NONLINEAR OBSERVABILITY AND IDENTIFIABILITY OF GENETIC CIRCUIT MODELS

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Abstract. A brief set of notes on nonlinear control. To be used as a base for building the theory of nonlinear controllability and observability for synthetic and systems biology.
Keywords. system composability, submanifolds, foliations, system identification, genetic circuits, machine learning, manifold learning

Contents

1. Introduction

2. Mathematical Preliminaries

We assume some familiarity with basic notions from differential geometry: smooth (more generally differentiable) manifolds, coordinate charts and local representatives of maps and points, derivatives as operators ([Boothby(2002)]), etc. We explicitly review some definitions and propositions we will need in subsequent sections. A more complete treatment can be found in any standard nonlinear control text, such as [Nijmeijer and Van Der Schaft(1990)].

In the following, we use the symbol N to denote an n-dimensional smooth (C^∞) manifold[1]. We will use the tuples (U, φ) and (V, ϕ) to denote example coordinate charts, where U and V are subsets of N and the bijections φ and ϕ map them to open sets in $\mathbb{R}^n$ (respectively). We also note the usual compatibility requirement on the differentiable structure of N: when $U \cap V$ has a nonempty interior, the map $\phi \circ \varphi^{-1} : \varphi(U \cap V) \to \phi(U \cap V)$ is a (smooth) diffeomorphism.

2.1. Lie Derivatives, distributions and Frobenius' Theorem.

Let

- vector field (briefly), local representative, so you dont need to define it for the lie derivative below.
- Lie deriv and flow (first draft done)
- lie bracket, involutivity
- frobenius thm

Let $X : N \to TN$ be a vector field on a manifold N, where TN is the tangent bundle on N. Recall the definition of the flow[2] generated by a vector field: given X, an initial point p in some

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[1] We reserve the more common symbol M to refer to models in this and related works.

[2] See [Boothby(2002)] for a comprehensive discussion of flow maps and one-parameter subgroups, and [Nijmeijer and Van Der Schaft(1990)] for a more focused definiton.

neighbourhood $U \subset N$, a maximal interval of existence (a, b) of solutions, and a value $t \in (a, b)$, the flow $\phi(t, \cdot) : U \to N$ generated by X maps $p \mapsto \phi(t, p) \in N$, and satisfies the set of differential equations $d\Phi(t, p)/dt = X(\Phi(t, p))$.[3]

Definition 2.1 (Lie Derivative of a real valued function)**.** Given a real-valued function $h : N \to \mathbb{R}$, and a vector field $X : N \to TN$, we define the Lie derivative of h with respect to X as

$$L_X h(p) \triangleq \left.\frac{d}{dt}\right|_{t=0} (h \circ \phi(t, p)), \tag{2.2}$$

It can be shown that $L_X h(p)$ is equivalent to $X(p)h$ (the vector $X(p)$ acting as a differential operator—or *derivation*—on h at point p), and motivates the alternate notation $L_X h(p) = X(h)(p)$

Given a chart (U, φ) around $p \in N$, we may write $X(p)$ in local coordinates as $(X_1(x), \ldots, X_n(x))^T$, where $x = \varphi(p)$

, where a vector field X may be written (with slight abuse of notation) as

and the Lie derivative might be written

$$L_X h(p) = X(h)(p) = \textit{stophere.} \tag{2.3}$$

This local representation illuminates that $X(p)h$, and therefore $L_X h(p)$, defines a real-valued function $X(h) : N \to \mathbb{R}$. This in turn allows for recursively defined Lie derivatives, $L_{X_1} L_{X_2} \cdots L_{X_r} h$ for some set of vector fields $\{X_1, \ldots, X_r\}$.

2.2. **One-forms and codistributions.** Let N denote a *smooth manifold*, T_pN its *tangent space* at point $p \in N$, and T_p^*N the dual *cotangent space.* If $\{x_1, \ldots, x_n\}$ are local coordinates around p, then $\{ \left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p \}$ is a basis for T_pN in local coordinates, with the corresponding dual basis denoted $\{dx_1|_p, \ldots, dx_n|_p\}$, with the defining relationship $dx_i|_p \left(\left.\frac{\partial}{\partial x_j}\right|_p \right) = \delta_{ij}$.

Let $s : N \to \mathbb{R}$ be a smooth function. At every point $p \in N$, the *differential* of s at p is an element of T_p^*N, defined via its action on vectors,

$$\begin{aligned} ds(p)(X_p) &\triangleq X_p(s)(p) \\ &= \left(\sum_{j=1}^n X_{p,j} \left.\frac{\partial}{\partial x_j}\right|_p \right) s(p) \\ &= \sum_{i=1}^n \left.\frac{\partial s}{\partial x_i}\right|_p X_{p,i}, \end{aligned} \tag{2.4}$$

where we note that vectors are differential operators on real valued smooth functions.

We can endow $ds(p)$ with a representation in terms of the basis of T_p^*N by considering its action on a basis of T_pN, which results[4] in $ds(p) = \sum_{i=1}^n \frac{\partial s}{\partial x_i}(p) dx_i|_p$. Then, (2.4) can be written as [5], $ds(p)(X_p) = \left(\sum_{i=1}^n \frac{\partial s}{\partial x_i}(p) dx_i \right) \left(\sum_{j=1}^n X_{p,j} \frac{\partial}{\partial x_j} \right)$.

The *cotangent bundle* of a manifold N is defined as $T^*N \triangleq \cup_{p \in N} T_p^*N$, and can be given a manifold structure. Then, a *smooth one-form* σ on a smooth manifold N is defined as a smooth map $\sigma : N \to T^*N$ satisfying

$$\pi \circ \sigma = \text{identity (on } N),$$

where $\pi : T^*N \to N$ is the natural projection. That is, a one-form is a smooth map assigning a cotangent vector to each point $p \in N$.

[3] Try to improve this flow part via one of the textbooks

[4] Consider $ds(p) \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} s(p)$

[5] Noting the linearity of dx_i and right-distributivity of the expression in the first parentheses.

In local coordinates x, one forms may be expressed[6] as $\sigma(x) = \sum_{i=1}^{n} \sigma_i(x) dx_i|_x$. Being dual objects of vector fields, one-forms act on them as expected: $\sigma(X)(p) = \sigma(p)X(p) \in \mathbb{R}$, and define smooth real-valued functions on N, $\sigma(X) : N \to \mathbb{R}$. Any smooth real-valued function s defines a one-form $ds \in T^*N$ via (2.4). Note the relationship with the Lie derivative, $ds(X) = X(s) = L_X s$. Note also that not every one form can be written as ds for some smooth real-valued function s. However, those that can, are called *exact*.

Recall that a (smooth) distribution add to that section! assigns a subspace of the tangent space to each point on the manifold in a smooth manner. Similarly, we can define a dual notion, the smooth codistribution. A smooth codistribution assigns, to each point on the manifold, a subspace of the corresponding cotangent space in a smooth manner (to be made precise below). Just as distributions and accessibility algebras play a fundamental role in describing nonlinear controllability, codistributions and *observation spaces* play a similar role in describing nonlinear observability (and by extension, identifiability). We discuss these ideas in the following sections.

Definition 2.5 (Smooth Codistribution)**.** Around any point p, let there exist a neighbourhood U of p and a set of smooth one-forms $\sigma_i \in T_p^*N, i \in I$ (I possibly infinite), such that for each $q \in U$, $P(q) = \text{span}\{\sigma_i(q); i \in I\}$. Then P is called a *smooth codistribution* on N.

In what follows, codistribution will always mean smooth codistribution. A one-form belongs to $P(p)$ if $\sigma(p) \in P(p)$ for any $p \in N$, and a codistribution is constant dimensional if the dimension of $P(p)$ does not depend on p. If a codistribution is constant dimensional of dimension l, then around each point p, there exist l independent one-forms (called the *local generators*) $\sigma_1, \dots, \sigma_l$ such that $P(q) = \text{span}\{\sigma_1(q), \dots, \sigma_l(q)\}$, for q near p.

Next, we define the notions of the kernel and annihilator of a codistribution and distribution respectively. Let P and D be a codistribution and distribution on N, respectively. Then,

$$\begin{aligned} \text{ker}P(p) &= \{X(p) \mid X \text{ is a vector field s.t. } \sigma(X)(p) = 0, \forall \sigma \in P\} \\ \text{ann}D(p) &= \{\sigma(p) \mid \sigma \text{ is a one-form s.t. } \sigma(X)(p) = 0, \forall X \in D\} \end{aligned} \tag{2.6}$$

If D and P are constant dimensional, then $D = \text{ker}(\text{ann}D)$ and $P = \text{ann}(\text{ker}P)$. If $\text{ker}P$ is involutive, then we call P an involutive codistribution. If P is generated by exact one forms, then it is easily shown that it must be involutive.

2.3. **Nonlinear Observability.** Consider the nonlinear system given by

$$\begin{aligned} \dot{x} &= f(x) + \sum_{j=1}^{m} g_j u_j, \qquad u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m), \\ y_i &= h_i(x), \qquad i = 1, \dots, p, \end{aligned} \tag{2.7}$$

where $h = (h_1, \dots, h_p)^T : N \to Y = \mathbb{R}^p$ and $y(t, x_0, u) = h(x(t, x_0, u))$

We define the notions of state indistinguishability, local observability, observation space and the observability codistribution. We can then state two versions of the nonlinear observability rank condition. These will allow us to talk about decomposing observability into observable and unobservable modes, analogously to the linear case. These results will be extended to include identifiability using a simple device: including system parameters as additional state variables with zero dynamics, and considering their observability.

Definition 2.8 (Nonlinear observability)**.** Two states $x_1, x_2 \in N$ are *indistinguishable* for system (2.7) if for every admissible input function u the output functions $t \mapsto y(t, x_1, u), t \geq 0$ and $t \mapsto y(t, x_2, u), t \geq 0$ are identical on their common domain of definition. The system is *observable* if the states being indistinguishable implies $x_1 = x_2$.

Definition 2.9 (Nonlinear local observability)**.** For an open set $V \subset M$, we say that $x_1, x_2 \in V$ are $V-$*indistinguishable*, denoted $x_1 I^V x_2$ if for every admissible *constant* control u such that $x(t, x_1, u)$

[6]With slight abuses of notation, like dropping the hat from the local representative $\hat{\sigma}$.

and $x(t, x_2, u)$ remain in V for $t \leq T > 0$, the output functions $y(t, x_1, u)$ and $y(t, x_2, u)$ are equal on their common domain of definition. The system (2.7) is called *locally observable* at x_0 if there exists a neighborhood W of x_0 such that for every neighborhood $V \subset W$ of x_0 the relation $x_0 I^V x_1$ implies $x_1 = x_0$. A system is locally observable if it is locally observable for all $x_0 \in M$.

A system is locally observable if every state x_0 can be distinguished from its neighbors by using system trajectories remaining close to x_0.

Definition 2.10 (Observation space)**.** The *observation space* $\mathscr{O}$ of the system (2.7) is the linear space (over field $\mathbb{R}$) of real valued smooth functions on N containing $h_i, i = 1, \dots, p$ and all repeated Lie derivatives

$$L_{X_1} \dots L_{X_k} h_j, \quad j = 1, \dots, p, \quad k = 1, 2, \dots \tag{2.11}$$

with $X_i \in \{f, g_1, \dots, g_m\}$, $i \in \{1, \dots, k\}$.

Proposition 2.12. *$\mathscr{O}$ is equivalent to the linear space of functions on N containing $h_1, \dots, h_p$ and all repeated Lie derivatives along system trajectories. These Lie derivatives can be written $L_{Z_1} L_{Z_2} \cdots L_{Z_k} h_j$, with $j \in \{1, \dots, p\}$ and $k = 1, 2, \dots,$ and Z_i, $i \in \{1, \dots, k\}$ of the form*

$$Z_i(x) = f(x) + \sum_{j=1}^{m} g_j(x) u_j^i, \tag{2.13}$$

for some point $u^i \in U$.

Proof. The linearity properties of the Lie derivative of a function, $L_{X_1+X_2} H = L_{X_1} H + L_{X_2} H$ and $L_X(H_1 + H_2) = L_X H_1 + L_X H_2$, together with the fact that the Z_i are linear combinations of $f, g_1, \dots, g_m$, imply $L_{Z_1} L_{Z_2} \cdots L_{Z_k} h_j \in \mathscr{O}$. Conversely, all vector fields $f, g_1, \dots, g_m$ can be written as linear combinations of vector fields of the form Z_i. To see this, note that $f = Z_i$ for $u^i = 0$ for any i, and defining $Z_j^+ = f + g_j$ and $Z_j^- = f - g_j$ (i.e., using $u = \pm(0, \dots, 0, 1, 0, \dots, 0)$ in (2.13), where the 1 is at the j-th coordinate), we have $g_j = \frac{1}{2}(Z_j^+ - Z_j^-)$. $\square$

Proposition 2.12 shows that $\mathscr{O}$ comprises the output functions and all derivatives of the output functions along system trajectories. In the case of systems with no inputs (or constant inputs), we can construct $\mathscr{O}$ using y_j and all repeated time derivatives, $\dot{y}_j = L_f h_j(x)$, $\ddot{y} = L_f L_f h_j(x)$, and so on. Initially, we will develop our results in this *autonomous* system setting.

Next, we define the central construct for gauging observability of a system.

Definition 2.14 (Observability Codistribution)**.** Given $\mathscr{O}$, the *observability codistribution $d\mathscr{O}$* is defined as

$$d\mathscr{O}(q) = \operatorname{span}\{dH(q) \mid H \in \mathscr{O}\} \tag{2.15}$$

Since $d\mathscr{O}$ is generated by exact one-forms, it is involutive[7].

Theorem 2.16. *Consider the system 2.7 with $dimN = n$. Assume that $dims d\mathscr{O}(x_0) = n$. Then the system is locally observable at x_0.*

The

References

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[7] Write out involutivity of distributions and codistributions, and the exact one form stuff, frobenius. Also, check what the involutivity of the observability codistribution is needed for.