Logic and Graph Theory

Traffic Lights and Graph Constraints

1. Traffic Lights (1+2+2) = 5 marks

1. At any given point in time, the light is either green, yellow, or red:

Definitions:

- $g_t \rightarrow$ Traffic light is green at time t
- $r_t \rightarrow$ Traffic light is red at time t
- $y_t \rightarrow$ Traffic light is yellow at time t

$$(g_t \leftrightarrow \neg(r_t \vee y_t)) \wedge (r_t \leftrightarrow \neg(g_t \vee y_t)) \wedge (y_t \leftrightarrow \neg(r_t \vee g_t))$$

The double-headed arrow $\leftrightarrow$

is used to express logical equivalence, meaning that the state of the traffic light being green, red, or orange is mutually exclusive and exhaustive.

- Exclusivity: Only one of the colors can be true at any given instant t.
- Exhaustiveness: The light must always be in one of these states (green, red, or orange).

OR

You can avoid $\leftrightarrow$ entirely by writing:

$$(g_k \vee r_k \vee o_k) \wedge (\neg(g_k \wedge r_k)) \wedge (\neg(g_k \wedge o_k)) \wedge (\neg(r_k \wedge o_k))$$

This formula ensures:

1. The light must be in at least one state:
$$g_k \vee r_k \vee o_k$$
2. The light cannot be in more than one state simultaneously:
$$\neg(g_k \wedge r_k), \quad \neg(g_k \wedge o_k), \quad \neg(r_k \wedge o_k)$$

2. Traffic lights switch from green to yellow, yellow to red, and red to green:
Case 1:

$$(g_{t-1} \rightarrow (y_t \vee g_t)) \wedge (y_{t-1} \rightarrow (y_t \vee r_t)) \wedge (r_{t-1} \rightarrow (r_t \vee g_t))$$

This case allows the traffic light to either stay in the same color or transition to the next color. This is a more realistic model since traffic lights often stay on the same color for multiple timesteps.

OR

Case 2:

$$(g_{t-1} \to y_t) \land (y_{t-1} \to r_t) \land (r_{t-1} \to g_t)$$

This case assumes the traffic light must change to the next color at every timestep. However, this is less realistic and partially correct as traffic lights usually remain in one color for a few timesteps before switching. [**A penalty of 0.5 marks**]

3. The light cannot remain in the same state for more than 3 consecutive cycles:

$$(g_{t-3} \land g_{t-2} \land g_{t-1} \to \neg g_t) \land (r_{t-3} \land r_{t-2} \land r_{t-1} \to \neg r_t) \land (y_{t-3} \land y_{t-2} \land y_{t-1} \to \neg y_t)$$

If the light has been green for the past 3 instants (t-3), (t-2), and (t-1), it cannot remain green at t. Similarly, this applies to red and orange.

2. Graph Theory (2*5)=10 marks

1. Connected nodes don't have the same color

Definitions:

- $\text{edge}(m, n) \to$ There is an edge between nodes m and n
- $\text{color}(m, c) \to$ Node m is of color c

$$\forall m\, \forall n\, \forall c\ (\text{edge}(m, n) \land \text{color}(m, c) \to \neg\text{color}(n, c))$$

2. Exactly 2 nodes are allowed to be yellow:

$$\exists m_1\, \exists m_0\ (\text{color}(m_1, \text{yellow}) \land \text{color}(m_0, \text{yellow}) \land m_1 \neq m_0 \land \forall n\, (\text{color}(n, \text{yellow}) \to (n = m_1 \lor n = m_0)))$$

OR

$$\exists n\, \exists n_0\, \Big(\text{color}(n, \text{yellow}) \land \text{color}(n_0, \text{yellow}) \land n \neq n_0 \land \forall m\, \big(m \neq n \land m \neq n_0 \to \neg\text{color}(m, \text{yellow})\big)\Big)$$

3. Starting from any red node, you reach out to green nodes in at most 4 steps:

$$\begin{gathered}\forall n\, \Big(\text{color}(n, \text{red}) \to \Big(\exists m_1\, (\text{edge}(n, m_1) \land \text{color}(m_1, \text{green}))\lor \\ \exists m_1, m_2\, (\text{edge}(n, m_1) \land \text{edge}(m_1, m_2) \land \text{color}(m_2, \text{green}))\lor \\ \exists m_1, m_2, m_3\, (\text{edge}(n, m_1) \land \text{edge}(m_1, m_2) \land \text{edge}(m_2, m_3) \land \text{color}(m_3, \text{green}))\lor \\ \exists m_1, m_2, m_3, m_4\, (\text{edge}(n, m_1) \land \text{edge}(m_1, m_2) \land \text{edge}(m_2, m_3) \land \text{edge}(m_3, m_4) \land \text{color}(m_4, \text{green}))\Big)\Big)\end{gathered}$$

4. For each color, there is at least 1 node:

$$\forall c\, \exists n\, \text{color}(n, c)$$

5. Nodes are divided into exactly $|C|$ disjoint non-empty cliques, one for each color:

$$\forall x \exists n\, \text{color}(n, x) \wedge$$

$$\forall n \exists x\, \text{color}(n, x) \wedge$$

$$\forall n \forall x\, (\text{color}(n, x) \rightarrow \neg \exists y\, (y \neq x \wedge \text{color}(n, y))) \wedge$$

$$\forall n \forall m \forall x\, (n \neq m \wedge \text{color}(n, x) \wedge \text{color}(m, x) \rightarrow \text{edge}(n, m) \vee \bigvee_{i=1}^{|N|} ((\exists n_1, \ldots, n_i : (\text{edge}(n, n_1) \wedge \bigwedge_{j=1}^{i-1} \text{edge}(x_j, x_{j+1}) \wedge \text{edge}(n_i, m)))))$$

The first part of the formula ensures that every color x is associated with at least one node n. This condition guarantees that no color is "empty." The second part ensures that every node n has at least one color x. The third part ensures that each node belongs to exactly one color and that there is no node n that has multiple colors. The last part specifies that for nodes n and m that share the same color x, they must form a clique, i.e., they must be connected either directly or indirectly through other nodes of the same color.

3. Statements (2.5 - PL and 2.5 - FOL)= 5 Marks

FOL

Let:

- Read(x): x can read.
- Literate(x): x is literate.
- Dolphin(x): x is Dolphin.
- Intelligent(x): x is Intelligent.

1. Whoever can read is literate:

$$\forall x\, (\text{Read}(x) \rightarrow \text{Literate}(x))$$

2. Dolphins are not literate:

$$\forall x\, (\text{Dolphin}(x) \rightarrow \neg \text{Literate}(x))$$

3. Some dolphins are intelligent:

$$\exists x\, (\text{Dolphin}(x) \wedge \text{Intelligent}(x))$$

4. Some who are intelligent cannot read:

$$\exists x\, (\text{Intelligent}(x) \wedge \neg \text{Read}(x))$$

5. A dolphin exists who is both intelligent and can read, but for every intelligent dolphin, if it can read, it must not be literate:

$$\exists x\, (\text{Dolphin}(x) \wedge \text{Intelligent}(x) \wedge \text{Read}(x)) \wedge \forall x\, (\text{Dolphin}(x) \wedge \text{Intelligent}(x) \wedge \text{Read}(x) \rightarrow \neg \text{Literate}(x)).$$

PL

Propositional logic does not support quantifiers ($\forall$, $\exists$) because it operates on atomic propositions, not individual objects or predicates. *(If this point has already been mentioned and expressions were not provided, the answer will still be considered correct).*

Each proposition is a single truth-valued entity, such as "R: Someone can read," without reference to specific individuals.

- R: Someone can read.
- L: Someone is literate.
- D: Someone is a dolphin.
- I: Someone is intelligent.
- R: Someone can read.

1. Whoever can read is literate:

$$R \rightarrow L$$

2. Dolphins are not literate:

$$D \rightarrow \neg L$$

3. Some dolphins are intelligent:

$$D \wedge I$$

4. Some who are intelligent cannot read:

$$I \wedge \neg R$$

5. A dolphin exists who is both intelligent and can read, but for every intelligent dolphin, if it can read, it must not be literate:

$$(D \wedge I \wedge R) \wedge (D \wedge I \wedge R \rightarrow \neg L)$$

Proving the Satisfiability of 4th and 5th sentence (5 Marks each)

To prove this, we will use proof by contradiction. This means we assume the negation of the statement we want to prove, and then show that this assumption leads to a contradiction.

We want to prove:

$$\exists x \, [Intelligent(x) \wedge \neg Read(x)]$$

Step 1: Negate the Theorem ——1 mark

Negate the theorem we want to prove:

$$\neg \left(\exists x \, [Intelligent(x) \wedge \neg Read(x)] \right)$$

By applying the negation to an existential quantifier, we get:

$$\forall x \, [Intelligent(x) \rightarrow Read(x)]$$

This is our negated theorem.

Conversion to Clause Form ——1 mark

Now, let's rewrite all the given statements and the negated theorem in clause form, which is the format used in automated theorem proving (such as resolution).

Axioms Converted to Clauses

Axiom 1:

$$\forall x\, [Read(x) \to Literate(x)]$$

This can be rewritten as (dropping universal quantifiers):

$$\neg Read(x) \lor Literate(x)$$

Clause Form:

$$C_1 : \neg Read(x) \lor Literate(x)$$

Axiom 2:

$$\forall x\, [Dolphin(x) \to \neg Literate(x)]$$

This can be rewritten as:

$$\neg Dolphin(x) \lor \neg Literate(x)$$

Clause Form:

$$C_2 : \neg Dolphin(x) \lor \neg Literate(x)$$

Axiom 3:

$$\exists x\, [Dolphin(x) \land Intelligent(x)]$$

By introducing a Skolem constant A to represent the existence of such an x, we have:

$$Dolphin(A) \quad \text{and} \quad Intelligent(A)$$

Clause Form:

$$C_3 : Dolphin(A)$$
$$C_4 : Intelligent(A)$$

Negated Theorem in Clause Form ——0.5 mark

The negated form was:

$$\forall x\, [Intelligent(x) \to Read(x)]$$

This can be rewritten as:

$$\neg Intelligent(x) \lor Read(x)$$

Clause Form:

$$C_5 : \neg Intelligent(x) \lor Read(x)$$

Applying Resolution to Derive a Contradiction ——2 marks

Now, we use the clauses to derive a contradiction:

From Clause $C_4 : Intelligent(A)$, we know that A is intelligent.

Substitute $z = A$ in Clause $C_5 : \neg Intelligent(z) \vee Read(z)$:

$$\neg Intelligent(A) \vee Read(A)$$

Since we know $Intelligent(A)$ from C_4, this simplifies to:

$$Read(A)$$

So, we have:

$$C_6 : Read(A)$$

Now, use Clause $C_1 : \neg Read(x) \vee Literate(x)$:
Substitute $x = A$:

$$\neg Read(A) \vee Literate(A)$$

Since we have $Read(A)$ from C_6, this simplifies to:

$$Literate(A)$$

So, we have:

$$C_7 : Literate(A)$$

Now, use Clause $C_2 : \neg Dolphin(y) \vee \neg Literate(y)$:
Substitute $y = A$:

$$\neg Dolphin(A) \vee \neg Literate(A)$$

Since we have C_2 and C_7,this simplifies to:

$$C_8 : \neg Dolphin(A)$$

Now, we have a contradiction between Clause $C_8 : Dolphin(A)$ and Clause $C_3 : Dolphin(A)$.

Conclusion ——0.5 mark

The assumption that:

$$\neg(\exists x\,[Intelligent(x) \wedge \neg Read(x)])$$

leads to a contradiction. Therefore, the negation of the theorem is false, and the original theorem:

$$\exists x\,[Intelligent(x) \wedge \neg Read(x)]$$

is true.
This proves that there exists an individual who is intelligent but cannot read.

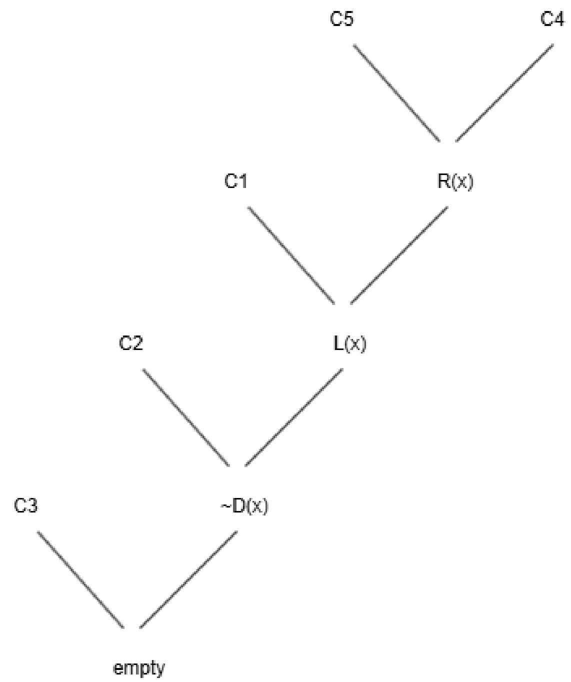

Figure 1: Resolution steps for Statement 4

Proving 5th statement (5 Marks)

Statement to Prove

First-order logic (FOL) representation:

$\exists x\,(Dolphin(x) \wedge Intelligent(x) \wedge Read(x)) \wedge \forall x\,(Dolphin(x) \wedge Intelligent(x) \wedge Read(x) \rightarrow \neg Literate(x))$.

Step 1: Negate the Theorem ——1 mark

Negate the theorem we want to prove:

$\neg\big(\exists x\,(Dolphin(x) \wedge Intelligent(x) \wedge Read(x)) \wedge \forall x\,(Dolphin(x) \wedge Intelligent(x) \wedge Read(x) \rightarrow \neg Literate(x))\big)$.

Thus, the negated theorem is:

$\forall x\,(\neg Dolphin(x) \vee \neg Intelligent(x) \vee \neg Read(x)) \vee \exists x\,(Dolphin(x) \wedge Intelligent(x) \wedge Read(x) \wedge Literate(x))$.

Let,

$$P1 : \forall x\,(\neg Dolphin(x) \vee \neg Intelligent(x) \vee \neg Read(x))$$

$$P2 : \exists x\,(Dolphin(x) \wedge Intelligent(x) \wedge Read(x) \wedge Literate(x)).$$

$$C1 : P1 \vee P2$$

1. From P1 (dropping the quantifier)

$$\neg Dolphin(x) \vee \neg Intelligent(x) \vee \neg Read(x).$$

2. From P2, introduce a Skolem constant C:

$$(Dolphin(C) \wedge Intelligent(C) \wedge Read(C) \wedge Literate(C))$$

Step 2: Conversion to Clause Form

We need to distribute the conjunction over the disjunction to get individual clauses.

Clauses from the Negated Theorem ———0.5 mark

$$C2 : \neg Dolphin(x) \vee \neg Intelligent(x) \vee \neg Read(x) \vee Dolphin(C)$$

$$C3 : \neg Dolphin(x) \vee \neg Intelligent(x) \vee \neg Read(x) \vee Intelligent(C)$$

$$C4 : \neg Dolphin(x) \vee \neg Intelligent(x) \vee \neg Read(x) \vee Read(C)$$

$$C5 : \neg Dolphin(x) \vee \neg Intelligent(x) \vee \neg Read(x) \vee Literate(C)$$

Clauses from the Axioms ———1 mark

1. **Whoever can read is literate:**

$$\forall x\,(Read(x) \rightarrow Literate(x)) \quad \text{becomes} \quad C6 : \neg Read(x) \vee Literate(x).$$

2. **Dolphins are not literate:**

$$\forall x\,(Dolphin(x) \rightarrow \neg Literate(x)) \quad \text{becomes} \quad C7 : \neg Dolphin(x) \vee \neg Literate(x).$$

3. **Some dolphins are intelligent:**

$$Dolphin(A), \quad Intelligent(A).$$

Clause form:

$$C8 : Dolphin(A), \quad C9 : Intelligent(A).$$

4. **Some who are intelligent cannot read:**

$$Intelligent(B), \quad \neg Read(B).$$

Clause form:

$$C10 : Intelligent(B), \quad C11 : \neg Read(B).$$

Resolving C2 to C5 :

$$C2 : \neg Dolphin(x) \vee \neg Intelligent(x) \vee \neg Read(x) \vee Dolphin(C)$$

$$C11 : \neg Intelligent(x) \vee \neg Read(x)$$

$$C3 : \neg Dolphin(x) \vee \neg Intelligent(x) \vee \neg Read(x) \vee Intelligent(C)$$

$$C12 : \neg Dolphin(x) \vee \neg Read(x)$$

$$C4 : \neg Dolphin(x) \vee \neg Intelligent(x) \vee \neg Read(x) \vee Read(C)$$

$$C13 : \neg Dolphin(x) \vee \neg Intelligent(x)$$

Step 3: Applying Resolution further ———2 marks

Resolving C11 with C10:

$$C14 : \neg Read(x)$$

There are multiple possibilities of reaching ¬ Read(x) and there is no further resolution possible

Conclusion ——0.5 mark

After systematically applying resolution to the negated theorem and the axioms, no empty clause is derived. It follows that the set of clauses is satisfiable. Therefore, there is no contradiction, and the set of clauses can be true under some interpretation.

Thus, the original statement:

$$\exists x\,(Dolphin(x) \land Intelligent(x) \land Read(x)) \land \forall x\,(Dolphin(x) \land Intelligent(x) \land Read(x) \rightarrow \neg Literate(x))$$

is **not valid**. This result shows that there is no guarantee of the existence of a dolphin satisfying both conditions simultaneously under the provided axioms.