Solution to Quiz 2

Discrete Structures

Problem 1

Consider the drawing below.

Let X be the set of circles in the drawing and define a binary relation $\sim$ on X by:

$$x \sim y \iff x \text{ and } y \text{ contain the same number of circles.}$$

The relation $\sim$ is an equivalence relation on X (this is given, no need to prove).

(a) How many elements are there in the set X?

There are **14 elements** in the set X.

(b) How many equivalence classes does the equivalence relation have?

Two elements will be in the same equivalence class if they contain same number of circles. There are either no circles, two circles, four circles, or five circles inside a given circle. So, there would be **4 equivalence classes**.

(c) How many elements does each equivalence class have?

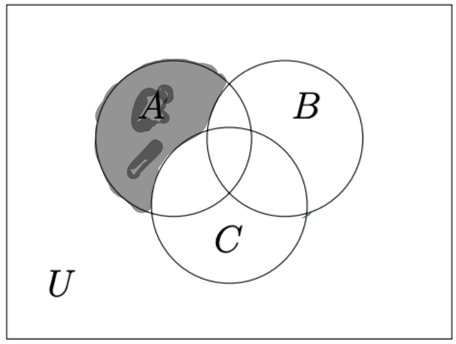
The number of elements in each equivalence class is:

- **9** circles contain 0 circles.
- **3** circles contain 2 circles.
- **1** circle contain 4 circles.
- **1** circle contain 5 circles.

Problem 2

Clearly shade the set $(A \cup B) \cap (A \cap C)^c \cap (B \cup C)^c$ in the Venn diagram.

Solution:



Problem 3

Determine whether or not each of the following relations is a partial order on $\mathbb{N} \times \mathbb{N}$. If it is not a partial order, give a counterexample. If it is a partial order, provide a proof, check if it is a total order, and find the maximal, minimal, greatest, and least elements.

(a) $(a, b) \leq (c, d) \iff a \leq c$

- **Reflexive:** For every $(a, b) \in \mathbb{N} \times \mathbb{N}$, we have $a \leq a$, so the relation is reflexive.

- **Antisymmetric:** If $(a, b) \leq (c, d)$ and $(c, d) \leq (a, b)$, then $a = c$, but we get no information about b and d, so this relation need not be antisymmetric. In fact, $(1, 1) \leq (1, 2)$ and $(1, 2) \leq (1, 2)$ in this relation but $(1, 1) \neq (1, 2)$. Hence, this relation is not antisymmetric.

  Conclusion: This relation is not a partial order.

(b) $(a, b) \leq (c, d) \iff a \leq c$ and $b \geq d$

- **Reflexive:** For every $(a, b) \in \mathbb{N} \times \mathbb{N}$, we have $a \leq a$ and $b \geq b$, so the relation is reflexive.

- **Antisymmetric:** If $(a, b) \leq (c, d)$ and $(c, d) \leq (a, b)$, then $a = c$ and $b = d$, so the relation is antisymmetric.

- **Transitive:** If $(a, b) \leq (c, d)$ and $(c, d) \leq (e, f)$, then $a \leq c \leq e$ and $b \geq d \geq f$, so $(a, b) \leq (e, f)$, making the relation transitive.

  Conclusion: This relation is a partial order.

  Total Order: No, it is not a total order because it only compares both components if $a \leq c$ and $b \geq d$. For example, $(1, 1)$ and $(2, 2)$ are non-comparable.

  Maximal and Minimal Elements: For a maximal element (a, b), a should be large and b should be small while for a minimal element (a, b),

a should be small and b should be large. Since, $\mathbb{N}$ has no maximal element in its order, this relation has no maximal or minimal elements.

Greatest and Least Elements: Since greatest or least element is also a maximal or minimal element, respectively. There doesn't exist a greatest or lest element in this partial order.