CS3050: Exercise Sheet — Proof Techniques

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To remind you of the definition of rational numbers: if a real number r can be written as the quotient $r = a/b$ of two integers a and b, where $b \neq 0$, then r is rational. If a real number r cannot be written in this form then r is irrational.

1. Prove that if x^2 is even, then x is even.
2. Prove that $\sqrt{2}$ is irrational.
3. Prove that for all integers n, $n(n+1)$ is even.
4. Prove that for every integer x, the remainder when x^2 is divided by 4 is either 0 or 1.
5. Prove that for all real numbers r, if r^2 is irrational, then r is irrational.
6. Prove or disprove that for all irrational numbers a and b, $a + b$ is irrational.
7. Suppose that A, B, and C are sets. Prove that if $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$.
8. What is wrong with the following "proof"?

   Claim. If $\forall x \in A(x \neq 0)$ and $A \subseteq B$ then $\forall x \in B(B \neq 0)$.

   "Proof". Let x be an arbitrary element of A. By using the condition $\forall x \in A(x \neq 0)$, we know that $x \neq 0$. Furthermore, since $A \subseteq B$, $x \in B$. Since we know $x \neq 0$, $x \in B$, and x is an arbitrary element, we conclude $\forall x \in B(B \neq 0)$. $\square$

Q8. To prove $\forall x \in A(x \neq 0) \Rightarrow$ Start with $x \in B$

$\hookrightarrow$ If not, it doesn't cover elements of $B \setminus A$

E.g. $A = \{1\}$, $B = \{0, 1\}$

$\Rightarrow$ Then $\forall x \in A(x \neq 0)$ and $A \subseteq B$

But $\forall x \in B(B \neq 0)$ false.

Q1. If x^2 is even $\Rightarrow$ x is even

so contrapos. x is odd $\Rightarrow$ x^2 is odd

express $x = 2k+1$, $k \in \mathbb{Z}$.

$$x^2 = (2k+1)^2 = 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1$$

is odd

$\therefore$ by contrapos:

x^2 is not odd $\Rightarrow$ x is not odd

ie x^2 is even $\Rightarrow$ x is even $\square$

Q2. Assume $\sqrt{2} \in \mathbb{Q}$

so $\sqrt{2} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$

where a & b are coprime

Now $\sqrt{2} = \frac{a}{b} \Rightarrow 2 = \frac{a^2}{b^2}$

$$2b^2 = a^2$$

so a^2 is even $\Rightarrow$ a is even

write $a = 2k \Rightarrow a^2 = 4k^2$

$\therefore$ $2b^2 = 4k^2 \Rightarrow b^2 = 2k^2 \Rightarrow b^2$ is even

$\Rightarrow b$ is even

$\because$ a, b are even, they have common factor 2

& are not coprime $\lightning$

$\therefore$ $\sqrt{2}$ is irrational $\square$

Q3. $n(n+1)$ is even $\forall\, n \in \mathbb{Z}$.

Case: n is odd

so write $n = 2k+1$, $k \in \mathbb{Z}$

$\Rightarrow n(n+1) = (2k+1)(2k+1+1)$

$= (2k+1)(2k+2)$

$= 2(2k+1)(k+1)$

is even

Case: n is even $\Rightarrow n = 2k$, $k \in \mathbb{Z}$

$\Rightarrow n(n+1) = 2k(2k+1)$

$= 2[k(2k+1)]$ is even

$\therefore n(n+1)$ is even $\forall\, n \in \mathbb{Z}$ $\square$

Q4. For all $x \in \mathbb{Z}$, we get rem. $= 0$ or 1 if x^2 is div. by 4.

write $x^2 = 4q + r$, $q \in \mathbb{Z}$, $r = 0$ or 1 (to prove)

Case: x is odd $\Rightarrow x = 2k+1$, $k \in \mathbb{Z}$

$x^2 = (2k+1)^2 = 4k^2+4k+1$

$= 4(k^2+k)+1$

$\Rightarrow q = k^2+k$, $r = 1$

Case: x is even $\Rightarrow x = 2k$, $k \in \mathbb{Z}$

$x^2 = (2k)^2 = 4k^2$ which divides 4 ($r=0$)

$\therefore x^2$ gives rem. 0 or 1 when div. by 4 $\forall\, x \in \mathbb{Z}$ $\square$

Q5. If $r^2 \notin \mathbb{Q} \Rightarrow r \notin \mathbb{Q} \quad \forall r \in \mathbb{R}$

show contrapos: If $r \in \mathbb{Q} \Rightarrow r^2 \in \mathbb{Q}$

So write $r = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$.

Then $r^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$ where $a^2, b^2 \in \mathbb{Z}$

so by definition, $r^2 \in \mathbb{Q}$

$\therefore$ By contrapos: $r^2 \notin \mathbb{Q} \Rightarrow r \notin \mathbb{Q}$ $\square$

Q6. Consider counterexample $a = -\pi$, $b = \pi$

$a + b = -\pi + \pi = 0 \in \mathbb{Q}$

Q7. If $A \subseteq C$, $B \subseteq C \Rightarrow A \cup B \subseteq C$.

Suppose we have an arbitrary element $x \in A$ and $x \in B$

So by definition $x \in A \cup B$

And also, $x \in C$ since $A \subseteq C$, $B \subseteq C$

$\Rightarrow$ $x \in A \cup B$ and $x \in C$

$\Rightarrow$ $A \cup B \subseteq C$ $\square$

↳ Consider cases instead:

1. $x \in A$
2. $x \in B$