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① ⓐ i) from fermats theorem we have

$$a^{P-1} = 1 \pmod P$$

Taking square root on both side we have,

$$a^{(P-1)/2} = \pm 1 \pmod P$$

$P \geq 3$ ensures that $(P-1)/2$ remains an integer

$\therefore$ we can write;

$$a^{(P-1)/2} \equiv 1 \pmod P \quad \text{or}$$

$$a^{(P-1)/2} \equiv -1 \pmod P \equiv P-1 \pmod P$$

ii) $30 \equiv (-1) \pmod{31}$

taking power of 15 to both side we have

$$30^{15} \equiv (-1)^{15} \pmod{31}$$

$$30^{15} \equiv (-1) \pmod{31}$$

$$30^{15} \equiv 30 \pmod{31}$$

$$\equiv 30 \text{ Ans}$$

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(1) (b) (i) $C(r,m)\,C(m,j) = C(r,j)\,C(r-j,\,m-j)$

$$C(n,k) = \frac{n!}{k!\,(n-k)!}$$

RHS

$C(r,j)\;C(r-1,\,m-j)$

$$= \frac{r!}{j!\,(r-j)!} \times \frac{(r-j)!}{(m-j)!\,(r-j-(m-j))!}$$

$$= \frac{r!}{j!\,\cancel{(r-j)!}} * \frac{\cancel{(r-j)!}}{(m-j)!\,(r-j-m+j)}$$

$$= \frac{r!}{j!} \times \frac{1}{(m-j)!\,(r-m)!} \times \frac{m!}{m!}$$ (introduce m! to numerator & denominator)

$$= \frac{r!}{m!\,(r-m)!} * \frac{m!}{j!\,(m-j)!}$$ rearrange

${}^{r}C_{m} \times {}^{m}C_{j}$

$= C(r,m) \times C(m,j) = \text{LHS}$ ~~Hence~~ ~~proved~~.

Hence LHS = RHS

Hence proved.

1 (b) (2) we Prove that

$$\sum_{K=0}^{n} \binom{n-K}{m-K} = 2^{m}\binom{n}{m}$$

we claim that both sides count the no of Pairs (x,y) of disjoint subsets of $\{1,2 \ldots n\}$ such that $|X \cup Y| = m$

On one hand there are $\binom{n}{m}$ ways to chose the set $X \cup Y$ and for each of these there 2^{m} ways to select subset of these m element to form the subset X (rest element in subset Y)

thus by the rule of Product, the no of Pairs by $2^{m}\binom{n}{m}$ which is RHS

On the other hand the set X must have some numbers K of elements where $0 \leq K \leq m$ for each such K there are $\binom{n}{K}$ ways to select the element of X and then $\binom{n-K}{m-K}$ ways to select the element of Y from $n-K$ element.

thus for each possible K the no of Pairs (x,y) with $|X| = K$ and satisfying the given

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Condition is $\binom{n}{k}\binom{n-k}{m-k} = 2^m \binom{n}{m}$. By the rule of sum the total no Pairs is $\sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{m-k}$ which is LHS.

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(2) (a) $$S(n,r) = \frac{1}{r!}\sum_{i=0}^{r}(-1)^i \binom{r}{i}(r-i)^n$$

$$= \frac{1}{r!}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i} i^n$$

$$S(n,2) = \sum_{i=0}^{n-2} 2^i S(n-1-i,1) = 2^{n-1}-1 , \quad n \geq 2$$

$$S(n,3) = \sum_{i=0}^{n-3} 3^i S(n-1-i,2) , \quad n \geq 3$$

$$= \sum_{i=0}^{n-3} 3^i \left(2^{n-2-i}-1\right) = \frac{1}{6}\left[3^n - 3(2^n) + 3\right]$$

$$S(n,4) = \sum_{i=0}^{n-4} 4^i S(n-1-i,3) , \quad n \geq 4$$

$$= \sum_{i=0}^{n-4} 4^i \left[\frac{1}{6}\left\{3^{n-1-i} - 3\left(2^{n-1-i}\right) + 3\right\}\right]$$

$$= \frac{1}{24}\left[4^n - 4(3^n) + 6(2^n) - 4\right]$$

$$= \frac{1}{4!}\sum_{i=0}^{4}(-1)^i\binom{4}{i}(4-i)^n$$

now we assume identity (2,2) is true & try to prove the identity in (2,3) the identity (2,3) holds for $S(n,0)=0$ & $S(n,1)=1$ it may be proved by induction for all real number r

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (r-i)^n = n!$$

Applying induction hypothesis for $n-1$ $1 < r < n$

$$S(n-1, r-1) = \frac{1}{(r-1)!} \sum_{i=0}^{r-1} (-1)^{r-1-i} (r-1)\, i^{n-1}$$

$$= \frac{1}{(r-1)!} \sum_{i=0}^{r-1} (-1)(-1)^{r-1} (1 - i/r) \binom{r}{i} i^{n-1}$$

$$= \frac{-1}{(r-1)!} \sum_{i=0}^{r-1} (-1)^{r-1} \frac{i}{r} \binom{r}{i} i^{n-1}$$

$$= r\,(S(n-1, r)) = r \frac{1}{r!} \sum_{i=0}^{r} (-1)^{r-1} \binom{r}{i} i^{n-1}$$

$$= \frac{1}{(r-1)!} \sum_{i=0}^{r-1} (-1)^{r-1} \binom{r}{i} i^{n-1} + \frac{r^{n-1}}{(r-1)!}$$

So that

$$S(n, r) = S(n-1, r-1) + r\, S(n-1, r)$$

now putting $n = 7$ and $r = 3$
we get

$$\boxed{S(7, 3) = S(6, 2) + 3\,(6, 3)}$$

Proved.

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(2) (b) i) In a complete bipartite graph with partition of size K, $n-K$ the vertices in part with K vertices have order $n-K$ and vertices in part $n-K$ has order K

$\therefore$ Sum of degrees is $2K(n-K)$ so by handshaking lemma no of edge $= K(n-K)$

$$e = K(n-K)$$

so to get max value of K we differiante

$$nK - K^2 = e = f(K)$$

$$f'(K) = n - 2K$$

$$f''(K) = -2$$

$\therefore$ it is maxima at $n - 2K = 0$

$$K = \frac{n}{2}$$

$$\therefore e_{max} = \frac{n^2}{2} - \frac{n^2}{4} = \frac{n^2}{4}$$

$\therefore$ max possible value of e in bipatite graph is $\frac{n^2}{4}$ (n is no of vertex)

(ii) Both the given graph have:

(1) 8 vertices

(2) 12 edges

(3) All vertices of degree 3

(4) Both have 6 circuits of same length

So we can define a function

$f: V \rightarrow V'$ from G to G', so that
if $\{e_1, e_3\} \in E$ then $\{f(e_1), f(e_2)\} \in E'$

$\therefore$ these two graphs are isomorphic to each other.