CS3050: Logic and Reasoning

Exercise 1: Propositional Logic I

Adam D. Barwell

19 September 2025

Proof by Natural Deduction

Exercises taken from Huth and Ryan, §1.7.

1. The formulæ below follow the binding and precedence conventions highlighted in lectures. Re-insert as many brackets as possible.

   (a) $(P \to Q) \wedge \neg(R \vee O \to Q)$

   (b) $P \to Q \wedge \neg R \vee O \to Q$

   (c) $P \vee (\neg Q \to p \wedge R)$

2. Prove the validity of the following sequents using natural deduction.

   (a) $(P \wedge Q) \wedge R, S \wedge T \vdash Q \wedge S$

   (b) $P \wedge Q \vdash Q \wedge P$

   (c) $(P \wedge Q) \wedge R \vdash P \wedge (Q \wedge R)$

   (d) $P \to P \to Q, P \vdash Q$

   (e) $Q \to P \to R, \neg R, Q \vdash \neg P$

   (f) $\vdash P \wedge Q \to P$

   (g) $P \vdash Q \to P \wedge Q$

   (h) $(P \to R) \wedge (Q \to R) \vdash P \wedge Q \to R$

Propositional Languages

From *Logics for Computer Science* by Wasilewska.

Preliminaries Recall that we define $\mathcal{L}_C = (\Sigma, \mathcal{F})$ for a set C of logical connectives. We assume that the set C is non-empty and finite, and that

$$C = C_1 \cup C_2$$

where C_1 is a finite set of unary connectives, and C_2 is a finite set of binary connectives of the language $\mathcal{L}_C$. An *extensional* connective $\triangledown \in C_1$ or $\circ \in C_2$ is defined by a respective function

$$\triangledown : \{\mathsf{T}, \mathsf{F}\} \to \{\mathsf{T}, \mathsf{F}\} \qquad \text{or} \qquad \circ : \{\mathsf{T}, \mathsf{F}\} \to \{\mathsf{T}, \mathsf{F}\}$$

The set $\mathcal{F}$ of all well-formed formulæ of a propositional language $\mathcal{L}_C$ is the smallest set such that the following conditions hold.

1. $\textsc{Var} \in \mathcal{F}$
2. $\triangledown\phi \in \mathcal{F}$ if $\phi \in \mathcal{F}$ and $\triangledown \in C_1$
3. $(\phi \circ \psi) \in \mathcal{F}$ if $\phi, \psi \in \mathcal{F}$ and $\circ \in C_2$

Given a language $\mathcal{L}_C = (\Sigma, \mathcal{F})$. For any connectives $\triangledown \in C_1$ and $\circ \in C_2$, $\triangledown$ is called a *main connective* of $\triangledown P \in F$ and $\circ$ is a main connective of $(P \circ Q) \in \mathcal{F}$. A *direct sub-formula* is defined: 1. ϕ where $\triangledown\phi \in \mathcal{F}$; 2. ϕ, ψ where $(\phi \circ \psi) \in \mathcal{F}$. The formula ϕ is a *proper sub-formula* of ψ if there is a sequence of formulæ, beginning with ϕ and ending with ψ, where each formula is a direct sub-formula of of the next. A *sub-formula* of a given formula ψ is any proper sub-formula of ψ, or ψ itself. Let the *degree* of a formula be the number of occurrences of logical connectives in the formula.

Exercises

1. Let the symbols $\Diamond$ and $\Box$ be modal, unary connectives. A formula $\Diamond\phi$ reads: *it is possible that* ϕ; a formula $\Box\phi$ reads: *it is necessary that* ϕ.

   Given a language $\mathcal{L}_1 = \mathcal{L}_{\{\neg, \Diamond, \Box, \vee, \wedge, \to\}}$ and the following set $\mathcal{S}$.

   $$\mathcal{S} = \{\Diamond\neg A \to (A \vee B),\ (\Diamond(\neg A \to (A \vee B))),\ \Diamond\neg(A \to (A \vee B))\}$$

   Determine which of the elements of $\mathcal{S}$ are, and which are not well-formed formulæ of $\mathcal{L}_1$. If $\phi \in \mathcal{S}$ is not a correct formula write its corrected version. For each correct or corrected formula determine its main connective, its degree, and state what it says in the natural language.

2. Given a set $\mathcal{S}$ of formulæ:

   $$\mathcal{S} = \{((A \to \neg B) \to \neg A),\ \Box(\neg \Diamond A \to \neg A),\ (A \vee \neg(A \to B))\}$$

   Define a formal language $\mathcal{L}_C$ to which all formulæ in $\mathcal{S}$ belong.

3. Give the set $\mathcal{S}_1$ of all sub-formulæ of

   $$\Diamond((P \vee \neg P) \wedge Q)$$

   and the set $\mathcal{S}_2$ of all proper sub-formulæ of

   $$\neg(P \to (Q \to R))$$