1 Inductive Proofs

Prove each of the following claims by induction

Claim 1. *The sum of the first n odd numbers is n^2. That is, $\sum_{i=1}^{n}(2i-1) = n^2$.*

1. Claim: The sum of the first n odd numbers is n^2. That is, $\sum_{i=1}^{n}(2i-1) = n^2$.
2. Mathematical Property: $P(n) =$ The sum of the first n odd numbers is n^2.
3. Base case: $P(1)$ is true since the sum of the first 1 odd numbers is $1^2 = 1$.
4. Induction step: Suppose $P(n)$ is true for $n \geq 1$. We must show $P(n+1)$ is true. By assumption, $\sum_{i=1}^{n}(2i-1) = n^2$, and we must show $\sum_{i=1}^{n+1}(2i-1) = (n+1)^2$. Note that

$$\begin{aligned} \sum_{i=1}^{n+1}(2i+1) &= \\ (2(n+1)-1) + \sum_{i=1}^{n}(2i-1) &= \\ (2(n+1)-1) + n^2 &= \\ n^2 + 2n + 1 &= \\ (n+1)^2, & \end{aligned}$$

   as desired.
5. PMI: We have proven the base and inductive cases so $P(n)$ is true for all $n \geq 1$.

Claim 2. $\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$

(1) Claim: (See above)

(2) Mathematical Property: $P(n) = \sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$.

(3) $P(1)$ is true since $\sum_{i=1}^{1} \frac{1}{2^i} = \frac{1}{2} = 1 - \frac{1}{2^1}$.

(4) Induction Step: Suppose $P(n)$ is true for $n \geq 1$. We must show $P(n+1)$ is true. By assumption,

$\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$, and we must show $\sum_{i=1}^{n+1} \frac{1}{2^i} = 1 - \frac{1}{2^{n+1}}$. Note that

$$\begin{aligned} \sum_{i=1}^{n+1} \frac{1}{2^i} &= \\ \frac{1}{2^{n+1}} + \sum_{i=1}^{n} \frac{1}{2^i} &= \\ \frac{1}{2^{n+1}} + (1 - \frac{1}{2^n}) &= \\ \frac{1}{2} \cdot \frac{1}{2^n} + (1 - \frac{1}{2^n}) &= \\ 1 - \frac{1}{2} \cdot \frac{1}{2^n} &= \\ 1 - \frac{1}{2^{n+1}}&, \end{aligned}$$

as desired.

(5) PMI: We have proven the base and inductive cases so $P(n)$ is true for all $n \geq 1$.

Claim 3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(1) Claim: (See above)

(2) Mathematical Property: $P(n) =$ the sum of the first n perfect squares is $\frac{n(n+1)(2n+1)}{6}$.

(3) $P(1)$ is true since $\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6} = 1$

(4) Induction Step: Suppose $P(n)$ is true for $n \geq 1$. We must show $P(n+1)$ is true. By assumption, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, and we must show

$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)((n+1)+1)+(2(n+1)+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}.$$

Note that

$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \\
\left(\sum_{i=1}^{n} i^2\right) + (n+1)^2 &= \\
\frac{n(n+1)(2n+1)}{6} + (n+1)^2 &= \\
\frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} &= \\
\frac{n(n+1)(2n+1) + 6(n+1)^2}{6} &= \\
\frac{2n^3 + 3n^2 + n + 6n^2 + 12n + 6}{6} &= \\
\frac{2n^3 + 9n^2 + 13n + 6}{6} &= \\
\frac{(n+1)(2n^2 + 7n + 6)}{6} &= \\
\frac{(n+1)(n+2)(2n+3)}{6}
\end{aligned}$$

as desired.

(5) PMI: We have proven the base and inductive cases so $P(n)$ is true for all $n \geq 1$.

2 Recursive Invariants

The function `minEven`, given below in pseudocode, takes as input an array A of size n of numbers. It returns the smallest *even* number in the array. If no even numbers appear in the array, it returns positive infinity ($+\infty$). Using induction, prove that the `minEven` function works correctly. Clearly state your recursive invariant at the beginning of your proof.

```
Function minEven(A,n)
  If n = 0 Then
    Return +∞
  Else
    Set best To minEven(A,n-1)
    If A[n-1] < best And A[n-1] is even Then
      Set best To A[n-1]
    EndIf
    Return best
  EndIf
EndFunction
```

Claim 4. *The function minEven(A,n) returns the smallest even number in an array sized n, for all $n \geq 0$. If no even numbers appear in the array, it returns positive infinity ($+\infty$).*

(1) Claim: (see above)

(2) Mathematical Property/Recursive invariant: $P(n) =$ `minEven(A,n)` returns the smallest even number in the first n elements in an array, or returns ($+\infty$) if there are no even numbers in the first n elements in an array.

(3) Base Cases:

P(0) is true: `minEven(A,0)` returns $+\infty$ since $n = 0$ (see the initial `if` statement). This is a valid return value, because the first zero entries of an array contains no numbers, and therefore no even numbers exist there.

P(1) is true: When `minEven(A,1)` runs, `best` is set to `minEven(A,1-1)` = `minEven(A,0)`. In turn, `minEven(A,0)`, returns $+\infty$ as shown in the previous base case. The program then sets `best` to $+\infty$. Then, the program checks if `A[1-1]` = `A[0]` (the first element in an array) is both even and smaller than ($+\infty$). If the first element in `A` is even, it must also be smaller than ($+\infty$), as integers are finite. Thus, we execute `Set best To A[n-1]` in the `if` statement, and then return `A[0]`, the smallest even number that is the first element of `A`, as desired. If the integer in `A` is *not* even, we would not go into the `if` statement, and we would return $+\infty$, the value of `best`. This would be valid as there would be even elements in the first 1 elements of `A`.

(4) Induction Step: Let a number $n > 1$ be given.

Suppose $P(n)$ is true for $n \geq 0$. We must show $P(n+1)$ is true. By assumption, `minEven(A,n)` returns the smallest even number in the first n elements elements of `A`, or $+\infty$ if there are no even numbers in the first n elements elements of `A`. When calling `minEven(A,n+1)`, first, `best` is set to the value returned by `minEven(A,(n+1)-1)`=`minEven(A,n)`. By assumption, this value is $+\infty$ if there are no even numbers in the first n elements elements of `A`, or otherwise, the smallest even element of the first n elements of `A`.

If the $(n+1)$st element of `A` is even and less than `best`[1], we go into the `if` block and set `best` the $(n+1)$st element of `A`. If this is the case, `A[n]` is the smallest even number in the first $(n+1)$ elements of the array, and the function returns `A[n]`, as desired.

[1]Which is either the smallest element in the first n elements of A, or positive infinity if no such element exists

If we did *not* go into the `if` block, then (1) the $(n+1)$st element of `A` is not even or (2) the last element of `A` is greater than or equal to `best`.

In case 1, there are either no even elements in the first n elements of `A` (meaning the first $(n+1)$ elements of `A` contains no even element), in which case `best` $= +\infty$ is returned, as desired, or `best` equals a smallest even element in the first n elements in `A`, and that value is returned, as desired. (This is desired as `A[n]` is not even.)

In case 2, `best` equals a smallest even element in the first n elements in `A`. Also, `best` must be smaller than the $(n+1)$st element of `A`. Then `best` is the smallest even integer in the first $(n+1)$ elements of `A` and is returned, as desired. (Note that `best` cannot equal $+\infty$ in this case, as the finite number that is the last element of `A` cannot be greater than or equal to $+\infty$.)

(5) PMI/Conclusion: We have proven the base and inductive cases so $P(n)$ is true for all $n \geq 1$. Since $P(n)$ is true for all n, it must be the case that, for an array `A` of size n, `minEven` gives the smallest even number in the first n elements of `A`, or the entire array, itself, as desired.