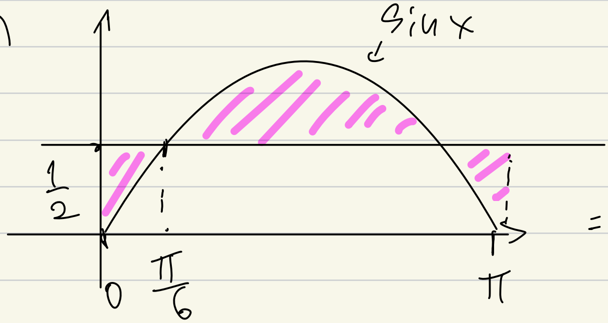
Зад. Да се пресметне лицето на фигурата, ограничена от:

а) $y=\frac{1}{2}$, $y=\sin x$, $x\in[0,\pi]$

б) $x^2=4y$, $y=\frac{8}{x^2+4}$

в) $y=\sqrt{8x-x^2}$, $y=2\sqrt{x}$ г) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

а)



$$S=\int_0^{\pi}\left|\frac{1}{2}-\sin x\right|dx$$

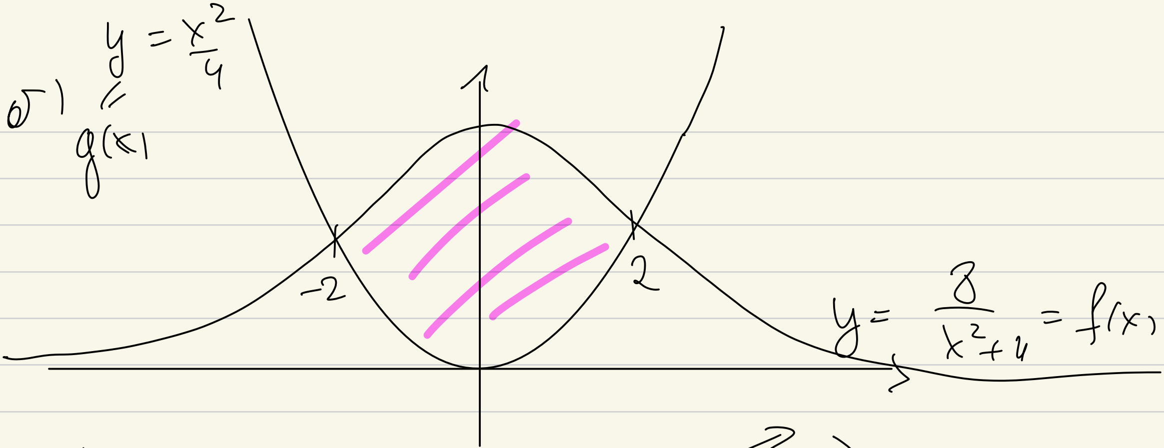
$$=\int_0^{\pi/6}\frac{1}{2}-\sin x\,dx+\int_{\pi/6}^{5\pi/6}\sin x-\frac{1}{2}dx+\int_{5\pi/6}^{\pi}\frac{1}{2}-\sin x$$

$$=\frac{1}{2}x+\cos x\Big|_0^{\pi/6}+\cos x-\frac{1}{2}\Big|_{\pi/6}^{5\pi/6}+\frac{1}{2}x+\cos x\Big|_{5\pi/6}^{\pi}=$$

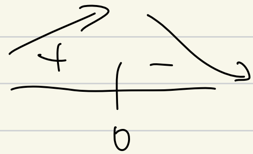
$$=\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1+\frac{\sqrt{3}}{2}-\frac{1}{2}+\frac{\sqrt{3}}{2}+\frac{1}{2}+\frac{\pi}{2}+0-\frac{5\pi}{12}+\frac{\sqrt{3}}{2}$$

$$=2\sqrt{3}+1+{}^{3)}\frac{\pi}{2}-{}^{2)}\frac{2}{3}\pi=2\sqrt{3}+1-\frac{\pi}{6}$$

б)



$f'(x) = \frac{-16x}{(x^2+4)^2} \rightarrow$



$f(x) = g(x) \Leftrightarrow \quad 8.4 = x^2(x^2+4)$

$x^4 + 4x^2 - 32 = 0 \qquad x^2 =: t$

$(t-4)(t+8) = 0$

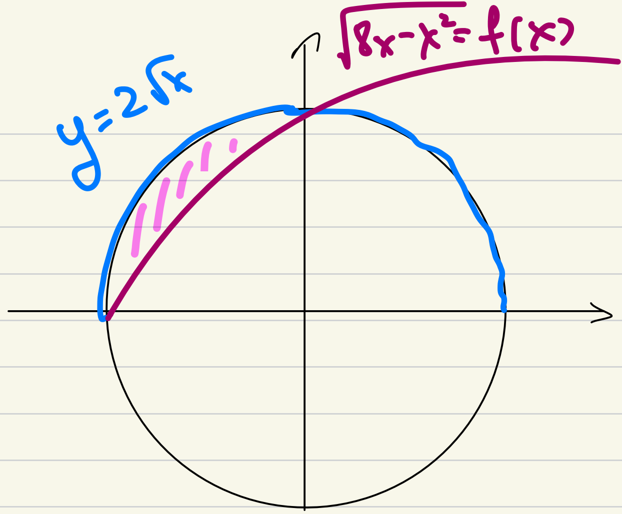
$\Rightarrow t = 4 \Rightarrow x = \pm 2 \ (t \geq 0)$

$$S = \int_{-2}^{2} \underbrace{\frac{8}{x^2+4} - \frac{x^2}{4}}_{\text{четна}} = 2\int_{0}^{2} \frac{8}{x^2+4}\,dx + 2\int_{0}^{2} \frac{x^2}{4}\,dx$$

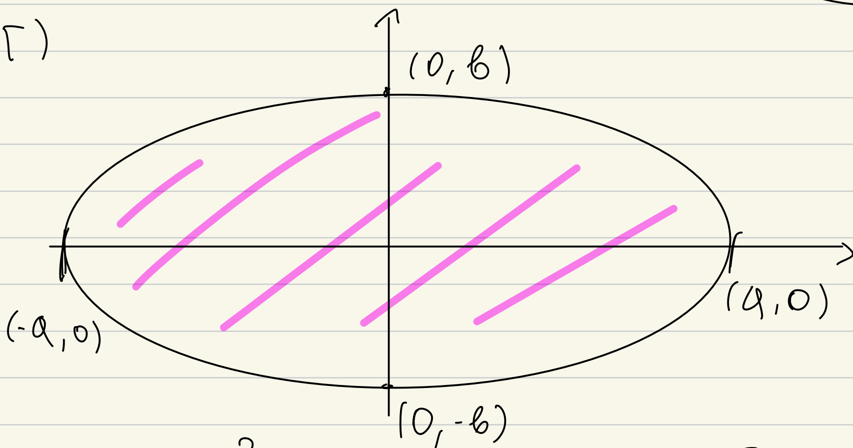
$$= 16\int_{0}^{2} \frac{1}{x^2+4}\,dx + \frac{1}{2}\int_{0}^{2} x^2\,dx = \frac{16}{2}\operatorname{arctg}\left(\frac{x}{2}\right)\Big|_{0}^{2} -$$

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\operatorname{arctg}\left(\frac{x}{a}\right) + C} \qquad \frac{1}{2}\frac{x^3}{3}\Big|_{0}^{2} = \dots = 2\pi - \frac{4}{3}$$

в) За упражнение:



г)



$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \qquad y^2 = b^2 - \frac{b^2x^2}{a^2}$$

$|y| = |b|\sqrt{1-\frac{x^2}{a^2}}$ (ще сметнем само горната половина)

$$S(X) = 2\int_{-a}^{a} |b|\sqrt{1-\frac{x^2}{a^2}} = 4|b|\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx \overset{t=\frac{x}{|a|}}{=}$$

$$4|b|\int_0^1\sqrt{1-t^2}\,d\,a\,t \overset{?\ \text{четна}}{=} 4ab\int_0^1\sqrt{1-t^2}\,dt = 4ab\frac{\pi}{4} = ab\pi$$

Зад. Да се намери дължината на кривата

а) $f(x) = \frac{2x^{3/2}}{3}$ (ф-ла: $\int_a^b \sqrt{1+f'(x)^2}\,dx$)

в $[0,1]$

б) $f(x) = e^{x/2} + e^{-x/2}$ в $[0,2]$

а)

$$l(f)_{x\in[0,1]} = \int_0^1 \sqrt{1+f'(x)^2}\,dx = \int_0^1 \sqrt{1+x}\,dx = \frac{2}{3}(x+1)^{\frac{3}{2}}\Big|_0^1 =$$

$$f'(x) = \frac{3}{2}\cdot\frac{2}{3}\sqrt{x} = \sqrt{x}, \quad f'(x)^2 = x \qquad = \frac{4\sqrt{2}}{3} - \frac{2}{3}$$

б)

$$l(f)_{x\in[0,2]} = \int_0^2 \ldots$$

$$f'(x) = e^{\frac{x}{2}}\left(\frac{x}{2}\right)' + e^{-x/2}\left(-\frac{x}{2}\right)' =$$

$$= \frac{e^{x/2} - e^{-x/2}}{2}$$

$$1+f'(x)^2 = 1 + \frac{1}{4}e^{x} + \frac{1}{4}e^{-x} = \left(\frac{1}{2}e^{x/2} + \frac{1}{2}e^{-x/2}\right)^2$$

$\frac{1}{2}$

$$\int_0^2 \left| \frac{1}{2} e^{x/2} + \frac{1}{2} e^{-x/2} \right| dx = \frac{1}{2}\int_0^2 e^{x/2} dx + \frac{1}{2}\int_0^2 e^{-x/2} dx$$

$$= e^{x/2}\Big|_0^2 - e^{-x/2}\Big|_0^2 = (e-1) - \frac{1}{e} + 1 =$$

$$= e - \frac{1}{e} = \frac{e^2-1}{e}$$

b) $f(x) = \frac{1}{4} x^2 - \frac{1}{2} \ln x \quad , \quad x \in [1,2]$

$$l(f)\Big|_{x\in[1,2]} = \int_1^2 \sqrt{1+f'(x)^2} = \int_1^2 \left| \frac{\overset{\geq 0}{x^2+1}}{\underset{\geq 0}{2x}} \right| dx = \int_1^2 \frac{x^2+1}{2x}\, dx \overset{*}{=}$$

$$f'(x) = \frac{1}{4} 2x - \frac{1}{2} \cdot \frac{1}{x} = \frac{x^2-1}{2x}$$

$$1+f'(x)^2 = 1 + \left(\frac{x^2-1}{2x}\right)^2 = \frac{4x^2+x^4-2x^2+1}{4x^2}$$

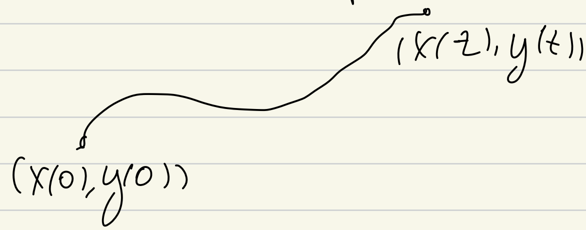
$$= \left(\frac{x-1}{2}\right)^2$$

$$\overset{*}{=} \frac{1}{2}\left(\frac{x^2}{2} + \ln x\right)\Big|_1^2 = \frac{1}{2}\left(2 + \ln 2 - \frac{1}{2}\right) = \frac{3}{4} + \frac{\ln 2}{2}$$

Зад. Да се пресметне дължината на кривата

$$G_1 : \begin{cases} x(t) = t - \sin t \\ y(t) = 1 - \cos t \end{cases}$$

$t \in [0, 2\pi]$



$$l(G_1) = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2} = \int_0^{2\pi} 2\left|\sin\left(\frac{t}{2}\right)\right| dt = 2\int_0^{2\pi} \sin\frac{t}{2}\, dt$$

$\frac{t}{2} \in [0, \pi]$

$$= 4 - \cos\frac{t}{2}\Big|_0^{2\pi} = 8$$

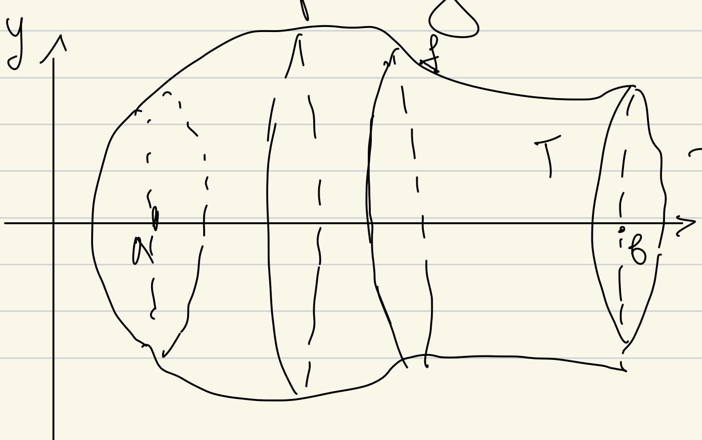
$x'(t) = 1 - \cos t$

$y'(t) = \sin t$

$$x'(t)^2 + y'(t)^2 = 1 - 2\cos t + \cos^2 t + \sin^2 t$$

$$= 2 - 2\cos t = 2(1 - \cos t) = 4\sin^2\left(\frac{t}{2}\right)$$

Обем на ротационно тяло



$$V(T) = \pi \int_a^b f(x)\, dx$$

$\tau : a < x_0 < \dots < x_n = b$

$$V = hR^2\pi$$

$$h_i = (x_i - x_{i-1})$$

$$r = f(\xi_i) \qquad \xi_i \in (x_{i-1}, x_i)$$

$$V(T_i) = (x_i - x_{i+1})\, f^2(\xi_i)\cdot\pi$$

$$V(T) = \sum_{i=1}^{n} V(T_i) = \pi \underbrace{\sum_{i=1}^{n} f^2(\xi_i)(x_i - x_{i-1})}_{\text{Риманова сума}}$$

$$= \pi \int_a^b f^2(x)\,dx$$

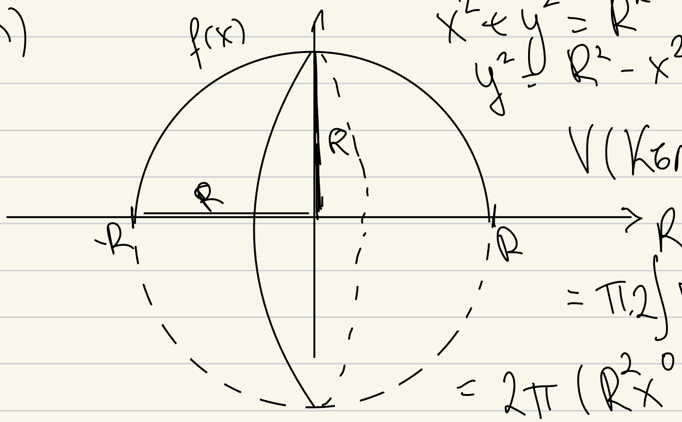
Задачи. Изведете формулите за намиране на обем на:

а) кълбо с радиус R

б) цилиндър с р. R и височина h

в) конус с р. R и височина h

а)



$$x^2 + y^2 = R^2 \Rightarrow y = \pm\sqrt{R^2 - x^2}$$

$$y^2 = R^2 - x^2$$

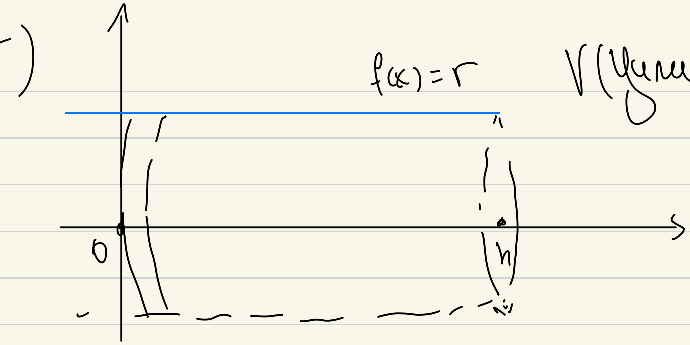
$$V(\text{Кълбо}) = \pi \int_{-R}^{R} \left(\sqrt{R^2 - x^2}\right)^2 dx = \quad \text{четна}$$

$$= \pi \cdot 2 \int^{R} R^2 - x^2\, dx =$$

$$= 2\pi \left(R^2 x^0 - \frac{x^3}{3}\right)\Big|_0^R = 2\pi\left(R^3 - \frac{R^3}{3}\right) =$$

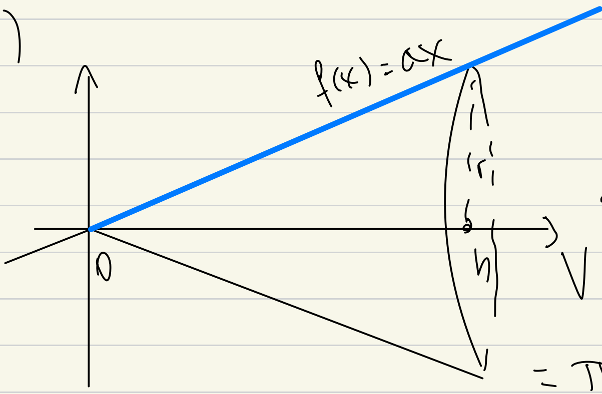
$$= \frac{4}{3}\pi R^3$$

б)



$$V(\text{цилиндър}) = \pi \int_0^h r^2\,dx = \pi r^2 x \Big|_0^h = \pi r^2 h$$

в)



$f(h) = r$?

$\Rightarrow ah = r \Rightarrow a = \frac{r}{h}$

$$f(x) = \frac{r}{h}x$$

$$V(\text{конус}) = \pi \int_0^h \frac{r^2}{h^2}x^2\,dx = \pi \frac{r^2}{h^2}\,\frac{x^3}{3}\Big|_0^h = \frac{\pi r^2 h}{3}$$