Propositional logic: review

- **Propositional logic:** a formal language for representing knowledge and for making logical inferences
- **A proposition is** a statement that is either true or false.
- **A compound proposition** can be created from other propositions using logical connectives
- **The truth of a compound proposition** is defined by truth values of elementary propositions and the meaning of connectives.
- **The truth table for a compound proposition:** table with entries (rows) for all possible combinations of truth values of elementary propositions.

Compound propositions

- Let p: 2 is a prime ….. **T**
  q: 6 is a prime ….. **F**
- Determine **the truth value** of the following statements:

  $\neg p$: **F**

  $p \wedge q$: **F**

  $p \wedge \neg q$: **T**

  $p \vee q$: **T**

  $p \oplus q$: **T**

  $p \rightarrow q$: **F**

  $q \rightarrow p$: **T**

Computer representation of True and False

We need to encode two values True and False:

- **Computers represents data and programs using 0s and 1s**
- Logical truth values – True and False
- A bit is sufficient to represent two possible values:
  - 0 (False) or 1(True)

- A variable that takes on values 0 or 1 is called a **Boolean variable**.

- **Definition**: A **bit string** is a sequence of zero or more bits. The **length** of this string is the number of bits in the string.

Bitwise operations

- T and F replaced with 1 and 0

p	q	$p \vee q$	$p \wedge q$
1	1	1	1
1	0	1	0
0	1	1	0
0	0	0	0

p	$\neg p$
1	0
0	1

Bitwise operations

- **Examples:**

 1011 0011	 1011 0011	 1011 0011
$\vee$ <u>0110 1010</u>	$\wedge$ <u>0110 1010</u>	$\oplus$ <u>0110 1010</u>
1111 1011	0010 0010	1101 1001

Applications of propositional logic

- **Translation of English sentences**
- **Inference and reasoning:**
  - new true propositions are inferred from existing ones
  - Used in Artificial Intelligence:
    - Rule based (expert) systems
    - Automatic theorem provers
- **Design of logic circuit**

Translation

Assume a sentence:

If you are older than 13 or you are with your parents then you can attend a PG-13 movie.

Parse:

- **If (you are older than 13 or you are with your parents) then (you can attend a PG-13 movie)**

Atomic (elementary) propositions:

- A= you are older than 13
- B= you are with your parents
- C=you can attend a PG-13 movie

- **Translation: $A \vee B \rightarrow C$**

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

Step 1 find logical connectives

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

Step 1 find logical connectives

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

Step 2 break the sentence into elementary propositions

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

$\underbrace{\text{You can have free coffee}}_{a}$ if $\underbrace{\text{you are senior citizen}}_{b}$ and $\underbrace{\text{it is a Tuesday}}_{c}$

Step 2 break the sentence into elementary propositions

Translation

- **General rule for translation .**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

$\underbrace{\text{You can have free coffee}}_{a}$ if $\underbrace{\text{you are senior citizen}}_{b}$ and $\underbrace{\text{it is a Tuesday}}_{c}$

Step 3 rewrite the sentence in propositional logic

$$\mathbf{b \wedge c \rightarrow a}$$

Translation

- Assume two elementary statements:
  - **p: you drive over 65 mph** ; **q: you get a speeding ticket**
- **Translate each of these sentences to logic**
  - you do not drive over 65 mph. $(\neg p)$
  - you drive over 65 mph, but you don't get a speeding ticket. $(p \wedge \neg q)$
  - you will get a speeding ticket if you drive over 65 mph. $(p \rightarrow q)$
  - if you do not drive over 65 mph then you will not get a speeding ticket. $(\neg p \rightarrow \neg q)$
  - driving over 65 mph is sufficient for getting a speeding ticket. $(p \rightarrow q)$
  - you get a speeding ticket, but you do not drive over 65 mph. $\mathbf{(q \wedge \neg p)}$

Application: inference

Assume we know the following sentences are true:

If you are older than 13 or you are with your parents then you can attend a PG-13 movie. You are older than 13.

Translation:

- **If (**you are older than 13 **or** you are with your parents **) then (**you can attend a PG-13 movie**)** . (You are older than 13).
  - A= you are older than 13
  - B= you are with your parents
  - C=you can attend a PG-13 movie
- $\mathbf{(A \vee B \rightarrow C), A}$
- $\mathbf{(A \vee B \rightarrow C) \wedge A}$ **is true**
- **<u>With the help of the logic we can infer the following statement (proposition):</u>**
  - You can attend a PG-13 movie or C **is True**

Tautology and Contradiction

- Some propositions are interesting since their values in the truth table are always the same

Definitions:

- A compound proposition that is always true for all possible truth values of the propositions is called a **tautology**.
- A compound proposition that is always false is called a **contradiction**.
- A proposition that is neither a tautology nor contradiction is called a **contingency**.

Example: $p \vee \neg p$ is a **tautology.**

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Tautology and Contradiction

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Example: $p \wedge \neg p$ is a **contradiction.**

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Equivalence

- We have seen that some of the propositions are equivalent. Their truth values in the truth table are the same.
- Example: $\mathbf{p \rightarrow q}$ is equivalent to $\mathbf{\neg q \rightarrow \neg p}$ **(contrapositive)**

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

- Equivalent statements are important for **logical reasoning** since they can be substituted and can help us to:

(1) make a logical argument and (2) infer new propositions

Logical equivalence

<u>Definition</u>: The propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology (alternately, if they have the same truth table). The notation **p <=> q** denotes p and q are logically equivalent.

Example of important equivalences

- **DeMorgan's Laws:**
- 1) $\neg(p \vee q) \iff \neg p \wedge \neg q$
- 2) $\neg(p \wedge q) \iff \neg p \vee \neg q$

Example: Negate "The summer in Mexico is cold and sunny" with DeMorgan's Laws

Solution: ?

Equivalence

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- 2) $\neg(p \wedge q) <=> \neg p \vee \neg q$

Example: Negate "The summer in Mexico is cold and sunny" with DeMorgan's Laws

Solution: "The summer in Mexico is not cold or not sunny."

Equivalence

Example of important equivalences

- **DeMorgan's Laws:**
- 1) $\neg(p \vee q) <=> \neg p \wedge \neg q$
- 2) $\neg(p \wedge q) <=> \neg p \vee \neg q$

To convince us that two propositions are logically equivalent use the truth table

p	q	$\neg p$	$\neg q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F		
T	F	F	T		
F	T	T	F		
F	F	T	T		

Equivalence

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- 1) $\neg(p \vee q) \iff \neg p \wedge \neg q$
- 2) $\neg(p \wedge q) \iff \neg p \vee \neg q$

To convince us that two propositions are logically equivalent use the truth table

p	q	¬p	¬q	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F	**F**	
T	F	F	T	**F**	
F	T	T	F	**F**	
F	F	T	T	**T**	

Equivalence

Example of important equivalences

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To convince us that two propositions are logically equivalent use the truth table

p	q	¬p	¬q	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F	**F**	**F**
T	F	F	T	**F**	**F**
F	T	T	F	**F**	**F**
F	F	T	T	**T**	**T**

Equivalence

Example of important equivalences

- **DeMorgan's Laws:**
- 1) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
- 2) $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

To convince us that two propositions are logically equivalent use the truth table

p	q	¬p	¬q	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F	**F**	**F**
T	F	F	T	**F**	**F**
F	T	T	F	**F**	**F**
F	F	T	T	**T**	**T**

Important logical equivalences

- **Identity**
  - $p \wedge T \Leftrightarrow p$
  - $p \vee F \Leftrightarrow p$
- **Domination**
  - $p \vee T \Leftrightarrow T$
  - $p \wedge F \Leftrightarrow F$
- **Idempotent**
  - $p \vee p \Leftrightarrow p$
  - $p \wedge p \Leftrightarrow p$

Important logical equivalences

- **Double negation**
  - $\neg(\neg p) \iff p$
- **Commutative**
  - $p \vee q \iff q \vee p$
  - $p \wedge q \iff q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \iff p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$

Important logical equivalences

- **Distributive**
  - $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
  - $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$
- **De Morgan**
  - $\neg(p \vee q) \iff \neg p \wedge \neg q$
  - $(p \wedge q) \iff \neg p \vee \neg q$
- **Other useful equivalences**
  - $p \vee \neg p \iff T$
  - $p \wedge \neg p \iff F$
  - **$p \rightarrow q \iff (\neg p \vee q)$**

Using logical equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example:** **Show $(p \wedge q) \rightarrow p$ is a tautology**.
- **Proof:** (we must show $(p \wedge q) \rightarrow p \iff T$)

$(p \wedge q) \rightarrow p \iff \neg(p \wedge q) \vee p$ Useful

- $\iff [\neg p \vee \neg q] \vee p$ DeMorgan
- $\iff [\neg q \vee \neg p] \vee p$ Commutative
- $\iff \neg q \vee [\neg p \vee p]$ Associative
- $\iff \neg q \vee [T]$ Useful
- $\iff T$ Domination

Using logical equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example:** Show $(p \wedge q) \rightarrow p$ is a tautology.
- Alternate proof:

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	**T**
T	F	F	**T**
F	T	F	**T**
F	F	F	**T**