Introduction to Cryptography
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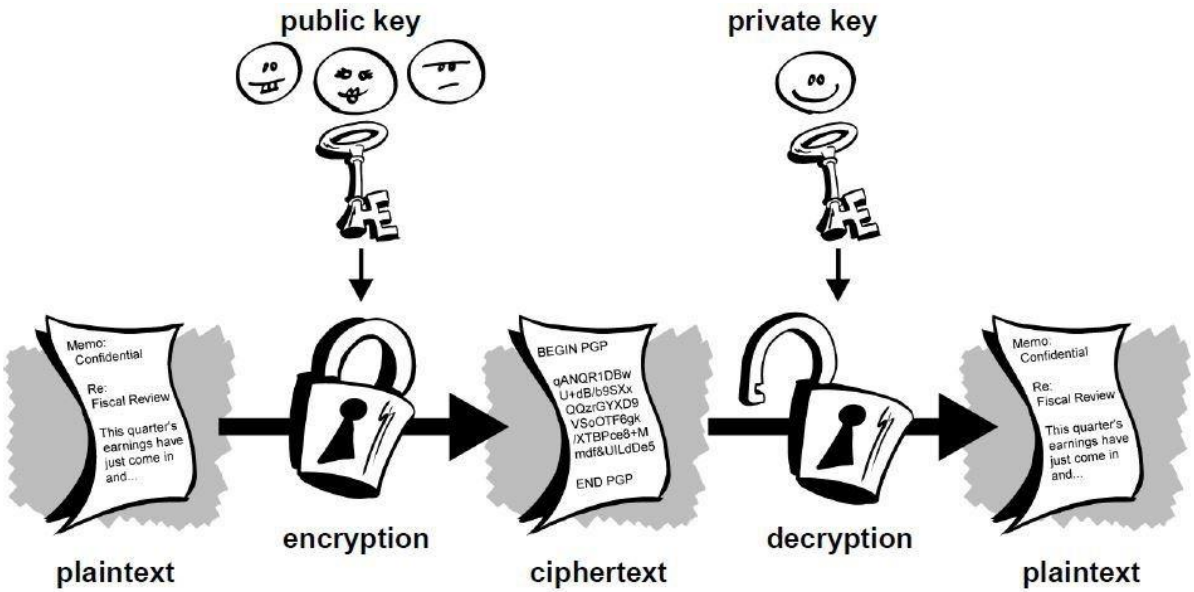


Hard Problems in Cryptography

Recap: Hard Problem 1: Discrete Log Problem

Discrete Log problem (DLP):

given g, N and $g^x \bmod N$, output x

DLP considered to hard (for carefully chosen g, N)

We will see how use DLP to build:

- Public-key encryption
- Private-key encryption (with reusable short key)
- Digital Signatures

Going forward: all arithmetic will be mod N.

Will not write mod N explicitly

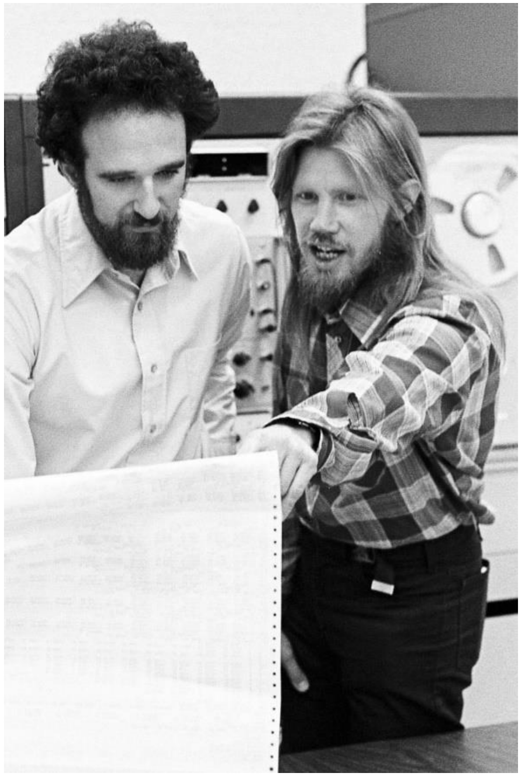
HP2: Computational Diffie-Hellman (CDH) Problem

Given (suitably chosen) g,

$A = g^a$, and $B = g^b$

Find C, s.t. (such that)

$C = g^{ab}$

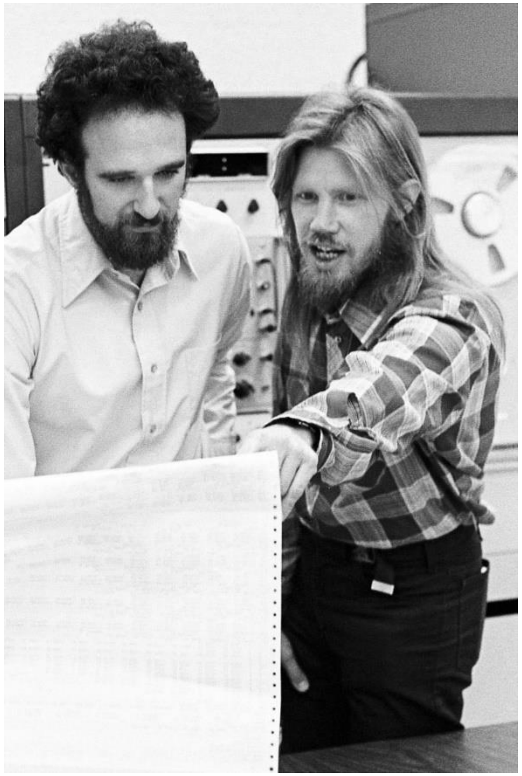


- Note: $A.B = g^a.g^b = g^{a+b}$
- Most natural way of solving CDH:
  - Step1: Find a from g^a
  - Step2: Compute $(g^b)^a = g^{ab}$
  - However Step1 is a hard problem (might be other ways)

HP3: Decisional Diffie-Hellman (DDH) Problem

Given (suitably chosen) g,

$A = g^a$, and $B = g^b$



And either

(1) $C = g^{ab}$, or,

(2) $R = g^r$ (for random r)

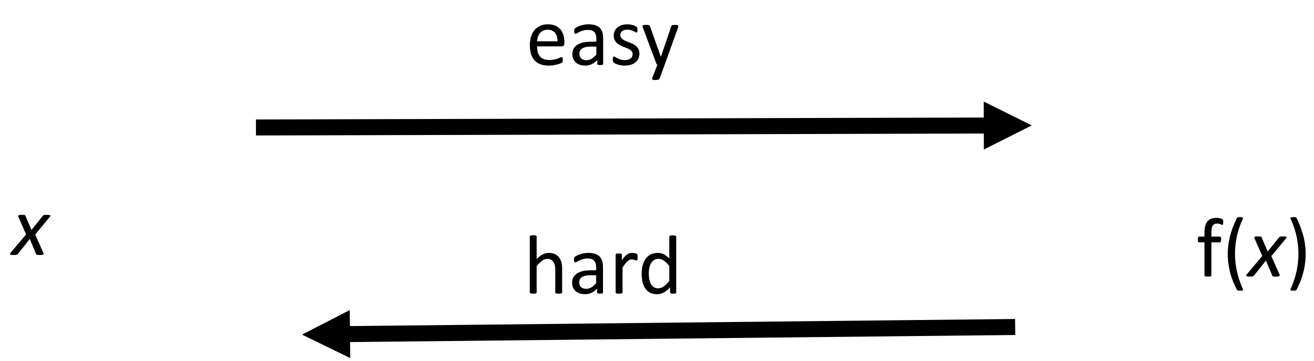
Tell whether its (1) or (2)

Still hard with better than ½

(even given the answer, hard to distinguish it from random

Building on Hard Problems: One-Way Functions (or One-Way Hash Functions)

One-Way Functions (OWF)



Sample random x, compute y = f(x)   (x called pre-image of y)

Given y, hard for any adversary to compute x

In fact, hard to compute any x’ s.t. y = f(x’)

(hard to compute any valid pre-image)

(will see an application of this to password storage)

Building a OWF?

Given y, hard for any adversary to compute any x' s.t. y = f(x')

(hard to compute any valid pre-image)

Attempt1: f(x) = x

Easy to invert, given output, can find input

Attempt2: f(x) = 0 (or some other constant)

Every string x' is a valid pre-image because f(x') = f(x) = 0

Hence, easy to invert

Attempt3: f(x) = 2x mod N

To recover x, simply compute 2^{-1} and multiply

OWF based on Discrete Log Assumption

Define OWF f as:

$f(x) = g^x \bmod N$

(description of g, N is public)

DLA: Given $f(x)$, hard to compute x

Other OWF: SHA-256, SHA-1, MD-5.

More complex to understand. Have additional properties.

Applications of OWF: Storing Passwords

Client

Server

p

p

p

Problem: Adv hacks into server. Can steal millions of passwords.

Solution:

1) Server only stores f(p)
2) Client still sends p, server computes f(p) and matches
3) Even if adv learns f(p) by hacking, computing p is hard

Applications of OWF: Storing Passwords

Client

p

p

Server

f(p)

- This solution is used everywhere on the Internet today! Servers should not store your password, only a hash of it.
- That’s why if you forget password: server can’t give it back to you. You can reset instead.

Offline Dictionary Attack on Passwords

Client

Server

p

p

f(p)

- If adv obtains f(p), it can guess millions of different passwords, hash them and check if they match f(p). If match, adv wins!
- Typically: adv checks words from dictionary and common patterns
- Hence: your password should have special characters.

Back to OWF Definition

Client

Server

p

p

f(p)

- Given f(p), say adv can compute p’ s.t. f(p’) = f(p). Server will also accept p’ as valid. Hence, adv still wins!
- That why in OWF: given f(x), finding any x’ s.t. f(x’) = f(x) must be hard.

Passwords Over the Internet?

Client

Server

p

p

f(p)

Problem: Adv watching the network?

Solution:

1) Use HTTPS to open a secure connection first

   (Client encrypts all messages under Server's public key)

2) Later: will see how HTTPS works!

Applications of OWF: Digital Signatures

Digital Signatures

VK

M, σ

SK M

sign

σ (signature)

VK, M, σ

verify

0/1

Note: we don't care about hiding M

Defining Digital Signatures

1) Gen: Takes no input, outputs VK and SK
2) Sign: Takes input *SK* and *M*. Outputs σ.
3) Verify: Takes input (M, σ, VK). Outputs 0/1.

Correctness: If (VK, SK) are output of Gen, must have

Verify(M, Sign(M, SK), VK) = 1

Security

Adv is even given:

1) Verification key VK
2) Signatures (σ_1, σ_2,…, σ_q) on messages (M_1, M_2,…, M_q) chosen by him

Adv still can't output a valid signature on a new message

(That is, can't output (σ, M) s.t. Verify(M, σ, VK) = 1 and M is different from all M_i)

One-Time (Digital) Signatures

- Easier to design: we need a scheme which is secure only for signing a single message

Adv is given:

1) Verification key VK
2) Signature σ_1 on any one message M_1 of his choice

Adv can't output a valid signature on a new message

(That is, can't output (σ, M) s.t. $\mathrm{Verify}(M, \sigma, VK) = 1$ and M is different from M_1)

One-Time Signatures: Intuition

- Only two possible messages. Hence only two possible signatures. These are random strings (x_0, x_1). They are generated at random and part of secret key SK.
    - If message is 0, signature is x_0
    - If message is 1, signature is x_1

- How to verify signature? Can't make it a part of the verification key
    - Use an idea similar to storing passwords
    - VK has $f(x_0)$ and $f(x_1)$
    - VK can be used for verification of signature, not for computation

One-Time Signatures

Message length: 1 bit

Gen: pick random (x_0, x_1)

$SK = (x_0, x_1)$

$VK = (f(x_0), f(x_1))$

sanity check

Sign: If $m = 0$, $\sigma = x_0$

If $m = 1$, $\sigma = x_1$

Verify: compute $f(\sigma)$

If $m = 0$, match with $f(x_0)$

If $m = 1$, match with $f(x_1)$

Security

Adv given: VK, and signature on m = 0

Wants to compute signature on m = 1

Given: $f(x_0)$, $f(x_1)$, x_0

Compute x_1

- x_1 is unrelated to x_0. Hence, $f(x_0)$ and x_0 are not relevant for computing x_1
- Thus, given $f(x_1)$, adv needs to compute x_1
- Needs to invert OWF (hard)!

One-Time Signatures: longer messages

Message length: n bit

For all i, $1 \leq i \leq n$

Gen: pick random ($x_0[i]$, $x_1[i]$) ([i] is the i-th number picked)

SK = ($x_0[i]$, $x_1[i]$)

VK = ($f(x_0[i])$, $f(x_1[i])$)

Sign: If m[i] = 0, include $x_0[i]$ in σ as σ[i] (m[i] = i-th bit of m)

If m[i] = 1, include $x_1[i]$ in σ as σ[i]

Verify: compute f(σ[i])

If m[i] = 0, match with $f(x_0[i])$

If m[i] = 1, match with $f(x_1[i])$

Security: say m_1 and m differ in i-th bit

Adv has $x_0[i]$, needs to compute $x_1[i]$

Needs to invert OWF

Digital Signature Schemes

What about:

- Signing multiple messages with the same key?
- *Can key size be shorter than message?*

Answer: Yes! We will see RSA signatures later

Digital Signature Applications

- Signing documents/forms digitally:

  Adobe PDF and others

- Online Contract Signing: Two parties can sign a contract over internet

- Digital Degrees and Marksheets

- Bitcoin/Cryptocurrencies and Smart Contracts

Questions?