Worksheet #2 Solution

Problem 1.

$$e^{3\theta}\,dr + 3re^{3\theta}\,d\theta = 0$$

$$M(r,\theta) = e^{3\theta}, \quad N(r,\theta) = 3re^{3\theta}$$

$$\frac{\partial M}{\partial \theta} = 3e^{3\theta} = \frac{\partial N}{\partial r} = 3e^{3\theta} \Rightarrow \text{exact}$$

$$\frac{\partial u}{\partial r} = e^{3\theta} \Rightarrow u(r,\theta) = re^{3\theta} + f(\theta)$$

$$\Rightarrow \frac{\partial u}{\partial \theta} = 3re^{3\theta} + f'(\theta) = N = 3re^{3\theta}$$

$$\Rightarrow f'(\theta) = 0 \Rightarrow f(\theta) = c$$

$$\Rightarrow u(r,\theta) = re^{3\theta} + c$$

Hence solution is $\boxed{re^{3\theta} = C}$.

Problem 2.

$$y\,dx + (y + \tan(x+y))\,dy = 0$$

Given that $\cos(x+y)$ is an integrating factor.
Multiplying by it we get

$$y\cos(x+y)\,dx + \left(y\cos(x+y) + \sin(x+y)\right)dy = 0$$

(We don't need to check for exactness as it is given that $\cos(x+y)$ is an integrating factor)

$$\frac{\partial u}{\partial x} = y\cos(x+y) \Rightarrow u(x,y) = y\sin(x+y) + f(y)$$

$$\Rightarrow \frac{\partial u}{\partial y} = \sin(x+y) + y\cos(x+y) + f'(y)$$

$$= N = y\cos(x+y) + \sin(x+y)$$

$$\Rightarrow f'(y) = 0 \Rightarrow f(y) = C$$

$$\Rightarrow u(x,y) = y\sin(x+y) + C$$

Therefore, solution is $\boxed{y\sin(x+y) = C}$.

Problem 3. $2\cosh(x)\cos(y)\,dx - \sinh(x)\sin(y)\,dy = 0$

$M = 2\cosh(x)\cos(y)$ $\qquad N = -\sinh(x)\sin(y)$

$$\frac{\partial M}{\partial y} = -2\cosh(x)\sin(y) \neq \frac{\partial N}{\partial x} = -\cosh(x)\sin(y)$$

$\Rightarrow$ Not exact

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\cosh(x)\sin(y)$$

$$\Rightarrow \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{\cosh(x)\sin(y)}{2\cosh(x)\cos(y)} = \frac{\tan(y)}{2}$$ is a function of y only. So, there is a integrating factor only a function of y, $F(y)$.

$$F(y) = \exp\left(\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\right) = \exp\left(\int \frac{1}{2}\tan(y)\,dy\right)$$

$$= \exp\left(\ln\left(\sqrt{\sec(y)}\right)\right) = \sqrt{\sec(y)} = \frac{1}{\sqrt{\cos(y)}}$$

Multiplying in eq. by $F(y)$, we get

$$2\cosh(x)\sqrt{\cos(y)}\,dx - \sinh(x)\frac{\sin(y)}{\sqrt{\cos(y)}}\,dy = 0$$

$$\frac{\partial u}{\partial x} = 2\cosh(x)\sqrt{\cos(y)} \Rightarrow u(x,y) = 2\sinh(x)\sqrt{\cos(y)} + f(y)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\cancel{2}\sinh(x)\frac{\sin(y)}{\cancel{2}\sqrt{\cos(y)}} + f'(y)$$

$$= N = -\sinh(x)\frac{\sin(y)}{\sqrt{\cos(y)}}$$

$$\Rightarrow f'(y) = 0 \quad \Rightarrow f(y) = C$$

$$\Rightarrow u(x,y) = 2\sinh(x)\sqrt{\cos(y)} + C$$

Hence solution is $\boxed{2\sinh(x)\sqrt{\cos(y)} = C}$.

Problem 4. $xy' = 2y + x^3 e^x$

$\Rightarrow y' - \frac{2}{x} y = x^2 e^x$

Linear with $p(x) = -\frac{2}{x}$, $r(x) = x^2 e^x$

Solution is $y(x) = e^{-h}\left(\int r e^h \, dx + C\right)$

where $h(x) = \int p(x)\, dx = -2\ln(x)$

So, $y(x) = e^{+2\ln(x)}\left(\int x^2 e^x e^{-2\ln(x)} dx + C\right)$

$= x^2\left(\int e^x \, dx + C\right)$

$\Rightarrow \boxed{y(x) = x^2(e^x + C)}$

Problem 5. Let P be the total population.

Let $I(t)$ be the number of infected people at time t.

Let $N(t)$ " " " " non-infected " " " ".

Then $N(t) = P - I(t)$

Proportion of infected people, $y(t) = \frac{I(t)}{P}$

" " non-infected " $= \frac{P - I(t)}{P} = 1 - y(t)$

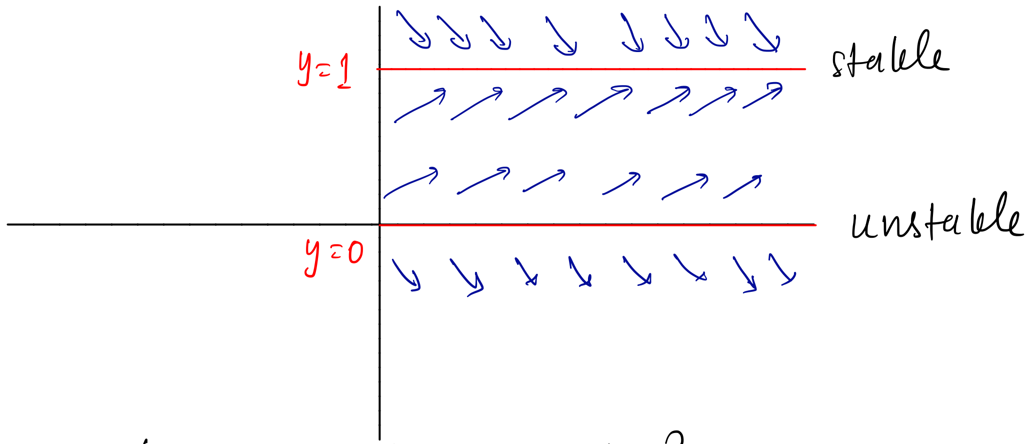
Given that $\frac{dI}{dt} \propto I(t)\, N(t)$

$\Rightarrow \frac{dI}{dt} = k_1 I(t)\,(P - I(t))$

$\Rightarrow \cancel{P}\frac{dy}{dt} = \underbrace{k_1 P^{\cancel{2}}}_{k} y(t)\,(1 - y(t))$

$\Rightarrow \frac{dy}{dt} = k\, y(t)\,(1 - y(t))$, $k > 0 \to$ model

Equilibrium solutions are $y = 0$ & $y = 1$.



$$y' = ky(1-y) = ky - ky^2$$

$$\Rightarrow y' - ky = -ky^2$$

Bernoulli with $p(t) = -k$, $g(t) = -k$, $a = 2$

$$u = y^{1-a} = y^{-1} \Rightarrow u' = -\frac{1}{y^2} y'$$

$$\Rightarrow u' = -\frac{1}{y^2}(ky - ky^2) = -\frac{k}{y} + k$$

$$\Rightarrow u' = -ku + k \Rightarrow u' + ku = k$$

So, $u = e^{-kt}\left(\int k e^{kt}\, dt + C\right)$

$$= e^{-kt}(e^{kt} + C) = 1 + Ce^{-kt}$$

$$\Rightarrow \boxed{y(t) = \frac{1}{u(t)} = \frac{1}{1 + Ce^{-kt}}}$$

$\lim_{t \to \infty} y(t) = 1 \Rightarrow$ In a long time, the whole population would be infected.

Problem 6.

$$x^2 + (y-C)^2 = 1 + C^2$$

$$\Rightarrow 2x + 2(y-C)y' = 0 \Rightarrow y' = -\frac{x}{y-C}$$

$$x^2 + y^2 + \cancel{C^2} - 2cy = 1 + \cancel{C^2}$$

$$\Rightarrow C = \frac{x^2+y^2-1}{2y} \Rightarrow y - C = y - \frac{x^2+y^2-1}{2y} = \frac{y^2-x^2+1}{2y}$$

$$\Rightarrow y' = -\frac{2xy}{y^2-x^2+1} = f(x,y)$$

So, for orthogonal family, $y' = -\frac{1}{f(x,y)} = \frac{y^2-x^2+1}{2xy}$

$$y' = \frac{1}{2x}y + \frac{1-x^2}{2x}\frac{1}{y} \Rightarrow y' - \frac{1}{2x}y = \frac{1-x^2}{2x}\frac{1}{y}$$

Bernoulli with $p(x) = -\frac{1}{2x}$, $g(x) = \frac{1-x^2}{2x}$, $a = -1$

$$u = y^{1-a} = y^2 \Rightarrow u' = 2yy' = 2y\left(\frac{1}{2x}y + \frac{1-x^2}{2x}\cdot\frac{1}{y}\right)$$

$$\Rightarrow u' = \frac{1}{x}y^2 + \frac{1-x^2}{x} \Rightarrow u' - \frac{1}{x}u = \frac{1-x^2}{x}$$

$$\Rightarrow u(x) = e^{-\int -\frac{1}{x}dx}\left(\int \frac{1-x^2}{x} e^{\int -\frac{1}{x}dx} dx + k\right)$$

$$= x\left(\int \frac{1-x^2}{x^2}dx + c\right) = x\left(-\frac{1}{x} - x + k\right)$$

$$\Rightarrow u(x) = -1 - x^2 + kx = y^2$$

$$\Rightarrow \quad x^2 - kx + y^2 = -1 \Rightarrow \left(x - \frac{k}{2}\right)^2 + y^2 = \frac{k^2}{4} - 1$$

$$\Rightarrow \boxed{(x-k)^2 + y^2 = k^2 - 1} \quad \text{(renaming } \tfrac{k}{2} \text{ as } k\text{)}$$

Problem 7.

$$(x-2)\,y' = y, \quad y(2) = 1$$

$$\int \frac{dy}{y} = \int \frac{dx}{x-2} \Rightarrow \ln(y) = \ln(x-2) + C$$

$$\Rightarrow y(x) = C(x-2) \Rightarrow y(2) = 0$$

So, above IVP has no solution.

$f(x,y) = \frac{y}{x-2}$, $f(x,y)$ is not defined on line $x=2$ & hence $f(x,y)$ is not defined in any rectangle containing $(2,1)$. Therefore, the result doesn't contradict theorems.