Комплексни числа

$\mathbb{N} \to$ естествени числа $1, 2, 3 \ldots$

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$\mathbb{Z} \to$ цели числа $\mathbb{N} \subseteq \mathbb{Z}$

$\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\} \to$ остатъци при делене на p

$\mathbb{Z}_p \subseteq \mathbb{Z}$

$\mathbb{Q} \to$ рационални числа $\frac{1}{2}, \frac{3}{4} \ldots$

$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, (p,q) = 1\}$ ← взаимно прости

$\mathbb{R} \to$ реални числа $\sqrt{2}, \pi, e$

$\mathbb{C} \to$ комплексни числа

$\mathbb{C} = \{(x, y) \mid x, y \in \mathbb{R}\}$ ← двойка реални числа

Запис на комплексни числа:

1) алгебричен вид:

$x^2 + 1 = 0 \qquad x = i = \sqrt{-1}$

$z = a + bi$

a - реална част $\quad Re(z) = a$

b - имагинерна част $\quad Im(z) = b$

деф: събиране: $a + bi$

$z_1 = a_1 + b_1 i \,, \quad z_2 = a_2 + b_2 i$

$z_1 + z_2 = a_1 + b_1 i + a_2 + b_2 i = (a_1 + a_2) + (b_1 + b_2) i$

Пример: $(5 - 3i) + (-2 + i) = 3 - 2i$

изваждане:

$z_1 - z_2 = a_1 + b_1 i - (a_2 + b_2 i) = a_1 + b_1 i - a_2 - b_2 i =$

$= (a_1 - a_2) + (b_1 - b_2) i$

Пример: $(2 - 3i) - (4 - 6i) = -2 + 3i$

$i^2=-1 \quad i^3=-i \quad i^4=1 \quad i^5=i$

умножение:

$z_1 \cdot z_2=(a_1+b_1 i)(a_2+b_2 i)=a_1a_2+a_1b_2 i$
$+a_2b_1 i\underbrace{-b_1b_2}_{i^2=-1}=(a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)i$

Пример: $(1+2i)(-3+i)=-5-5i$

деление:

$$\frac{z_1}{z_2}=\frac{a_1+b_1 i}{a_2+b_2 i}\cdot\frac{(a_2-b_2 i)}{(a_2-b_2 i)}=\frac{(a_1a_2+b_1b_2)+(b_1a_2-a_1b_2)i}{a_2^2+b_2^2}$$

$a^2+b^2\geq 0, \mathbb{R}$

$$=\left(\frac{a_1a_2-b_1b_2}{a_2^2+b_2^2}\right)+\left(\frac{b_1a_2-a_1b_2}{a_2^2+b_2^2}\right)i$$

Пример: $\frac{2+3i}{3+4i}\cdot\frac{3-4i}{3-4i}=\frac{18+i}{3^2+4^2}=\frac{18}{25}+\frac{1}{25}i$

комплексно спрегнато на $z = a + bi$:

$\bar{z} = \overline{a+bi} = a - bi$

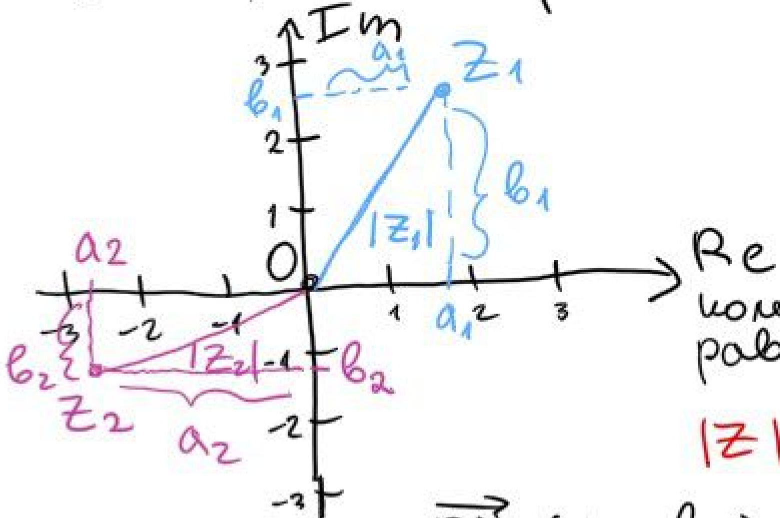
$z \cdot \bar{z} = a^2 + b^2$

модул на z: $|z| = \sqrt{a^2+b^2} \geq 0 \in \mathbb{R}$

2) тригонометричен вид

геометрично представяне

комплексна равнина:



$z_1 = a_1 + b_1 i$

$z_2 = a_2 + b_2 i$

$|z|$ - разстоянието на компл. число до центъра на равнината

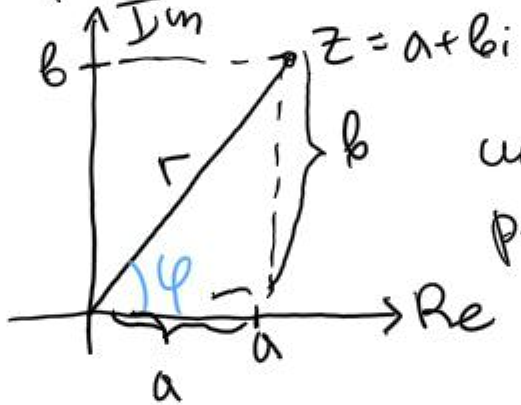
$|z| = 0 \Leftrightarrow z = (0,0)$

$\overrightarrow{OZ_1}(a_1, b_1)$ нар. радиус вектор

Пример: $z = 3+4i$ $\quad |z| = \sqrt{3^2+4^2} = 5$

Озн. $r := |z|$

аргумент:



φ - аргумент на z

(ъгълът, който радиус векторът сключва с полож. посока на реалната ос)

$$Arg(z) = \varphi$$

виждаме, че $a = r.\cos\varphi \quad b = r.\sin\varphi$

$\Rightarrow z = a+bi = r(\cos\varphi + i\sin\varphi)$ това ще го ползвате по ДИС2

тригонометричен вид на комплексно число

Привеждане в триг. вид: $z = a+bi$

1) $r = \sqrt{a^2+b^2}$ $\quad$ 2) $Arg\, z = \varphi = \begin{cases} \cos\varphi = \frac{a}{r} \\ \sin\varphi = \frac{b}{r} \end{cases}$

$\Rightarrow z = r(\cos\varphi + i\sin\varphi)$

Умножение и деление:

$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$

$z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$

* $z_1 z_2 = r_1 r_2 [\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)]$

* $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2)]$

Примери: $z_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$z_2 = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

$z_1 z_2 = 2\ (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 2i$

$\dfrac{z_1}{z_2} = \dfrac{1}{2}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \dfrac{1}{2}(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i) =$

$= \dfrac{\sqrt{3}}{4} + \dfrac{1}{4}i$

формули на Моавър: $z = r(\cos\varphi + i\sin\varphi)$

$z^n = r^n(\cos(n\varphi) + i\sin(n\varphi))$

Пример: $(-1+i)^{543}$, прив. в триг вид: $z = -1+i$

$r = \sqrt{1+1} = \sqrt{2}$

$\cos\varphi = \frac{a}{r} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

$\sin\varphi = \frac{b}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\Rightarrow -1+i = z = \sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)$

$\Rightarrow z^{543} = \sqrt{2}^{543}\left(\cos\frac{543\cdot 3\pi}{4} + i\sin\frac{543\cdot 3\pi}{4}\right) =$

$= 2^{271}\sqrt{2}\left(\cos\left(406\pi + \frac{5\pi}{4}\right) + i\sin\left(406\pi + \frac{5\pi}{4}\right)\right) =$

$= 2^{271}\sqrt{2}\left(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi\right) =$

$= 2^{271}\sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = -2^{271}(1+i)$

$* \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right) \quad k=0,1,\dots,n-1$

Пример: $\sqrt[12]{1}$

$1 = 1(\cos 0 + i\sin 0)$

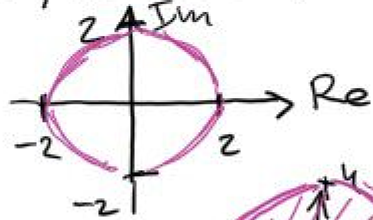
$\sqrt[12]{1} = \cos\frac{2\pi k}{12} + i\sin\frac{2\pi k}{12} = \quad k=0,1,\dots,11$

$= \cos\frac{\pi k}{6} + i\sin\frac{\pi k}{6}$

Задачи

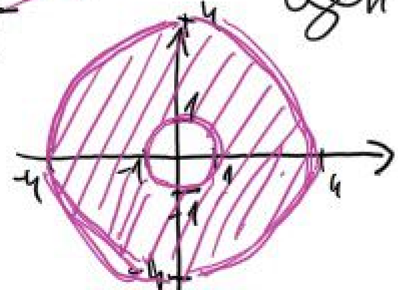
зад. Да се определи мн-вото от точките в компл. равнина, на които съотв z, за които:

$|z| = 2$ $\longrightarrow$



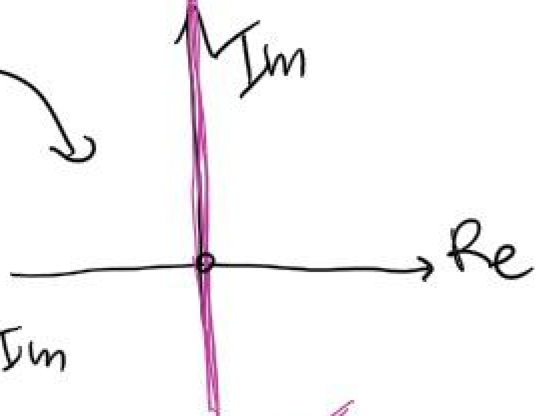
точките на разстояние 2 от центъра

$1 \le |z| \le 4$

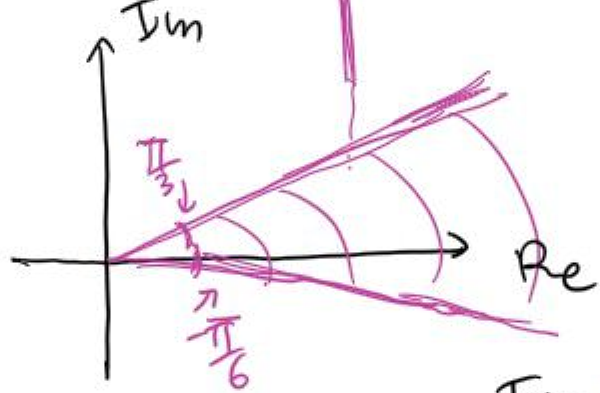


$\arg z = \frac{\pi}{2}$

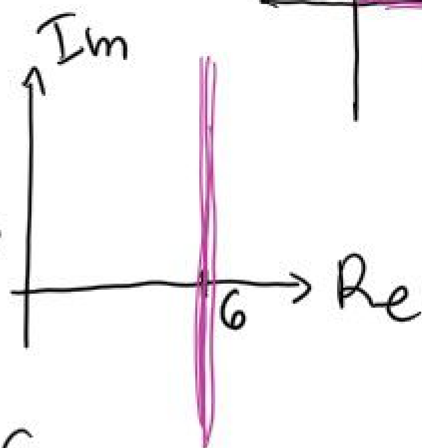
$\Rightarrow \mathrm{Re}(z) = 0 \quad (\cos\frac{\pi}{2} = 0)$



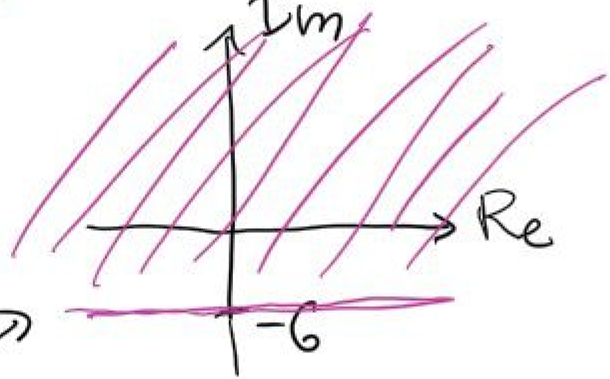
$-\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{3}$



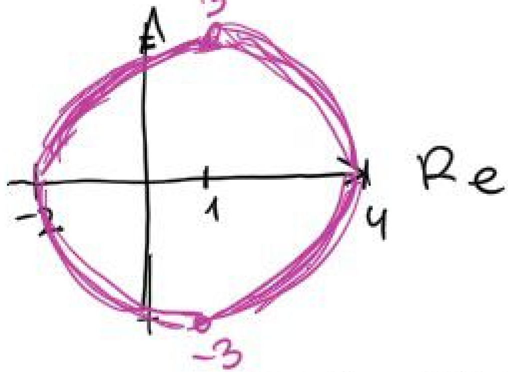
$\mathrm{Re}\, z = 6$



$\mathrm{Im}\, z \geq -6$



$|z-1|=3$



Зад. Решете у-ето $|z|+(1-i)z=4+7i$

Представяме $z\in\mathbb{C}$ като: $z=x+yi$ $x,y\in\mathbb{R}$

$$\Rightarrow \sqrt{x^2+y^2}+\underbrace{x+y+(y-x)i}_{(1-i)(x+yi)}=4+7i$$

сега остава да решим системата:

$$\begin{cases}\sqrt{x^2+y^2}+x+y=4\\ y-x=7 \to y=x+7\end{cases}$$

заместване

$\vdots$

$x=-\frac{9}{2}$ $y=\frac{5}{2}$

Зад. Да се запише в триг вид: (0.5)

б) $2+\sqrt{3}+i$

търсим точен квадрат

$$r=\sqrt{(2+\sqrt{3})^2+1}=\sqrt{8+4\sqrt{3}}=\sqrt{2+4\sqrt{3}+6}=$$

$$=\sqrt{\sqrt{2}^2+2\cdot\sqrt{2}\cdot\sqrt{6}+\sqrt{6}^2}=\sqrt{(\sqrt{2}+\sqrt{6})^2}=\sqrt{2}+\sqrt{6}$$

Hint: търсим а, в такива, че $\begin{cases}2ab=4\sqrt{3}\\ a^2+b^2=8\end{cases}$

$$z=(\sqrt{2}+\sqrt{6})\left(\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{6}}\cdot\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{6}}\cdot\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}\,i\right)=\ \dots\ \text{(сметки)}$$

$$=(\sqrt{2}+\sqrt{6})\left(\underbrace{\frac{\sqrt{6}+\sqrt{2}}{4}}_{\cos\frac{\pi}{12}}+\underbrace{\frac{\sqrt{6}-\sqrt{2}}{4}}_{\sin\frac{\pi}{12}}\,i\right)$$

Лесен начин за намиране, ако сме забравили: $\cos^2 x=\frac{1+\cos 2x}{2}$

$$\cos^2\frac{\pi}{12}=\frac{1+\cos\frac{\pi}{6}}{2}=\frac{2+\sqrt{3}}{4}\ \dots$$

b) $1+\cos\varphi + i\sin\varphi$

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = 2\cos^2 x - 1$$
$$1+\cos 2x = 2\cos^2 x$$

$r = |z| =$

$= \sqrt{1+2\cos\varphi + \cos^2\varphi + \sin^2\varphi} = \sqrt{2+2\cos\varphi} =$

$= \sqrt{2+4\cos^2\frac{\varphi}{2} - 2} = 2\cos\frac{\varphi}{2}$

$$z = 2\cos\frac{\varphi}{2}\left(\frac{1+\cos\varphi}{2\cos\frac{\varphi}{2}} + i\frac{\sin\varphi}{2\cos\frac{\varphi}{2}}\right) =$$

$$= 2\cos\frac{\varphi}{2}\left(\frac{2\cos^2\frac{\varphi}{2}}{2\cos\frac{\varphi}{2}} + i\frac{2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}}{2\cos\frac{\varphi}{2}}\right) =$$

$$= 2\cos\frac{\varphi}{2}\left(\cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2}\right)$$

g) $\dfrac{(\sqrt{3}-i)^{15}}{(1+i)^8}$ $\quad \sqrt{3}-i = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 2\left(\cos\left(-\frac{\pi}{6}\right) - i\sin\left(-\frac{\pi}{6}\right)\right)$

$1+i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$\frac{(\sqrt{3}-i)^{15}}{(1+i)^8} = \frac{2^{15}\left(\cos\left(-\frac{5\pi}{2}\right) + i\sin\left(-\frac{5\pi}{2}\right)\right)}{2^4(\cos 2\pi + i\sin 2\pi)} = 2^{11}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

e) $(1-i)^n + (1+i)^n$

$$1-i = \sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

$$\sqrt{2}^n\left(\cos\left(-\frac{\pi . n}{4}\right) + i\sin\left(-\frac{\pi . n}{4}\right)\right) + \sqrt{2}^n\left(\cos\frac{\pi . n}{4} + i\sin\frac{\pi . n}{4}\right) =$$

$$= \sqrt{2}^{n+2}\cos\frac{\pi . n}{4}$$

Зад. Да се намерят корените на у-ето

$\boxed{8}x^4 + 6x^2 + 10x + \boxed{3} = 0$

Възможни корени
делители на $\frac{\boxed{3}}{\boxed{8}}$

$\pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{8}, \pm\frac{3}{4}, \pm\frac{3}{2}, \pm 3$

Схема на Хорнер:

	8	0	6	10	3
$-\frac{1}{2}$	8	−4	8	6	0
$-\frac{1}{2}$	8	−8	12	0	

$8\cdot\left(-\frac{1}{2}\right) + 4$ $\quad -8\left(-\frac{1}{2}\right) + 8$ $\quad 12\cdot\left(-\frac{1}{2}\right) + 6$

$$8x^4 + 6x^2 + 10x + 3 = 4\left(x + \frac{1}{2}\right)^2(2x^2 - 2x + 3)$$

$2x^2-2x+3$

$D=4-24=-20$

$$x_{3,4}=\frac{2\pm\sqrt{-20}}{4}=\frac{2\pm 2\sqrt{5}i}{4}=\frac{1\pm\sqrt{5}i}{2}$$

Бонус: $x^2-(3-2i)x+(5-5i)=0$

$D=9-12i-4-20+20i=-15+8i=z$

$|z|=\sqrt{15^2+8^2}=17$

$z=17\left(-\frac{15}{17}+\frac{8}{17}i\right)\longrightarrow \cos\varphi=-\frac{15}{17}$, $\sin\varphi=\frac{8}{17}$ $\Rightarrow \operatorname{tg}\varphi=-\frac{8}{15}$

$\Rightarrow \varphi=\operatorname{arctg}\left(-\frac{8}{15}\right)$ ← рис-1

$$\Rightarrow\sqrt{D}=\sqrt{z}=\sqrt{17}\left(\cos\left(\frac{\operatorname{arctg}\left(-\frac{8}{15}\right)+2k\pi}{2}\right)+i\sin\left(\frac{\operatorname{arctg}\left(-\frac{8}{15}\right)+2k\pi}{2}\right)\right)$$

$k=0,1$

и т.к. ...

Зад. Да се докаже, че:

С $\omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ означаваме n-ти корени на единицата $(k=0,1,\dots,n-1)$

а) $\omega_1^k = \omega_k$

Д-во: следва директно от формулите на Моавър

б) $\omega_1^s = 1 \Leftrightarrow$ n дели s

Д-во: $\omega_1^0 = 1, \omega_1, \omega_2, \dots, \omega_{n-1}$ са различни и n е най-малкото ест. число r, че

$\omega_1^n = 1$

остатък при деление на n

Нека $s = nq + r \quad 0 \leq r \leq n-1$

$\omega_1^s = \omega_1^{nq+r} = (\omega_1^n)^q \cdot \omega_1^r = \omega_1^r$

$\Rightarrow \omega_1^s = \omega_1^r = 1 \Leftrightarrow r = 0$, т.е. когато n дели s { ще означ. с $n|s$ (по висша алгебра)

Зад. ДСД, че ако m е цяло число, то
$$\omega_0^m + \omega_1^m + \dots + \omega_{n-1}^m \begin{cases} n, & \text{ако } n \text{ дели } m \\ 0, & \text{ако } n \text{ не дели } m \end{cases}$$

Д-во: ако n дели m, то
$$\omega_k^m = 1 \quad (k = 0, 1, \dots, n-1) \text{ и}$$
$$\omega_0^m + \dots + \omega_{n-1}^m = n$$

ако n не дели m (ще означ. с $n \nmid m$ (по Висша алгебра))
$$\omega_0^m + \omega_1^m + \dots + \omega_{n-1}^m = 1 + \omega_1^m + \dots + \omega_1^{(n-1)m} =$$
$$= \underbrace{\frac{\overset{1}{\omega_1^{mn}} - 1}{\omega_1^m - 1}}_{\text{сума на геометрична прогресия}} = 0 \quad (\omega_1^m \neq 1)$$