Детерминанти 2

1 Рекурентни детерминанти

* Рекурентни уравнения - алгоритъм за решаване

- общ вид

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \dots c_k \cdot a_{n-k}$$

1) съставяме характ. у-е

$$x^n = c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_k x^{n-k} \quad /:x^{n-k}$$

$$x^{n-k} - c_1 x^{n-k-1} - \dots - c_k = 0$$

Пример: $x^n = x^{n-1} + x^{n-2}$

$$x^2 - x - 1 = 0$$

2) Намираме всички корени

$$\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$$

I случай: Всички k корена са еднакви, тогава:

$$a_n = A_1 \cdot \lambda_1^n + A_2 \cdot \lambda_2^n + \dots + A_k \cdot \lambda_k^n$$

$A_1, A_2, \dots, A_k$ - константи, които се определят от началните условия

II случай: Има повтарящи се корени

$$\underbrace{\lambda_1 \dots \lambda_1}_{m \text{ пъти}} \quad \lambda_2, \dots, \lambda_{k-m}$$

$$a_n = (A_{11} \cdot \lambda_1^n + A_{12} \cdot n \cdot \lambda_1^n + \dots A_{1m} n^{m-1} \lambda_1^n$$
$$A_2 \lambda_2^n + A_3 \lambda_3^n + \dots A_{k-m} \lambda_{k-m}^n)$$

Зад. ← развиване по първи ред

$$\begin{vmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & 1 \\ 0 & \dots & 0 & 1 & 2 \end{vmatrix}_{n \times n} = 2 \begin{vmatrix} 2 & 1 & & 0 \\ 2 & \ddots & \ddots & \\ & \ddots & & 2 & 1 \\ 0 & & & 1 & 2 \end{vmatrix}_{n-1} - 1 \begin{vmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 2 & 1 & \ddots & 0 \\ \vdots & 1 & \ddots & & 1 \\ 0 & \dots & 0 & 1 & 2 \end{vmatrix}_{n-1}$$

$$= 2 \Delta_{n-1} - 1 \cdot \begin{vmatrix} 2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & 2 & 1 \\ & & 1 & 2 \end{vmatrix}$$

$$= 2\Delta_{n-1} - \Delta_{n-2}$$

$$a_n = 2a_{n-1} - a_{n-2}$$

$$\downarrow$$

$$x^n = 2x^{n-1} - x^{n-2} \quad /: x^{n-2}$$

$$x^2 = 2x - 1$$

$$x^2 - 2x + 1 = 0$$

$$x_{1,2} = 1$$

$$\Delta_n = n.A.1^n + B.1^n = n.A + B$$

$$\left.\begin{aligned} \Delta_1 &= |2| = 2 = A + B \\ \Delta_2 &= \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 = 2A + B \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 1 \\ B &= 1 \end{aligned}$$

$$\Rightarrow \Delta_n = n + 1$$

Зад.

$$\begin{vmatrix} 3 & 2 & & 0 \\ 1 & 3 & \ddots & \\ & \ddots & \ddots & 2 \\ 0 & & 1 & 3 \end{vmatrix} = 3\begin{vmatrix} 3 & 2 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 2 \\ 0 & & 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 & & 0 \\ 0 & 3 & \ddots & \\ & 1 & \ddots & 2 \\ 0 & & 1 & 3 \end{vmatrix} =$$

$$= 3\Delta_{n-1} - 2\Delta_{n-2}$$

$$x^2 = 3x - 2$$

$$x_1 = 1 \quad x_2 = 2$$

$$\Delta_n = A.1^n + B.2^n$$

$$\left.\begin{array}{l} \Delta_1 = |3| = 3 = A + 2B \\ \Delta_2 = \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = 7 = A + 4B \end{array}\right\} \Rightarrow \begin{array}{l} A = -1 \\ B = 2 \end{array}$$

$$\Rightarrow \Delta_n = -1 + 2.2^n = 2^{n+1} - 1$$

За упражнение:

$$\begin{vmatrix} 7 & 2 & & & 0 \\ 5 & 7 & 2 & & \\ & 5 & \ddots & \ddots & \\ & & \ddots & \ddots & 2 \\ 0 & & & 5 & 7 \end{vmatrix} = ?$$

3ag.

$$\begin{vmatrix} \alpha+\beta & \alpha & & & 0 \\ \beta & \alpha+\beta & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \alpha \\ 0 & & & \beta & \alpha+\beta \end{vmatrix} =$$

$$= (\alpha+\beta)\Delta_{n-1} - \alpha \begin{vmatrix} \beta & \alpha & & & 0 \\ 0 & \alpha+\beta & \alpha & & \\ & \beta & \ddots & \ddots & \\ & & \ddots & \ddots & \alpha \\ 0 & & & \beta & \alpha+\beta \end{vmatrix} =$$

$$= (\alpha+\beta)\Delta_{n-1} - \alpha\beta\,\Delta_{n-2}$$

$$x^2 = (\alpha+\beta)x - \alpha\beta$$

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

ф-ли на Виет:

$$x_1 x_2 = \alpha\beta \qquad \Rightarrow \qquad x_1 = \alpha$$
$$x_1 + x_2 = \alpha + \beta \qquad\qquad x_2 = \beta$$

$$\Delta_n = A\alpha^n + B.\beta^n$$

$$\Delta_1 = \alpha+\beta = A\alpha + B.\beta$$
$$\Delta_2 = \alpha^2 + \alpha\beta + \beta^2 = A.\alpha^2 + B.\beta^2$$

$$\left(\begin{array}{cc|c} \alpha & \beta & \alpha+\beta \\ \alpha^2 & \beta^2 & \alpha^2+\alpha\beta+\beta^2 \end{array}\right) \cdot(-\alpha) \sim$$

$$\left(\begin{array}{cc|c} \alpha & \beta & \alpha+\beta \\ 0 & \beta^2-\alpha\beta & \beta^2 \end{array}\right) \sim \quad /: \beta^2-\alpha\beta \ (\neq 0)$$

иначе $\alpha = \beta$

$$\sim \left(\begin{array}{cc|c} \alpha & 0 & \alpha+\beta - \frac{\beta^2}{\beta-\alpha} \\ 0 & 1 & \frac{\beta}{\beta-\alpha} \end{array}\right) \sim$$

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{\beta^2-\alpha^2-\beta^2}{\beta-\alpha} \\ 0 & 1 & \frac{\beta}{\beta-\alpha} \end{array}\right)$$

$$\Rightarrow A = \frac{\alpha}{\beta-\alpha}$$

$$B = \frac{\beta}{\beta-\alpha}$$

$$\Rightarrow \Delta_n = \frac{\alpha^{n+1}-\beta^{n+1}}{\beta-\alpha}$$

$$\alpha \neq \beta$$

$\alpha = \beta$:

$$\Delta_n = n \cdot \alpha^n \cdot A + \alpha^n \cdot B$$

$$\Delta_1 = 2\alpha = \alpha A \Rightarrow A = 1$$

$$\Delta_2 = 3\alpha^2 = 2\alpha A + \alpha \cdot B \Rightarrow B = 1$$

$$\Rightarrow \Delta_n = n\alpha^n + \alpha^n = (n+1)\alpha^n$$

Зад.

$$\begin{vmatrix} a & & & & & b \\ & a & 0 & & b & \\ & 0 & \ddots & \ddots & 0 & \\ & & \ddots & \ddots & & \\ & b & & 0 & a & \\ b & & & & & a \end{vmatrix} = a \begin{vmatrix} a & 0 & b & 0 \\ 0 & \ddots & 0 & \vdots \\ b & 0 & \ddots & 0 \\ 0 & \cdots & 0 & a \end{vmatrix}_{2n} - b \begin{vmatrix} 0 & a & \cdots & b \\ \vdots & & 0 & 0 \\ 0 & 0 & \ddots & a \\ b & 0 & \cdots & 0 \end{vmatrix} =$$

$$= a^2 \Delta_{2n-2} - b^2 \Delta_{2n-2}$$

$$\Rightarrow \Delta_n = (a^2 - b^2)\Delta_{2n-2}$$

Зад.

$$\begin{vmatrix} a_0 & a_1 & \cdots & & a_n \\ -1 & x & & 0 & \\ & -1 & \ddots & & \\ & 0 & & \ddots & \\ & & & -1 & x \end{vmatrix} = a_0 \begin{vmatrix} x & & 0 \\ -1 & \ddots & \\ & 0 & \ddots & x \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ x & & & \\ -1 & x & 0 & \\ 0 & -1 & \ddots & x \end{vmatrix} =$$

$$= a_0 x^n + a_1 x^{n-1} + \ldots a_n$$

2. Детерминанта на Вандермонд

$$\to W(x_1, \dots, x_n) = \begin{array}{c} \\ (-x_1) \\ (-x_1) \\ \vdots \\ (-x_1) \end{array} \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & & \dots & x_n \\ x_1^2 & x_2^2 & & \ddots & x_n^2 \\ \vdots & \vdots & & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & & x_n^{n-1} \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & \dots & x_n - x_1 \\ 0 & x_2^2 - x_2 x_1 & & & x_n^2 - x_n x_1 \\ \vdots & \vdots & & \ddots & \\ 0 & x_2^{n-1} - x_2^{n-2} x_1 & & & x_n^{n-1} - x_n^{n-2} x_1 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & \dots & x_n - x_1 \\ x_2(x_2 - x_1) & \dots & x_n(x_n - x_1) \\ \vdots & \ddots & \vdots \\ x_2^{n-2}(x_2 - x_1) & \dots & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$

$$= (x_2 - x_1) \dots (x_n - x_1) \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_2 & & & x_n \\ \vdots & & \ddots & \vdots \\ x_2^{n-1} & & & x_n^{n-1} \end{vmatrix} = \prod_{1 < i \le n} (x_i - x_1) W(x_2, \dots, x_n)$$

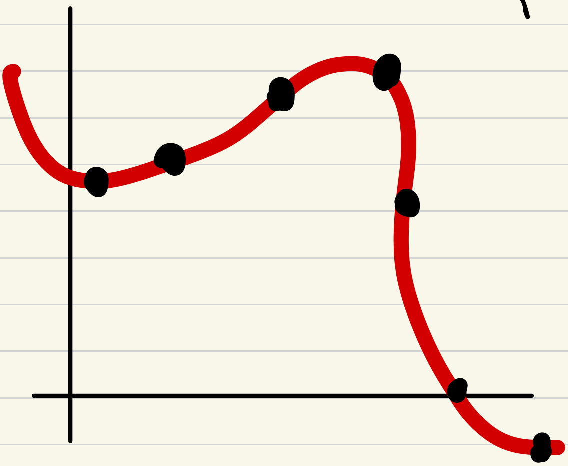
$(x_0, y_0), (x_1, y_1) \dots (x_n, y_n)$

Има ли полином от n-та степен, който минава през точките?

$P(x_0) = y_0, P(x_1) = y_1 \dots P(x_n) = y_n$

$P(x) = a_0 + a_1 x + \dots + a_n x^n$

$? \; a_0 \dots a_n$



$$\begin{cases} a_0 + a_1 x_0 \dots + a_n x_0^n = y_0 \\ \vdots \\ a_0 + a_1 x_n + \dots a_n x_n^n = y_n \end{cases}$$

Матрицата на системата е

$$\begin{pmatrix} 1 & x_0 & \dots & x_0^n \\ \vdots & \vdots & & \\ 1 & x_n & \dots & x_n^n \end{pmatrix}$$

$W \neq 0 \iff x_1 \neq x_2 \neq \dots \neq x_n$

3. Произведение на детерминанти.

$A, B \in M_n(\mathbb{R})$ $\quad \det AB = \det A . \det B$

$$\begin{vmatrix} a_1+b_1 & a_1+b_2 & \dots & a_1+b_n \\ a_2+b_1 & a_2+b_2 & \dots & a_2+b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n+b_1 & a_n+b_2 & \dots & a_n+b_n \end{vmatrix}$$

$$a_i + b_j = a_i . 1 + 1 . b_j + 0 . 0$$

$$\begin{pmatrix} a_1 & 1 & \\ \vdots & \vdots & \Theta \\ a_n & 1 & \end{pmatrix} \begin{pmatrix} 1 & 1 & \dots & 1 \\ b_1 & b_2 & & b_n \\ & & \Theta & \end{pmatrix}$$

$\Rightarrow \Delta_n = 0 \Leftrightarrow n > 2$

$$\Delta_1 = |a_1 + b_1| = a_1 + b_1$$

$$\Delta_2 = \begin{vmatrix} a_1 + b_1 & a_1 + b_2 \\ a_2 + b_1 & a_2 + b_2 \end{vmatrix} = (a_1 + b_1)(a_2 + b_2) - (a_1 + b_2)(a_2 + b_1)$$

$$\Delta_3 = 0$$

Зад.

$$\begin{vmatrix} \cos(\alpha_1 - \beta_1) & \cos(\alpha_1 - \beta_2) & \cdots & \cos(\alpha_1 - \beta_n) \\ \cos(\alpha_2 - \beta_1) & \cos(\alpha_2 - \beta_2) & \cdots & \cos(\alpha_2 - \beta_n) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(\alpha_n - \beta_1) & \cos(\alpha_n - \beta_2) & \cdots & \cos(\alpha_n - \beta_n) \end{vmatrix}$$

$$\cos(\alpha_i - \beta_j) = \cos\alpha_i \cos\beta_j - \sin\alpha_i \sin\beta_j$$

$$\begin{pmatrix} \cos\alpha_1 & \sin\alpha_1 & \\ \cos\alpha_2 & \sin\alpha_2 & \Theta \\ \vdots & \vdots & \\ \cos\alpha_n & \sin\alpha_n & \end{pmatrix} \begin{pmatrix} \cos\beta_1 & \cdots & \cos\beta_n \\ \sin\beta_1 & \cdots & \sin\beta_n \\ & \Theta & \end{pmatrix}$$

$$\Delta = 0, n \geq 3$$

4. Минори. Ранг на матрица

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

ранг по редовете = ранг на стълбовете = ранг на матрица

Минор от ред k: поддетерминанта с k реда и k стълба

Ранг: Максималният ред на ненулев минор

ранг по редове = ранг по стълбове = ранг по минори

Зад. ? ранг на матр. в зав. от парам

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{pmatrix}$$

$r = 4 \iff \det A \neq 0$

$\det A$ е полином на A

при $r < 4$ разгл. отделно:

$$\begin{vmatrix} 0 & 1 & 2 & 1 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{vmatrix} = (-1)(-1)^{1+2} \begin{vmatrix} 1 & 2 & 1 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{vmatrix}^{(1)} =$$

$$= \begin{vmatrix} 1 & 2 & 1 \\ 0 & \lambda-2 & 1 \\ 0 & -1 & \lambda \end{vmatrix} = (\lambda-2)\lambda + 1 =$$

$$\lambda^2 - 2\lambda + 1 = (\lambda-1)^2$$

$\lambda \neq 1 \Rightarrow r(A) = 4$

$\lambda = -1 \Rightarrow r(A) < 4$

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

ненулев минор от ред 3 $\Rightarrow r(A) = 3$

$$\Rightarrow r(A) = \begin{cases} 3, & \lambda = -1 \\ 4, & \lambda \neq -1 \end{cases}$$

Зад. $r(B) = ?$

$$\begin{pmatrix} 1-b & 0 & 2 & -1 \\ 0 & 1-b & 4 & -2 \\ 2 & -1 & -b & 1 \\ 2 & -1 & -1 & 2-b \end{pmatrix}$$

$$\det B = \begin{vmatrix} 1-b & 0 & 2 & -1 \\ 0 & 1-b & 4 & -2 \\ 2 & -1 & -b & 1 \\ 2 & -1 & -1 & 2-b \end{vmatrix} \begin{matrix} (-2) \\ \swarrow \\ (-1) \\ \swarrow \end{matrix} =$$

$$= \begin{vmatrix} 1-b & 0 & 2 & -1 \\ -2(1-b) & 1-b & 0 & 0 \\ 2 & -1 & -b & 1 \\ 0 & 0 & b-1 & 1-b \end{vmatrix} = (1-b)^2 \begin{vmatrix} 1-b & 0 & 2 & -1 \\ -2 & 1 & 0 & 0 \\ 2 & -1 & -b & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix}$$

$$= (1-b)^2 \begin{vmatrix} 1-b & 0 & 2 & -1 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & -b & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix} =$$

$$= (1-b)^2 \begin{vmatrix} 1-b & 2 & -1 \\ 0 & -b & 1 \\ 0 & -1 & 1 \end{vmatrix} = (1-b)^3 \begin{vmatrix} -b & 1 \\ -1 & 1 \end{vmatrix}$$

$$= (1-b)^3(1-b) = (1-b)^4$$

$$b \neq 1 \Rightarrow r(B) = 4$$

$b = 1$:

$$B = \begin{pmatrix} 0 & 0 & 2 & -1 \\ \cancel{0} & \cancel{0} & \cancel{4} & \cancel{-2} \\ 2 & -1 & -1 & 1 \\ \cancel{2} & \cancel{-1} & \cancel{-1} & \cancel{1} \end{pmatrix} \Rightarrow r = 2$$

$$\Rightarrow r(B) = \begin{cases} 2, & b = 1 \\ 4, & b \neq 1 \end{cases}$$