Assignment-7

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Abstract—**This document explains the concept of tracing central conics using Affine transformation and Eigenvalue Decomposition.**

Download all python codes from

https://github.com/vipulmalik8569/MT−EE5609

and latex-tikz codes from

https://github.com/vipulmalik8569/MT−EE5609

1 Problem

Trace the following central conic :

$$x^2 + y^2 + xy + x + y = 1 \tag{1.0.1}$$

2 Solution

General equation of second degree is given by :

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.1}$$

In the vector form (1.0.1) can be written as :

$$\mathbf{x}^T\begin{pmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}^T\mathbf{x} - 1 = 0 \tag{2.0.2}$$

By comparing (2.0.1) and (2.0.2) we get :

$$\mathbf{V} = \begin{pmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{pmatrix}, \mathbf{u} = \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}, f = -1 \tag{2.0.3}$$

Eigen values for matrix $\mathbf{V}$ can be calculated by solving :

$$\begin{vmatrix}1-\lambda & \frac{1}{2}\\ \frac{1}{2} & 1-\lambda\end{vmatrix} = 0 \tag{2.0.4}$$

$$\lambda^2 - 2\lambda + \frac{3}{4} = 0 \tag{2.0.5}$$

$$\lambda_1 = \frac{3}{2}, \lambda_2 = \frac{1}{2} \tag{2.0.6}$$

By doing Eigenvalue Decomposition and Affine Transformation we get :

$$\mathbf{P}^{-1}\mathbf{V}\mathbf{P} = \mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \tag{2.0.7}$$

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{2.0.8}$$

Where the matrix $\mathbf{P}$ is normalised eigenbasis and $\mathbf{c}$ is the center.
By putting the value of $\mathbf{x}$ from (2.0.8) in (2.0.1) we get :

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^T\mathbf{V}(\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.9}$$

Further solving this we get :

$$\mathbf{V}\mathbf{c} + \mathbf{u} = 0 \implies \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.0.10}$$

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{2.0.11}$$

As

$$\left|\mathbf{V}\right| = \begin{vmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{vmatrix} = \frac{3}{4} > 0 \tag{2.0.12}$$

Equation (2.0.11) forms an ellipse centered at origin with major and minor axis given as :

$$a = \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \tag{2.0.13}$$

$$b = \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} \tag{2.0.14}$$

Using Gauss Jordan Elimination on matrix $\mathbf{V}$:

$$\xleftrightarrow{R_2\leftarrow\frac{1}{2}R_1-R_2}\begin{pmatrix}1 & \frac{1}{2} & : & 1 & 0\\ 0 & \frac{-3}{4} & : & \frac{1}{2} & -1\end{pmatrix} \tag{2.0.15}$$

$$\xleftrightarrow{R_2\leftarrow\frac{-4}{3}R_2}\begin{pmatrix}1 & \frac{1}{2} & : & 1 & 0\\ 0 & 1 & : & \frac{-2}{3} & \frac{4}{3}\end{pmatrix} \tag{2.0.16}$$

$$\xleftrightarrow{R_1\leftarrow R_1-\frac{1}{2}R_2}\begin{pmatrix}1 & 0 & : & \frac{4}{3} & \frac{-2}{3}\\ 0 & 1 & : & \frac{-2}{3} & \frac{4}{3}\end{pmatrix} \tag{2.0.17}$$

Therefore,

$$\mathbf{V}^{-1} = \begin{pmatrix}\frac{4}{3} & \frac{-2}{3}\\ \frac{4}{3} & \frac{-2}{3}\end{pmatrix} \tag{2.0.18}$$

Using (2.0.10) and (2.0.18) we get :

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = -\begin{pmatrix}\frac{4}{3} & \frac{-2}{3}\\ \frac{4}{3} & \frac{-2}{3}\end{pmatrix}\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{-3}{10}\\ \frac{-3}{10}\end{pmatrix} \tag{2.0.19}$$

By putting the values of $\mathbf{u}$, $\mathbf{V}^{-1}$, f, λ_1 and λ_2 in (2.0.13) and (2.0.14) respectively we get :

$$a = \sqrt{\frac{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{4}{3} & \frac{-2}{3} \\ \frac{4}{3} & \frac{-2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} - 1}{\frac{3}{2}}} = \frac{9}{10} \tag{2.0.20}$$

$$b = \sqrt{\frac{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{4}{3} & \frac{-2}{3} \\ \frac{4}{3} & \frac{-2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} - 1}{\frac{1}{2}}} = \frac{8}{5} \tag{2.0.21}$$

In the transformed space with Eigenbasis, an ellipse centered at origin with major and minor axis as a and b is traced as 'Standard Ellipse' in the plot.

And after doing Affine Transformation on $\mathbf{y}$ as in (2.0.8) we get our 'Actual Ellipse' centered at $\mathbf{c}$ shown in the plot.

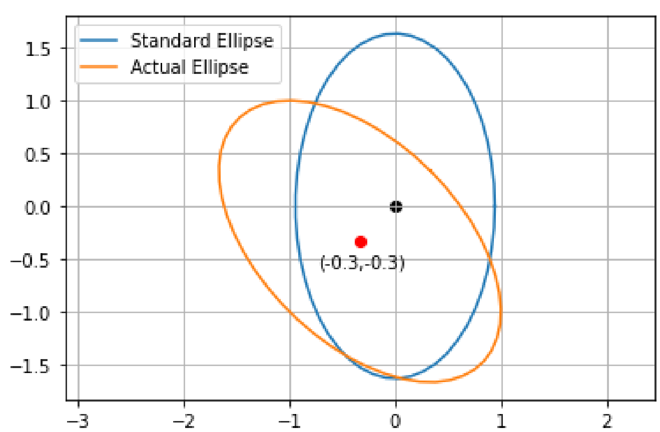

Fig. 0: Standard Ellipse centered at origin and Actual Ellipse centered at $(-0.3, -0.3)$.