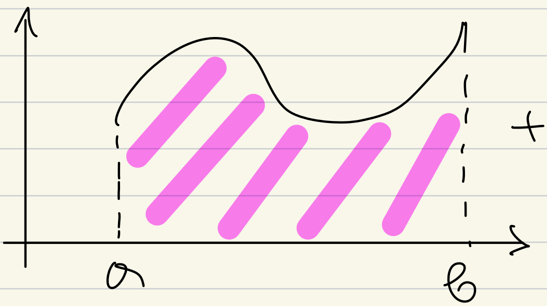
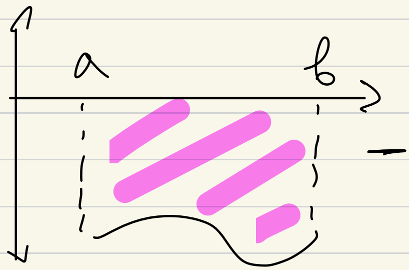
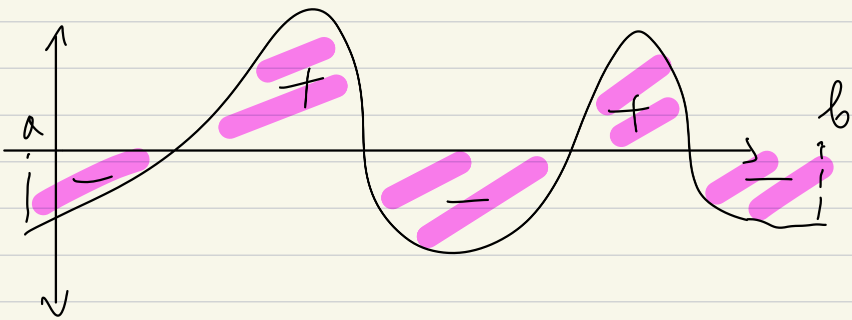
1.2 Определени интеграли

$$\int_a^b f(x)\,dx = \text{число}$$

Ориентирано лице на $f(x)$ в $[a, b]$





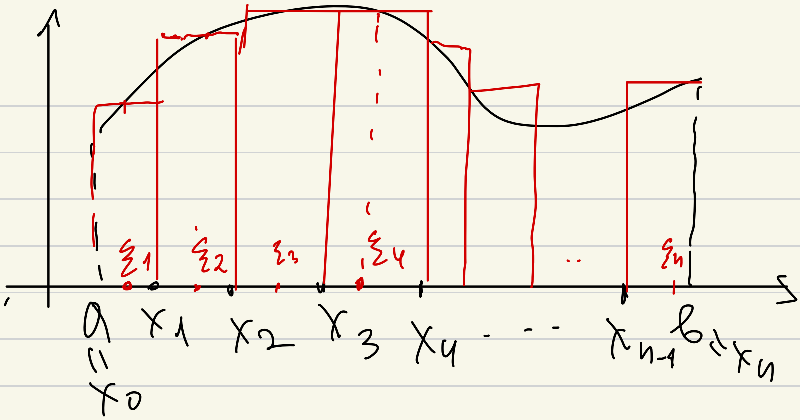


$a < b$

$\tau: a = x_0 < x_1 < \ldots < x_n = b$ - разбиване на интервала

Риманова сума

$$R(f, \tau) = \sum_{k=0}^{n-1} f(\xi_k)(x_{k+1} - x_k), \quad \xi_k \in [x_k, x_{k+1}]$$



$$\operatorname{diam}(\tau) := \max(x_{k+1} - x_k)$$

(най-големият интервал)

$$\lim_{\operatorname{diam}(\tau)\to 0} R(f,t) = \int_a^b f(x)\,dx$$

Зад.

$$\int_0^1 x\,dx = ?$$

$$f(x) = x$$

$$\tau: a = x_0 < x_1 \dots < x_n = b$$

$$x_k = \frac{k}{n}, \quad x_{k+1} - x_k = \frac{k+1}{n} - \frac{k}{n} = \frac{1}{n}$$

$$\xi_k \in [x_k, x_{k+1}] \quad k \in [0, n], \quad \xi_k = x_{k+1}$$

$$R(f,t) = \sum_{k=0}^{n-1} f(\xi_k)(x_{k+1} - x_k) =$$

$$= \sum_{u=0}^{n-1} f(x_u)(x_{u+1} - x_u) = \sum_{u=0}^{n} \frac{u+1}{n} \cdot \frac{1}{n} =$$

$$= \sum_{u=0}^{n-1} u+1 \cdot \frac{1}{n^2} = \frac{1}{n^2} \sum_{u=0}^{n} u+1 =$$

$$= \frac{1}{n^2} \sum_{u=1}^{n} u = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n^2+n}{2n^2}$$

$$R(f, \tau) = \frac{n^2+n}{2n^2}$$

$$\int_0^1 x \, dx = \lim_{n \to \infty} R(f, t) = \lim_{n \to \infty} \frac{n^2+n}{2n^2} = \frac{1}{2}$$

Зад. $\int_0^1 x^2 dx = \frac{1}{3}$

$$R(f, t) = \sum_{u=0}^{n-1} f(\xi_u)(x_{u+1} - x_u) =$$

$$= \sum_{u=0}^{n-1} f(x_{u+1})(x_{u+1} - x_u) = \sum_{u=0}^{n-1} \left(\frac{u+1}{n}\right)^2 \frac{1}{n}$$

$$= \frac{1}{n^3} \sum_{u=0}^{u-1} (u+1)^2 = \frac{1}{n^3} \cdot \sum_{u=1}^{n} u^2 =$$

$$= \left(\frac{\frac{n(n+1)(2n+1)}{6}}{n^3} \right) = \frac{(n^2+n)(2n+1)}{6n^3}$$

$$= \frac{2n^3 + 3n^2 + n}{6n^3}$$

$$\int_0^1 x^2 dx = \lim_{n\to\infty} R(f,t) = \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3} = \frac{1}{3}$$

Формула на Лайбниц-Нютон

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$F'(x) = f(x)$ (F - примитивна на f) (произволна)

$\phi'(x) = f(x)$

$(\phi(x) - F(x))' = 0$ $\phi(x) - F(x) = c$

$\phi(a) = c + F(a)$ $\phi(b) = c + F(b)$

$\Rightarrow \phi(b) - \phi(a) = F(b) - F(a)$

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Зад.

$$\int_0^1 x\,dx = F(1) - F(0)$$

$$F(x) = \int x\,dx = \frac{x^2}{2} + C_1$$

$$\int_0^1 x\,dx = F(1) - F(0) = \frac{1}{2} + C_1 - 0 - C_1 = \frac{1}{2}$$

Записваме така:

$$\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

Зад.

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

Деф) $f(x)$ е четна, ако $f(x) = f(-x)\ \forall x$

Пример: $f(x) = \cos x$, $f(x) = x^2$

Деф) $f(x)$ е нечетна, ако $f(x) = -f(-x)\ \forall x$

Пример: $f(x) = \sin x$, $f(x) = x^3$

Тв) Ако $f(x)$, $g(x)$ - четни, $h(x) = f(x)\,g(x)$ - четна

Д-во: $h(x) = f(x) \cdot g(x) = f(-x)\,g(-x) = h(-x)$

Тв] Ако $f(x)$ – четна, $g(x)$ – нечетна, то
$h(x) = f(x)\,g(x)$ – нечетна

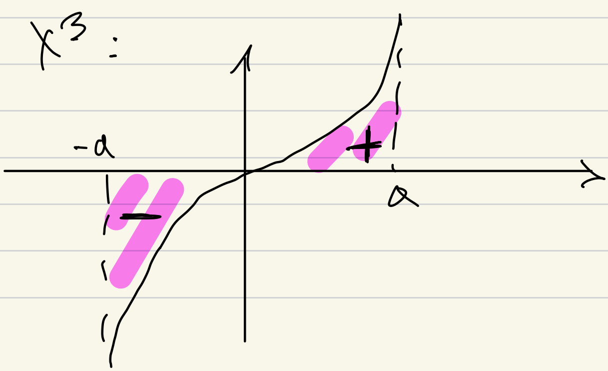
Д-во: $h(x) = f(-x) \cdot -g(-x) = -h(-x)$

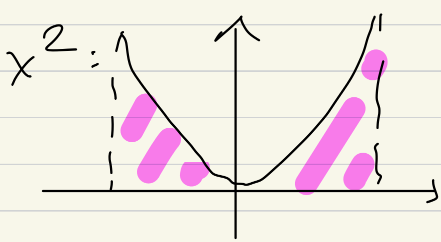
Тв] Ако $f(x)$, $g(x)$ – нечетни, то
$h(x) = f(x)\,g(x)$ – четна

Д-во: $h(x) = f(x)\,g(x) = -f(-x) \cdot -g(-x) =$
$= f(-x)\,g(-x) = h(-x)$

Св-во] $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
$c \in (a, b)$

Тв] $\forall a \in \mathbb{R} \setminus \{0\}$:
$$\int_{-a}^{a} f(x) = \begin{cases} 2\int_0^a f(x)\,dx, & \text{ако } f(x) \text{ – четна} \\ 0, & f(x) \text{ – нечетна} \end{cases}$$





Д-во: $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx =$

$\overset{x:=-t}{=} \int_{a}^{0} f(-t)\, d(-t) + \int_{0}^{a} f(x)\, dx =$

$a \rightarrow$ $\begin{array}{c|c|c|} x & -a & 0 \\ \hline t & a & 0 \end{array}$

$= \int_{a}^{0} f(-t)\,(-t)'\,dt + \int_{0}^{a} f(x)\, dx =$

$= -\int_{a}^{0} f(-t)\, dt + \int_{0}^{a} f(x)\, dx =$

$= \int_{0}^{a} f(-x)\, dx + \int_{0}^{a} f(x)\, dx = \begin{cases} 2\int_{a}^{b} f(x)dx, & f\text{ - четна} \\ 0, & f\text{ - нечетна} \end{cases}$

Зад. а)

$\int_{-1}^{1} x^2 dx = \begin{cases} \text{I н.} \quad 2\int_{0}^{1} x^2 dx = 2\frac{x^3}{3}\Big|_0^1 = 2\left(\frac{1^3}{3} - 0\right) = \frac{2}{3} \\ \text{II н.} \quad \frac{x^3}{3}\Big|_{-1}^{1} = \left(\frac{1^3}{3} - \frac{(-1)^3}{3}\right) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \end{cases}$

четна

симетричен интервал

б) $\int_{1}^{2} \frac{1}{x} dx = \ln|x|\Big|_1^2 = \ln|2| - \ln|1|$

$$= \ln 2 - 0 = \ln 2$$

$$\int_{-2}^{-1} \frac{1}{x}\, dx = \ln|x| \Big|_{x=-2}^{-1} = \ln|-1| - \ln|-2|$$
$$= -\ln 2$$

в) $$\int_0^1 \sqrt{x+1}\, dx = \int_0^1 (x+1)^{1/2}\, d(x+1) =$$

$$= \frac{(x+1)^{3/2}}{3/2} \Big|_0^1 = \frac{2 \cdot 2^{\frac{3}{2}}}{3} - \frac{2 \cdot 1^{3/2}}{3} = \frac{2}{3}(\sqrt{8} - 1)$$

г) $$\int_0^2 |1-x|\, dx = ?$$

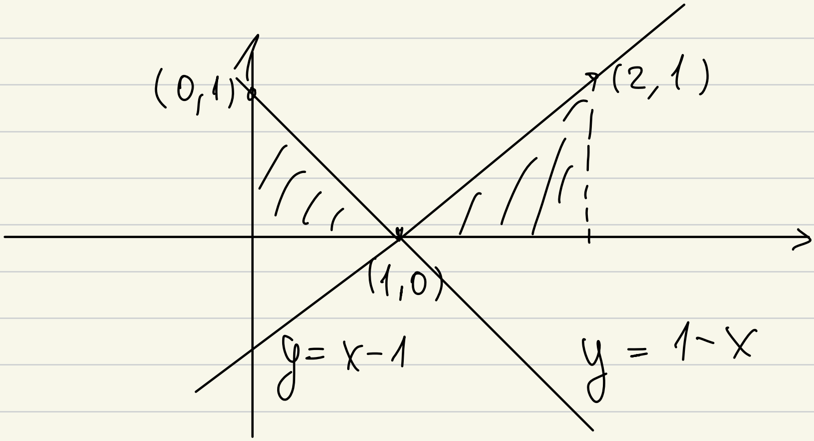
$$f(x) = |1-x| = \begin{cases} 1-x, & 1-x \geq 0 \\ x-1, & 1-x \leq 0 \end{cases} = \begin{cases} 1-x, & x \leq 1 \\ x-1, & x \geq 1 \end{cases}$$

$$\int_0^2 |1-x| = \int_0^1 (1-x)\, dx + \int_1^2 (x-1)\, dx =$$

$$= \left(x - \frac{x^2}{2}\right) \Big|_0^1 + \left(\frac{x^2}{2} - x\right) \Big|_1^2 =$$

$$= \left(\left(1 - \frac{1^2}{2}\right) - \left(0 - \frac{0^2}{2}\right)\right) + \left(\left(\frac{2^2}{2} - 2\right) - \left(\frac{1^2}{2} - 1\right)\right)$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$



Алтернативно, със смяна на променливата:

$$\int_0^2 |1-x|\,dx \overset{t:=1-x}{=} \int_1^{-1} |t|\,d(1-t) =$$

x	0	2
t	1	-1

$$= -\int_1^{-1} |t|\,dt = \int_{-1}^{1} |t|\,dt \underset{\text{четна}}{=} 2\int_0^1 t\,dt =$$

$$= 2\frac{t^2}{2}\Big|_0^1 = 1$$

Т-ма на Лагранж

Ако f - непрекъсната в $[a, b]$, $\exists \xi \in (a, b)$

$$\int_a^b f(x)\,dx = f(\xi).(b-a)$$

$$\int_a^b x^2\,dx = \frac{x^3}{3}\Big|_a^b = \frac{1}{3}(b^3 - a^3) = \underbrace{f(\xi)}_{=\xi^2}(b-a)$$

$$\Rightarrow \xi^2 = \frac{1}{3}(b^2 + ab + a^2)$$

$$\xi = \frac{1}{\sqrt{3}}\sqrt{b^2 + ab + a^2}$$

Определени интеграли

Зад.

$$\int_0^1 x(2-x^2)^2\,dx = -\frac{1}{2}\int_0^1 (2-x^2)^2\,d(2-x^2) =$$

$2-x^2$

$$= -\frac{1}{2}\,\frac{(2-x^2)^3}{3}\Big|_0^1 = -\frac{1}{2}\left(\frac{(2-1^2)^3}{3} - \frac{(2-0^2)^3}{3}\right) =$$

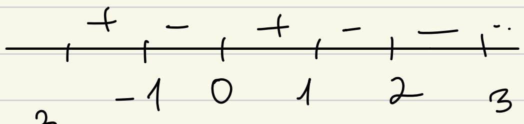
$$= -\frac{1}{2}\left(\frac{1}{3}-\frac{8}{3}\right) = -\frac{7}{6}$$

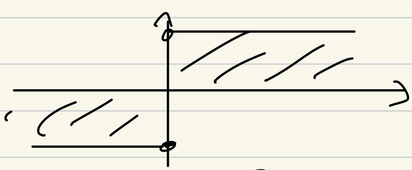
Зад.

$$\int_2^3 \operatorname{sign}(x-x^3)\,dx = ?$$

$x - x^3 = x(1-x^2) = x(1-x)(1+x)$

$$\operatorname{sign}(x) = \begin{cases} 1, & x>0 \\ -1, & x<0 \\ 0, & x=0 \end{cases}$$





$$\int_2^3 \operatorname{sign}(x-x^3) = \int_2^3 -1\,dx = -x\Big|_2^3 = -1$$

Зад.

$$\int_1^2 \frac{1}{x+x^3}\,dx = ?$$

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} = \frac{A(1+x^2)+(Bx+C)x}{x(1+x^2)}$$

$$1 = A(1+x^2) + x(Bx+C)$$

$$= \underbrace{Ax^2+Bx^2}_{A+B=0} + \underbrace{Cx}_{C=0} + \underbrace{A+C}_{A+C=1}$$

$$\Rightarrow A=1,\ B=-1$$

$$\int_1^2 \frac{1}{x+x^3}\,dx = \int_1^2 \frac{1}{x} - \frac{x}{1+x^2}\,dx =$$

$$= \int_1^2 \frac{1}{x} - \int_1^2 \frac{x}{1+x^2}\,dx =$$

$$= \ln|x|\Big|_1^2 - \int_1^2 \frac{1}{1+x^2}\,d\left(\frac{x^2}{2}\right) \quad -$$

$$= \ln 2 - \frac{1}{2}\int_1^2 \frac{1}{1+x^2}\,d(x^2+1) = \ln 2 - \frac{1}{2}\ln|1+x^2|\Big|_1^2$$

$$= \ln 2 - \frac{1}{2}(\ln 5 - \ln 2) =$$

$$= \ln 2 - \frac{1}{2}\ln\frac{5}{2} = \ln 2 - \ln\sqrt{\frac{5}{2}} =$$

$$= \ln\left(\frac{2\sqrt{2}}{\sqrt{5}}\right) = \ln\left(\frac{2\sqrt{10}}{5}\right)$$

$$\int_0^1 x\sqrt{1+x}\,dx = , \text{ con} \quad 1+x = t^2$$

$$x = t^2 - 1$$

$$t = \sqrt{1+x}$$

$$= \int_1^{\sqrt{2}} (t^2-1)\, t \; d(t^2-1) =$$

$$= \int_1^{\sqrt{2}} (t^3 - t) \cdot 2t \; dt = 2\int_1^{\sqrt{2}} t^4 - t^2 \, dt =$$

$$= 2\left(\frac{t^5}{5} - \frac{t^3}{3} \Big|_1^{\sqrt{2}}\right) =$$

$$= 2\left(\left(\frac{4\sqrt{2}}{5}^{(3} - \frac{2\sqrt{2}}{3}^{(5}\right) - \left(\frac{1}{5}^{(3} - \frac{1}{3}^{(5}\right)\right) =$$

$$= 2\left(\frac{2}{15}\sqrt{2} - \frac{2}{15}\right) = \frac{4}{15}(\sqrt{2} - 1)$$

Зад.

$$\underbrace{\int_{-\pi}^{\pi} \underset{\text{четна}}{e^{x^2}} \underset{\text{нечетна}}{\sin x}}_{\text{нечетна}}\, dx = 0$$

$$\int_{-\pi}^{\pi} \sqrt[3]{\sin x} = 0$$

$\hookrightarrow$ неч: $\sqrt[3]{\sin(-x)} = \sqrt[3]{-\sin x} = -\sqrt[3]{\sin x}$

$$\int_{-\pi}^{\pi} \underbrace{\frac{e^x + e^{-x}}{\cot g x}}_{\text{неч.}} dx = 0$$

$$f(x) := \frac{e^x + e^{-x}}{\cot g(x)}$$

$$f(-x) = \frac{e^{-x} + e^x}{\frac{\cos(-x)}{\sin(-x)}} = \frac{e^x + e^{-x}}{-\frac{\cos x}{\sin x}} = \frac{e^x + e^{-x}}{-\cot g x} = -f(x)$$

$$\int_{-1}^{1} \sqrt{1 - x^2}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \sin^2 t}\, d\sin t =$$

$x := \sin t$

$t = \arcsin t$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \sqrt{\cos^2 t}\, \cos t \, dt =$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} |\cos t| \cos t \, dt = \int_{-\frac{\pi}{2}}^{\pi/2} \cos^2 t \, dt =$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\cos(2t)+1}{2} \, dt = \frac{1}{2}\left(\frac{\sin(2t)}{2} + t \right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$\cos(2t) = \cos^2 t - \sin^2 t = \cos^2 t - (1 - \cos^2 t)$$

$$= 2\cos^2 t - 1$$

$$= \frac{1}{2}\left(\frac{\sin(\pi)}{2} + \frac{\pi}{2} - \frac{\sin(-\pi)}{2} + \frac{\pi}{2} \right)$$

$$= \frac{\pi}{2}$$