Time: 30 minutes Max. Marks: 10

Name and Roll No.: ______________________________

Instructions:
- Do not plagiarize. Do not assist your classmates in plagiarism.
- Show your full solution for the questions to get full credit.
- Attempt all questions that you can.
- True/False questions will get full credit only if the justification and answer are both correct.
- In the unlikely case that you find a question ambiguous, discuss it with an invigilating TA/invigilator. Please ensure that you clearly write any assumptions you make, even after clarification from the invigilator.

1. (3 points) State True or False. Mathematically prove it if True else provide a counter example.
   (a) (1 point) If $P(a|b,c) = P(b|a,c)$, then $P(a|c) = P(b|c)$
   (b) (1 point) If $P(a|b) = P(a)$ then $P(a|b,c) = P(a|c)$
   (c) (1 point) The expression $\sum_{b',c',d'} P(A)P(d'|A,b',c')P(c'|A,b')P(b'|A)$ is equivalent to P(A).

Solution:

(a) (1 point) **True**. By the product rule we know $P(b,c)P(a|b,c) = P(a,c)P(b|a,c)$, which by assumption reduces to $P(b,c) = P(a,c) \Rightarrow P(b|c)P(c) = P(a|c)P(c)$. Dividing through by $P(c)$ gives the result.

(b) (1 point) **False**. While the statement $P(a|b) = P(a)$ implies that a is independent of b, it does not imply that a is conditionally independent of b given c. A counter-example: a and b record the results of two independent coin flips, and c equals the `xor` of a and b.

(c) (1 point) **True**.

$$
\begin{aligned}
&\sum_{b',c',d'} P(A)P(d'|A,b',c')P(c'|A,b')P(b'|A) \\
&= \sum_{b',c',d'} P(A)P(d'|A,b',c')P(c',b'|A) \\
&= \sum_{b',c',d'} P(A)P(d'|A,b',c')P(c',b'|A) \\
&= \sum_{b',c',d'} P(A)P(b',c',d'|A) \\
&= P(A)\sum_{b',c',d'} P(b',c',d'|A) \\
&= P(A)
\end{aligned}
$$

since $P(b',c',d'|A)$ is a conditional probability mass distribution and its sum should be equal to one, when summed over all the random variables (b',c',d')

2. (2 points) Let A and B be Boolean random variables. You are given: $P(A = true) = 0.5$, $P(B = true|A = true) = 1$, and $P(B = true) = 0.75$. What is $P(B = true|A = false)$?

Solution: The simplest way to solve this is to realize that

$$P(B = true) = P(B = true|A = true)P(A = true) + P(B = true|A = false)P(A = false)$$

Using this fact, you can solve for $P(B = true|A = false)$:

$$(1)\left(\frac{1}{2}\right) + P(B = true|A = false)\left(\frac{1}{2}\right) = \frac{3}{4}$$
$$\Rightarrow P(B = true|A = false) = \frac{1}{2}$$

3. (3 points) We have a bag of three biased coins a, b, and c with probabilities of coming up heads of 30%, 60%, and 75%, respectively. One coin is drawn randomly from the bag (with all coins equally likely to be drawn), and then the coin is flipped three times to generate the outcomes X_1, X_2, and X_3.
   (a) (1 point) Draw the Bayesian network corresponding to this setup and define the necessary CPTs.
   (b) (2 points) Calculate which coin was most likely to have been drawn from the bag if the observed flips come out heads twice and tails once.

Solution:

(a) (1 point) With the random variable C denoting which coin $\{a, b, c\}$ we drew, the network has C at the root and X_1, X_2, and X_3 as children. The CPT for C is: $P(C = a) = P(C = b) = P(C = c) = 1/3$ The CPT for X_i is the same for $i = 1, 2, 3$ and is given by: $P(X_i = heads|C = a) = 0.3$, $P(X_i = heads|C = b) = 0.6$, $P(X_i = heads|C = c) = 0.75$. The probability of $X_i = tails$ can be computed as $1 - P(X_i = heads|C)$ from above.

(b) (1 point) The coin most likely to have been drawn from the bag given this sequence is the value of C with greatest posterior probability $P(C|2\ heads, 1\ tails)$. Now,

$$\begin{aligned}
P(C|2\ heads, 1\ tails) &= P(2\ heads, 1\ tails|C)P(C)/P(2\ heads, 1\ tails) && (1)\\
&\propto P(2\ heads, 1\ tails|C)P(C) && (2)\\
&\propto P(2\ heads, 1\ tails|C) && (3)
\end{aligned}$$

where in the second line we observe that the constant of proportionality $1/P(2\ heads, 1\ tails)$ is independent of C, and in the last we observe that $P(C)$ is also independent of the value of C since it is, by hypothesis, equal to $1/3$. From the Bayesian network we can see that X_1, X_2, and X_3 are conditionally independent given C, so,

$P(X_1 = tails, X_2 = heads, X_3 = heads|C = a) = P(X_1 = tails|C = a)P(X_2 = heads|C = a)P(X_3 = heads|C = a) = 0.7 \times 0.3 \times 0.3 = 0.063$

Note that since the CPTs for each coin are the same, we would get the same probability above for any ordering of 2 heads and 1 tails. Since there are three such orderings, we have

$P(2\ heads, 1\ tails|C = a) = 3 \times 0.063 = 0.189$

Similar calculations to the above find that

$P(2\ heads, 1\ tails|C = b) = 0.432$

$P(2\ heads, 1\ tails|C = c) = 0.422$

showing that coin b is most likely to have been drawn.