Assignnment-2

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Abstract—**This document explain the concept of finding the shortest distance between lines.**

Download all python codes from

https://github.com/vipulmalik8569/MT−EE5609

and latex-tikz codes from

https://github.com/vipulmalik8569/MT−EE5609

1 Problem

Find the shortest distance between the lines :

$$L_1 : \mathbf{x} = \begin{pmatrix} 1-t \\ t-2 \\ 3-2t \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \tag{1.0.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix} s+1 \\ 2s-1 \\ -2s-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \tag{1.0.2}$$

2 Solution

We have,

$$L_1 : \mathbf{x} = \mathbf{a_1} + t\mathbf{b_1} \tag{2.0.1}$$

$$L_2 : \mathbf{x} = \mathbf{a_2} + s\mathbf{b_2} \tag{2.0.2}$$

where, $\mathbf{a_i}$, $\mathbf{b_i}$ are positional and slope vectors of line L_i respectively.

As $\mathbf{b_1} \neq \lambda\mathbf{b_2}$, lines L_1 and L_2 are not parallel to each other.

Now, let us assume that L_1 and L_2 are intersecting at a point. Therefore,

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \tag{2.0.3}$$

$$s\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} \tag{2.0.4}$$

$$\begin{pmatrix} -1 & -1 \\ -2 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} \tag{2.0.5}$$

Using Gaussian elimination method:

$$E = E_{32}E_{31}E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \frac{-1}{2} & 1 & \frac{3}{4} \end{pmatrix} \tag{2.0.6}$$

$$E\begin{pmatrix} -1 & -1 & : & 0 \\ -2 & 1 & : & 1 \\ 2 & -2 & : & -4 \end{pmatrix} = \begin{pmatrix} -1 & -1 & : & 0 \\ 0 & 3 & : & 1 \\ 0 & 0 & : & -2 \end{pmatrix} \tag{2.0.7}$$

From (2.0.7) it is clear that the system of linear equations are inconsistent. Therefore L_1 and L_2 are not intersecting at any point.

Hence our assumption was wrong, L_1, L_2 are **skew lines**.

Let d be the shortest distance between L_1, L_2 and $\mathbf{p_1}$, $\mathbf{p_2}$ be the positional vectors of its end points.

For d to be the shortest, we know that,

$$\mathbf{b_1}^T(\mathbf{p_2} - \mathbf{p_1}) = 0 \tag{2.0.8}$$

$$\mathbf{b_2}^T(\mathbf{p_2} - \mathbf{p_1}) = 0 \tag{2.0.9}$$

$$\mathbf{b_1}^T\left((\mathbf{a_2} - \mathbf{a_1}) + \begin{pmatrix} \mathbf{b_2} & \mathbf{b_1} \end{pmatrix}\begin{pmatrix} s \\ -t \end{pmatrix}\right) = 0 \tag{2.0.10}$$

$$\mathbf{b_2}^T\left((\mathbf{a_2} - \mathbf{a_1}) + \begin{pmatrix} \mathbf{b_2} & \mathbf{b_1} \end{pmatrix}\begin{pmatrix} s \\ -t \end{pmatrix}\right) = 0 \tag{2.0.11}$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{b_2} & \mathbf{b_1} \end{pmatrix}, \mathbf{B^T} = \begin{pmatrix} \mathbf{b_2}^T \\ \mathbf{b_1}^T \end{pmatrix} \tag{2.0.12}$$

By combining equation (2.0.10) and (2.0.11) and writing in terms of $\mathbf{B}$ and $\mathbf{B}^T$ using (2.0.12) we get:

$$\mathbf{B}^T\mathbf{B}\begin{pmatrix} s \\ -t \end{pmatrix} = \mathbf{B}^T(\mathbf{a_1} - \mathbf{a_2}) \tag{2.0.13}$$

By putting the values of $\mathbf{a_1}, \mathbf{a_2}, \mathbf{b_1}$ and $\mathbf{b_2}$ in equation (2.0.13) we get:

$$\begin{pmatrix} 5 & 6 \\ 9 & 5 \end{pmatrix}\begin{pmatrix} s \\ -t \end{pmatrix} = \begin{pmatrix} -9 \\ -10 \end{pmatrix} \tag{2.0.14}$$

Solving equation (2.0.14) we get:

$$s = \frac{-15}{29}, t = \frac{31}{29} \tag{2.0.15}$$

By putting the values of t and s in equation (2.0.1) and (2.0.2) respectively we get:

$$\mathbf{p_1} = \begin{pmatrix} \frac{-17}{250} \\ \frac{-93}{100} \\ \frac{43}{50} \end{pmatrix}, \mathbf{p_2} = \begin{pmatrix} \frac{12}{25} \\ -2 \\ \frac{17}{500} \end{pmatrix} \tag{2.0.16}$$

Hence the shortest distance d between the two skew lines is :

$$d = \|\mathbf{p_2} - \mathbf{p_1}\| = 1.4855$$

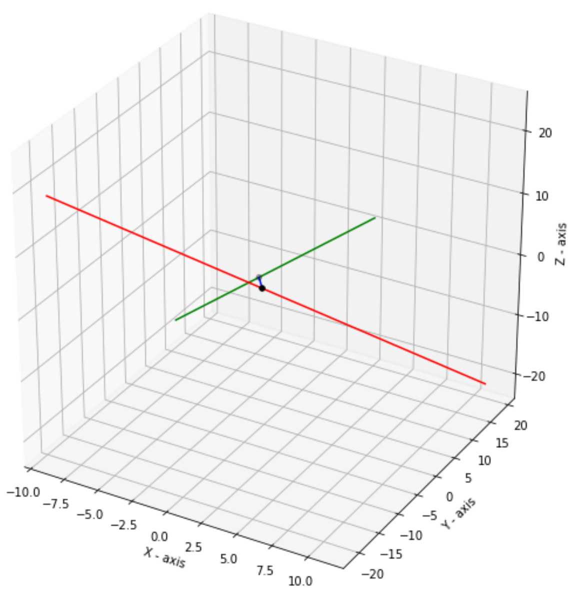

Fig. 0: 3-D plot for the skew lines and the shortest distance between them.