Assignment-8

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***Abstract*—This document explains the concept of finding the QR decomposition of a 2×2 matrix.**

Download all python codes from

https://github.com/vipulmalik8569/MT−EE5609

and latex-tikz codes from

https://github.com/vipulmalik8569/MT−EE5609

1 Problem

Find the QR decomposition of the matrix

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \tag{1.0.1}$$

2 Solution

Any matrix say $\mathbf{V}$ with linearly independent column vectors can be factorised as :

$$\mathbf{V} = \mathbf{QR} \tag{2.0.1}$$

where $\mathbf{Q}$ is a orthogonal matrix and $\mathbf{R}$ is a upper triangular matrix with non zero diagonal elements.

From (1.0.1) let $\mathbf{a}$ and $\mathbf{b}$ be the column vectors of $\mathbf{V}$ written as :

$$\mathbf{a} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{2.0.2}$$

(2.0.1) can also be written as,

$$\mathbf{V} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \begin{pmatrix} u_1 & u_3 \\ 0 & u_2 \end{pmatrix} \tag{2.0.3}$$

where,

$$\mathbf{Q} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \quad \mathbf{R} = \begin{pmatrix} u_1 & u_3 \\ 0 & u_2 \end{pmatrix} \tag{2.0.4}$$

Now,

$$u_1 = \|a\| = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2} \tag{2.0.5}$$

$$\mathbf{p_1} = \frac{\mathbf{a}}{u_1} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{2.0.6}$$

$$u_3 = \frac{\mathbf{p_1}^T \mathbf{b}}{\|\mathbf{p_1}\|^2} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \frac{2}{\sqrt{5}} \tag{2.0.7}$$

$$\mathbf{p_2} = \frac{\mathbf{b} - u_3 \mathbf{p_1}}{\|\mathbf{b} - u_3 \mathbf{p_1}\|} = \begin{pmatrix} \frac{-2}{5} \\ \frac{9}{10} \end{pmatrix} \tag{2.0.8}$$

$$u_2 = \mathbf{p_2}^T \mathbf{b} = \begin{pmatrix} \frac{-2}{5} & \frac{9}{10} \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \frac{7}{10} \tag{2.0.9}$$

By putting the values of $\mathbf{p_1}$, $\mathbf{p_2}$, u_1, u_2 and u_3 in (2.0.4) we get :

$$\mathbf{Q} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-2}{5} \\ \frac{1}{\sqrt{5}} & \frac{9}{10} \end{pmatrix} \quad \mathbf{R} = \begin{pmatrix} \frac{\sqrt{5}}{2} & \frac{2}{\sqrt{5}} \\ 0 & \frac{7}{10} \end{pmatrix} \tag{2.0.10}$$

As,

$$\mathbf{Q}^T \mathbf{Q} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-2}{5} \\ \frac{1}{\sqrt{5}} & \frac{9}{10} \end{pmatrix}^T \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-2}{5} \\ \frac{1}{\sqrt{5}} & \frac{9}{10} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.11}$$

Shows $\mathbf{Q}$ is an Orthogonal matrix.

Hence using (2.0.3) the QR Decomposition of $\mathbf{V}$ can be written as :

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-2}{5} \\ \frac{1}{\sqrt{5}} & \frac{9}{10} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{5}}{2} & \frac{2}{\sqrt{5}} \\ 0 & \frac{7}{10} \end{pmatrix} \tag{2.0.12}$$