Solution to Quiz #1: Discrete Structures

01/09/2024

Problem 1

[7+1+2 points]

Consider the following hypotheses (H1, H2, H3) and conclusion (C):

- (H1) If Puneet Superstar insults Deepak Kalal, then Deepak will make a reaction video.
- (H2) If Deepak Kalal makes a reaction video, then Puneet Superstar will post a response on Instagram.
- (H3) Puneet Superstar did not post a response on Instagram.
- (C) Deepak Kalal did not make a reaction video.

(a) **Defining the logical statements:**

Let:

- p: Puneet Superstar insults Deepak Kalal.
- q: Deepak Kalal makes a reaction video.
- r: Puneet Superstar posts a response on Instagram.

Then the hypotheses and conclusion can be written as:

- $H1$: $p \to q$
- $H2$: $q \to r$
- $H3$: $\neg r$

- C: $\neg q$

(b) **Logical statement to check validity:**

To check if the argument is valid, we need to determine if the following logical statement is a tautology:

$$((p \to q) \land (q \to r) \land \neg r) \to \neg q$$

(c) **Validity of the argument:**

The argument is valid because the logical statement

$$((p \to q) \land (q \to r) \land \neg r) \to \neg q$$

is a tautology. This can be shown by constructing a truth table or by using logical equivalences.

Reason: If $q \to r$ is true and $\neg r$ is true, then $\neg q$ must also be true for the hypotheses to hold. Hence, the conclusion $\neg q$ follows logically from the given hypotheses.

Problem 2

[5 points]

A function f defined on real numbers is said to be continuous at x_0 if the proposition

$$p : (\forall \epsilon > 0)(\exists \delta > 0)(\forall x)\,[(|x - x_0| < \delta) \to (|f(x) - f(x_0)| < \epsilon)]$$

is true.

Proposition for discontinuity:

To define that f is discontinuous at x_0, we write the negation of p:

$$\neg p : (\exists \epsilon > 0)(\forall \delta > 0)(\exists x)\,[(|x - x_0| < \delta) \land (|f(x) - f(x_0)| \geq \epsilon)]$$

This proposition states that there exists an $\epsilon > 0$ such that for every $\delta > 0$, there is some x within δ of x_0 where the function's value does not fall within ϵ of $f(x_0)$, hence indicating discontinuity.

Problem 3

[5 points]

Proof by contrapositive:

We need to prove that if $a^2 + b^2$ is irrational, then at least one of a or b is irrational.

Contrapositive statement:

The contrapositive of the given statement is:

If both a and b are rational, then $a^2 + b^2$ is rational.

Proof:

Assume that both a and b are rational. Then we can write $a = \frac{m}{n}$ and $b = \frac{p}{q}$, where m, n, p, q are integers and $n, q \neq 0$.

The sum of their squares is:

$$a^2 + b^2 = \left(\frac{m}{n}\right)^2 + \left(\frac{p}{q}\right)^2 = \frac{m^2}{n^2} + \frac{p^2}{q^2}$$

Since the sum of two rational numbers is rational, $a^2 + b^2$ is rational. This completes the proof of the contrapositive, and hence the original statement is proven true.