

A PRIMER ON NONLINEAR OBSERVABILITY AND IDENTIFIABILITY

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ABSTRACT. A brief set of notes on nonlinear observability and identifiability. We begin with a description of basic ideas from differential geometry (manifolds, Frobenius theorem, distributions and codistributions), and move on to describing the observability-identifiability codistribution. We end with a method for identifying state and parameter variables that are unobservable or non-identifiable. This geometric viewpoint can be used to describe uncertainty in models of gene expression in systems and synthetic biology, which we will explore in a subsequent work.

Optionally (not sure yet) I will describe the accessibility algebra and nonlinear controllability. We can then unify everything into *minimal realizations* of nonlinear control systems.

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1. INTRODUCTION

Nonlinear control is a beautiful area within mathematics, with a myriad of applications in fields ranging from electrical and aerospace engineering to biology and machine learning (cite). Add more intro.

We assume some familiarity with basic notions from differential geometry: smooth manifolds, coordinate charts, local representatives of objects (maps, vectors, etc.), vectors as differential operators, tangent spaces, and flows generated by vector fields. We will briefly review some of these definitions in the next section. A more complete treatment can be found in any standard nonlinear control or differential geometry

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text, such as Nijmeijer and Van Der Schaft [1990], Boothby [2002]. Another excellent source is Abraham et al. [1988], though this is better used as a second course, since its simultaneous focus on depth and breadth can make it feel like one is taking too many detours from the “goal” of getting to core results like Frobenius’ theorem.

2. DIFFERENTIAL GEOMETRIC PRELIMINARIES

2.1. Manifolds, charts and local representatives. Roughly speaking, a smooth manifold is a set which is locally Euclidean—in the sense that neighborhoods of points can be modeled using Euclidean spaces—along with a certain *differentiable structure*. We use the symbol N to denote an n -dimensional smooth (C^∞) manifold, and the tuples (U, φ) and (V, ϕ) to denote example coordinate charts, where U and V are open sets in N (which exist by virtue of N being a Hausdorff topological space) and the homeomorphisms φ and ϕ map them (respectively) onto open sets in \mathbb{R}^n . The key component of N ’s differentiable structure is the following compatibility requirement for coordinate changes: whenever $U \cap V$ has a nonempty interior, the map $\phi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \phi(U \cap V)$ is a (smooth) diffeomorphism.

Given a point $p \in U$, its local representative is given by its local coordinates, $\varphi(p) = (x^1(p), \dots, x^n(p))$, the functions $p \rightarrow x^i(p)$ are called the *coordinate functions*, coordinate neighborhoods can also be denoted (U, x_1, \dots, x_n) , where we drop the explicit dependence on p . The local representative of a map $f : U \rightarrow \mathbb{R}$ is constructed as $\hat{f} \triangleq f \circ \varphi^{-1}$, so that

$$(2.1) \quad f(p) = \hat{f}(x_1(p), \dots, x_n(p)).$$

Per convention, we will drop the caret from f and the explicit dependence of the coordinate functions on p , resulting in the RHS of Equation 2.1 being written $f(x_1, \dots, x_n)$.

2.2. The Inverse and Implicit Function Theorems, and the Rank Theorem. In this section, we describe the inverse function theorem, which is one of the pillars on which analysis stands. We also describe two corollaries of the inverse function theorem—the implicit function and rank theorems—which are widely used in calculus and differential geometry.

Consider two manifolds N_1 and N_2 of dimension n_1 and n_2 respectively, with respective coordinate charts (U, φ) and (V, ϕ) around points $p \in U \subset N_1$ and $F(p) \in V \subset N_2$, where $F : N_1 \rightarrow N_2$ is a map between the manifolds. Its local representative is given by $\hat{F}(x_1, \dots, x_{n_1}) = F \circ \varphi^{-1}(x_1, \dots, x_{n_1})$, where $(x_1, \dots, x_{n_1}) = \varphi(p)$. As is customary, we will drop the caret, and overload F by using it to refer to both the map between manifolds, and its local representative, and relying on context to make its meaning clear. F is called a *diffeomorphism* if it is bijective and both (the local representatives of) F and F^{-1} are smooth, and if such a diffeomorphism exists then (i) $n_1 = n_2$ and (ii) N_1 and N_2 are called diffeomorphic. Note that $\varphi : U \rightarrow \mathbb{R}^{n_1}$ is a diffeomorphism, and this is what it means for a manifold to be locally Euclidean—each U is diffeomorphic to Euclidean space of appropriate dimension.

- describe the three theorems
- can also briefly mention the few propositions (without proof.)
- and its local representative
- submersion
- immersion

2.3. Tangent bundles, vectors as operators, and the pushforward map. In this section, we review the tangent space to a manifold, vectors as differential operators on functions, and the idea of the pushforward of a vector from one manifold to another.

Denote by $C^\infty(p)$ the set of all smooth real-valued functions, whose domain of definition contains a neighbourhood of p , *identifying* those functions that agree on an open set containing p . Let $T_p\mathbb{R}^n$ be a vector space with basis $\{E_{ip}\}_{i=1}^n$, so that a vector may be written, for $\alpha_i \in \mathbb{R}$, $X_p = \sum_{i=1}^n \alpha_i E_{ip}$. Now define the action of a vector on a real-valued smooth function at p as the directional derivative of that function in the direction of that (not necessarily unit norm) vector,

$$(2.2) \quad X_p(f) \triangleq \sum_{i=1}^n \alpha_i \left(\frac{\partial f}{\partial x_i} \right)_p.$$

Notice that interpreting X_p as a directional derivative means that it can be thought of as a map from $C^\infty(p)$ to \mathbb{R} . This motivates us to define a particular type of operator on real-valued smooth functions, called a derivation:

Definition 2.3 (Derivations). The space of *derivations*, denoted $\mathcal{D}(p)$, is the vector space of all mappings $C^\infty(p) \rightarrow \mathbb{R}$ such that for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(p)$ the following two properties hold:

$$(2.4) \quad \begin{aligned} X_p(\alpha f + \beta g) &= \alpha(X_p f) + \beta(X_p g) && \text{(linearity),} \\ X_p(fg) &= (X_p f)g(p) + f(p)(X_p g) && \text{(Leibniz rule),} \end{aligned}$$

with the vector space operations satisfying

$$(2.5) \quad \begin{aligned} (X_p + Y_p)(f) &= X_p(f) + Y_p(f), \\ (\alpha X_p)(f) &= \alpha X_p(f), \quad \alpha \in \mathbb{R}. \end{aligned}$$

These ideas are tied together by the following theorem.

Theorem 2.6. *The vector space $T_p\mathbb{R}^n$ is isomorphic to the vector space $\mathcal{D}(p)$ of all derivations of $C^\infty(p)$ into \mathbb{R} . This isomorphism is given by making each X_p correspond to the directional derivative in the direction of X_p .*

This theorem allows us to *identify* $T_p\mathbb{R}^n$ with $\mathcal{D}(p)$. Using this isomorphism, we write vectors as linear combinations of basis vectors, $X_p = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i} \Big|_p$, so that the E_{ip} are identified with $\frac{\partial}{\partial x_i} \Big|_p$, and $E_{ip}f = \frac{\partial f}{\partial x_i} \Big|_p$. Note that $X_p x_i = \alpha_i$, so that X_p is completely determined by its action on the coordinate functions, x_i . This equivalence between derivations on functions and the tangent space will be especially useful in defining the tangent space to manifolds, a topic we cover next.

Consider a smooth manifold N and a point $p \in N$, we may define $C^\infty(p)$ as the algebra of smooth real-valued functions whose domain of definition contains some neighbourhood of p . The tangent space to N at p may then be defined using derivations.

Definition 2.7 (Tangent Space of N). The *tangent space* to N at point p , denoted by $T_p N$, is defined as the space of derivations on $C^\infty(p)$, i.e., the vector space of mappings $X_p : C^\infty(p) \rightarrow \mathbb{R}$ satisfying the linearity and Leibniz rule properties defined in Equation 2.4. Elements of this space, denoted $X_p \in T_p N$, are called *tangent vectors*.

2.4. Vector Fields and Distributions. The tangent bundle associated with a manifold is defined as the disjoint union of individual tangent spaces (i.e., keeping

track of the base points in addition to the tangent spaces themselves),

$$\begin{aligned}
 TN &= \bigsqcup_{p \in N} T_p N \\
 (2.8) \quad &= \bigcup_{p \in N} \{p\} \times T_p N \\
 &= \bigcup_{p \in N} \{(p, v) \mid v \in T_p N\}.
 \end{aligned}$$

One defines the natural projection $\pi : TN \rightarrow N : (p, X_p) \mapsto p$. Note that some authors define the tangent bundle without the base point explicitly included, $TN = \cup_{p \in N} T_p N$, with projection $X_p \mapsto \pi(X_p) = p$. The tangent bundle is useful when defining (smooth) vector fields on manifolds.

A vector field $X : N \rightarrow TN$ assigns, in a smooth way (to be made precise below), a vector to each point on a manifold. Note that $\pi \circ X = i_N$, the identity on N . Smoothness of this assignment is defined as follows. For (U, φ) a chart around a point $p \in N$, and $\{E_{ip}\}_{i=1}^n$ the basis of $T_p N$, a vector may be written $X_p = \sum_{i=1}^n \alpha_i(p) E_{ip}$. If p is varied in U , so that we consider different vectors in the vector field, the components α_i are real-valued functions on N , which we require to be smooth. In local coordinates, we require

$$(2.9) \quad \alpha^i = \alpha^i(x^1, \dots, x^n), \quad i = 1, \dots, n \text{ on } \varphi(U) \subset \mathbb{R}^n.$$

to be $C^\infty(\varphi(U))$ (smooth on $\varphi(U) \in \mathbb{R}^n$).

2.5. Lie Derivatives, distributions and Frobenius' Theorem.

- vector field (briefly), local representative, so you dont need to define it for the lie derivative below.
- Lie deriv and flow (first draft done)
- lie bracket, involutivity
- frobenius thm

Let $X : N \rightarrow TN$ be a vector field on a manifold N , where TN is the tangent bundle on N . Recall the definition of the flow generated by a vector field: given X , an initial point p in some neighborhood $U \subset N$, a maximal interval of existence (a, b) of solutions, and a value $t \in (a, b)$, the flow $\phi(t, \cdot) : U \rightarrow N$ generated by X maps $p \mapsto \phi(t, p) \in N$, and satisfies the set of differential equations $d\phi(t, p)/dt = X(\phi(t, p))$.¹

Definition 2.10 (Lie Derivative of a real valued function). Given a real-valued function $h : N \rightarrow \mathbb{R}$, and a vector field $X : N \rightarrow TN$, we define the Lie derivative of h with respect to X as

$$(2.11) \quad L_X h(p) \triangleq \left. \frac{d}{dt} \right|_{t=0} (h \circ \phi(t, p)),$$

It can be shown that $L_X h(p)$ is equivalent to $X(p)h$ (the vector $X(p)$ acting as a differential operator—or *derivation*—on h at point p), and motivates the alternate notation $L_X h(p) = X(h)(p)$

Given a chart (U, φ) around $p \in N$, we may write $X(p)$ in local coordinates as $(X_1(x), \dots, X_n(x))^T$, where $x = \varphi(p)$

, where a vector field X may be written (with slight abuse of notation) as
and the Lie derivative might be written

$$(2.12) \quad L_X h(p) = X(h)(p) = \lim_{h \rightarrow 0} \frac{f(\phi(h, p)) - f(p)}{h} \text{ *stophere.*}$$

¹Try to improve this flow part via one of the textbooks

This local representation illuminates that $X(p)h$, and therefore $L_X h(p)$, defines a real-valued function $X(h) : N \rightarrow \mathbb{R}$. This in turn allows for recursively defined Lie derivatives, $L_{X_1} L_{X_2} \cdots L_{X_r} h$ for some set of vector fields $\{X_1, \dots, X_r\}$.

2.6. One-forms, codistributions and pullbacks. Let N denote a *smooth manifold*, $T_p N$ its *tangent space* at point $p \in N$, and $T_p^* N$ the dual *cotangent space*. If $\{x_1, \dots, x_n\}$ are local coordinates around p , then $\left\{ \frac{\partial}{\partial x_1} \Big|_p, \dots, \frac{\partial}{\partial x_n} \Big|_p \right\}$ is a basis for $T_p N$ in local coordinates, with the corresponding dual basis denoted $\{dx_1|_p, \dots, dx_n|_p\}$, with the defining relationship $dx_i|_p \left(\frac{\partial}{\partial x_j} \Big|_p \right) = \delta_{ij}$.

Let $s : N \rightarrow \mathbb{R}$ be a smooth function. At every point $p \in N$, the *differential* of s at p is an element of $T_p^* N$, defined via its action on vectors,

$$(2.13) \quad \begin{aligned} ds(p)(X_p) &\triangleq X_p(s)(p) \\ &= \left(\sum_{j=1}^n X_{p,j} \frac{\partial}{\partial x_j} \Big|_p \right) s(p) \\ &= \sum_{i=1}^n \frac{\partial s}{\partial x_i} \Big|_p X_{p,i}, \end{aligned}$$

where we note that vectors are differential operators on real valued smooth functions.

We can endow $ds(p)$ with a representation in terms of the basis of $T_p^* N$ by considering its action on a basis of $T_p N$, which results² in $ds(p) = \sum_{i=1}^n \frac{\partial s}{\partial x_i}(p) dx_i|_p$. Then, (2.13) can be written as³, $ds(p)(X_p) = \left(\sum_{i=1}^n \frac{\partial s}{\partial x_i}(p) dx_i \right) \left(\sum_{j=1}^n X_{p,j} \frac{\partial}{\partial x_j} \right)$.

The *cotangent bundle* of a manifold N is defined as $T^*N \triangleq \cup_{p \in N} T_p^* N$, and can be given a manifold structure. Then, a *smooth one-form* σ on a smooth manifold N is defined as a smooth map $\sigma : N \rightarrow T^*N$ satisfying

$$\pi \circ \sigma = \text{identity (on } N),$$

where $\pi : T^*N \rightarrow N$ is the natural projection. That is, a one-form is a smooth map assigning a cotangent vector to each point $p \in N$.

In local coordinates x , one forms may be expressed⁴ as $\sigma(x) = \sum_{i=1}^n \sigma_i(x) dx_i|_x$. Being dual objects of vector fields, one-forms act on them as expected: $\sigma(X)(p) = \sigma(p)X(p) \in \mathbb{R}$, and define smooth real-valued functions on N , $\sigma(X) : N \rightarrow \mathbb{R}$. Any smooth real-valued function s defines a one-form $ds \in T^*N$ via (2.13). Note the relationship with the Lie derivative, $ds(X) = X(s) = L_X s$. Note also that not every one form can be written as ds for some smooth real-valued function s . However, those that can, are called *exact*.

2.7. Pushforwards are derivatives. Add here.

2.8. Relationship between one forms, pushforwards, and differentials. !! verify. time derivative of a real valued function is the pushforward of the unit vector by that function. More generally, if you want to differentiate a C^∞ map $F : N \rightarrow Q$ along some direction and at some point, given by the derivation $X_p \in T_p N$, then you simply pushforward this derivation via F , as follows: $F_* X_p \in T_{F(p)} Q$. Note that this can be verified by observing that $(F_* X_p)(q)|_{F(p)} = (X_p)(q \circ F)|_p$. So in the

²Consider $ds(p) \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} s(p)$

³Noting the linearity of dx_i and right-distributivity of the expression in the first parentheses.

⁴With slight abuses of notation, like dropping the hat from the local representative $\hat{\sigma}$.

case of time derivatives, let's see what happens. Consider a curve $F : (a, b) \rightarrow N$, the unit vector in (a, b) , namely $\partial_t \in T_p(a, b)$ for some $p \in (a, b)$ and its pushforward $F_*\partial_t \in T_{F(p)}N$ defined by

$$(2.14) \quad (F_*\partial_t) f = \partial_t(f \circ F)$$

$$(2.15) \quad = \left. \frac{\partial f}{\partial s} \right|_{s=F(t)} \left. \frac{\partial F}{\partial t} \right|_t,$$

where $f \in C^\infty(N)$ is a real-valued smooth function on N . Let's consider what this looks like in local coordinates.

$$(2.16) \quad (F_*\partial_t) f = \partial_t(f \circ F) \Big|_t = \left. \frac{\partial f}{\partial s} \right|_{s=F(t)} \left. \frac{\partial F}{\partial t} \right|_t =$$

2.9. Codistributions. Recall that a (smooth) distribution **add to that section!** assigns a subspace of the tangent space to each point on the manifold in a smooth manner. Similarly, we can define a dual notion, the smooth codistribution. A smooth codistribution assigns, to each point on the manifold, a subspace of the corresponding cotangent space in a smooth manner (to be made precise below). Just as distributions and accessibility algebras play a fundamental role in describing nonlinear controllability, codistributions and *observation spaces* play a similar role in describing nonlinear observability (and by extension, identifiability). We discuss these ideas in the following sections.

Definition 2.17 (Smooth Codistribution). Around any point p , let there exist a neighborhood U of p and a set of smooth one-forms $\sigma_i \in T_p^*N, i \in I$ (I possibly infinite), such that for each $q \in U$, $P(q) = \text{span}\{\sigma_i(q); i \in I\}$. Then P is called a *smooth codistribution* on N .

In what follows, codistribution will always mean smooth codistribution. A one-form belongs to $P(p)$ if $\sigma(p) \in P(p)$ for any $p \in N$, and a codistribution is constant dimensional if the dimension of $P(p)$ does not depend on p . If a codistribution is constant dimensional of dimension l , then around each point p , there exist l independent one-forms (called the *local generators*) $\sigma_1, \dots, \sigma_l$ such that $P(q) = \text{span}\{\sigma_1(q), \dots, \sigma_l(q)\}$, for q near p .

Next, we define the notions of the kernel and annihilator of a codistribution and distribution respectively. Let P and D be a codistribution and distribution on N , respectively. Then,

$$(2.18) \quad \begin{aligned} \ker P(p) &= \{X(p) \mid X \text{ is a vector field s.t. } \sigma(X)(p) = 0, \forall \sigma \in P\} \\ \text{ann} D(p) &= \{\sigma(p) \mid \sigma \text{ is a one-form s.t. } \sigma(X)(p) = 0, \forall X \in D\} \end{aligned}$$

If D and P are constant dimensional, then $D = \ker(\text{ann} D)$ and $P = \text{ann}(\ker P)$. If $\ker P$ is involutive, then we call P an involutive codistribution. If P is generated by exact one forms, then it is easily shown that it must be involutive.

3. NONLINEAR OBSERVABILITY

Consider the nonlinear system given by

$$(3.1) \quad \begin{aligned} \dot{x} &= f(x) + \sum_{j=1}^m g_j u_j, & u &= (u_1, \dots, u_m) \in U \subset \mathbb{R}^m, \\ y_i &= h_i(x), & i &= 1, \dots, p, \end{aligned}$$

where $h = (h_1, \dots, h_p)^T : N \rightarrow Y = \mathbb{R}^p$ and $y(t, x_0, u) = h(x(t, x_0, u))$

We define the notions of state indistinguishability, local observability, observation space and the observability codistribution. We can then state two versions of the nonlinear observability rank condition. These will allow us to talk about decomposing observability into observable and unobservable modes, analogously to the linear case. These results will be extended to include identifiability using a simple device: including system parameters as additional state variables with zero dynamics, and considering their observability.

Definition 3.2 (Nonlinear observability). Two states $x_1, x_2 \in N$ are *indistinguishable* for system (3.1) if for every admissible input function u the output functions $t \mapsto y(t, x_1, u), t \geq 0$ and $t \mapsto y(t, x_2, u), t \geq 0$ are identical on their common domain of definition. The system is *observable* if the states being indistinguishable implies $x_1 = x_2$.

Definition 3.3 (Nonlinear local observability). For an open set $V \subset M$, we say that $x_1, x_2 \in V$ are *V-indistinguishable*, denoted $x_1 I^V x_2$ if for every admissible *constant* control u such that $x(t, x_1, u)$ and $x(t, x_2, u)$ remain in V for $t \leq T > 0$, the output functions $y(t, x_1, u)$ and $y(t, x_2, u)$ are equal on their common domain of definition. The system (3.1) is called *locally observable* at x_0 if there exists a neighborhood W of x_0 such that for every neighborhood $V \subset W$ of x_0 the relation $x_0 I^V x_1$ implies $x_1 = x_0$. A system is locally observable if it is locally observable for all $x_0 \in M$.

A system is locally observable if every state x_0 can be distinguished from its neighbors by using system trajectories remaining close to x_0 .

Definition 3.4 (Observation space). The *observation space* \mathcal{O} of the system (3.1) is the linear space (over field \mathbb{R}) of real valued smooth functions on N containing $h_i, i = 1, \dots, p$ and all repeated Lie derivatives

$$(3.5) \quad L_{X_1} \dots L_{X_k} h_j, \quad j = 1, \dots, p, \quad k = 1, 2, \dots$$

with $X_i \in \{f, g_1, \dots, g_m\}, i \in \{1, \dots, k\}$.

Proposition 3.6. \mathcal{O} is equivalent to the linear space of functions on N containing h_1, \dots, h_p and all repeated Lie derivatives along system trajectories. These Lie derivatives can be written $L_{Z_1} L_{Z_2} \dots L_{Z_k} h_j$, with $j \in \{1, \dots, p\}$ and $k = 1, 2, \dots$, and $Z_i, i \in \{1, \dots, k\}$ of the form

$$(3.7) \quad Z_i(x) = f(x) + \sum_{j=1}^m g_j(x) u_j^i,$$

for some point $u^i \in U$.

Proof. The linearity properties of the Lie derivative of a function, $L_{X_1+X_2}H = L_{X_1}H + L_{X_2}H$ and $L_X(H_1+H_2) = L_XH_1 + L_XH_2$, together with the fact that the Z_i are linear combinations of f, g_1, \dots, g_m , imply $L_{Z_1}L_{Z_2} \dots L_{Z_k}h_j \in \mathcal{O}$. Conversely, all vector fields f, g_1, \dots, g_m can be written as linear combinations of vector fields of the form Z_i . To see this, note that $f = Z_i$ for $u^i = 0$ for any i , and defining $Z_j^+ = f + g_j$ and $Z_j^- = f - g_j$ (i.e., using $u = \pm(0, \dots, 0, 1, 0, \dots, 0)$ in (3.7), where the 1 is at the j -th coordinate), we have $g_j = \frac{1}{2}(Z_j^+ - Z_j^-)$. \square

Proposition 3.6 shows that \mathcal{O} comprises the output functions and all derivatives of the output functions along system trajectories. In the case of systems with no inputs (or constant inputs), we can construct \mathcal{O} using y_j and all repeated time derivatives, $\dot{y}_j = L_f h_j(x)$, $\ddot{y}_j = L_f L_f h_j(x)$, and so on. Initially, we will develop our results in this *autonomous* system setting.

Next, we define the central construct for gauging observability of a system.

Definition 3.8 (Observability Codistribution). Given \mathcal{O} , the *observability codistribution* $d\mathcal{O}$ is defined as

$$(3.9) \quad d\mathcal{O}(q) = \text{span}\{dH(q) \mid H \in \mathcal{O}\}$$

Since $d\mathcal{O}$ is generated by exact one-forms, **it is involutive**⁵.

Theorem 3.10. *Consider the system 3.1 with $\dim N = n$. Assume that $\dim d\mathcal{O}(x_0) = n$. Then the system is locally observable at x_0 .*

The

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⁵Write out involutivity of distributions and codistributions, and the exact one form stuff, frobenius. Also, check what the involutivity of the observability codistribution is needed for.