Задачи $\int_0^1 x(2-x^2)^2\,dx$; $\int_1^3 sign(x-x^3)\,dx$; $\int_1^2 \frac{1}{x+x^3}\,dx$

$\int_0^1 x\sqrt{1+x}\,dx$; $\int_{-\pi}^{\pi}\sqrt[3]{\sin x} = 0$; $\int_{-1}^{1}\sqrt{1-x^2}\,dx$

Реш:

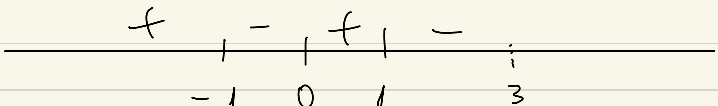
$$\int_0^1 x(2-x^2)^2\,dx = \int_0^1 (2-x^2)^2\,d\frac{x^2}{2} = -\frac{1}{2}\int_0^1 (2-x^2)^2\,d(2-x^2)$$

$$= -\frac{1}{2}\,\frac{(2-x)^3}{3}\Big|_0^1 = \frac{7}{6}$$

$$\int_1^3 sign(x-x^3)\,dx$$

$$sign(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases}$$

$$x - x^3 = x(1-x^2) = x(1-x)(1+x)$$



$$\Rightarrow \quad = \int_1^3 -1\,dx = -x\Big|_1^3 = -2$$

$$\int_1^2 \frac{1}{x+x^3}\,dx = ?$$

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} = \frac{A(1+x^2)+Bx^2+Cx}{x(1+x^2)}$$

$$\Rightarrow \left|\begin{array}{l} A = 1 \\ B = -1 \\ C = 0 \end{array}\right. \qquad 1 = \underbrace{(A+B)}_{0}x^2 + \underbrace{C}_{0}x + \underbrace{A}_{1}$$

$$\int_1^2 \frac{1}{x+x^3}\,dx = \int_1^2 \frac{1}{x} + \int_1^2 \frac{-x}{1+x^2}\,dx =$$

$$= \ln x \Big|_1^2 - \int_1^2 \frac{1}{1+x^2}\,d\left(\frac{x^2}{2}\right) =$$

$$= \ln 2 - \frac{1}{2}\int_1^2 \frac{1}{1+x^2}\,d(1+x^2) =$$

$$= \ln 2 - \frac{1}{2}\ln(1+x^2)\Big|_1^2 = \ln 2 - \left(\frac{1}{2}\ln 5 - \frac{1}{2}\ln 2\right) = \ln 2 - \frac{1}{2}\ln\frac{5}{2} = \ln 2 - \ln\sqrt{\frac{5}{2}} =$$

$$= \ln\left(\frac{2\sqrt{10}}{5}\right)$$

$$\int_0^1 x\sqrt{1+x}\,dx \overset{1+x:=t^2}{=} \int_1^{\sqrt{2}} (t^2-1)\,t\,d(t^2-1)$$

$$= \int_1^{\sqrt{2}} (t^3-t)\,2t\,dt$$

$$1+x=t^2$$
$$t=\sqrt{1+x} \longrightarrow \begin{array}{c|c|c|} x & 0 & 1 \\ t & 1 & \sqrt{2} \end{array}$$
$$x=t^2-1$$

$$=2\int_1^{\sqrt{2}} t^4\,dt - 2\int_1^{\sqrt{2}} t^2\,dt =$$

$$=2\,\frac{t^5}{5}\Big|_1^{\sqrt{2}} - 2\,\frac{t^3}{3}\Big|_1^{\sqrt{2}} = \frac{4}{15}\left(\sqrt{2}-1\right)$$

Пресмятане на лица с определени интеграли

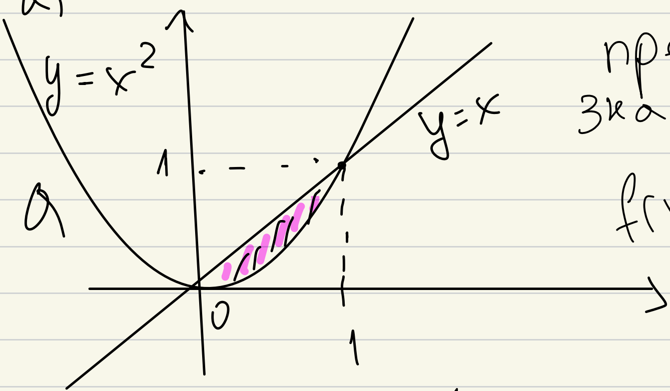
Зад. Да се пресметне лицето на фигурата ограничена от

а) $y = x$ и $y = x^2$ б) $y = x^2 - 4x + 3$, $x = 0$, $y = 0$

в) $\sqrt{1 - x^2}$ и $x = 0$ (огр сме и от $y = 1$, $y = -1$)

г) $y = -x^2 + 6x - 7$ и $y = x - 3$

а)



$f(x) = g(x)$?

пресечни точки - 0, 1

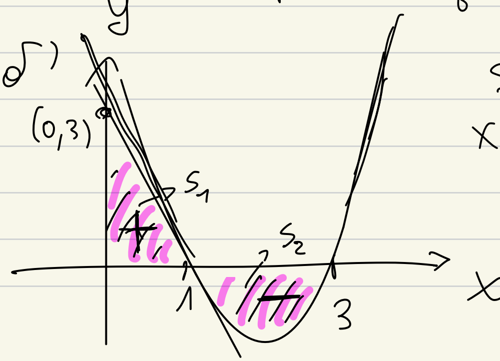
значи $\int_0^1 \ldots$

$f(x) \geq g(x)$?

$x > x^2$ в $(0, 1)$

Значи търсим $\int_0^1 x - x^2 \, dx = \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1 = \frac{1}{6}$

б)



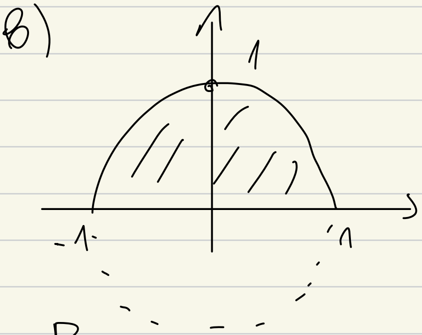
$S = S_1 + S_2$

$x^2 - 4x + 3 = (x-1)(x-3)$

$S_1 = \int_0^1 x^2 - 4x + 3 \, dx = \frac{4}{3}$

$$S_2 = -\int_1^3 x^2-4x+3 = \frac{4}{3} \Rightarrow S = S_1+S_2 = \frac{8}{3}$$

в)



Ще пок., че лицето на кръг е πR^2 (с радиус R)

$$\int_{-R}^{R} \sqrt{R^2-x^2}\,dx = \quad (x^2+y^2=R^2)$$

$$= 2\int_0^R \sqrt{R^2-x^2}\,dx = 2R\int_0^R \sqrt{1-\frac{x^2}{R^2}}\,dx =$$

$$= 2\int_0^R \sqrt{1-\frac{x^2}{R^2}}\,d\left(\frac{x}{R}\right) \underset{t:=\frac{x}{R}}{=} 2\int_0^1 \sqrt{1-t^2}\,dt =$$

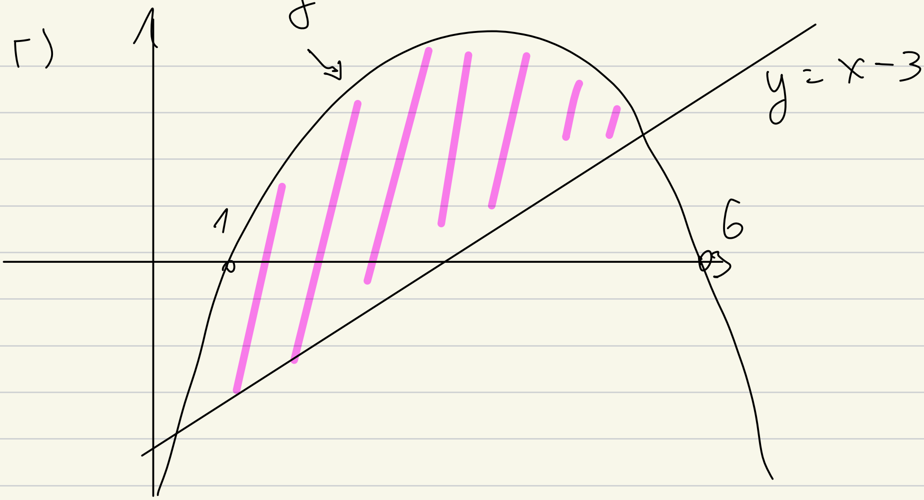
$$\overset{t:=\sin\varphi}{=} 2\int_0^{\pi/2} \sqrt{1-\sin^2\varphi}\,d(\sin\varphi) = 2\int_0^{\pi/2} |\cos\varphi|\cdot\cos\varphi\,d\varphi$$

$$= 2\int_0^{\pi/2} \cos^2\varphi\,d\varphi = \int_0^{\pi/2} \cos 2\varphi + 1\,d\varphi = \frac{R^2\pi}{2}$$

$$\cos^2\varphi = \frac{\cos 2\varphi}{2} + 1$$

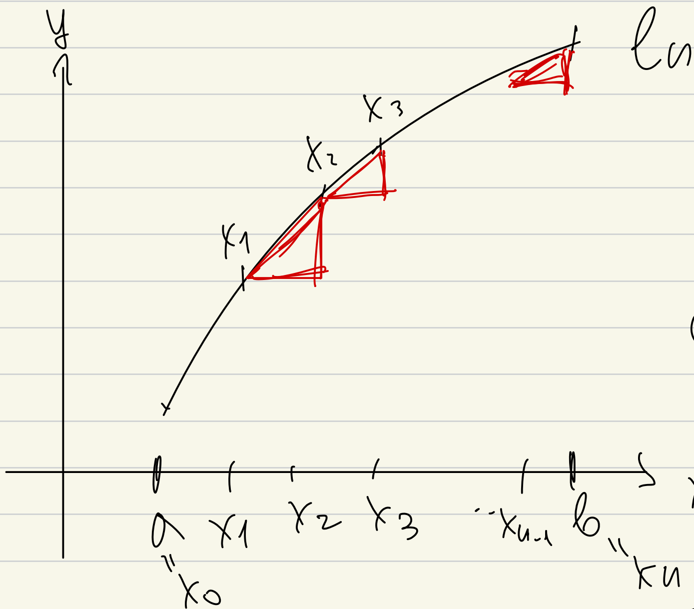
г)

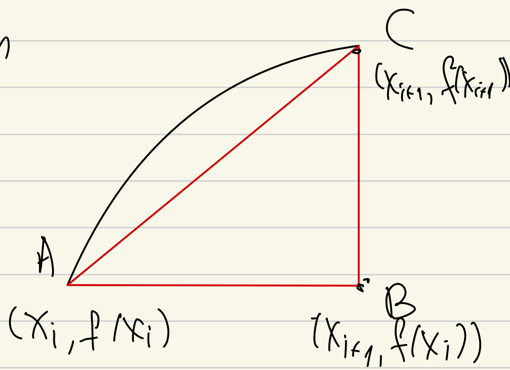
$y = -x^2+6x-7 = -(x-1)(x-6)$



$$\int_1^4 -x^2+6x-7-(x-3)\,dx = \frac{9}{2}$$

Дължина на крива (от a до b)





$AB = x_{i+1} - x_i$

$BC = f(x_{i+1}) - f(x_i)$

$r_i = AC = \sqrt{AB^2+BC^2}$

$$\ell(f)_{x\in[a,b]} = \sum_{i=1}^{n} r_i$$

$$\sum_{k=1}^{n} r_k = \sqrt{(x_{k+1}-x_k)^2+(f(x_{k+1})-f(x_k))^2} =$$

$$= \sum_{k=1}^{n} |x_{k+1}-x_k| \cdot \underbrace{\sqrt{1+f'(\xi_k)^2}}_{f(\xi_k)} = R_\tau\left(\sqrt{1+f'(x)^2}\right)$$

$\xi_k \in (x_k, x_{k+1})$

$$\lim_{d(\tau)\to 0} \ell(f)\Big|_{x\in[a,b]} = \int_a^b \underbrace{\sqrt{1+f'(x)^2}\,dx}_{\text{дължина на крива } f(x) \text{ в } [a,b]}$$

Зад. $f(x) = \frac{1}{3}(x^2+2)^{3/2}$ $\quad f'(x)^2 = x^2(x^2+2) = x^4+2x^2$

$$\int_0^1 \sqrt{1+f'(x)^2}\,dx = \int_0^1 \sqrt{1+2x^2+x^4}\,dx = \int_0^1 \sqrt{(x^2+1)^2}\,dx =$$

$$= \int_0^1 |x^2+1|\,dx = \int_0^1 x^2+1\,dx = \left(\frac{x^3}{3}+x\right)\Big|_0^1 = \frac{4}{3}$$