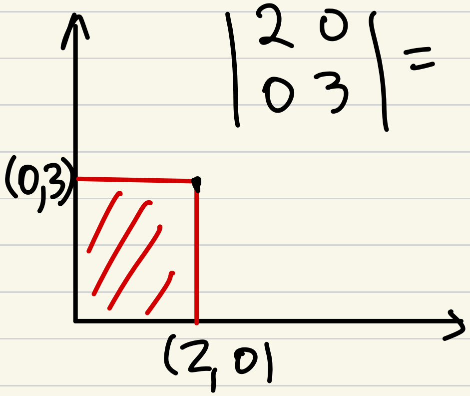
Детерминанти
Ранг на матрица

Детерминанти

Геометричен смисъл

* лица
* обеми



$$\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

Алгебричен смисъл

$$\Delta_n = \sum_{i_1, i_2, \dots i_n} (-1)^{[i_1, \dots, i_n]} \cdot a_{1 i_1} a_{2 i_2} \dots a_{n i_n}$$

където

→ $i_1, \dots, i_n$ - пермутация на $1, \dots, n$

→ $[i_1, \dots, i_n]$ - четност на пермутацията

Четност на пермутация - брой размени (четността им) необходими, за да наредим числата в нея до $1, 2, \dots, n$

Примери:

$[2, 3, 1] \rightarrow 2\ 3\ 1 \rightarrow 2\ 1\ 3 \rightarrow$
$\rightarrow 1\ 2\ 3$ ✓

бројот инверсии (размени) е четен (2) $\Rightarrow$ четна

$[3, 5, 2, 1, 4] \rightarrow 3\ 5\ 2\ 1\ 4$
$\rightarrow 3\ 5\ 1\ 2\ 4 \rightarrow 3\ 1\ 5\ 2\ 4 \rightarrow$
$\rightarrow 1\ 3\ 5\ 2\ 4 \rightarrow 1\ 3\ 2\ 5\ 4 \rightarrow$
$\rightarrow 1\ 2\ 3\ 5\ 4 \rightarrow 1\ 2\ 3\ 4\ 5$

$\Rightarrow$ 6 (четна)

Нека изведем формулата за пресметање на детерминанта за $n=2$:

$$\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (-1)^{[1,2]} a_{11} a_{22} + (-1)^{[2,1]} a_{12} a_{21} =$$

$$= (-1)^0 a_{11}a_{22} + (-1)^1 a_{12}a_{21} =$$
$$= a_{11}a_{22} - a_{12}a_{21}$$

Сега за $n = 3$ (набързо)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{aligned} & a_{11}a_{22}a_{33} + \\ & a_{12}a_{23}a_{31} + \\ & a_{13}a_{21}a_{32} - \\ & a_{13}a_{22}a_{31} - \\ & a_{12}a_{21}a_{33} - \\ & a_{11}a_{23}a_{32} \end{aligned}$$

Правило на Сарус:

$$\begin{array}{ccc|cc} + & + & + & & \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\ - & - & - & & \end{array}$$

Зад. Да се пресметне дет.

$$\begin{vmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{vmatrix} = \sin^2\alpha + \cos^2\alpha =$$
$$= 1$$

$$\begin{vmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & 1 \end{vmatrix} = 2+12+5-10-4-3$$
$$= 2$$

Свойства на детерминантите

* когато имаме ⋀3 редове в матр. A $\Rightarrow \det A = 0$
(A - квадратна)

|Дет. намираме само на квадратни матрици !

* при триъгълна матрица:

$$\begin{vmatrix} a_1 & & \theta \\ & a_2 & \\ * & & \ddots \\ & & & a_n \end{vmatrix} = a_1 a_2 \dots a_n = \begin{vmatrix} a_1 & & * \\ & a_2 & \\ \theta & & \ddots \\ & & & a_n \end{vmatrix}$$

* елементарни преобразувания

1. Прибавяне на ред към число, умножен с друг

$$\begin{vmatrix} - & A_1 & - \\ - & A_2 & - \\ - & - & - \\ - & - & - \end{vmatrix} = \begin{vmatrix} - & A_1 & - \\ - & \lambda A_1 + A_2 & - \\ - & - & - \\ - & - & - \end{vmatrix}$$

- det се запазва

2. Умножаване на ред с число λ ($\lambda \neq 0$)

$$\begin{vmatrix} - & A_1 & - \\ - & \lambda \cdot A_2 & - \\ - & - & - \\ - & - & - \end{vmatrix} = \lambda \begin{vmatrix} - & A_1 & - \\ - & A_2 & - \\ - & - & - \\ - & - & - \end{vmatrix}$$

- det се умн. по λ

3. Смяна на редове

$$\begin{vmatrix} - & A_2 & - \\ - & A_1 & - \\ - & - & - \\ - & - & - \end{vmatrix} = - \begin{vmatrix} - & A_1 & - \\ - & A_2 & - \\ - & - & - \\ - & - & - \end{vmatrix}$$

- det се умн. по (-1)

Ще разгледаме доста "специални" дет., които си имат удобни формули за пресмятане. Ще ги отбелязвам така

Зад. Да се пресметне det

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \begin{matrix} (-a) & (-a^2) \end{matrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} (b+a)$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(b+a)(c-a) \end{vmatrix} = (b-a)(c^2-a^2-(b+a)(c-a))$$

$$= (b-a)(c-a)(c-b)$$

Дет. от вида $\begin{vmatrix} 1 & 1 & & 1 \\ x_1 & x_2 & & x_n \\ x_1^2 & x_2^2 & & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & & x_n^{n-1} \end{vmatrix}$ нар.

Детерминанта на Вандермонд

и се пресмята по формулата

$$W(x_1, \dots, x_n) = \prod_{1 \le i < j \le n} (x_j - x_i)$$

Зад.

$$\begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix} \begin{array}{l} (-1) \\ \\ \\ \\ \end{array} =$$

$$= \begin{vmatrix} 1+a & b & c & d \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix} \begin{array}{l} \\ (-b) \\ (-c) \\ (-d) \end{array} =$$

$$= \begin{vmatrix} 1+a+b+c+d & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix} =$$

$$= 1+a+b+c+d$$

Зад. $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix} \begin{array}{l} (1) \\ \\ \\ \\ \end{array} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{vmatrix} =$$

$$= 8 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} \begin{array}{l} \\ (-1) \\ (-1) \\ (-1) \end{array} = 8 \begin{vmatrix} -2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = -16$$

Зад.

$$\begin{vmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{vmatrix} \begin{matrix} (-t)\ (-t^2)\ (-t^3) \\ \\ \\ \\ \end{matrix} =$$

$$= \begin{vmatrix} 1 & t & t^2 & t^3 \\ 0 & 1-t^2 & t-t^3 & t^2-t^4 \\ 0 & t-t^3 & 1-t^4 & t-t^5 \\ 0 & t^2-t^4 & t-t^5 & 1-t^6 \end{vmatrix} \begin{matrix} \\ /:(1-t^2) \\ /:(1-t^2) \\ /:(1-t^2) \end{matrix}$$

$$= (1-t^2)^3 \begin{vmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & t & t^2 \\ 0 & t & 1+t^2 & t+t^3 \\ 0 & t^2 & t+t^3 & 1+t^2+t^4 \end{vmatrix} \begin{matrix} \\ (-t)\ (-t^2) \\ = \\ \\ \end{matrix}$$

$$= (1-t^2)^3 \begin{vmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & t & t^2 \\ 0 & 0 & 1 & t \\ 0 & 0 & t & 1+t^2 \end{vmatrix} \begin{matrix} \\ \\ (-t) \\ \\ \end{matrix} =$$

$$= (1-t^2)^3 \begin{vmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & t & t^2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{vmatrix} = (1-t^2)^3$$

Свойство на дет:

$\det(A) = \det(A^t)$

$\Rightarrow$ можем да джуркаме и по **стълбове**

Зад. **Матрица тип "почти край"**

$$\begin{vmatrix} a_0 & b_1 & b_2 & \dots & b_n \\ c_1 & a_1 & 0 & \dots & 0 \\ c_2 & 0 & a_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ c_n & 0 & \dots & 0 & a_n \end{vmatrix} \begin{matrix} \\ \left(-\frac{b_1}{a_1}\right) \\ \vdots \\ \\ \left(-\frac{b_n}{a_n}\right) \end{matrix} =$$

$$= \begin{vmatrix} * & 0 & 0 & \dots & 0 \\ c_1 & a_1 & 0 & \dots & 0 \\ \vdots & 0 & a_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ c_n & 0 & \dots & 0 & a_n \end{vmatrix} = * \, a_1 \cdot a_2 \cdot \ldots \cdot a_n$$

¥ $= a_0 - \frac{c_1 b_1}{a_1} - \frac{c_2 b_2}{a_2} - \dots - \frac{c_n b_n}{a_n} =$

$$= a_0 - \sum_{i=1}^{n} \frac{a_i b_i}{c_i}$$

$$\Delta = a_0 \dots a_n - (b_1 c_1 a_2 \dots a_n + \dots + a_1 a_2 \dots a_n + b_n c_n)$$

важи и за $a_1, \dots, a_n = 0$

Зад.

$$\begin{vmatrix} 1 & 1 & \dots & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ \vdots & & \ddots & 0 & 1 \\ 1 & \dots & \dots & 1 & 0 \end{vmatrix} (-1) = \begin{vmatrix} 1 & 1 & \dots & \dots & 1 \\ 0 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & -1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & -1 \end{vmatrix} =$$

$$= (-1)^{n-1}$$

Зад.

$$\begin{vmatrix} a & b & \dots & b \\ b & a & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \dots & b & a \end{vmatrix} (-1) = \begin{array}{c} \begin{matrix} \quad & (1) & \quad & (1) & (1) \end{matrix} \\ \begin{vmatrix} a & b & \dots & \dots & b \\ b-a & a-b & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & 0 \\ b-a & 0 & \dots & 0 & a-b \end{vmatrix} \end{array} =$$

$$= \begin{vmatrix} a+(n-1)b & b & \cdots & b \\ 0 & a-b & 0 \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ & \vdots & \ddots & 0 \\ 0 & 0 & \cdots\ 0 & a-b \end{vmatrix} =$$

$$= (a+(n-1)b)(a-b)^{n-1}$$

Зад.

$$\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -x_1 & x_2 & \cdots & x_{n-1} & x_n \\ 1 & x_1 & -x_2 & \cdots & x_{n-1} & x_n \\ \vdots & \vdots & \vdots & \cdots & & \vdots \\ \vdots & x_1 & x_2 & & -x_{n-1} & x_n \\ 1 & x_1 & x_2 & & x_{n-1} & -x_n \end{vmatrix} =$$

$(-x_1)$

$(-x_2)$

$\vdots$

$(-x_n)$

$$= \begin{vmatrix} 0 & 1 & \cdots & \cdots & 1 \\ 1 & -2x_1 & 0 & \cdots & 0 \\ \vdots & 0 & -2x_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & -2x_n \end{vmatrix} =$$

$$= (-2)^{n-1} \begin{vmatrix} 0 & 1 & \cdots & \cdots & 1 \\ 1 & -x_1 & 0 & \cdots & 0 \\ \vdots & 0 & -x_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & -x_n \end{vmatrix} \begin{matrix} \left(\frac{1}{x_1}\right) \\ \vdots \\ \vdots \\ \left(\frac{1}{x_n}\right) \end{matrix} =$$

$$= (-2)^{n-1} \begin{vmatrix} \sum_{i=1}^{n} \frac{1}{x_i} & 0 & \cdots & \cdots & 0 \\ 1 & -x_1 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & -x_n \end{vmatrix} =$$

$$= (-2)^{n-1}(-1)^{n-1}\sum_{i=1}^{n}\frac{1}{x_i}\prod_{i=1}^{n} x_i$$

НО

$n=1 \quad |0| = 0$

$n=2 \quad \begin{vmatrix} 0 & 1 \\ 1 & -x_1 \end{vmatrix} = -1$

Развитие на детерминанта по ред и по стълб

* по ред

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{n1} & \dots & \dots & \dots & a_{nn} \end{vmatrix} =$$

Δ_{ij} – $(n-1)\times(n-1)$ детерминанта на матрицата без ред i и стълб j

$$= (-1)^{1+1} a_{11}\Delta_{11} + (-1)^{1+2} a_{12}\Delta_{12} + \dots + (-1)^{1+n} a_{1n}\Delta_{1n} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j}\Delta_{1j}$$

по стълб – аналогично

Зад.

$$\begin{vmatrix} 2 & 0 & 1 & 2 & 0 \\ 2 & -1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ -2 & 0 & 2 & -1 & 2 \\ 2 & 0 & 0 & 1 & 1 \end{vmatrix} \overset{(1)}{=} \begin{vmatrix} 2 & 0 & 1 & 2 & 0 \\ 2 & 0 & 2 & 2 & 3 \\ 0 & 1 & 2 & 1 & 2 \\ -2 & 0 & 2 & -1 & 2 \\ 2 & 0 & 0 & 1 & 1 \end{vmatrix} =$$

развиваме по втори стълб

$$= (-1)^{3+2} \cdot 1 \cdot \begin{vmatrix} 2 & 1 & 2 & 0 \\ 2 & 2 & 2 & 3 \\ -2 & 2 & -1 & 2 \\ 2 & 0 & 1 & 1 \end{vmatrix} = (-2) \begin{vmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 2 & 3 \\ -1 & 2 & -1 & 2 \\ 1 & 0 & 1 & 1 \end{vmatrix} \overset{(1)}{=}$$

2

$$= (-2) \begin{vmatrix} 0 & 3 & 1 & 2 \\ 0 & 4 & 1 & 5 \\ -1 & 2 & -1 & 2 \\ 0 & 2 & 0 & 3 \end{vmatrix} = (-2)(-1)(-1)^{3+1} \begin{vmatrix} 3 & 1 & 2 \\ 4 & 1 & 5 \\ 2 & 0 & 3 \end{vmatrix}$$

развиваме по първи стълб

$$= 2 \begin{vmatrix} 3 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 0 & 3 \end{vmatrix} =$$

$$= 2 \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 2(1 \cdot 3 - 2 \cdot 3) = -6$$

Зад.

$$\begin{vmatrix} x & y & 0 & \cdots & \cdots & 0 \\ 0 & x & y & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & y \\ 0 & & & & & \\ y & 0 & \cdots & \cdots & 0 & x \end{vmatrix} = (-1)^{1+1} \cdot x \begin{vmatrix} x & y & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & x & y \\ 0 & \cdots & \cdots & 0 & x \end{vmatrix}$$

развиване

$$+ (-1)^{n+1} y \begin{vmatrix} y & 0 & \cdots & 0 \\ x & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 \quad x & y \end{vmatrix} = x \cdot x^{n-1} + (-1)^{n+1} y \cdot y^{n-1} =$$

$$= x^n + (-1)^{n+1} y^n =$$

$$= x^n + (-1)(-1)^n y^n =$$

$$= x^n - (-y)^n$$

Зад.

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & & n \\ 1 & 3 & 3 & \cdots & \cdots & n \\ 1 & 2 & 5 & 4 & \cdots & n \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & n \\ 1 & 2 & \cdots & \cdots & n-1 & 2n-1 \end{vmatrix} \begin{array}{l} (-1) \\ \\ \\ \\ \\ \\ \end{array} =$$

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & n-1 \end{vmatrix} = (n-1)!$$

Зад.

$$\underbrace{\begin{vmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 2 & \cdots & \cdots & 2 \\ \vdots & \vdots & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{vmatrix}}_{\Delta_n} \begin{matrix} (-1) \\ \\ \\ \\ \end{matrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & 1 & 2 & \cdots & 2 \\ \vdots & 1 & \vdots & \ddots & \vdots \\ 0 & 1 & 2 & \cdots & n-1 \end{vmatrix} =$$

$$= \underbrace{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2 \\ \vdots & \vdots & & \\ \vdots & \vdots & & \\ 1 & 2 & & n-1 \end{vmatrix}}_{\Delta_{n-1}} \Rightarrow \Delta_n = \Delta_{n-1} = \Delta_{n-2} = \ldots = \Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 = \Delta_2$$

$$= \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 = \Delta_1 = |1| = 1$$

Овде пази крак:

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 6 \\ 1 & 1 & 1 & 1 & 6 & 1 \\ 1 & 1 & 1 & 6 & 1 & 1 \\ 1 & 1 & 6 & 1 & 1 & 1 \\ 1 & 6 & 1 & 1 & 1 & 1 \\ 6 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} \; (-1) =$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 & 5 & -5 \\ \vdots & & & 5 & 0 & -5 \\ \vdots & & \iddots & & \vdots & \vdots \\ 0 & \iddots & & & \vdots & \vdots \\ 5 & 0 & \cdots & \cdots & 0 & -5 \end{vmatrix} =$$

$$= 5^5 \begin{vmatrix} 1 & \cdots & 1 & 6 \\ 0 & \cdots\, 0 & 1 & -1 \\ \vdots & \iddots\; 1 & 0 & \vdots \\ 0 & \iddots & \vdots & \vdots \\ 1 & 0 \cdots\, 0 & & -1 \end{vmatrix} \; (-1) =$$

$$= 5^5 \begin{vmatrix} 0 & \cdots & 0 & 11 \\ 0 & \cdots & 1 & -1 \\ \vdots & & 0 & \vdots \\ 0 & \cdots & \vdots & \vdots \\ 1 & 0 & \cdots\ 0 & -1 \end{vmatrix} = 5^5 \cdot 11$$

Зад. 9. $(a \neq 0)$

$$\begin{vmatrix} x+1 & x & \cdots & x \\ x & x+a & \cdots & \vdots \\ \vdots & x & x+a^2 & x \\ \vdots & \vdots & \ddots & \vdots \\ x & x & \cdots\ x & x+a^n \end{vmatrix} \begin{matrix} (-1) \\ \downarrow \\ \\ \\ \downarrow \end{matrix} =$$

$$= \begin{vmatrix} x+1 & x & \cdots & x \\ -1 & a & 0 \cdots & 0 \\ \vdots & 0 & a^2 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ -1 & 0 & \cdots\ 0 & a^n \end{vmatrix} =$$

$$\left(\tfrac{1}{a}\right) \qquad \left(\tfrac{1}{a^n}\right)$$

$$= \begin{vmatrix} x+1+\sum_{i=1}^{n}\frac{x}{a^i} & x & \cdots & x \\ 0 & a & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a^n \end{vmatrix} =$$

$$= a^{\frac{n(n+1)}{2}} \left(x + 1 + \sum_{i=1}^{n} \frac{x}{a^i} \right) =$$

$$= a^{\frac{n(n+1)}{2}} \left(1 + x \frac{a^{n+1} - 1}{a^n (a-1)} \right)$$