1.1 Определени интеграли – кратък преговор (задачи)

Зад. $\int \frac{1}{(1+\sqrt[4]{x})^2}\,dx \underset{x=t^4}{\overset{t:=\sqrt[4]{x}}{=}} \int \frac{1}{(1+t)^2}\,dt^4$

$dx = dt^4 = (t^4)' = 4t^3$

$= 4\int \frac{t^3}{(1+t)^2}\,dt = 4\int t\,dt - 4\int 2\,dt + \int \frac{12t+8}{(1+t)^2}\,dt \overset{*}{=}$

$$t^3 : t^2+2t+1 = t-2$$
$$-\;(t^3+2t^2+t)$$
$$-2t^2-t$$
$$-\;(-2t^2-4t-2)$$
$$3t+2$$

$$\frac{3t+2}{(1+t)^2} = \frac{A}{1+t} + \frac{B}{(1+t)^2} = \frac{A(1+t)+B}{(1+t)^2}$$

$\Rightarrow A = 3$, $B = -1$

$\overset{*}{=} 4\cdot\frac{t^2}{2} - 4\cdot 2t + 4\int \frac{\overset{=A}{3}}{1+t}\,dt + 4\int \frac{-1}{(1+t)^2}\,dt$

$= 2t^2 - 8t + 12\ln|1+t| + 4\,\frac{1}{1+t} + C =$

$= 2\sqrt{x} - 8\sqrt[4]{x} + 12\ln|1+\sqrt[4]{x}| + \frac{4}{1+\sqrt[4]{x}} + C$

Зад.

$$\int \frac{dx}{x^2\sqrt{1+x^2}} \; \vdots \; \begin{matrix} x := tg\, u \\ u = arctg\, x \end{matrix} \; = \int \frac{\overbrace{d(tg\, u)}^{=\frac{1}{\cos^2 u}}}{tg^2 u \underbrace{\sqrt{1+tg^2 u}}_{=\frac{1}{\cos u}}}$$

$$= \int \frac{1}{\cancel{\cos^2 u}\, \frac{\sin^2 u}{\cancel{\cos^2 u}}\, \frac{1}{\cos u}}\, du =$$

$1 + tg^2 u = \frac{\cos^2 u + \sin^2 u}{\cos^2 u} = \frac{1}{\cos^2 u}$

$\Rightarrow \sqrt{1+tg^2 u} = \frac{1}{\cos u}$

$$= \int \frac{\cos u}{\sin^2 u}\, du \overset{t := \sin u}{=} \; = \int \frac{1}{t^2}\, dt$$

Ако разгледаме $\cos u\, du$ с тази субституция, забелязваме, че $dt = \cos u\, du$

$$= -\frac{1}{t} + C = -\frac{1}{\sin u} + C =$$

но $x = tg\, u$, и значи $x^2 = tg^2 u$, и $1 + x^2 = 1 + tg^2 u = \frac{1}{\cos^2 u} \Rightarrow \cos u = \frac{1}{\sqrt{1+x^2}}$

$\Rightarrow \sin u = x \cos u = \frac{x}{\sqrt{1+x^2}}$

$$\Rightarrow \int - \cdot = -\frac{\sqrt{1+x^2}}{x} + C =$$

$$= -\sqrt{1+\frac{1}{x^2}} + C$$

$$\int u\,dv = uv - \int v\,du$$ (интегр. по части)

Зад. $\int x^2 e^x\,dx = ?$

$$\int x^2 e^x\,dx = \int x^2\,d e^x =$$

$$= x^2 e^x - \int e^x\,dx^2 =$$

$$= x^2 e^x - \int 2x e^x\,dx = x^2 e^x - 2\int x\,de^x$$

$$= x^2 e^x - 2\left[x e^x - \int e^x dx\right] + C$$

$$= x^2 e^x - 2x e^x + 2e^x + C$$

$$= e^x (x^2 - 2x + 2) + C$$

1.1 Ойлеров интеграл

$$\int R(\sqrt{ax^2+bx+c},\ x)\,dx \quad , \ a\neq 0$$

Ойлерови смени. → което е по-удобно

1) $a>0$, $\pm\sqrt{a}\,x \pm t = \sqrt{ax^2+bx+c} \quad / \uparrow 2$

2) $c>0$, $\pm t \pm \sqrt{c} = \sqrt{ax^2+bx+c} \quad / \uparrow 2$

3) $D \geq 0$, т.е. $ax^2+bx+c = a(x-x_1)(x-x_2)$

Полагаме: $\sqrt{ax^2+bx+c} = \pm t(x-x_i)$, $i=\overline{1,2}$ $x_i \in \mathbb{R}$

Правейки тази смяна преминаваме към интеграл от рационална функция

$$\int \frac{P(t)}{Q(t)}\,dt \qquad P,Q - \text{полиноми}$$

Зад. $\displaystyle\int \frac{1}{x+\sqrt{x^2+x+1}}\,dx$

x^2+x+1, $D<0$, правим смяната

$x+t = \sqrt{x^2+x+1} \quad / \uparrow 2$

$x^2+2xt+t^2 = x^2+x+1$

$\Rightarrow x(2t-1) = 1-t^2 \Rightarrow x = \dfrac{1-t^2}{2t-1}$

Възможно ли е $2t-1=0$? $t=\frac{1}{2}$?

Тогава от $x(2t-1) = 1-t^2$ ще имаме, че $1-\frac{1}{4} = x\cdot 0$ ↯

(Друг начин: да изследваме ф-ята $1-2t$ и да видим дали графиката пресича $\frac{1}{2}$)

$$\int \frac{1}{x+\sqrt{x^2+x+1}}\,dx = \int \frac{1}{x+x+t}\,dx =$$

$$= \int \frac{1}{2x+t}\,dx \overset{x:=\frac{1-t^2}{2t-1}}{=} \int \frac{1}{2\cdot\frac{1-t^2}{2t-1}+t}\,d\left(\frac{1-t^2}{2t-1}\right) =$$

$$= \int \frac{2t-1}{2-2t^2+2t^2-t}\left(\frac{1-t^2}{2t-1}\right)'dt =$$

$$= \int \frac{\cancel{2t-1}\left((1-t^2)'(2t-1)-(2t-1)'(1-t^2)\right)}{2-t\cdot(2t-1)^{\cancel{2}}}\,dt$$

$$= \int \frac{2-4t^2-2+2t^2}{(2-t)(2t-1)}\, dt = \int \frac{-2t^2+2t-2}{(2-t)(2t-1)}$$

$$= \int \frac{2t^2-2t+2}{2t^2-5t+2}\, dt = \underbrace{\int \frac{3t}{(2t-1)(t-2)}\, dt + \int 1\, dt}_{(*)}$$

$$\frac{3t}{(2t-1)(2-t)} = \frac{A}{2t-1} + \frac{B}{t-2}$$

$$\Leftrightarrow$$

$$3t = A(t-2) + B(2t-1) =$$

$$= At - 2A + B.2t - B$$

$$\Rightarrow \left| \begin{array}{l} A + 2B = 3 \\ 2A + B = 0 \Rightarrow B = -2A \Rightarrow A = -1 \\ \qquad\qquad\qquad\qquad\qquad\qquad B = 2 \end{array} \right.$$

$$\overset{(*)}{=} \int \frac{-1}{2t-1}\, dt + \int \frac{2}{t-2}\, dt + \int 1\, dt =$$

$$= \frac{1}{2}\int \frac{1}{2t-1}\, d(2t-1) + 2\int \frac{1}{t-2}\, d(t-2) + t + C$$

$$= \frac{1}{2}\ln|2t-1| + 2\ln|t-2| + t + C$$

Заместваме с $t = \sqrt{x^2+x+1} - x$

$$= \frac{1}{2}\ln|2\sqrt{x^2+x+1} - x - 1| +$$

$$2\ln|\sqrt{x^2+x+1} - x| + \sqrt{x^2+x+1} - x$$

Зад. $\displaystyle\int \frac{1}{(2x-3)\sqrt{4x-x^2}}\, dx = ?$

$a < 0$ $\cancel{1)}$ $c = 0$ $\cancel{2)}$ $D \geq 0$ ✓

$\Rightarrow$ 3) : $4x - x^2 = x(4-x)$ $x_1 = 0, x_2 = 4$

$\sqrt{4x-x^2} = tx$ $4x - x^2 = t^2x^2$

$4 - x = t^2x$ $4 = x(1+t^2)$

$\Rightarrow x = \dfrac{4}{1+t^2} \neq 0$

$$\int \frac{1}{(2x-3)\sqrt{4x-x^2}}\,dx = \int \frac{1}{\left(2\,\frac{4}{1+t^2}-3\right)t\,x}\,dx =$$

$$= \int \frac{1}{\left(\frac{8}{1+t^2}-3\right)t\cdot\frac{4}{1+t^2}}\,d\left(\frac{4}{1+t^2}\right) =$$

$$= 4\int \frac{1}{\left(\frac{8-3-3t^2}{1+t^2}\right)t\,\frac{4}{1+t^2}}\left(\frac{1}{1+t^2}\right)'\,dt =$$

$$= -4\int \frac{(1+t^2)^2}{(5-3t^2)\cdot t\cdot 4}\cdot\frac{(2t)}{(1+t^2)^2}\,dt =$$

$$= -\frac{4\cdot 2}{4}\int \frac{t}{(5-3t^2)\,t}\,dt = -2\int \frac{1}{5-3t^2}\,dt$$

$$= 2\int \frac{1}{3t^2-5}\,dt = 2\int \frac{1}{(\sqrt{3}\,t-\sqrt{5})(\sqrt{3}\,t+\sqrt{5})}\,dt =$$

(може и да го решим като интеграл от вида $\int \frac{1}{t^2-a^2}\,dt$)

$$\frac{1}{(\sqrt{3}t-\sqrt{5})(\sqrt{3}t+\sqrt{5})} = \frac{A}{\sqrt{3}t-\sqrt{5}} + \frac{B}{\sqrt{3}t+\sqrt{5}}$$

$$\Rightarrow \left|\begin{array}{l} \sqrt{3}A + \sqrt{3}B = 0 \\ \sqrt{5}A - \sqrt{5}B = 1 \end{array}\right.$$

$$\left(\begin{array}{cc|c} \sqrt{3} & \sqrt{3} & 0 \\ \sqrt{5} & -\sqrt{5} & 1 \end{array}\right) \sim \begin{array}{c} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ \sqrt{5} & -\sqrt{5} & 1 \end{array}\right) \end{array}\begin{array}{l} (-\sqrt{5}) \\ \swarrow \end{array} \sim$$

$$\sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2\sqrt{5} & 1 \end{array}\right) \Rightarrow B = \frac{1}{-2\sqrt{5}} = \frac{-\sqrt{5}}{10}$$

$$A = \frac{\sqrt{5}}{10}$$

$$\overset{*}{=} \frac{2\sqrt{5}}{10}\left(\int \frac{1}{\sqrt{3}t-\sqrt{5}}\,dt - \int \frac{1}{\sqrt{3}t+\sqrt{5}}\,dt\right) =$$

$$= \frac{2\sqrt{5}}{10}\left(\frac{1}{\sqrt{3}} \ln|\sqrt{3}t-\sqrt{5}| - \frac{1}{\sqrt{3}} \ln|\sqrt{3}t+\sqrt{5}| + C\right)$$

Заместване с t...

в) $\int \frac{1}{x-\sqrt{x^2-1}}\,dx = ?$

$x^2-1 = (x-1)(x+1)$, под:

$\sqrt{x^2-1} = t(x-1) \;/\uparrow 2$, $\quad t = \frac{\sqrt{x^2-1}}{x-1}$

$x^2-1 = t^2(x-1)^2$

$\overset{\shortparallel}{(x-1)(x+1)}$

$x+1 = t^2(x-1)$

$x(1-t^2) = -(1+t^2) \Rightarrow \quad x = \frac{t^2+1}{t^2-1}$

$t=1$? не може

$$\overset{*}{=} \int \frac{1}{\frac{t^2+1}{t^2-1} - \frac{t^3+t}{t^2-t} + t}\, d\left(\frac{t^2+1}{t^2-1}\right) =$$

$$= \int \frac{-4t}{(t-1)^3(t+1)}\,dt = -4\int \frac{A}{t-1}$$

$$\frac{t}{(t-1)^3(t+1)} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{(t-1)^3} + \frac{D}{t+1}$$

$$\vdots \quad A = -1, \quad B = -\frac{1}{4} \quad C = \frac{1}{2} \quad D = \frac{1}{8}$$

$$\overset{4}{=} -4 \int \frac{-1}{t-1} + \frac{1}{4(t-1)^2} + \frac{1}{2(t-1)^3} + \frac{1}{8(t+1)} \, dt$$

$$= -4 \ln|t-1| - \frac{1}{t-1} + \frac{2}{(t-1)^2} - \frac{1}{2}\ln|t+1| + C$$

За упражнение:

$$\int \frac{\sqrt{2x-3}}{\sqrt[3]{2x-3}+1} \, dx \qquad \int \frac{2}{(2-x)^2} \sqrt[3]{\frac{2-x}{2+x}} \, dx$$

$$\int \frac{1}{\sqrt{1+e^x+e^{2x}}} \, dx \to \text{Hint}: \; p = e^x, \; x = \ln p$$