Definition Graph

A Graph consist of

A set $V = V(G)$ whose elements are called vertices.

A set $E = V(E)$ whose elements are called Edges.

The Graph (G) with vertices and edges is written as $G = (V, E)$ or $G(V, E)$

The set of vertices (V) of Graph (G) is may be finite or infinite.

A graph with finite vertice set and finite edges set are called finite graph.

Types of Graph (Based on the Edges)

- Simple Graph
- Pseudo Graph
- Multi Graph

Types of Graph (Based on the direction)

- Directed Graph
- Undirected Graph
- Mixed Graph

Types of Graph (Based on Connectivity)

- Connected Graph
- Disconnected Graph
- Strongly connected Graph
- Weakly connected Graph

→ Basic Terminologies
Adjacent vertices
Adjacent Edges
The handshaking Theory
Degree of Vertex
↳ Undirected Graph
↳ Directed Graph (indegree & outdegree)
Isolated vertices (degree = 0)
Pendant vertices (degree = 1)
Empty Graph
Incident vertex

→ The Handshaking Theory.

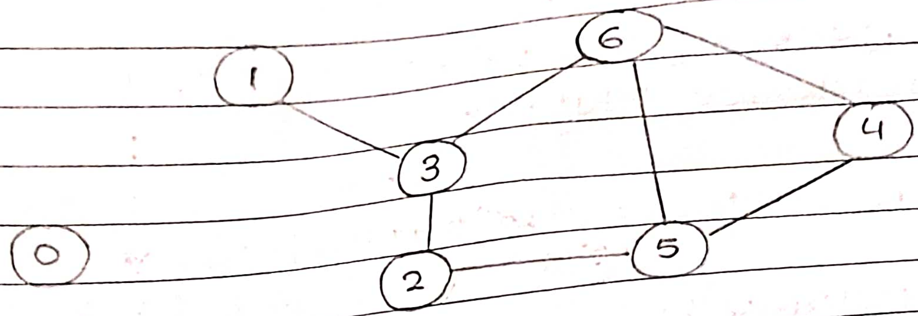
let G(V,E) be an undirected graph with vertice (V) and Edges (E). Then
$\Sigma \deg(v) =$

let G(V,E) be an directed graph with vertice (V) and Edge (E).
Σ indegree

→ Types of Graph (Based on the degree)
complete Graph
non-complete Graph
Regular Graph

→ Type of Graph
Bipartite Graph
complete bipartite graph
Weighted Graph
Unweighted Graph

Q.) construct the adjacency matrix for the given graph.

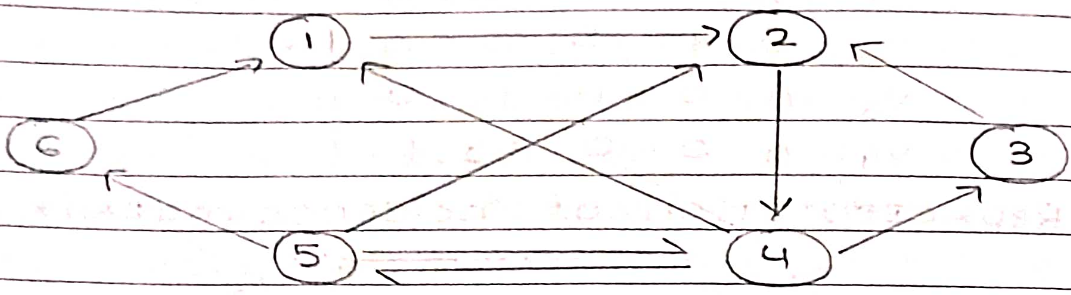


Solution :-
This given graph is an undirected graph with 7 vertices. So the adjacency matrix is a 7×7 matrix.

$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0
\end{bmatrix}
\end{array}
$$

- Note :-
The given matrix is a symmetric matrix. The row or the column total corresponding to each vertex is

Q.) Construct the adjacency matrix of a given graph.

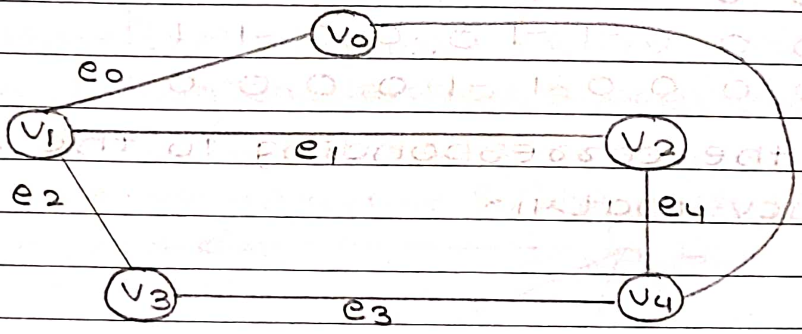


Solution :-

The given graph is directed with 6 vertice Thus, it is 6×6 adjacency matrix.

$$\therefore A = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1,-1 & 0 \\ 0 & -1 & 0 & 1,-1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Q.) Construct an incident matrix of following graph :-



Solution :-

The given graph has 5 vertices and 6 edges
The incident matrix will be of 5×6.

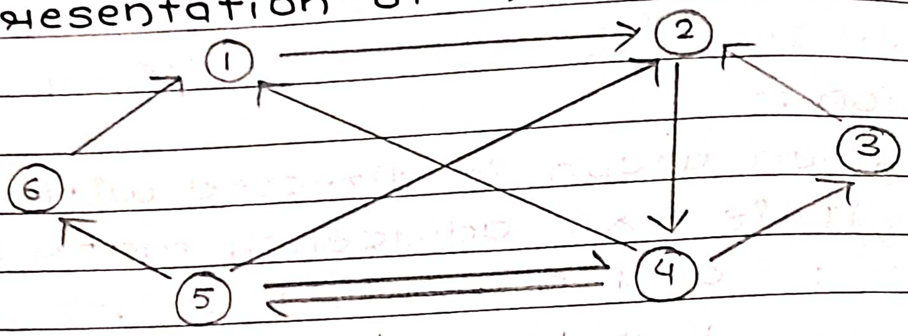
Note :-

Sum of each column will be 2.

Sum of each row varies.

$$\begin{array}{c|c} & \begin{matrix} e_0 & e_1 & e_2 & e_3 & e_4 \end{matrix} \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Q.) Representation of incidence matrix

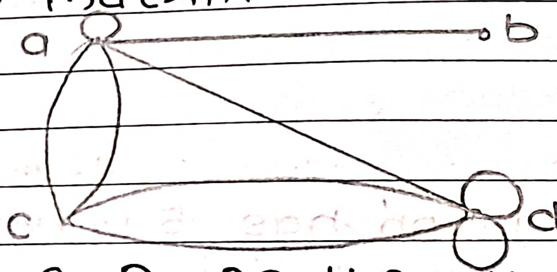


No. of vertices :- 6

No. of Edges :- 10

$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e & f & g & h & i & j \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Q.) Draw the corresponding to the given adjacency matrix



$$\begin{array}{c|c} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \\ 1 & 0 & 2 & 2 \end{bmatrix} \end{array}$$

As the adjacency matrix is order of 4 vertices, the corresponding graph has 4 vertices.

→ Connectivity in graphs

- Walk

i.) An alternate sequence of vertice and edges are called in a graph is called a walk.

ii.) In a walk, the vertices and edges may or maynot be repeated. The starting and ending vertex may or may not be repeated.

iii.) The number of edges in the walk is called the length of the walk.

- Trail

i.) An alternate sequence of vertice and edge in a graph, in which no edge should be repeated is called trail.

ii.) In a trail, vertice may or maynot be repeated

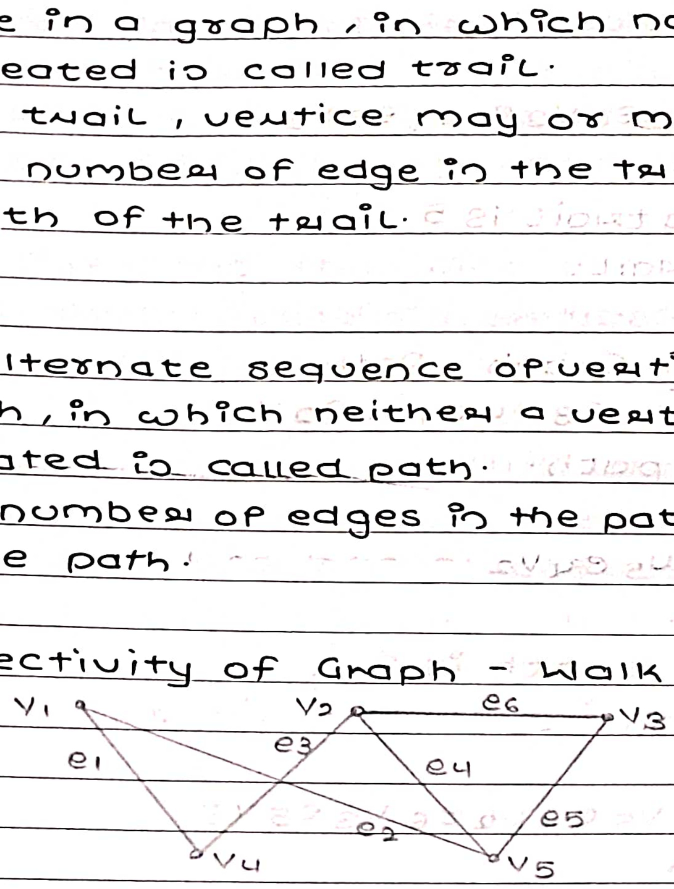
iii.) The number of edge in the trail is called the length of the trail.

- Path

i.) An alternate sequence of vertices and edge in a graph, in which neither a vertex nor an edge is repeated is called path.

ii.) The number of edges in the path is called length of the path.

→ Connectivity of Graph - Walk



/ /

1.) $v_1 e_1 v_4 e_3 v_2 e_3 v_4 e_1 v_1 e_2 v_5$ length of walk is 5 open walk.	2.) $v_2 e_6 v_3 e_5 v_5 e_4 v_2$ length of walk is 3 close ~~walk~~ walk

→ connectivity Graph - trail

1.) $v_1 e_1 v_4 e_3 v_2 e_3 v_4 e_1 v_1 e_2 v_5$
is not a trail

2.) $v_2 e_6 v_3 e_5 v_5 e_4 v_2$
is a trail
length of a trail is 3
closed trail

3.) $v_4 e_1 v_1 e_2 v_5 e_4 v_2 e_6 v_3 e_5 v_5$
is a trail
length of a trail is 5
close ~~ao~~ trail

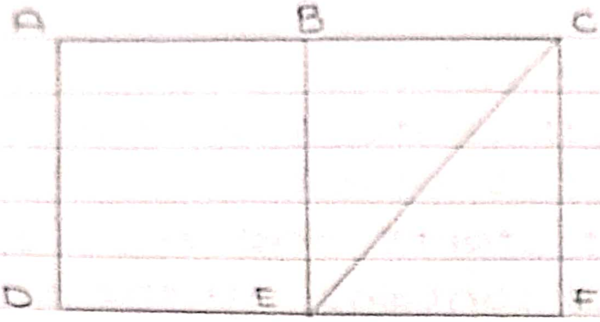
→ connectivity Graph - Path

1.) $v_1 e_1 v_4 e_3 v_2 e_3 v_4 e_1 v_1 e_2 v_5$
is not a path.

2.) $v_2 e_6 v_3 e_5 v_5 e_4 v_2$
is a path.
length of a path is 3
closed path

3.) $v_4 e_1 v_1 e_2 v_5 e_4 v_2 e_6 v_3 e_5 v_5$
is a path
length of path is 5
closed path

Q.) Consider a Graph and answer the following



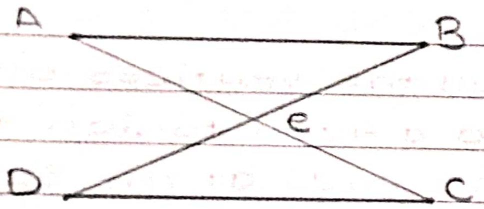
i.) list all cycle of length 3

→ we have 12 cycles of length 3.
BCEB, BECB, CEBC, CBEC, EBCE, ECBE, CEFC, FECF,

ii.) list all closed path in the given graph that start with vertice A.

→ There are 6 closed paths that starts with A
ABEDA, ABCEDA, ABCFEDA...

Q.) iii) Consider the following graphs. mention a walk that covers all vertices exactly at once and a walk that covers all edges exactly once.



→ A walk that covers all the edges exactly once abedcea.

__/__/__

→ Konigsberg bridge pattern

• Eulerian Graphs

i.) A walk that contains all the edges of graph exactly once is called an eulerian trials.

ii.) A walk that starts and ends at the same vertice and contains all the edges of the given graph exactly once is called an eulerian circuit.

iii.) A Graph that contains an eulerian trail and eulerian circuit is called an eulerian Graph.

iv.) A Graph $G = (V, E)$ is eulerian, if there is a sequence $(x_0, x_1, x_2, \ldots x_n)$ of vertice from G. Such that

$x_0 = x_t$

for every $i = 0, 1 \ldots t-1$, $x_i \ldots x_{i+1}$ is an edge of G,

for every edge, there is an unique integer i with $0 \leq i \leq t$ for which $e = x_i x_{i+1}$.

• Hamiltonian Graph

i.) A walk that contains all the vertices of the graph exactly once is called a Hamiltonian trail.

ii.) A walk that starts and ends at the same vertex and contain all the vertices of the graph exactly once is called a hamiltonian circuit.

iii.) A graph that contains a hamiltonian trail or hamiltonian circuit is called hamiltonian graph.

iv.) A graph ~~ca~~ $G = (V, E)$ is hamiltonian, if there is a sequence $(x_0, x_1, x_2 \cdots x_t)$ of vertices from G,

→ Properties of Eulerian Graph
Eulerian trail contains walk that contains all the edges of graph exactly once.
Eulerian circuit contains walk that starts and ends at the same vertex and contains all the edges of the graph exactly once.
Eulerian Graph contain both trail and circuit.

→ Properties of Hamiltonian Graphs
Hamiltonian trail contains ~~walk~~ walk that contains all the vertices of the graph exactly once.
Hamiltonian circuit contains walk that starts and ends at the same vertex and contains all vertices of graph exactly once.

→ Condition to check if graph is eulerian.
i.) Every vertex in graph must have even degree and should be connected.
ii.) A graph $G = (V, E)$ is eulerian, if there is sequence $(x_0, x_1, x_2, x_3 \dots x_n)$ of vertices from G, such that
- $x_0 = x_t$,
- for every $i = 0, 1 \dots t-1$, $x_i \dots x_{i+1}$, is an edge of G.
- for every edge $e \in E$, there is an unique integer i with $0 \leq i \leq t$, for which $e = x_i x_{i+1}$.

- Planar Graphs

A simple or multi graph that can be drawn on a page without crossing edges is called a planar graph. Such a drawing is called the planar representation of the graph.

- Euler Formula for planar graphs.

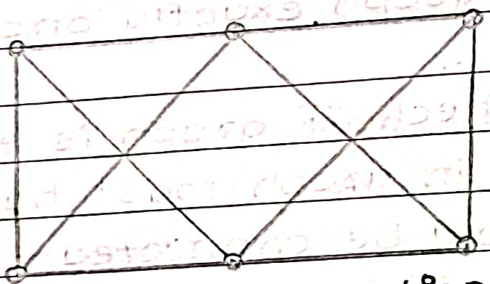
Let G be a connected planar graph with n vertice and m edges. Every planar drawing of G has faces, where G satifies $n-m+f=2$

OR

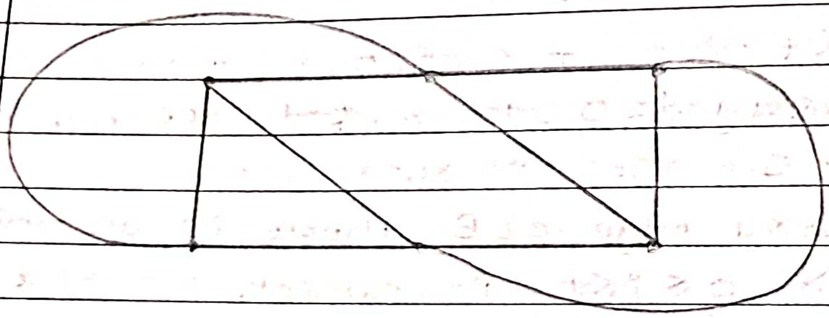
e → edge v → vertice r → region no.

$\therefore r = e - v + 2$

Q.) Check if the graph is planar



→ Its planar representation is



no. of edge = 10

no. of vertice = 6

no. of region = 6

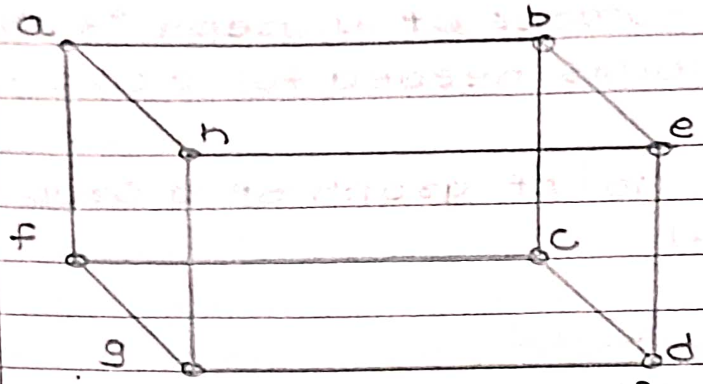
$\therefore r = e - v + 2$

$r = 10 - 6 + 2$

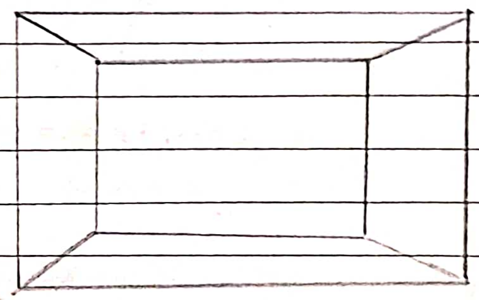
$\therefore r = 6$

$\therefore$ L.H.S = R.H.S

Q.) Check if the graph is planar



→ Its planar representation is



no. of vertice $= 8$

no. of edge $= 12$

no. of region $= 6$

$\therefore r = e - v + 2$

$= 12 - 8 + 2 = 6$

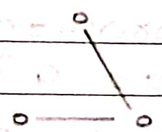
$\therefore$ L.H.S = R.H.S

- Is K_4 a plane? Yes
- Is K_5 a plane? NO
- K_5 also known as Kuratowski's first graph.
- $K_{3,3}$ is also known as Kuratowski's second graph.
- K_5 and $K_{3,3}$ are the smallest non-planar graphs.

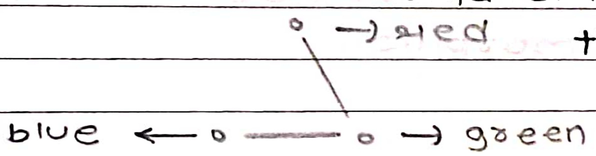
- Graph Colouring

A colouring of graph is the assignment of a colouring to each vertex of the graph so that no two ag adjacent vertices are assigned the same colour.

Example :- colour the given graph.



Solution :- colouring of graph refer to colouring vertice such that adjacent vertice do not repeat the colour.

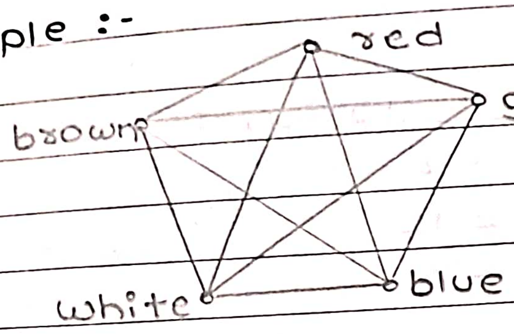


• Chromotic Number

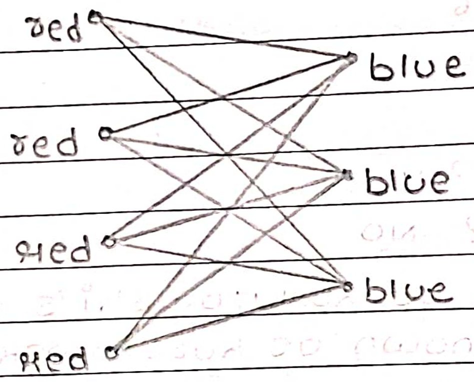
The chromatic number of a graph is the least no. of colours needed for a colouring of this graph.

The chromatic no. of graph of a Graph (G) is denoted by $\chi(G)$.

Example :-



$\therefore \chi(G) = 5$



$\therefore \chi(G) = 2$

• Isomorphic Graphs

→ Two simple graphs $G=(V,E)$ and $G'=(V',E')$ are isomorphic if there exists as a one to one and onto function f from V to V' with the property that v_1 and v_2 are adjacent in G if only $f(v_1)$ and (v_2) are adjacent $f(v_1) = v_1'$ and $f(v_2) = v_2'$ are adjacent in G', for all v_1 and v_2 in V.

→ Such a function f is called an isomorphism between G and G' and is denoted as $G \cong G'$

→ Two simple graphs that are not isomorphic are called non - isomorphic.

- If two graphs are isomorphic then the following conditions hold true

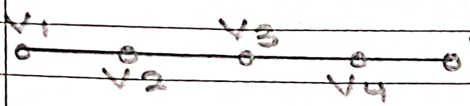
i.) Both graphs have the same number of vertices.

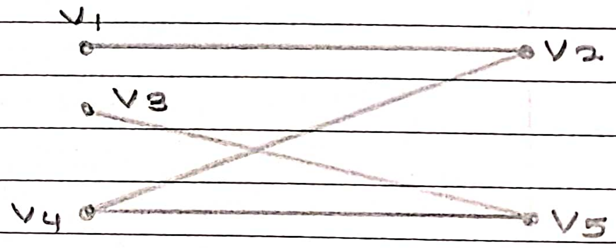
ii.) Both graphs have the same number of edges.

iii.) Both graphs have the same degree of sequence and same adjacency structure.

iv.) There exist a biject function $f : G \to G'$ mapping the vertices.

Q.) Determine whether the following pair of graphs are isomorphic. If not, justify. :-





→ Solution :-

Graph G1 | Graph G2

- No. of vertices = 5 | No. of vertices = 5

∴ Thus, Both the graph have same.

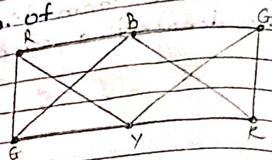
- No. of Edges = 4 | No. of Edges = 4

∴ Thus, Both the graph have same no. of edges.

Graph G1		Graph G2	
deg (v_1)	= 1	deg (v_1)	= 1
deg (v_2)	= 2	deg (v_2)	= 2
deg (v_3)	= 2	deg (v_3)	= 1
deg (v_4)	= 2	deg (v_4)	= 2
deg (v_5)	= 1	deg (v_5)	= 2

degree of Sequence = 2, 2, 2, 1, 1 | degree of Sequence = 2, 2, 2, 1, 1

Find the cromatic No. of



$k(S) = 3$

Properties of cromatic Number.

- cromatic number of $K_n = n$.
- chromatic number of a bipartite graph = 2
- The four colour Problem :- chromatic number of a planer graph is not greater than 4.

Isomorphic graph.

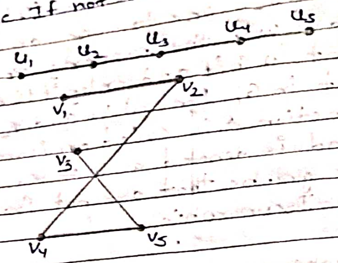
- Two Sample graphs $G = (V, E)$ and (G', E') are isomorphic if their exists a one to-one and onto function f from V to V' with the property that v_1 and v_2 are adjecent in G if and only if $f(v_1) = v_1'$ and $(v_2) = v_2'$ are adjecent in G', for all v_1 and v_2 in V.
- Such a function f is called an isomorphism betuen G and G' and is denoted as $G \cong G$.
- Two simple graphs that are not isomorphic are called non-Isomorphic.

- If two graphs are isomorphic then the following Condition holds true
  - Both the graphs have the same number of vertices.
  - Both the graphs have the same number of edges.
  - Both the graph have the same degree sequence and same adjacency structure.
  - There exist a bjective function $f : G \rightarrow G'$ mapping vertices.
  - If any of the above condition is not satisfied, then the graphs are not isomorphic.

Q. Determine whether the following pairs of graph are isomorphic If not Justify.

1)

2)



	Graph 1	Graph 2
Number of Vertices	No. of vertices $G1 = 5$	No. of vertices $G2 = 5$
	Thus, Both the graph have the same no. of vertices	
No. of edges	No. of edges in $G1 = 4$	No. of edges in $G2 = 4$
	Thus, Both the graph have the same no. of edges.	
Degree Sequence	$Deg(u_1) = 1$, $Deg(u_2) = 2$, $Deg(u_3) = 2$, $Deg(u_4) = 2$, $Deg(u_5) = 1$	$Deg(v_1) = 1$, $Deg(v_2) = 2$, $Deg(v_3) = 1$, $Deg(v_4) = 2$, $Deg(v_5) = 2$
	Degree of sequence of G1 is 2,2,2,1,1	Degree sequence of G2 is 2,2,2,1,1

Thus the both graphs have the same degrees sequence

Graph 1 :-
Adjancency structure :-

Vertex	Degree	Adjecent Vertices	Degree of adjacent vertices
u_1	1	u_2	2
u_2	2	u_1, u_3	1,2
u_3	2	u_2, u_4	2,2
u_4	2	u_3, u_5	2,1
u_5	1	u_4	2

Graph 2 :-
Adjacency structure.

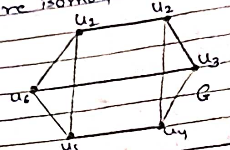
Vertex	Degree	Adjecent Vertices	Degee of adjecent vertices.
V_1	1	V_2	2
V_2	2	V_1, V_4	1,2
V_3	1	V_5	2
V_4	2	V_2, V_5	2,2
V_5	2	V_3, V_4	1,2.

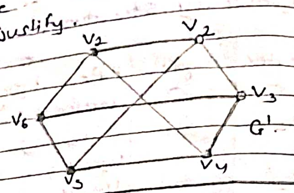
Bijective function :-
$f: G1 \to G2$
$f(u1) = V1$
$F(U2) = V2$
$f(U_3) = V4$
$F(U4) = VS$
$F(US] = V3$

Raigh work.
V1 (U2] → V_1 (V2) OR ~~V5~~
V2 ($U1, U_3$) → V2($V1, V4$)
U3 (U2, U4) → (V_1, (V2, V5)
U4 (U3, U5) → V5(V3, V4)
U5(U4) → V3 (V5)

As there is a unique mapping g_1 ~~of~~ and G_2, Hence they are isomorphic.

Q) Determine whether the following pair of graphs are isomorphic. If not justify.





	Graph 1	Graph 2
No. of Vertices	No. of Vertices G1 :- 6	No. of Vertices G2 :- 6
	Thus both graph is have the same no. of vertices	
No. of edges	No. of edges G1 = 9	No. of edges G2 = 9
	Thus both graph is have the same no. of edges	
Degree of Sequence	Deg (u_1) = 3 Deg (u_2) = 3 Deg (u_3) = 3 Deg (u_4) = 3 Deg (u_5) = 3 Deg (u_6) = 3	Deg (V_1) = 3 Deg (V_2) = 3 Deg (V_3) = 3 Deg (V_4) = 3 Deg (V_5) = 3 Deg $V(V_6)$ = 3
	Degree of Sequence of G1 is 3, 3, 3, 3, 3, 3	Degree of Sequence of G2 is 3, 3, 3, 3, 3, 3

Thus, the both graphs have the Same degree sequence

Adjacency structure :-

Graph 1 :-

vertex	Degree	Adjective Vertices	Degree of adjecent vertices
u1	3		
u2	3		
u3	3		
u4	3		
u5	3		

Graph 2

Vertex	Degree	Adjective Vertices	Degree of adjecent vertices
V1	3		
V2	3		
V3	3		
V4	3		
V5	3		

Bijective function :-

Graph Traversals.

- To solve problems on graph, ~~as~~ we need a mechanism for traversing the graph.
- Graph traversal algorithums, can be thought of as starting at some source vertex in a graph & searching the graph by doing through the edges and Marking the vertices.

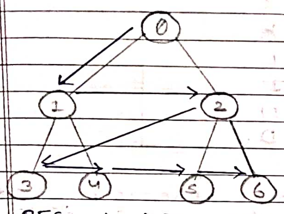
* Algorithums for transversing the graph.
  - Dept first Search (DFS)
  - Bredth first search (BFS).

- Breadth First search (BFS)
- The general idea behind a breadth-first search begining at a starting vertex A is a follow.
  - First we process the starting vertex A.
  - Then we process all the neighbors of A.
  - Then we process all the neighbors of A.
  - And so on.

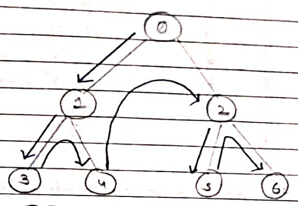
We need to keep track of the neighbors of a vertex and We need to gaurentee that no vertex is processed twice This is accumplished by using a QUEUE to hold vertices that are waiting to be processed and by a feild status which Tells us The current status of a vertex

- Depth first search [DFS].
- The gernal idea behind a depth-first search begining at a starting vertex A is as follows:
  - first we process the starting vertex A.
  - Then we process each vertex N along ~~it~~ a path P which begins at A; that is, we process a neighbor of A, then a neighbor of A, and so on.
  - After coming to a "dead end" that is to a vertex with no unprocessed neighbour, we backtrack on the path P until we can continue along another Path P.
  - And so on.
- The backtracking is accumplished by using a STACK to hold the initial values of future possible Paths. we also need a field STATUS which tells us the current status of any vertex so that no vertex is processed more than once.

Example on BFS and DFS.



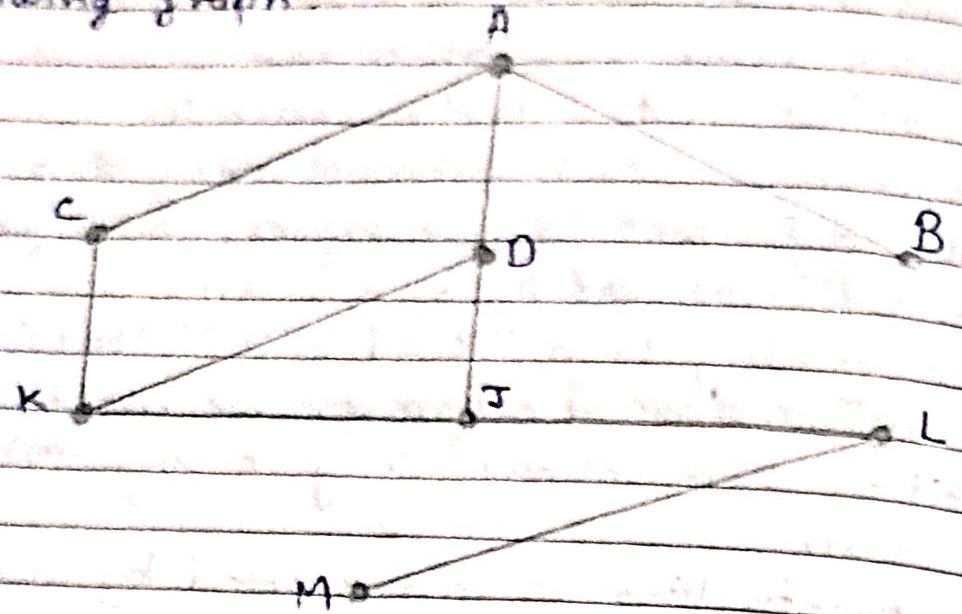
BFS order: 0123456.



DFS order :- 0134256.

(2)

Q) Construct the BFS & DFS traversal order for the following graph.



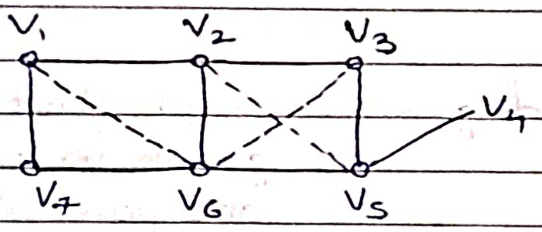
Step	OPEN	CLOSED.
1	A	∅
2	B, C, D.	A
3	C, D	B
4	K, D	C
5	L, D	K
6	J, D	L
7	M, D	J
8	D	M
9	∅	D.

- The order in which the vertices of G are processed using a BFS algorithum begining at vertex A is "ABCDKLJM".

Types of Graph.

1) Bipartite graph :- If a vertex set V of a graph G can be partishoned can be partishoned into two non.empty disjoint subsets X and y in such a way that edge of G has one end in X and one end Y.

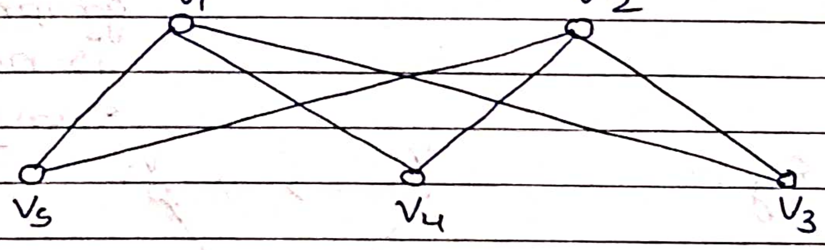
Eg.



Here $V = \{V_1, V_2, V_3, V_4, V_5, V_6, V_7\}$
$X = \{V_1, V_6, V_3, V_4\}$
$Y = \{V_7, V_2, V_5\}$.

2) Complete Bipartaite graph :- Graph in which every vertex of any one of the set is connected to the every vertex of another set.

Eg.



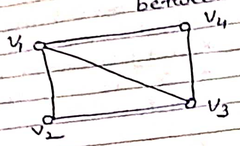
$x = \{V_1, V_4, V_2\}$

$Y = \{V_5, V_3\}$

Types of graphs (based on connectivity).

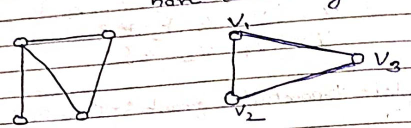
1) Connected graph :- An unidirected graph is said to be connected if there is a path between every two vertices.

Eg :-



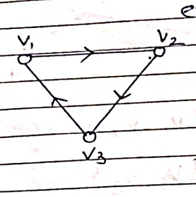
2) Disconnected graph :- graph that is not connected or graph in which there is atleast one pair of vertices which do not have connecting path.

Eg :-



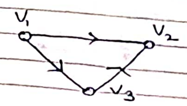
3) Strongly connected graphs :- A graph is said to be as strongly connected graph. if every Vertex is reachable from every another Vertex.

Eg :-



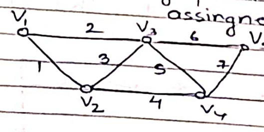
4) Weakly Connected graph :- A graph is said as weakly connected if every vertex is not reachable from every other vertex.

Eg :-



5) Weighted graphs :- graph in which a number is assingned to each edges.

Eg :-



6) Unweighted graph :- graph in which there is no such number (weights, costs) assinged to e edges.

Example :-