Mid-Semester Exam Solutions

Discrete Structures

Problem 1

Let $X = \{0, 1, 2, 3, 4\}$ and define an equivalence relation R on X by:

$$nRm \quad \text{if} \quad n^2 - m^2 \text{ is divisible by 5}$$

Solution:

We need to determine the number of equivalence classes and list the elements in each class.

First, calculate $n^2 \mod 5$ for each $n \in X$:

$$\begin{aligned} 0^2 &= 0 \mod 5 \\ 1^2 &= 1 \mod 5 \\ 2^2 &= 4 \mod 5 \\ 3^2 &= 9 \equiv 4 \mod 5 \\ 4^2 &= 16 \equiv 1 \mod 5 \end{aligned}$$

The equivalence classes are determined by the values of $n^2 \mod 5$:

$$\begin{aligned} \{0\} &= \{0\}, \\ \{1, 4\} &= \{1, 4\}, \\ \{2, 3\} &= \{2, 3\}. \end{aligned}$$

Thus, the equivalence classes are $\{0\}, \{1, 4\}, \{2, 3\}$, and there are **3 equivalence classes**.

Problem 2

Solve the recurrence relation:

$$a_n - 4a_{n-1} + 5a_{n-2} - 2a_{n-3} = 0$$

subject to the initial conditions $a_0 = 0, a_1 = 1, a_2 = 2$.

Solution:

The characteristic equation is:

$$r^3 - 4r^2 + 5r - 2 = 0$$

Using the rational root theorem, we find that $r = 1$ is a root. We then factor the characteristic equation as:

$$(r - 1)(r^2 - 3r + 2) = 0$$

Solving $r^2 - 3r + 2 = 0$, we get $r = 1, r = 2$. So the general solution is:

$$a_n = c_1 + c_2 n + c_3 2^n$$

Using the initial conditions:

$$\begin{gathered} a_0 = 0 \implies c_1 + c_2 \cdot 0 + c_3 = 0 \implies c_1 + c_3 = 0 \\ a_1 = 1 \implies c_1 + c_2 + 2c_3 = 1 \\ a_2 = 2 \implies c_1 + 2c_2 + 4c_3 = 2 \end{gathered}$$

Solving this system, we find $c_1 = 0, c_2 = 1, c_3 = 0$. Therefore, the solution is:

$$a_n = n.$$

Problem 3

Let $P(x), Q(x), R(x)$ be abbreviations for the following predicates:

- $P(x)$: x is perfect.
- $Q(x)$: x failed the quiz.
- $R(x)$: x read a lot of books.

Write these propositions using $P(x), Q(x), R(x)$, logical connectives, and quantifiers.

Solution:

- (a) Everyone who is perfect read a lot of books:
$$(\forall x)(P(x) \rightarrow R(x))$$
- (b) No one who read a lot of books failed the quiz:
$$(\forall x)(R(x) \rightarrow \neg Q(x))$$
- (c) Someone who failed the quiz isn't perfect:
$$(\exists x)(Q(x) \wedge \neg P(x))$$
- (d) If everyone reads a lot of books, then no one will fail the quiz:
$$(\forall x)(R(x)) \rightarrow (\forall x)(\neg Q(x))$$

Problem 4

Use a truth table to show that $(p \rightarrow q) \wedge (p \rightarrow r)$ is logically equivalent to $p \rightarrow (q \wedge r)$.

Solution:

We construct the truth table:

p	q	r	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \wedge (p \rightarrow r)$	$p \rightarrow (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	F	F
T	F	F	F	F	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

Since the columns for $(p \rightarrow q) \wedge (p \rightarrow r)$ and $p \rightarrow (q \wedge r)$ are identical, they are logically equivalent.

Problem 5

Prove that at a party where there are at least two people, there are at least two people who know the same number of people. Assume that if person A knows person B, then person B also knows person A. Furthermore, assume that each person knows him/herself.

Solution:

Suppose there are n people at the party. Each person can know anywhere from 0 to $n-1$ other people. If everyone knows a different number of people, then one person must know 0 people, and another person must know $n-1$ people.

However, this is impossible because if someone knows $n-1$ people, that means they know everyone else, which contradicts the assumption that someone knows 0 people. Therefore, by the pigeonhole principle, at least two people must know the same number of people.

Problem 6

Determine the truth value of the following statements if the universe of discourse for each variable consists of all real numbers:

Solution:

- (a) $\forall x \exists y (y = 3x + 2)$: **True**. For every x, we can find a y such that $y = 3x + 2$.
- (b) $\exists y \forall x (y = 3x + 2)$: **False**. There is no single y that satisfies $y = 3x + 2$ for all x.
- (c) $\exists x \exists y (x^2 + y^2 = 1)$: **True**. There exist x and y (e.g., $x = 1, y = 0$) such that $x^2 + y^2 = 1$.
- (d) $\forall y \forall x ((x + y)^2 = x^2 + y^2)$: **False**. This is generally false because $(x + y)^2 = x^2 + 2xy + y^2$, which includes a cross term.

Problem 7

Given statements 1 and 2, find an argument to conclude statement n. For each intermediate statement you make, state what rule of inference you use and the number(s) of the previous lines that rule uses.

Statements:

- 1. Samay, a student in this class, has a Coldplay concert ticket.
- 2. Everyone who has a Coldplay concert ticket is traveling to Mumbai in January.
- n. Someone in this class is traveling to Mumbai in January.

Solution:

1. Samay has a Coldplay concert ticket. (Given)
2. Everyone who has a Coldplay concert ticket is traveling to Mumbai in January. (Given)
3. Therefore, Samay is traveling to Mumbai in January. (Modus Ponens on 1, 2)
4. Therefore, someone in this class is traveling to Mumbai in January. (Existential Generalization on 3)

Problem 8

For each statement, mention if it is "True" or "False". Justify your answer in each case.

Solution:

- (a) There is no one-to-one correspondence between the set of all positive integers and the set of all odd positive integers because the second set is a proper subset of the first. **False**. There is a one-to-one correspondence, since we can map every positive integer n to $2n - 1$, which is an odd integer.
- (b) If f is a function $A \to B$, and S and T are subsets of A, then $f(S \cap T) = f(S) \cap f(T)$. **False**. This is not true in general unless f is injective.
- (c) $\lfloor 14.85 \rfloor + \lceil 14.85 \rceil = 30$. **False**. $\lfloor 14.85 \rfloor = 14$ and $\lceil 14.85 \rceil = 15$, so $14 + 15 = 29$.
- (d) $\sum_{k=0}^{10} 7 = 70$. **False**. The sum is $7 \times 11 = 77$.
- (e) If $f(n) = (3n^6 + 5n^2 - 6)(n + \log n)$, then f is $O(n^6 \log n)$. **False**. It is $O(n^6)$.
- (f) If the product $A \times B$ of two sets A and B is the empty set $\emptyset$, then both A and B must be $\emptyset$. **False**. Even if one out of A or B is empty, the product $A \times B$ will be empty.

Problem 9

Solve the recurrence relation:

$$a_n - 4a_{n-1} + 4a_{n-2} = 2^n$$

with initial conditions $a_0 = 1$ and $a_1 = 6$.

Solution:

The homogeneous part of the recurrence is:

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

The characteristic equation is:

$$r^2 - 4r + 4 = 0$$

which has a repeated root $r = 2$. Therefore, the solution to the homogeneous part is:

$$a_n^{(h)} = (c_1 + c_2 n)2^n$$

Next, we find a particular solution to the non-homogeneous equation $a_n - 4a_{n-1} + 4a_{n-2} = 2^n$. We try a particular solution of the form:

$$a_n^{(p)} = An^2 2^n$$

Substituting this into the recurrence and solving for A, we get $A = 1/2$. Therefore, the particular solution is:

$$a_n^{(p)} = \frac{1}{2}n^2 2^n$$

Thus, the general solution is:

$$a_n = (c_1 + c_2 n)2^n + \frac{1}{2}n^2 2^n$$

Using the initial conditions $a_0 = 1$ and $a_1 = 6$, we get $c_1 = 1$ and $c_2 = \frac{3}{2}$, yielding the final solution:

$$a_n = \left(1 + \frac{3}{2}n\right)2^n + \frac{1}{2}n^2 2^n$$