CS3050: Logic and Reasoning

Exercise 1: Propositional Logic I

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Proof by Natural Deduction

Exercises taken from Huth and Ryan, §1.7.

1. The formulæ below follow the binding and precedence conventions highlighted in lectures. Re-insert as many brackets as possible.

   (a) $(P \to Q) \wedge \neg(R \vee O \to Q)$

   (b) $P \to Q \wedge \neg R \vee O \to Q$

   (c) $P \vee (\neg Q \to p \wedge R)$

a) $((P \to Q) \wedge (\neg(R \vee O) \to Q)))$

b) $((((P \to Q) \wedge \neg R) \vee O) \to Q)$

c) $(P \vee ((\neg Q \to p) \wedge R))$

2. Prove the validity of the following sequents using natural deduction.

   (a) $(P \wedge Q) \wedge R, S \wedge T \vdash Q \wedge S$

   (b) $P \wedge Q \vdash Q \wedge P$

   (c) $(P \wedge Q) \wedge R \vdash P \wedge (Q \wedge R)$

   (d) $P \to P \to Q, P \vdash Q$

   (e) $Q \to P \to R, \neg R, Q \vdash \neg P$

   (f) $\vdash P \wedge Q \to P$

   (g) $P \vdash Q \to P \wedge Q$

   (h) $(P \to R) \wedge (Q \to R) \vdash P \wedge Q \to R$

a)

1.	$(P \wedge Q) \wedge R$	premiss
2.	$S \wedge T$	premiss
3.	$P \wedge Q$	[∧-E$_1$] 1
4.	S	[∧-E$_1$] 2
5.	Q	[∧-E$_2$] 3
6.	$Q \wedge S$	[∧-I] 5,4

b)

1.	$P \wedge Q$	premiss
2.	P	[∧-E$_1$] 1
3.	Q	[∧-E$_2$] 2
4.	$Q \wedge P$	[∧-I] 3,2

c)

1.	$(P \wedge Q) \wedge R$	premiss
2.	$P \wedge Q$	[∧-E$_1$] 1
3.	R	[∧-E$_2$] 1
4.	P	[∧-E$_1$] 2
5.	Q	[∧-E$_2$] 2
6.	$Q \wedge R$	[∧-I] 5,3
7.	$P \wedge (Q \wedge R)$	[∧-I] 4,6

d)

1.	$P \to P \to Q$	premiss
2.	P	premiss
3.	$P \to Q$	[MP] 1,2
4.	Q	[MP] 3,2

e)

1.	$Q \to P \to R$	premiss
2.	$\neg R$	premiss
3.	Q	premiss
4.	$P \to R$	[MP] 1,3
5.	$\neg P$	[MT] 2,4

f)

1.	$P \wedge Q$	assumption
2.	P	[∧-E$_1$] 1
3.	$(P \wedge Q) \to P$	[→-I] 1-2

g)

1.	P	premiss
2.	Q	assumption
3.	$P \wedge Q$	[∧-I] 1,2
4.	$P \to (P \wedge Q)$	[→-I] 1-3

h)

1.	$(P \to R) \wedge (Q \to R)$	premiss
2.	$P \wedge Q$	assumption
3.	$Q \to R$	[∧-E$_2$] 1
4.	Q	[∧-E$_2$] 2
5.	R	[MP] 3,4
6.	$(P \wedge Q) \to R$	[→-I] 2-5

Propositional Languages

From *Logics for Computer Science* by Wasilewska.

Preliminaries Recall that we define $\mathcal{L}_C = (\Sigma, \mathcal{F})$ for a set C of logical connectives. We assume that the set C is non-empty and finite, and that

$$C = C_1 \cup C_2$$

where C_1 is a finite set of unary connectives, and C_2 is a finite set of binary connectives of the language $\mathcal{L}_C$. An *extensional* connective $\triangledown \in C_1$ or $\circ \in C_2$ is defined by a respective function

$$\triangledown : \{\mathsf{T}, \mathsf{F}\} \to \{\mathsf{T}, \mathsf{F}\} \qquad \text{or} \qquad \circ : \{\mathsf{T}, \mathsf{F}\} \to \{\mathsf{T}, \mathsf{F}\}$$

The set $\mathcal{F}$ of all well-formed formulæ of a propositional language $\mathcal{L}_C$ is the smallest set such that the following conditions hold.

1. $\textsc{Var} \in \mathcal{F}$
2. $\triangledown\phi \in \mathcal{F}$ if $\phi \in \mathcal{F}$ and $\triangledown \in C_1$
3. $(\phi \circ \psi) \in \mathcal{F}$ if $\phi, \psi \in \mathcal{F}$ and $\circ \in C_2$

Given a language $\mathcal{L}_C = (\Sigma, \mathcal{F})$. For any connectives $\triangledown \in C_1$ and $\circ \in C_2$, $\triangledown$ is called a *main connective* of $\triangledown P \in F$ and $\circ$ is a main connective of $(P \circ Q) \in \mathcal{F}$. A *direct sub-formula* is defined: 1. ϕ where $\triangledown\phi \in \mathcal{F}$; 2. ϕ, ψ where $(\phi \circ \psi) \in \mathcal{F}$. The formula ϕ is a *proper sub-formula* of ψ if there is a sequence of formulæ, beginning with ϕ and ending with ψ, where each formula is a direct sub-formula of of the next. A *sub-formula* of a given formula ψ is any proper sub-formula of ψ, or ψ itself. Let the *degree* of a formula be the number of occurrences of logical connectives in the formula.

Exercises

1. Let the symbols $\Diamond$ and $\Box$ be modal, unary connectives. A formula $\Diamond\phi$ reads: *it is possible that* ϕ; a formula $\Box\phi$ reads: *it is necessary that* ϕ.

   Given a language $\mathcal{L}_1 = \mathcal{L}_{\{\neg, \Diamond, \Box, \vee, \wedge, \to\}}$ and the following set $\mathcal{S}$.

   $$\mathcal{S} = \{\Diamond\neg A \to (A \vee B),\ (\Diamond(\neg A \to (A \vee B))),\ \Diamond\neg(A \to (A \vee B))\}$$

   Determine which of the elements of $\mathcal{S}$ are, and which are not well-formed formulæ of $\mathcal{L}_1$. If $\phi \in \mathcal{S}$ is not a correct formula write its corrected version. For each correct or corrected formula determine its main connective, its degree, and state what it says in the natural language.

2. Given a set $\mathcal{S}$ of formulæ:

   $$\mathcal{S} = \{((A \to \neg B) \to \neg A),\ \Box(\neg \Diamond A \to \neg A),\ (A \vee \neg(A \to B))\}$$

   Define a formal language $\mathcal{L}_C$ to which all formulæ in $\mathcal{S}$ belong.

3. Give the set $\mathcal{S}_1$ of all sub-formulæ of

   $$\Diamond((P \vee \neg P) \wedge Q)$$

   and the set $\mathcal{S}_2$ of all proper sub-formulæ of

   $$\neg(P \to (Q \to R))$$