CS3050: Exercise Sheet — Induction II and First Order Theories

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1. Prove that the number of nodes $m(t)$ in a binary tree t is at most $2^{height(t)+1} - 1$ where height of the tree is defined as:
   - Base step: $\text{height}(t) = 0$ for a binary tree t, consisting of a leaf only
   - Recursive step: If the left and right children of the root node of t, t_L and t_R, are binary trees, then $\text{height}(t) = 1 + \max(\text{height}(t_L), \text{height}(t_R))$
2. Consider the set of strings S defined as follows (a and b are letters, and . is the concatenation operator on strings):
   - Base step: the empty string $\epsilon \in S$.
   - Recursive steps: If μ and ν are strings in the set S, then the strings $a.\mu.b$, $b.\mu.a$, and $\mu.\nu$ are also in the set S.

   Show that for any string s in S, s has the same number of a's and b's.
3. Let S be the subset of the set of ordered pairs of positive integers such that $(0, 0) \in P$ and if $(a, b) \in S$, then $(a + 2, b + 3) \in S$ and $(a + 3, b + 2) \in S$.

   Prove that 5 divides $(a + b)$ whenever $(a, b) \in S$.
4. Define the *reverse* function for strings as follows:
   - Base case: $reverse(\epsilon) = \epsilon$
   - Recursive case: $reverse(a \cdot \lambda) = concat(reverse(\lambda), (a \cdot \epsilon))$

   Prove by structural induction that for any string λ,

   $$length(reverse(\lambda)) = length(\lambda)$$
5. Prove in the theory of Dense Linear Orders (also called the theory of ordering on real numbers in the slides of Lecture 17):

   $$\forall x.\forall y.\forall z.\forall w.((x = y) \rightarrow (((y < z) \wedge (z < w)) \rightarrow (x < w)))$$
6. Prove in the theory of Dense Linear Orders (also called the theory of ordering on real numbers in the slides of Lecture 17):

   $$\forall x.\neg\exists y.(x < y \wedge y < x)$$