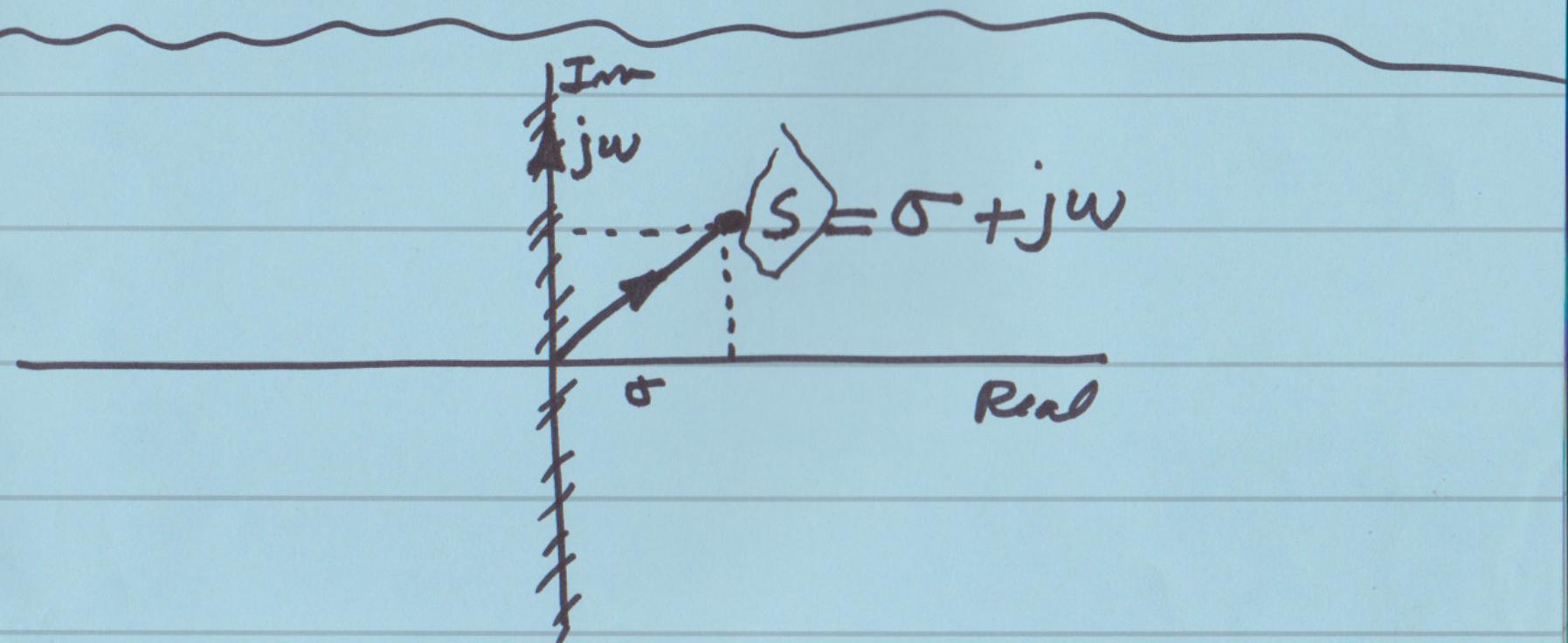


## Chapter 5

3

### Laplace Transform and its application in System Analysis



3.5

Application of L- Transform

Solv of Diff. Egn.

e.g.

$$\left\{ \begin{array}{l} \frac{dx}{dt} + 2x = 3u(t) \\ x(0) = 5 \end{array} \right.$$

 $\mathcal{L}$ 

$$\rightarrow (sX(s) - 5) + 2X(s) = 3 \times \frac{1}{s}$$

$$(s+2)X(s) = \frac{3}{s} + 5$$

$$X(s) = \frac{\frac{3}{s} + 5}{s+2} = \frac{3+5s}{s(s+2)}$$

$$x(t) \xleftarrow{\mathcal{L}}$$

Given  $f(t) : 0 \rightarrow \infty$

4

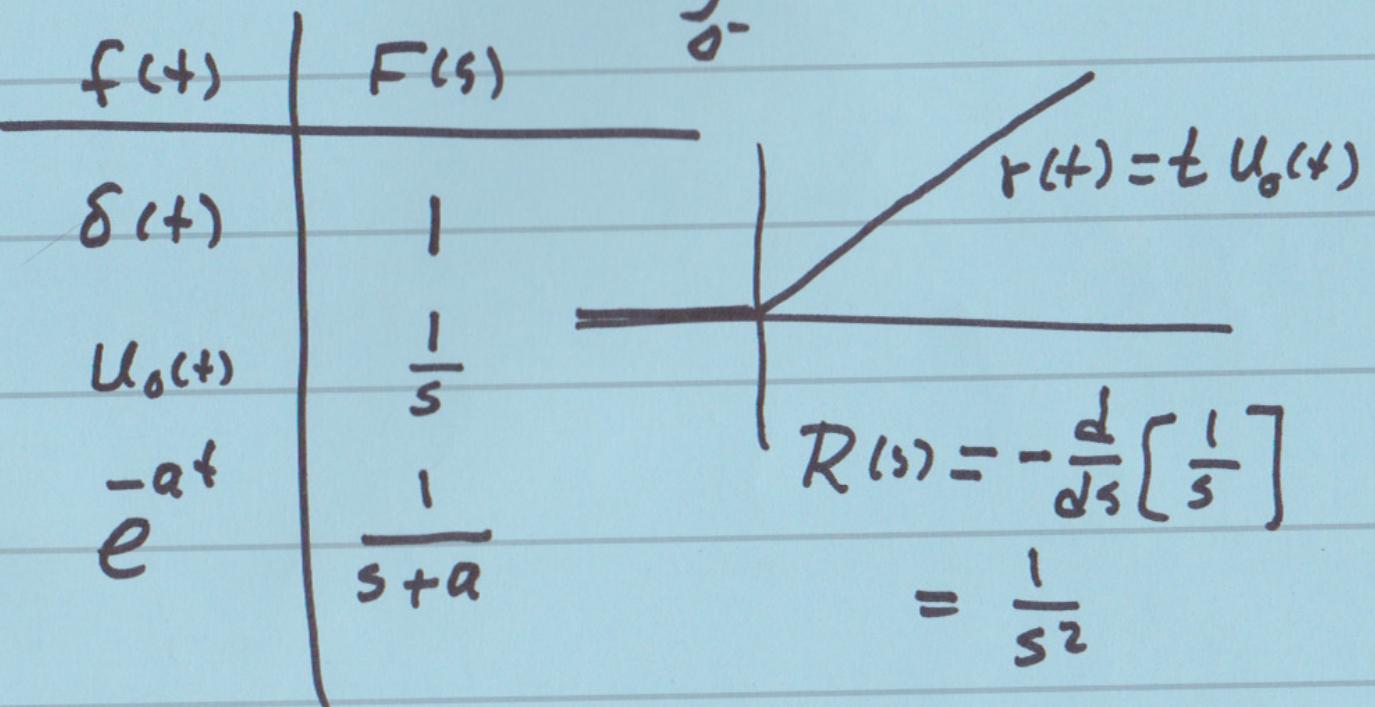
Then

$$\bar{F}(s) = \int_{0^-}^{\infty} f(t) \cdot e^{-st} dt$$

e.g.

$$f(t) = \delta(t)$$

$$\bar{F}(s) = \int_{0^-}^{0^+} \delta(t) e^{-st} dt = 1$$



$$e^{-j\omega t}$$

$$e^{j\omega t}$$

$$\sin \omega t$$

$$\cos \omega t$$

$$e^{-at} \cos \omega t$$

$$t u_0(t)$$

$$\frac{1}{s + j\omega}$$

$$\frac{1}{s - j\omega}$$

$$\frac{\omega}{s^2 + \omega^2}$$

$$\frac{s}{s^2 + \omega^2}$$

$$\frac{s+a}{(s+a)^2 + \omega^2}$$

$$\frac{1}{s^2}$$

$$4.5$$
$$\sin \omega t = \frac{1}{2j} [ e^{j\omega t} - e^{-j\omega t} ]$$

$$\frac{1}{2j} \left[ \frac{1}{s+j\omega} - \frac{1}{s-j\omega} \right]$$

$$\frac{1}{2j} \left[ \frac{s+j\omega - s-j\omega}{s^2 + \omega^2} \right]$$

$$\frac{2j\omega}{2j(s^2 + \omega^2)}$$

5

## Properties of L-Transform

$$1) \quad \mathcal{L} \left\{ k_1 f_1(t) + k_2 f_2(t) \right\} = k_1 \bar{F}_1(s) + k_2 \bar{F}_2(s)$$

$$2) \quad \mathcal{L} \left\{ \frac{d}{dt} f(t) \right\} = s \bar{F}(s) - f(0)$$

Proof , integration by parts of def. of L.T.

$$3) \quad \mathcal{L} \left\{ \frac{d^2}{dt^2} f(t) \right\} = s^2 \bar{F}(s) - sf(0) - f'(0)$$

$$\mathcal{L} \left\{ \frac{d^n}{dt^n} f(t) \right\} = s^n \bar{F}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

$$3) \mathcal{L} \left\{ \int_{0^-}^t f(\sigma) d\sigma \right\} = \frac{1}{s} \bar{F}(s) + \frac{1}{s} \int_0^{\bar{o}^-} f(t) dt$$

$$4) \mathcal{L} \left\{ e^{-at} f(t) \right\} = \bar{F}(s+a)$$

$$5) \mathcal{L} \left\{ f(t-\tau) U_0(t-\tau) \right\} = e^{-sT} \bar{F}(s)$$

$$6) \mathcal{L} \left\{ t \cdot f(t) \right\} = - \frac{d}{ds} \bar{F}(s)$$

$$\mathcal{L} \left\{ t^n \cdot f(t) \right\} = (-1)^n \frac{d^n}{ds^n} \bar{F}(s)$$

$$\mathcal{L} \left\{ t^n \cdot \right\} = (-1)^n \frac{d^n}{ds^n} \left[ \frac{1}{s} \right] = - \frac{n!}{s^{n+1}}$$

$$1) \mathcal{L}\left\{ f(at) \right\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

---

Example Find  $\mathcal{L}\left\{ t \cos wt \right\} = -\frac{d}{ds} \left[ \frac{s}{s^2 + w^2} \right]$

$f(t) = \cos wt$  Rule #6  $\Rightarrow$

$$F(s) = \frac{\cancel{s}}{s^2 + w^2},$$

$\rightarrow - \frac{(s^2 + w^2) - 2s^2}{(s^2 + w^2)^2} = \frac{s^2 - w^2}{(s^2 + w^2)^2} \checkmark$

$$\frac{-s^2 - \omega^2 + 2s^2}{( )^2} = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2} \quad 8'$$

$$\{ (t-2) \cos(t-2) u_0(t-2) \} ; T$$

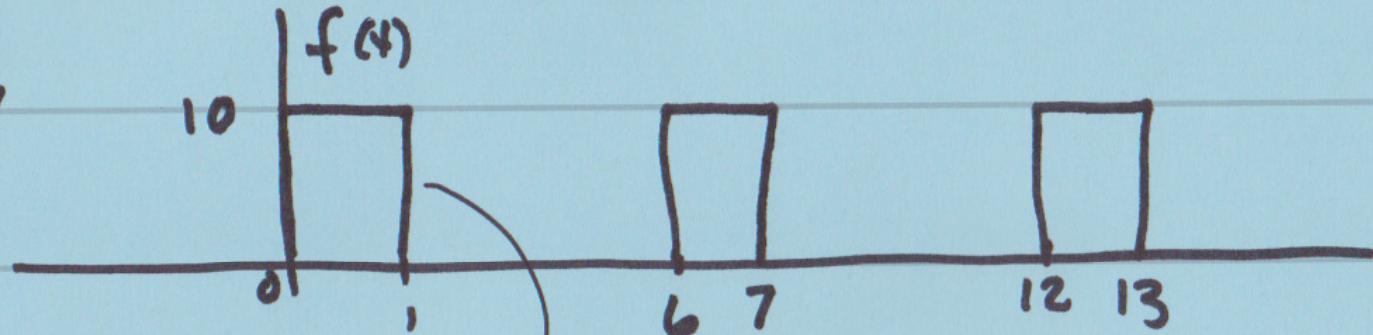
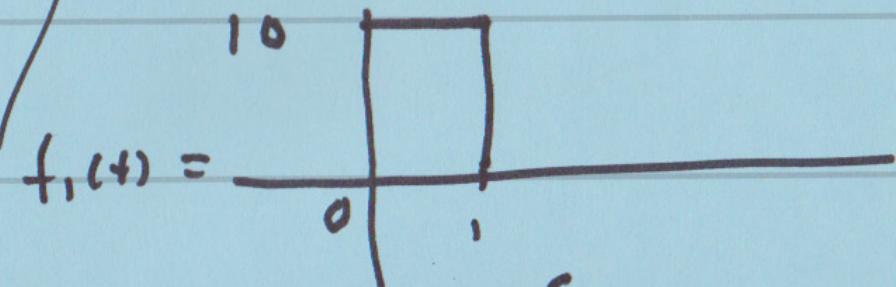
$$\{ t \cos \omega t u_0(t) \} = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$$

$$\mathcal{L} \{ f(t-T) u_0(t-T) \} = e^{-sT} F(s)$$

$$= e^{-2s} \cdot \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$$

8)

$$\bar{F}(s) = \frac{\bar{f}_1(s)}{1 - e^{-sT}}$$

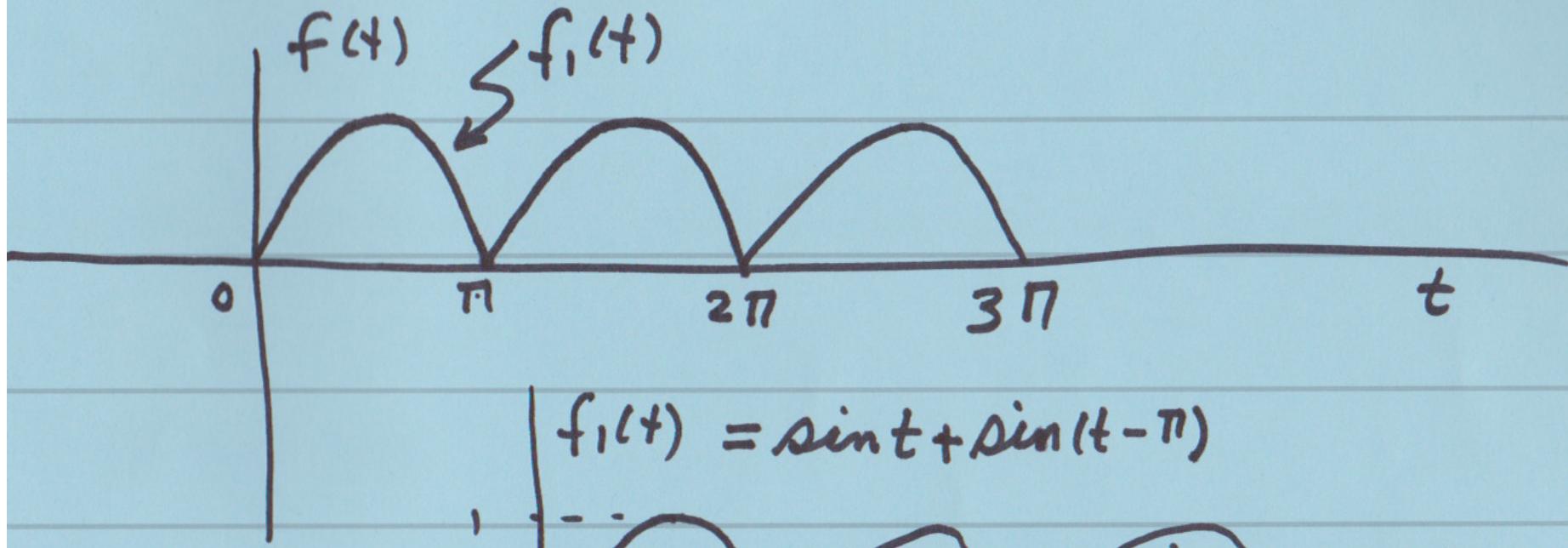
ExampleFind  $\bar{F}(s)$ 

$$\bar{f}_1(s) = \frac{10}{s} - \frac{10e^{-s}}{s}$$

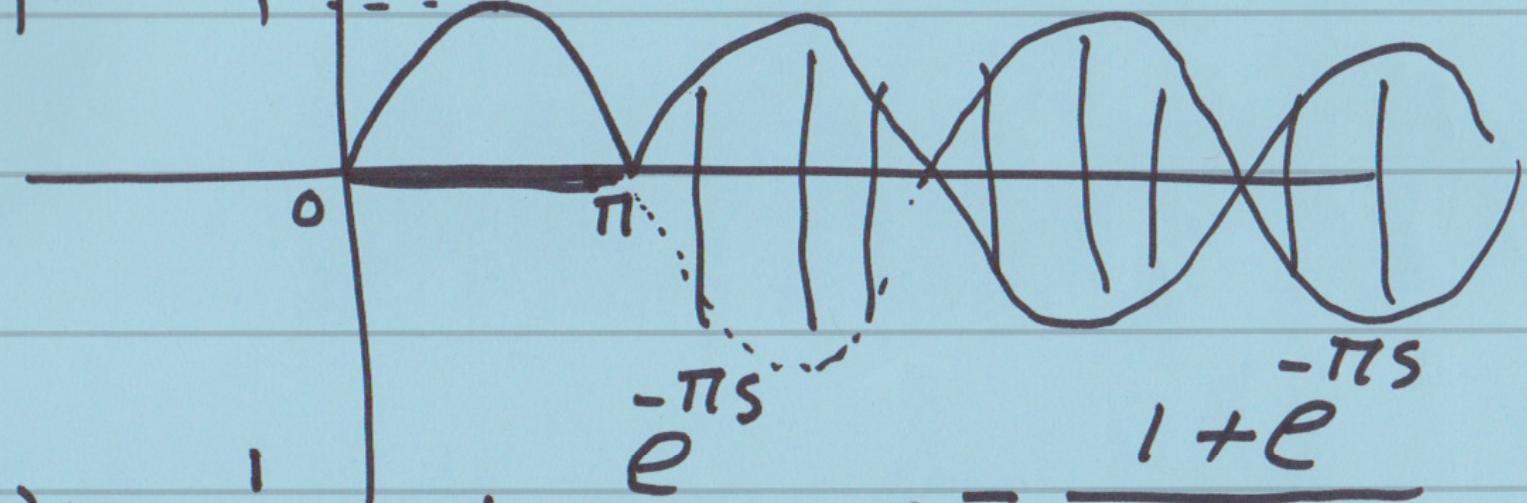
$$\text{Then } \bar{F}(s) = \frac{10}{s} \frac{1 - e^{-s}}{1 - e^{-6s}}$$

$$F(s) = \frac{10(1-e^{-s})}{s(1-e^{6s})}$$

10



$$f_1(t) = \sin t + \sin(t - \pi)$$

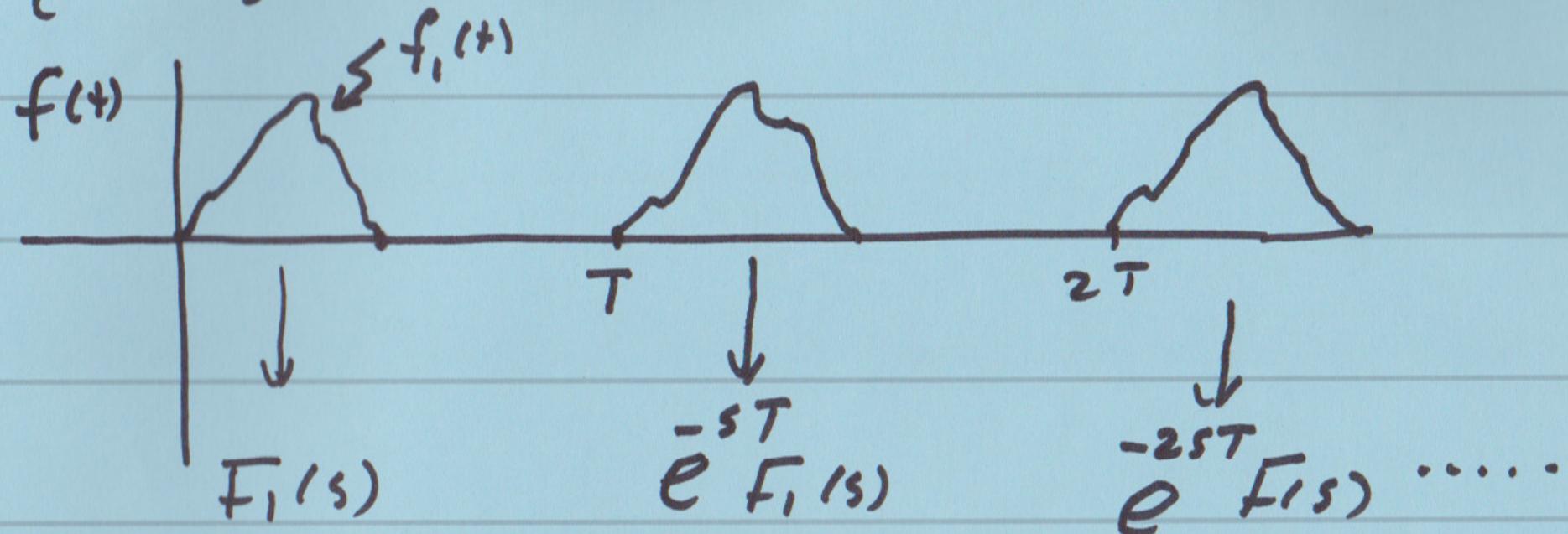


$$F_1(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}$$

$$\text{j.o. } \bar{F}(s) = \frac{\bar{F}_1(s)}{1 - \bar{e}^{-\pi s}}$$

$$\boxed{\bar{F}(s) = \frac{1 + e^{-\pi s}}{(s^2 + 1)(1 - \bar{e}^{-\pi s})}}$$

$$\mathcal{Z}\{f(t)\} ; \quad f(t) = f(t + kT) ; \quad k = 0, 1, 2, \dots$$



$$F(s) = F_1(s) \left[ 1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \dots \right]$$

$$\left[ 1 + x + x^2 + x^3 + x^4 + \dots \right]$$

$$\frac{1}{1-x}$$

$$\frac{3+5s}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$$

$$= \frac{(A+B)s + 2A}{s(s+2)} \equiv \frac{5s + 3}{s(s+2)}$$

$$2A = 3, A + B = 5$$

$$A = 1.5 \quad B = 3.5$$

$$X(s) = \frac{1.5}{s} + \frac{3.5}{s+2}$$

$$x(t) = 1.5u_0(t) + 3.5e^{-2t}u_0(t)$$

$$x(t) = (1.5 + 3.5e^{-2t})u_0(t)$$