Assignment-10

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***Abstract*—This document explains the relationship between the rank of matrix and the solution of the linear system of equations.**

Download all python codes from

https://github.com/vipulmalik8569/MT−EE5609

and latex-tikz codes from

https://github.com/vipulmalik8569/MT−EE5609

1 Problem

Let $\mathbf{A}$ be an m x n matrix with rank n and real entries. Which of the following statements are correct?

1) $\mathbf{Ax} = \mathbf{b}$ has a solution for any $\mathbf{b}$.
2) $\mathbf{Ax} = 0$ does not have a solution.
3) If $\mathbf{Ax} = \mathbf{b}$ has a solution then it is unique.
4) $\mathbf{y}^T\mathbf{A} = 0$ for some non zero vector $\mathbf{y}$.

2 solution

2.1 Option 1

A solution exist only if $\mathbf{b}$ lies in the column space of $\mathbf{A}$. Let us take an example

$$\mathbf{Ax} = \mathbf{b} \tag{2.1.1}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{2.1.2}$$

Here the system has no solution because the column space of matrix $\mathbf{A}$ is x-y plane and $\mathbf{b}$ lies outside it.

Thus, option 1 is incorrect.

2.2 Option 2

For the system $\mathbf{Ax} = 0$, $\mathbf{x} = 0$ is always the solution.

Therefore, option 2 is incorrect.

2.3 Option 3

There are two possibilities of matrices with dimension m x n and rank n :

1) Rectangular matrix with $m > n$.
2) Square matrix with $m = n$.

Therefore there is no possibility for the system to have infinitely many solutins.

Hence, option 3 is correct.

2.4 Option 4

Let us consider an example with $\mathbf{u}$ and $\mathbf{v}$ as the column vectors of $\mathbf{A}$:

$$\mathbf{y}^T \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix} = 0 \tag{2.4.1}$$

$$\begin{pmatrix} \mathbf{y}^T\mathbf{u} & \mathbf{y}^T\mathbf{v} \end{pmatrix} = 0 \tag{2.4.2}$$

This is possible only when vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{y}$ are orthogonal to each other.
Example :

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} \tag{2.4.3}$$

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.4.4}$$

Hence, option 4 is correct.