Robust Rank Correlation Coefficients on the Basis of Fuzzy Orderings

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Introduction

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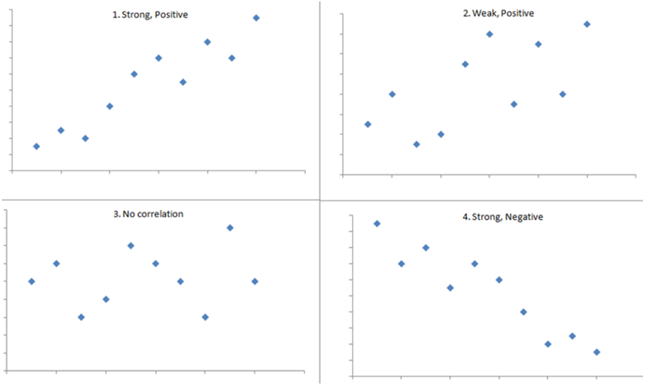

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Drawback - It can not be used to detect non-linear relationship.

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- Kendall's tau (rank correlation coefficient)

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Note - It did not take coinciding values (ties) into account.

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The basic variant of Kendall's tau which we denote with τ_a is computed as -

$$\boxed{\tau_a = \frac{C - D}{\frac{1}{2}n(n-1)}}$$

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> τ_a is best suited for detecting strictly monotonic relationships, but not ideally suited in the presence of ties.

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A simple and tie-robust rank correlation measure is the **gamma rank correlation** measure according to Goodman and Kruskal that is defined as

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Remark - τ_a, τ_b and γ coincide in all cases where no ties occur in the data.

Motivation

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The numerical data are almost always subject to **random perturbations (noise)** which the concepts introduced above do not take it into account.

This is because pairs are concordant or discordant only on the basis of **ordering relations**, but without taking into account that only minimal differences may decide whether a pair is concordant or discordant.

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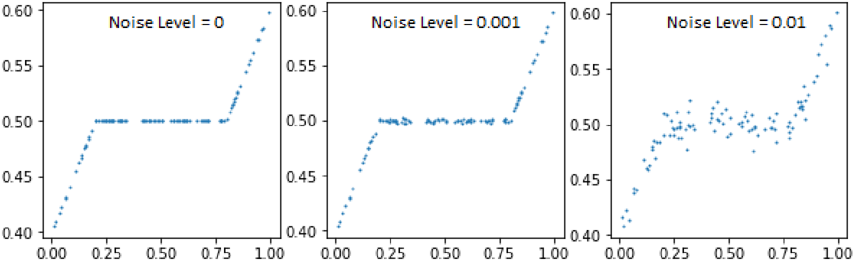
Noise Level = 0
Noise Level = 0.001
Noise Level = 0.01
0.60
0.55
0.50
0.45
0.40
0.00
0.25
0.50
0.75
1.00

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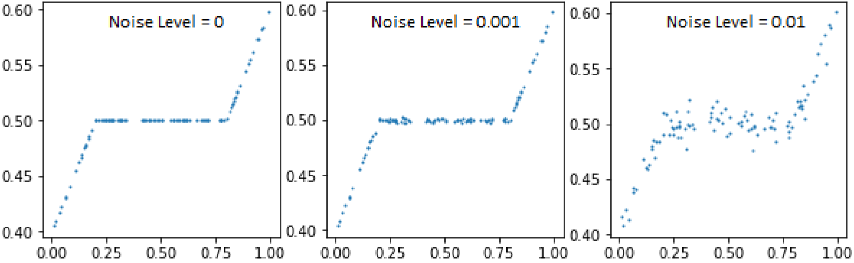

Noise level	0	0.001	0.01
ρ	0.8727	0.7730	0.7063
τ_a	0.61797	0.6254	0.5507
τ_b	0.7941	0.6318	0.5563
γ	1.0	0.6254	0.5507

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How we can define a **robust rank correlation** measure that depends continuously on the data by taking similarities into account, but still serves as a meaningful measure of rank correlation.

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- T-transitivity: $T(L(x,y), L(y,z)) \leq L(x,z)$

Moreover, we call a T-E-ordering **L strongly complete** if

$$max(L(x,y), L(y,x)) = 1$$

for all $x, y \in X$

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For the above E_r and T_L we can prove that

$$L_r(x, y) = min(1, max(0, 1 - \frac{1}{r}(x - y)))$$

is a strongly complete $T_L - E_r$-ordering on $\mathbb{R}$.

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Given a T-E-ordering $L : X^2 \to [0,1]$,

$$R(x,y) = min(L(x,y), N_T(L(y,x))) \qquad (1)$$

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$$min(R(x, y), R(y, x)) = 0 \tag{3}$$

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Fuzzy Ordering-Based rank correlation coefficient

Assume that the data is given as (x_i, y_i) with $x_i \in X$ and $y_i \in Y$ for all $i = 1, 2, ..., n$. Further assume that E_X and E_Y are two T_L-equivalences, L_X and L_Y a strong complete T_L-E_X-ordering, T_L-E_Y-ordering resp. Then we can define strict T_L-E_X-ordering on X as $R_X = 1 - L_X(x_2, x_1)$ and a strict T_L-E_Y-ordering on Y as $R_Y = 1 - L_Y(y_2, y_1)$

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Given a pair (i, j), we can compute the degree to which (i, j) is **concordant pair** as

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where $\bar{T}$ is some t-norm to aggregate the relationships of x and y components.

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So the fuzzy Ordering-Based rank correlation measure $\tilde{\gamma}$ as

$$\boxed{\tilde{\gamma} = \frac{\tilde{C} - \tilde{D}}{\tilde{C} + \tilde{D}}}$$

Experiment 1

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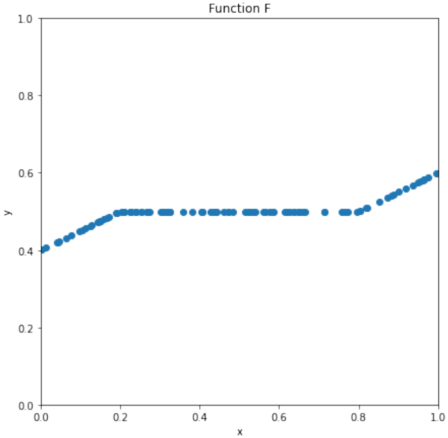

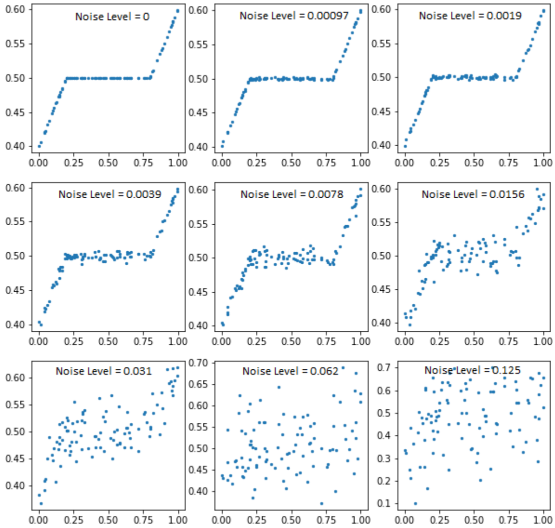

Figure- Different data sets obtained from contaminating a non-decreasing relationship by normally distributed noise with different standard deviations.

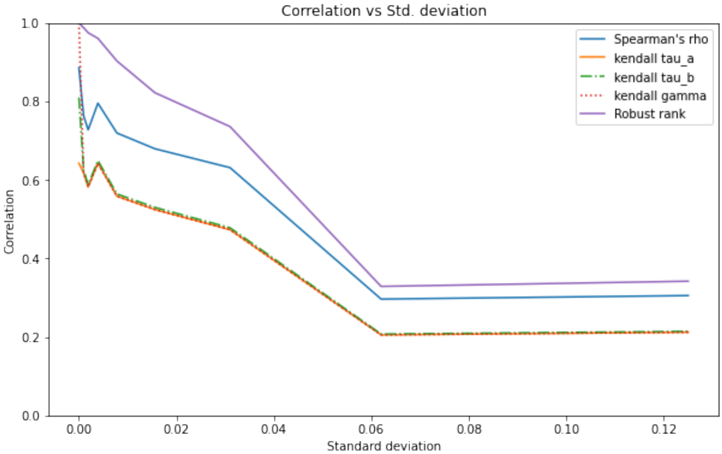

Figure: Results obtained by applying different correlation measures to the data sets

Result (Experiment 1)

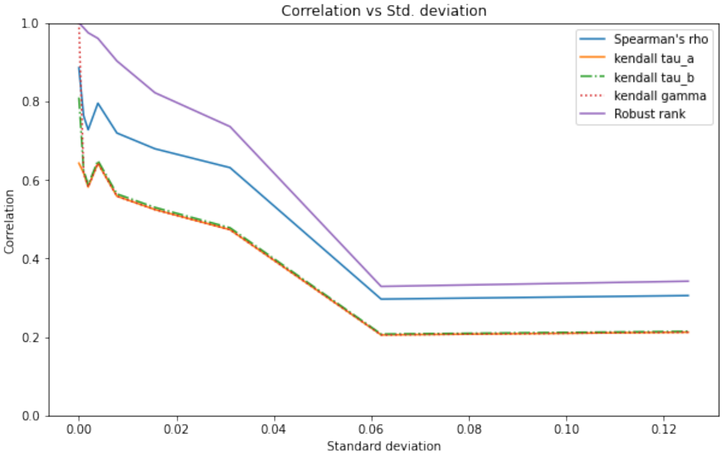

Figure: Results obtained by applying different correlation measures to the data sets

The three lines τ_a, τ_b and γ coincide except for no noise. Lines reveal that these measures react to noise in a **non-robust** way.

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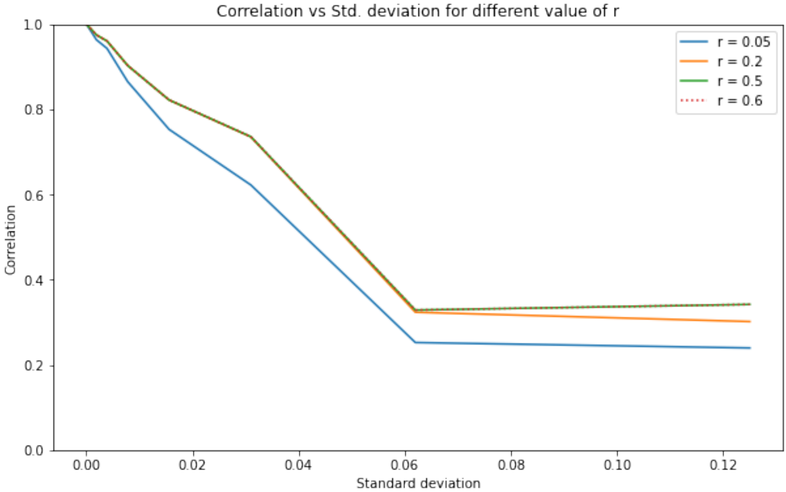

Figure: Results obtained by using different value of r in Robust rank correlation measures to the data sets

Result (Experiment 1)

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Note- larger the r, more difficult it is for $\tilde{\gamma}$ to find out whether there are slightly non-monotonic parts in the data.

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This time noise level is fixed $\sigma = 0.01$.

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Right of $x = 0.5$, we use $f(x) = \frac{x}{2} + \frac{1}{4}$ and, to the left of $x = 0.5$, we linearly interpolate between $(0, q)$.

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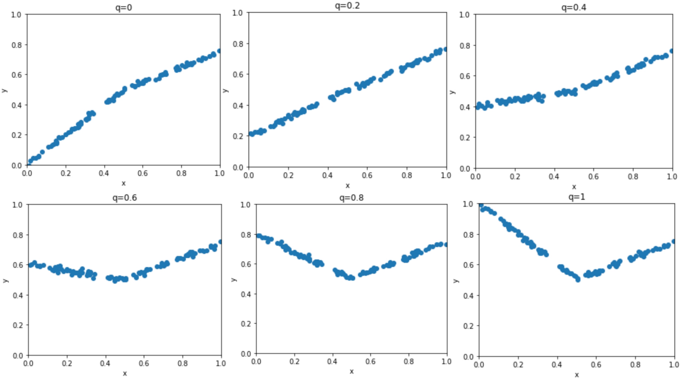

Figure: Noisy data sets that correspond to monotonic ($q \leq 0.5$) and non-monotonic relationships ($q > 0.5$).

Result (Experiment 2)

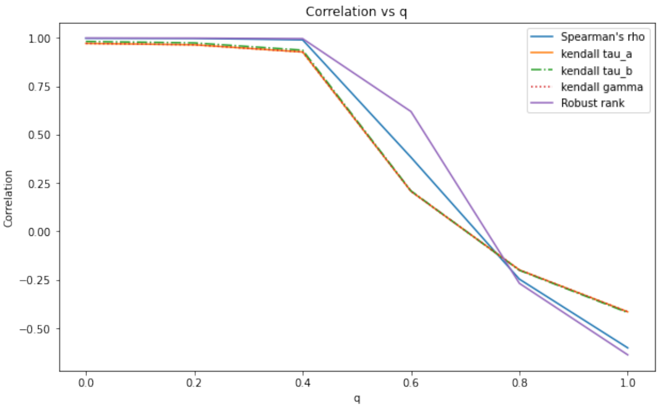

Figure: Results obtained by applying different correlation measures to the data sets.

Result (Experiment 2)

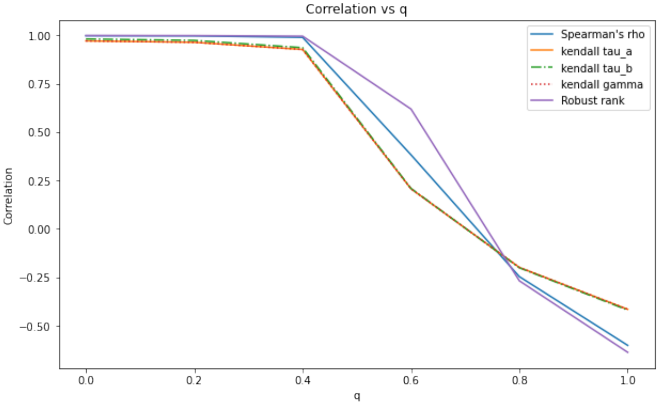

Figure: Results obtained by applying different correlation measures to the data sets.

ρ, τ_a, τ_b and γ again have problems to handle the noise in case of the large proportion of ties that occurs for $q = 0.5$.

Result (Experiment 2)

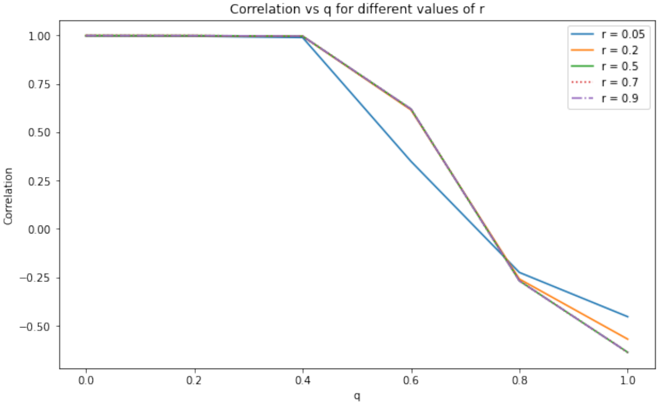

Figure: Results obtained by applying Robust rank correlation measures to the data sets for different values of r.

Result (Experiment 2)

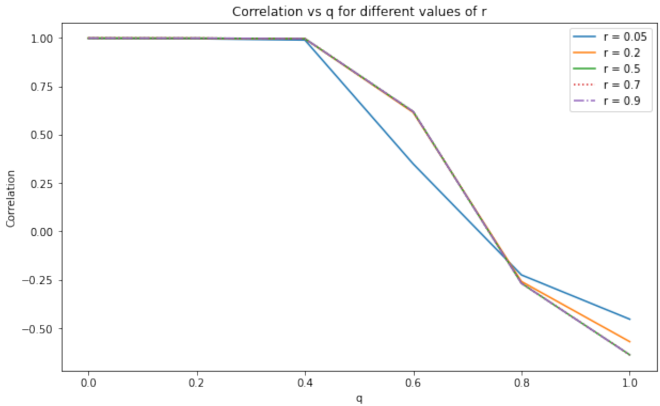

Figure: Results obtained by applying Robust rank correlation measures to the data sets for different values of r.

- All the variants of $\tilde{\gamma}$ show acceptable results for $q \leq 0.5$.

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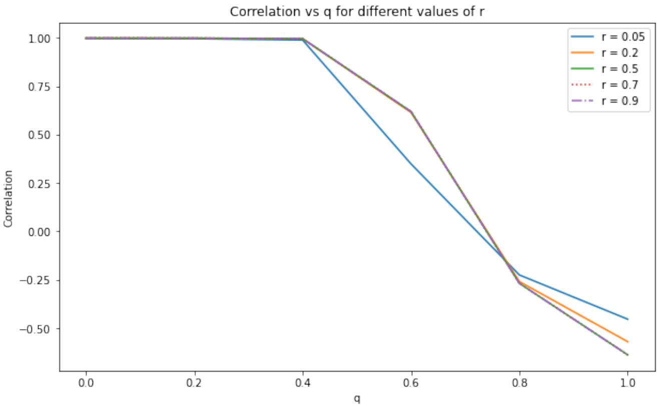

Figure: Results obtained by applying Robust rank correlation measures to the data sets for different values of r.

- All the variants of $\tilde{\gamma}$ show acceptable results for $q \leq 0.5$.
- $\tilde{\gamma}$ yields significantly lower values for $q = 0.6$ in the case $r = 0.05$ which is not in the case of large value of r.

Conclusion

- Robust rank coefficient perform better with noise as compare to other measure.

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- When choosing r, there is a trade-off between robustness (the larger r, the better) and sensitivity (the smaller r, the better).

References

- Robust Rank Correlation Coefficients on the Basis of Fuzzy Orderings: Initial Steps by U. Bodenhofer and F. Klawonn
- The Kendall rank correlation coefficient by H. Abdi. In N. J. Salkind, editor, Encyclopedia of Measurement and Statistics. Sage, Thousand Oaks, CA, 2007.