Matrices

Matrices

Definitions:

- **A matrix** is a rectangular array of numbers.
- A matrix with ***m*** **rows and** ***n*** **columns** is called an ***m* x *n*** matrix.

Note: The plural of matrix is *matrices*.

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Definitions:

- **A matrix** is a rectangular array of numbers.
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Note: The plural of matrix is *matrices*.

Definitions:

- A matrix with the same number of rows as columns is called a *square matrix*.
- Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

Matrices

- Let m and n be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & & . \\ . & . & & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- The ith row of $\mathbf{A}$ is the 1 x n matrix $[a_{i1}, a_{i2},\ldots,a_{in}]$. The jth column of $\mathbf{A}$ is the m x 1 *matrix:*

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{mj} \end{bmatrix}$$

- The (i,j)th *element* or *entry* of $\mathbf{A}$ is the element a_{ij}. We can use $\mathbf{A} = [a_{ij}]$ to denote the matrix with its (i,j)th element equal to a_{ij}.

Matrix addition

Defintion:

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be m x n matrices. The sum of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} + \mathbf{B}$, is the m x n matrix that has $a_{ij} + b_{ij}$ as its (i,j)th element. In other words, $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$.

Example:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

Note: matrices of different sizes can not be added.

Matrix multiplication

Definition:

- Let **A** be an m x k matrix and **B** be a k x n matrix. The *product* of **A** and **B**, denoted by **AB**, is the m x n matrix that has its (i,j)th element equal to the sum of the products of the corresponding elments from the ith row of **A** and the jth column of **B**. In other words, if

  AB = $[c_{ij}]$ then $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{jk}b_{kj}$.

Example:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

- The product is not defined when the number of columns in the first matrix is not equal to the number of rows in the second matrix

Matrix multiplication

The Product of **A** = $[a_{ij}]$ and **B** = $[b_{ij}]$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ . & . & & . \\ . & . & & . \\ a_{i1} & a_{i2} & \ldots & a_{ik} \\ . & . & & . \\ . & . & & . \\ a_{m1} & a_{m2} & \ldots & a_{mk} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & a_{12} & \ldots & b_{1j} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2j} & \ldots & b_{2n} \\ . & . & & . & & \\ . & . & & . & & \\ b_{k1} & b_{k2} & \ldots & b_{kj} & \ldots & b_{kn} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ . & . & & . \\ . & . & c_{ij} & . \\ . & . & & . \\ c_{m1} & c_{m2} & \ldots & c_{mn} \end{bmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

Matrix multiplication

- **Example**:

$$\begin{vmatrix} 2 & 4 & 3 \\ 1 & 1 & 2 \\ 2 & 4 & 6 \end{vmatrix} * \begin{vmatrix} 1 & 4 & 2 \\ 1 & 2 & 2 \\ 1 & 5 & 3 \end{vmatrix} = \begin{vmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{vmatrix}$$

Matrix multiplication

Properties of matrix multiplication:

- Does $\mathbf{AB} = \mathbf{BA}$?

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{AB} \neq \mathbf{BA}$$

Matrix multiplication

Properties of matrix multiplication:

- Does **AB** = **BA**?

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- **AB: ?**

Matrix multiplication

Properties of matrix multiplication:

- Does **AB** = **BA**?

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- **AB:**

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

Matrix multiplication

Properties of matrix multiplication:

- Does **AB** = **BA**?

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- **AB:** **BA: ?**

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

Matrix multiplication

Properties of matrix multiplication:

- Does **AB** = **BA**?

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- **AB:** **BA:**

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \qquad \mathbf{BA} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

- **Conclusion: AB ≠ BA**

Matrices

Definition:

- **The identity matrix (of order n)** is the $n \times n$ matrix $\mathbf{I}_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

$$\mathbf{I_n} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ . & . & & . \\ . & . & . & . \\ . & . & & . \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

Properties:

- Assume A is an $m \times n$ matrix. Then:

  $\mathbf{AI}_n = \mathbf{A}$ and $\mathbf{I}_m\mathbf{A} = \mathbf{A}$

- Assume A is an $n \times n$ matrix. Then: $\mathbf{A}^0 = \mathbf{I}_n$

Matrices

Definition: **Powers of square matrices**

- When A is an $n \times n$ matrix, we have:

$$\mathbf{A}^0 = \mathbf{I}_n \qquad \mathbf{A}^r = \underbrace{\mathbf{AAA}\cdots\mathbf{A}}_{r}$$

Matrix transpose

Definition:

- Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix. The **transpose** of $\mathbf{A}$, denoted by $\mathbf{A}^T$,is the $n \times m$ matrix obtained by interchanging the rows and columns of $\mathbf{A}$.

If $\mathbf{A}^T = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1,2,\ldots,n$ and $j = 1,2, \ldots,m$.

The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Matrix inverse

Definition:

- Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix. The **inverse** of $\mathbf{A}$, denoted by $\mathbf{A}^{-1}$, is the $n \times m$ matrix such that $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$

- Note: the inverse of the matrix $\mathbf{A}$ may not exist.