Solutions to Quiz #3

MTH210: Discrete Structures

Monsoon 2024

Problem 1

Problem: Use generating functions to find the number of non-negative integer solutions of:

$$x_1 + x_2 + x_3 + x_4 = 11,$$

where $x_1 \geq 0$, $x_2 \geq 2$, $2 \leq x_3 \leq x_4$, and $x_4 \geq 3$.

Solution:

1. Let $x_1' = x_1$, $x_2' = x_2 - 2$, $x_3' = x_3 - 2$, and $x_4' = x_4 - 3$. Substituting these into the equation gives:
$$x_1' + x_2' + x_3' + x_4' = 11 - 2 - 2 - 3 = 4,$$
where $x_1', x_2', x_3', x_4' \geq 0$ and $x_3' \leq x_4'$.

2. The generating functions for each variable are:
$$x_1' : \frac{1}{1-z}, \quad x_2' : z^2\frac{1}{1-z}, \quad x_3' : z^2\frac{1}{1-z}, \quad x_4' : z^3\frac{1}{1-z}.$$

3. Combining these, the total generating function is:
$$f(z) = \frac{z^2}{(1-z)} \cdot \frac{z^2}{(1-z)} \cdot \frac{z^3}{(1-z)} \cdot \frac{1}{1-z} = \frac{z^7}{(1-z)^4}.$$

4. The coefficient of z^4 in $\frac{1}{(1-z)^4}$ is given by:
$$\binom{4+4-1}{4-1} = \binom{7}{3} = 35.$$

Thus, the total number of solutions is $\boxed{35}$.

Problem 2

Problem: Four friends (Anna, Bob, Carol, and Dave) exchange gifts, but no one receives their own gift. How many ways can this happen?

Solution:

This is a derangement problem where no element maps to itself. The formula for the number of derangements D_n of n items is:

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

For $n = 4$:

$$D_4 = 4! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right).$$

$$D_4 = 24\left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right) = 24\left(\frac{12}{24} - \frac{4}{24} + \frac{1}{24}\right) = 24 \cdot \frac{9}{24} = 9.$$

Thus, the total number of derangements is $\boxed{9}$.

Problem 3

Problem: A school has 5 math teachers, 4 science teachers, and 3 language teachers. Form a committee of 3 teachers that includes at least one math teacher.

Solution:

1. Total ways to choose any 3 teachers:
$$\binom{12}{3} = 220.$$
2. Total ways to form a committee with no math teachers (only science and language teachers):
$$\binom{7}{3} = 35.$$
3. Committees with at least one math teacher:
$$220 - 35 = 185.$$

Thus, the total number of committees with at least one math teacher is $\boxed{185}$.

Problem 4

Problem: Prove that graphs G and H are isomorphic.

Solution:

Consider the map between vertices set of two graphs A-4, B-2, C-3, D-1, E-5 or alternatively, A-4, B-3, C-2, D-1, E-5. Then it is a graph isomorphism.