MTH210 – END-SEM EXAMINATION – 20221213

TIME: 2 HOURS **MAXIMUM MARKS: 80**

NB: You may use any known result (i.e. theorems, propositions and lemmas, and tutorial problems) without proof; however, it should be identified clearly. This does not apply if you have been asked to prove a known result. Marks will depend on the correctness and completeness of your proofs. <u>Start each question on a fresh page of the answer-book</u>. All questions have equal marks.

1. Consider the poset $P = \langle \mathcal{F}(\mathbb{N}), \subseteq \mid \rangle$, where $\mathcal{F}(\mathbb{N})$ = the set of all **<u>finite</u>** subsets of $\mathbb{N}$.

   a) Is P a lattice (YES/NO) ? Justify your answer in detail. *(3 marks)*

   b) Does P have a least (minimum) element (YES/NO) ? Does P have a largest (maximum) element (YES/NO) ? Justify your answers in detail. *(5 marks)*

   c) For $A \in \mathcal{F}(\mathbb{N})$, what can you say about the complement of A ? *(2 marks)*

2. Let R and S be binary relations on a set $X \neq \emptyset$. Show that $(R \bullet S)^{-1} = S^{-1} \bullet R^{-1}$. [*NB*: Here the symbol $\bullet$ indicates composition of relations.]

3. Let A be some fixed 10-element subset of $M = \{1,2,3,\ldots,50\}$. Show that A contains two different 5-element subsets the sums of whose elements are the same.

4. Alice has 10 balls, each one different. At the first step, she splits them into two piles. Then she takes any pile which has at least two balls and splits it into two piles. She repeats this until each pile has exactly one ball.

   a) How many steps does this take ? Justify your answer. *(3 marks)*

   b) Show that the number of ways in which she can carry out this procedure is B(10,2)B(9,2)…….B(2,2). *(7 marks)*

5. At the zoo there are two cages, ~~X~~ A and ~~Y~~ B. Following are the signs on these two cages:

   • On ~~X~~ A - In this cage, there is a deer, and in the other one, there is a lion.

   • On ~~Y~~ B - In one of these cages, there is a deer, and in one of them there is a lion.

One of these signs is true, the other false. In which cage is there a deer ?

[***Hint: You may assume that a deer and a lion cannot be in the same cage.***]

6. a) Does there exist a simple graph G with 6 vertices, such that 5 of the degrees are: 5, 5, 3, 2, 1 ? Justify your answer. *(6 marks)*

c) In the Havel-Hakimi Theorem (Theorem 7), there is no separate mention of the requirement that the sum of the terms in a given sequence should be even. Explain why this is not needed. In other words, show that if a sequence satisfies the recursive condition to be a graphic sequence (score of a simple graph), then the sum of its terms is always even.

(4 marks)

7. Recall that a **decomposition** of a graph is a list of subgraphs such that each edge appears in exactly one of the subgraphs in the list. Prove that a simple graph with more than six vertices of odd degree cannot be decomposed into three paths.

8. Let $V = \{0,1\}$, and let V_n = Cartesian product of n copies of V, i.e. V_n is the set of all ordered n-tuples with 0-1 entries, for $n \in \mathbb{Z}^+$. Consider the simple graph G_n such that $V(G_n) = V_n$, and vertex u is adjacent to vertex v if and only if u and v differ **in exactly two positions (coordinates)**. Determine the number of components of G_n. Justify your answer.

SOLUTIONS FOLLOW

(NOT IN SAME ORDER)

SOLUTIONS

Q 1: Consider the poset $P = \langle \mathcal{F}(\mathbb{N}), \subseteq \rangle$ where $\mathcal{F}(\mathbb{N})$ = the set of all finite subsets of $\mathbb{N}$.

a) Is $\mathcal{F}(\mathbb{N})$ a lattice (YES/NO)? Justify.

Answer: YES. Justification:- Note that $\mathcal{F}(\mathbb{N})$ is a subfamily of the power set of $\mathbb{N}$, which is a lattice under the $\subseteq$ relation with $A \vee B = A \cup B$, and $A \wedge B = A \cap B$. Since the union and intersection of finite sets A and B are again finite sets, $A \vee B$ and $A \wedge B \in \mathcal{F}(\mathbb{N})$. Hence $\mathcal{F}(\mathbb{N})$ is a lattice.

b) Does $\mathcal{F}(\mathbb{N})$ have a least (minimum) element and a largest (maximum) element? Justify.

Answer: Least element: YES.

The empty set, $\emptyset \in \mathcal{F}(\mathbb{N})$ and satisfies $\emptyset \subseteq A$ for all $A \in \mathcal{F}(\mathbb{N})$

Largest element: NO.

Suppose BWOC that $\mathcal{F}(\mathbb{N})$ has a largest element, ~~M, say~~ say M.

Q1 (cont'd). Since M is a finite set, ~~say~~ it has a largest member, say $k \in \mathbb{N}$. But then $\{k+1\} \subseteq M$ does not hold $\Rightarrow\Leftarrow$

(c) For $A \in \mathcal{F}(\mathbb{N})$, what can you say about its complement?

Ans: Since $\mathcal{F}(\mathbb{N})$ does not have a largest element, ~~the concept of~~ the complement is not defined.

Problem 2 ([redacted]). Suppose R and S are binary relations on a set $A \neq \emptyset$. Show that $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$.

Solution. Let $x, y \in A$ such that $(x, y) \in (R \circ S)^{-1}$.

Then $(y, x) \in R \circ S$. This implies that there exists a $z \in A$ such that $(y, z) \in R$ and $(z, x) \in S$.

Now, $(y, z) \in R$ implies $(z, y) \in R^{-1}$ and $(z, x) \in S$ implies $(x, z) \in S^{-1}$.

Then $(x, z) \in S^{-1}$ and $(z, y) \in R^{-1}$ implies that $(x, y) \in S^{-1} \circ R^{-1}$. Thus

$$(R \circ S)^{-1} \subseteq S^{-1} \circ R^{-1}. \tag{1}$$

Conversely, let $x, y \in A$ such that $(x, y) \in S^{-1} \circ R^{1}$.

Then there exists $z \in A$ such that $(x, z) \in S^{-1}$ and $(z, y) \in R^{-1}$.

Now, $(x, z) \in S^{-1}$ implies that $(z, x) \in S$ and $(z, y) \in R^{-1}$ implies that $(y, z) \in R$.

Then $(y, z) \in R$ and $(z, x) \in S$ implies that $(y, x) \in R \circ S$. So $(x, y) \in (R \circ S)^{-1}$. Thus

$$S^{-1} \circ R^{-1} \subseteq (R \circ S)^{-1} \tag{2}$$

Combining (1) and (2), we have $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$. $\square$

Q3. Let A be some fixed 10-element subset of $M = \{1, 2, \ldots, 50\}$. Show that A contains two different 5-element subsets the sum of whose elements is the same.

Ans: Let $\mathcal{S}$ be the family of 5-element subsets of A. For each $B \in \mathcal{S}$, let $f(B) =$ sum of the elements of B. Note that the following inequalities must hold for B:-

$$f(B) \geq 1+2+3+4+5 = 15 \quad (1)$$

and $$f(B) \leq 50+49+48+47+46 = 240 \quad (2).$$

Hence, f is a function on $\mathcal{S}$ to $T = \{15, 16, \ldots, 240\}$. Thus, $|T| = 226$ (3)

However, $$|\mathcal{S}| = \binom{10}{5} = 252 \quad (4)$$

The result now follows from PHP.

Remark: The pigeons are the 5-element subsets of A. The holes are the values of the sums $f(B)$ for $B \in \mathcal{S} =$ family of 5-element subsets of A.

Q4: Alice has 10 balls, each one different. At the first step, she splits them into 2 piles. Then, she takes any pile which has at least two balls, and splits it into two piles. She repeats this until each pile has exactly one ball.

(a) How many steps does this take? Justify.

Answer: It takes 9 steps.

Justification: At the start, there is one pile.

At the end of ~~the~~ step 1: 2 piles

At the end of step 2: 3 piles.

⋮

Arguing on these lines, we need 9 steps to achieve 10 piles – number of piles increases by one at each step.

Q4(b) ~~This~~ Show that this can be done in $\binom{10}{2}\binom{9}{2}\cdots\binom{2}{2}$ ways.

Ans: We argue backwards.

At the end, we had 10 piles:

o o o o o o o o o o

At the previous stage, we had 9 piles, obtained by taking 2 of the piles above and combining them into one pile: → So this could be done in $\binom{10}{2}$ ways. (*) (1)

Now, we have <u>9 piles</u>:

(o o) o o o o o o o o

To get to the previous stage, we have to take two of them, and combine into a pile. This can be done in $\binom{9}{2}$ ways. (**) (2)

We continue backwards.

Eventually, we have 2 piles, which have to be combined into 1 pile. This can be done in $\binom{2}{2}$ ways.

Since all the balls are different ways, we in all have:

$$\binom{10}{2}\cdot\binom{9}{2}\cdots\binom{2}{2} \text{ ways.}$$

Ans: The argument for both (a) and (b) is general, and could be applied to any $n \geq 2$.

Q5. At the zoo, there are two cages, A and B. Following are the signs on these two cages:

- On A – in this cage there is a deer, and in the other one, there is a lion.
- On B – In one of these cages, and in one of them, there is a lion.

One of these signs is true, the other false. In which cage is there a deer?

Hint: You may assume that a deer and a lion cannot be in the same cage.

Answer: Let L_A, L_B, T_A, and T_B denote the following statements:–

L_A: There is a deer in Cage A.

L_B: There is a deer in Cage B

T_A: There is a lion in Cage A.

T_B: There is a lion in Cage B.

The signs on the cages can now be expressed symbolically as follows:

Sign on Cage A: $\alpha_A :- (L_A \wedge T_B)$,

Sign on Cage B: $\alpha_B :- (L_A \vee L_B) \wedge (T_A \vee T_B)$

Q5. Continued :-

We now construct the following truth-table.

	L_A	L_B	T_A	T_B	α_A	α_B
1.	T	T	T	T	T	T
2.	T	T	T	F	F	T
3.	T	T	F	T	T	T
4.	T	T	F	F	F	F
5.	T	F	T	T	T	T
6.	T	F	T	F	F	T
7.	T	F	F	T	T	T
8.	T	F	F	F	F	F
9.	F	T	T	T	F	T
10.	F	T	T	F	F	T
11.	F	T	F	T	F	T
12.	F	T	F	F	F	F
13.	F	F	T	T	F	F
14.	F	F	T	F	F	F
15.	F	F	F	T	F	F
16.	F	F	F	F	F	F

We now eliminate the rows which say either both L_A and T_A are true or L_B and T_B are true, ~~because a lady and a tiger cannot be in the same room~~. We are thus left with the following rows.

because a deer and a lion cannot be in the same cage.

	L_A	L_B	T_A	T_B	α_A	α_B
4.	T	T	F	F	F	F
7.	T	F	F	T	T	T
8.	T	F	F	F	F	F
10.	F	T	T	F	F	T
12.	F	T	F	F	F	F
13.	F	F	T	T	F	F
14.	F	F	T	F	F	F
15.	F	F	F	T	F	F
16.	F	F	F	F	F	F

Now, the instructions mention that one of the signs is true and the other one false. So, we now eliminate the rows where α_A and α_B have identical truth values.

	L_A	L_B	T_A	T_B	α_A	α_B
10.	F	T	T	F	F	T

This leaves us with only one row of the truth table.

Thus, the sign on Cage B is true, and the sign on Cage A is not true.

The Deer is in Cage B. ✓ ANSWER

Q6 (a) Does there exist a simple graph with 6 vertices, such that 5 of the degrees are: 5, 5, 3, 2, 1 ? Justify.

Ans: NO.

Justify: Suppose the degree of the sixth vertex is x. Since the degree sum has to be even, and four of the degrees are odd, x has to be even.

Consider the three possibilities: $x = 0, 2, 4$.

(i) $x=0$. The degree sequence $d = 5\ 5\ 3\ 2\ 1\ 0$

$\Rightarrow d' = 4\ 2\ 1\ 0\ -1 \rightarrow$ Not possible

(ii) $x=2$. Then $d = 5\ 5\ 3\ 2\ 2\ 1$

$\Rightarrow d' = 4\ 2\ 1\ 1\ 0 \Rightarrow d'' = 1\ 0\ 0\ -1$

$\longrightarrow$ not possible

(iii) $x=4$. Then $d = 5\ 5\ 4\ 3\ 2\ 1$

$\Rightarrow d' = 4\ 3\ 2\ 1\ 0 \Rightarrow d'' = 2\ 1\ 0\ -1$

$\longrightarrow$ not possible.

(b) Show that if a sequence satisfies the recursive condition of Theorem 7, then the sum of the terms has to be even.

[Cont'd]

Q6(b) – Continued.

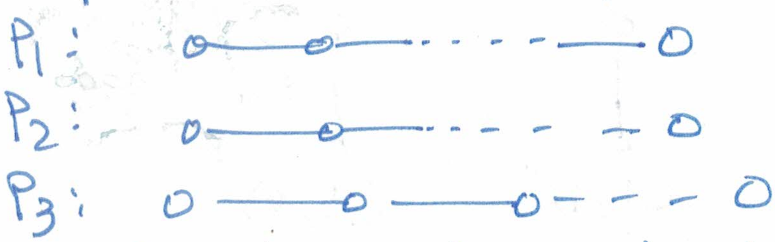
This follows from the two following statements. The sum of terms is even because:

The sum is reduced by an <u>even</u> number at each iteration, and at the final step, we have a single term = 0, which is even (1)

The reduction is <u>even</u> because we ~~do~~ delete the highest term Δ and reduce each of the next Δ terms by -1, i.e. we reduce by 2Δ at each step. (2)

Q7. Show that a simple graph with more than 6 vertices of odd degree cannot be decomposed into three paths.

Ans: Suppose BWOC that G ~~ha~~ can be decomposed into three paths:-



Let u be any vertex of odd degree. If u occurs <u>only as</u> an interior vertex of ~~a~~ some path, then it will get an even degree from each path. $\therefore$ u has to occur as an end-vertex of ~~ea~~ some path. But there are only 6 end-vertices. So, a contradiction!

Remark: Another application of PHP!

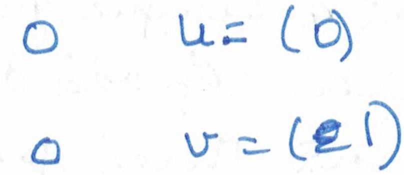
Q8) Given G_n with $V(G_n) = V_n$, and vertices u and v adjacent if and only if they differ in <u>exactly two coordinates</u>.
Determine the number of components of G_1 with justification.

Answer: For any $u \in V(G_n) = V_n$, we put $u = (u_1, \dots, u_n)$ and let parity(u) = odd or even according as $\sum u_i$ is odd or even.

Let O_n = set of vertices of odd parity and E_n = set of vertices of even parity.

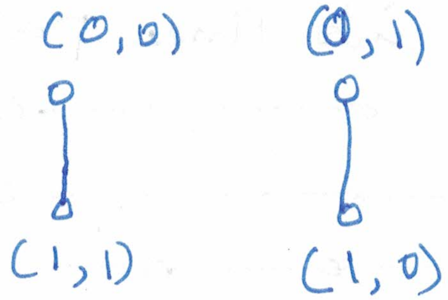
Let us inspect G_n for small n.

$n = 1$:- G_1 =



Two components

$n = 2$: G_2



We see that there are <u>two</u> components in each case, and O_n and E_n lie in separate components. $(*)$

Q8 – continued

We next observe: let u and v be adjacent. Then, they differ in exactly two coordinates, so they must have the same parity, i.e. parity(u) = parity(v) along any path in G_n. (1)

Thus, G_n must have ≥ 2 components (2)

In view of the above, we claim: G_n has exactly two components, one consisting of the subgraph induced by vertices of odd parity and one consisting of the subgraph induced by vertices of even parity. (3)

We will prove the claim (3) by induction on n. (strong induction)

Base Case(s): $n = 1$ and $n = 2$.
Shown by inspection, see (*).

NB: Both the base cases are needed.

Inductive Step:
Suppose that G_k has two components, for all k, $1 \leq k < n$, for some $n \geq 3$. (IH)

Q8 – continued.

We will denote the components (as well as their vertices) by O_k and E_k, respectively.

So, now consider G_n, and let

$u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$

have the same parity, $u \neq v$. If they differ in exactly one coordinate, they cannot have the same parity, so they differ in ≥ 2 coordinates. WOLOG, they differ in the last two coordinates. Put $k = n-2$, and let $u' = (u_1, \dots, u_k) \in V_k$

and $v' = (v_1, \dots, v_k) \in V_k$.

If $u' = v'$, then u and v are adjacent in G_n. If $u' \neq v'$, then since u and v differ in the last two coordinates, u' and v' have the same parity. Since u' and v' are in G_k, by the IH, they lie in the same component (since ~~$k \geq n-2$~~ $n > k = n-2$).

Thus, we have a path

$u' = w_0' \cdots w_j' = v'$ in G_k.

Let $u'' = (u_{n-1}, u_{n-2})$, i.e. the

Q8 – continued

last two coordinates.
For each of the k-tuples w_0' to w_j',
consider the n-tuple ~~w_i~~ obtained by adjoining
u'' in the final two positions,

i.e. $W_i = W_i' \; u_{n-1} u_n$

first k position — last two position.

Since the last two positions are the same,
W_i and W_{i+1} differ in exactly two
positions.

$\therefore$ We have a path

$u = W_0 W_1 \cdots W_j$ in G_n.

Note that $W_j = v' \; u_{n-1} u_n$

and $v = v' \; v_{n-1} v_n$

$\therefore$ W_j and v differ in exactly the last two
positions only, and so are adjacent in
G_n.

Hence, $u = W_0 \cdots W_j v$ is a u,v-path
in G_n.

Since u and v had the same parity,
this proves claim 3 by PMI.