Assignment-9

Vipul Kumar Malik

***Abstract*—This document explains the concept of finding the foot of the perpendicular to the line from the given point using Singular Value Decomposition (SVD).**

Download all python codes from

https://github.com/vipulmalik8569/MT−EE5609

and latex-tikz codes from

https://github.com/vipulmalik8569/MT−EE5609

1 Problem

Find the foot of the perpendicular to the line from the point $\mathbf{p}$.

Equation of the line and point $\mathbf{p}$ is given as :

$$\frac{3-x}{2} = \frac{y-1}{2} = \frac{z+6}{-1} = t \tag{1.0.1}$$

$$\mathbf{p} = \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix} \tag{1.0.2}$$

2 Solution

Vector form of the equation (1.0.1) in terms of slope vector $\mathbf{m}$ and positional vector $\mathbf{c}$ is :

$$\mathbf{x} = \mathbf{m}t + \mathbf{c} \tag{2.0.1}$$

Where,

$$\mathbf{m} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix} \tag{2.0.2}$$

After the Affine transformation of

$$\mathbf{x} = \mathbf{I}\mathbf{x}^* + \mathbf{c} \tag{2.0.3}$$

equation (2.0.1) and point $\mathbf{p}$ will become :

$$\mathbf{x}^* = \mathbf{m}t \tag{2.0.4}$$

$$\mathbf{p}^* = \mathbf{p} - \mathbf{c} = \begin{pmatrix} -5 \\ 2 \\ 14 \end{pmatrix} \tag{2.0.5}$$

Now let us solve :

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{2.0.6}$$

Where,

$$\mathbf{M} = \mathbf{m} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{b} = \mathbf{p}^* = \begin{pmatrix} -5 \\ 2 \\ 14 \end{pmatrix} \tag{2.0.7}$$

This is clearly the case of Over-determined system of equations.

From the Singular value Decomposition on $\mathbf{M}$ we get :

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{2.0.8}$$

By substituting the value of $\mathbf{M}$ from equation (2.0.8) to (2.0.6) we get :

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{2.0.9}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \tag{2.0.10}$$

Where,

$\mathbf{S}_+$ is the Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$.

Columns of $\mathbf{U}$ are eigenvectors of $\mathbf{M}\mathbf{M}^T$.

Columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^T\mathbf{M}$.

$\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

Now the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$ can be calculated as :

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} -2 & 2 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \end{pmatrix} \tag{2.0.11}$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{2.0.12}$$

$$9 - \lambda = 0 \tag{2.0.13}$$

$$\lambda = 9 \tag{2.0.14}$$

Hence the Normalised eigenvector corresponding to λ is,

$$\mathbf{V} = \frac{1}{9}\begin{pmatrix} 9 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \tag{2.0.15}$$

Eigenvectors corresponding to $\mathbf{M}\mathbf{M}^T$ can be calculated as :

$$\mathbf{MM}^T = \begin{pmatrix}-2\\2\\-1\end{pmatrix}\begin{pmatrix}-2 & 2 & -1\end{pmatrix} \tag{2.0.16}$$

$$\implies \begin{pmatrix}4 & -4 & 2\\-4 & 4 & -2\\2 & -2 & 1\end{pmatrix} \tag{2.0.17}$$

Eigenvalues corresponding to $\mathbf{MM}^T$ are,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{2.0.18}$$

$$\begin{vmatrix}4-\lambda & -4 & 2\\-4 & 4-\lambda & -2\\2 & -2 & 1-\lambda\end{vmatrix} = 0 \tag{2.0.19}$$

$$\lambda^2(\lambda-9) = 0 \tag{2.0.20}$$

Solving this we get :

$$\lambda_1 = 0 \quad \lambda_2 = 9 \tag{2.0.21}$$

Hence the eigenvectors corresponding to λ_1, λ_2 and λ_3 are as follows :

$$\mathbf{u_1} = \begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbf{u_2} = \begin{pmatrix}\frac{-1}{2}\\0\\1\end{pmatrix}, \mathbf{u_3} = \begin{pmatrix}2\\-2\\1\end{pmatrix} \tag{2.0.22}$$

Normalizing the eigenvectors we get,

$$\mathbf{u_1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{pmatrix} \tag{2.0.23}$$

$$\mathbf{u_2} = \frac{\sqrt{5}}{2}\begin{pmatrix}\frac{-1}{2}\\0\\1\end{pmatrix} = \begin{pmatrix}\frac{-\sqrt{5}}{4}\\0\\\frac{\sqrt{5}}{2}\end{pmatrix} \tag{2.0.24}$$

$$\mathbf{u_3} = \frac{1}{3}\begin{pmatrix}2\\-2\\1\end{pmatrix} = \begin{pmatrix}\frac{2}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{pmatrix} \tag{2.0.25}$$

$$\mathbf{U} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-\sqrt{5}}{4} & \frac{2}{3}\\\frac{1}{\sqrt{2}} & 0 & \frac{-2}{3}\\0 & \frac{\sqrt{5}}{2} & \frac{1}{3}\end{pmatrix} \tag{2.0.26}$$

$\mathbf{S}$ corresponding to eigenvalues λ_1 and λ_2 is as follows :

$$\mathbf{S} = \begin{pmatrix}3\\0\\0\end{pmatrix} \tag{2.0.27}$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix}\frac{1}{3} & 0 & 0\end{pmatrix} \tag{2.0.28}$$

Hence the singular value decomposition of $\mathbf{M}$ is as follows :

$$\mathbf{M} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-\sqrt{5}}{4} & \frac{2}{3}\\\frac{1}{\sqrt{2}} & 0 & \frac{-2}{3}\\0 & \frac{\sqrt{5}}{2} & \frac{1}{3}\end{pmatrix}\begin{pmatrix}3\\0\\0\end{pmatrix}\begin{pmatrix}1\end{pmatrix} \tag{2.0.29}$$

By putting the values of $\mathbf{V}$, $\mathbf{S}_+$, $\mathbf{U}$ and $\mathbf{b}$ in (2.0.10) we get :

$$\mathbf{x} = \begin{pmatrix}1\end{pmatrix}\begin{pmatrix}\frac{1}{3} & 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-\sqrt{5}}{4} & \frac{2}{3}\\\frac{1}{\sqrt{2}} & 0 & \frac{-2}{3}\\0 & \frac{\sqrt{5}}{2} & \frac{1}{3}\end{pmatrix}^T\begin{pmatrix}-5\\2\\14\end{pmatrix} \tag{2.0.30}$$

$$\implies \mathbf{x} = \begin{pmatrix}0\end{pmatrix} \tag{2.0.31}$$

On verifying by computing the least square solution we get :

$$\mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \tag{2.0.32}$$

$$\begin{pmatrix}-2 & 2 & -1\end{pmatrix}\begin{pmatrix}-2\\2\\-1\end{pmatrix}\mathbf{x} = \begin{pmatrix}-2 & 2 & -1\end{pmatrix}\begin{pmatrix}-5\\2\\14\end{pmatrix} \tag{2.0.33}$$

$$\begin{pmatrix}9\end{pmatrix}\mathbf{x} = \begin{pmatrix}0\end{pmatrix} \tag{2.0.34}$$

$$\mathbf{x} = \begin{pmatrix}0\end{pmatrix} \tag{2.0.35}$$

From equations (2.0.31) and (2.0.35) we can conclude that the solution is verified.

As our solution is in affine space. It is the foot of the perpendicular to the line (2.0.4) from the point $\mathbf{p}^*$.

Therefore by using (2.0.3) we get

$$\mathbf{x} = \mathbf{I}\begin{pmatrix}0\end{pmatrix} + \begin{pmatrix}1\end{pmatrix} \tag{2.0.36}$$

$$\mathbf{x} = \begin{pmatrix}1\end{pmatrix} \tag{2.0.37}$$

Hence equation (2.0.37) gives the foot of the perpendicular to the line (2.0.1) from the point $\mathbf{p}$.