Option Pricing in Classiq using Quantum Amplitude Estimation

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***Abstract*—This article presents an exploration of option pricing methodology using the Classiq Software Development Kit (SDK) by leveraging the Quantum Amplitude Estimation (QAE) Algorithm.. The primary objective of option pricing is to ascertain the future value of an option at its expiration date. Subsequently, this future value is adjusted to determine the fair value for the options contract, representing the optimal amount to initiate the contract. The utilization of the QAE algorithm offers a quantum advantage in option pricing, seamlessly integrating Amplitude Amplification (AA) and Quantum Phase Estimation (QPE). Within the QAE framework, AA encodes the desired probability amplitude as a phase, and QPE estimates that phase. This article provides practical insights into implementing these quantum methodologies using Classiq SDK, demonstrating a postprocessing technique that optimizes results while minimizing circuit depth and width.**

INTRODUCTION

Financial options are contracts granting the right to buy (*call*) or sell (*put*) a financial instrument at a predetermined price (called strike price, K) before or at expiration. In the time leading to the expiration, the price of the option changes due to the stochastic nature of the stock prices underlying it. In a call option scenario, where the stock's price (S) underlying the option contract surpasses the predetermined target (K) before the expiry date, the buyer profits $S - K$ (deducting the premium for net profit) if the option contract is exercised on or before the expiry date. If the stock price remains below the target, the buyer faces a loss of the premium as the option contract would not be exercised. The goal of option pricing is to determine the future value of an option at its maturity date. Subsequently, this future value is adjusted to ascertain the fair value of the options contract.
Numerical pricing, often using Monte Carlo methods, involves modeling the underlying asset's price as random variables $\mathbf{X} = \{X_1, X_2, ..., X_N\}$ (N stands for number of assets in the contract), generating P number of price paths $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_M\}$ for each asset from the probability distribution P followed by X. Note that $\mathbf{X_i}$ is a column vector with N rows and indicates a specific price path. Let the option's payoff on each price path i be denoted as $f(\mathbf{X}_i)$, the expectation value of the payoff $E_P[f(X)]$ as an average across all paths can be written as:

$$E_P[f(\mathbf{X})] = \sum_{i=1}^{M} p_i f(\mathbf{X}_i)$$

p_i is the probability of realizing the price path i. The final step is to discount the calculated expectation value to get the option's fair value.

While classical Monte Carlo methods exhibit appealing features in option pricing, they typically demand substantial computational resources for accurate option price estimates, especially for complex options. Quantum Amplitude Estimation algorithm provides an advantage of pricing options on a quantum computer. It exhibits a theoretical quadratic speed-up compared to classical Monte Carlo methods.

Options are categorized as path-independent or path-dependent and can involve a single asset or multiple assets. Path-independent options depend on the asset at a single point, making prior asset prices irrelevant. In contrast, path-dependent options rely on the asset's evolution and history. Pricing path-independent options on a single asset is straightforward, while path-independent options on multiple assets are slightly more complex. Path-dependent options are challenging due to the need for multiple expensive payoff calculations. Quantum computing is envisioned to have the most significant impact in pricing complex, path-dependent options.

In this article, section 1 will review the theory and gate based implementation of QAE algorithm which uses the QPE algorithm. In section 2 a demo of Classiq using these algorithms shall be shown along with post-processing technique for improved accuracy in the results. In the final section 2, option pricing methodology for a single asset class using Classiq will be demonstrated by applying smart post-processing to get to accurate results with lower circuit depth and width.

QPE and QAE Review

Quantum Phase Estimation (QPE) has been the key for many quantum algorithms. Suppose a unitary operator U has an eigenvector $|u\rangle$ with eigenvalue $e^{2\pi i\phi}$ where ϕ is unknown, that is, $U|u\rangle = e^{2\pi i\phi}|u\rangle$. The goal of QPE is to estimate ϕ. To perform this estimation, we assume that we have available black boxes (also called *oracles*) capable of preparing the state $|u\rangle$ and performing the controlled-U^{2^j} (CU^{2^j}) operation, for suitable non-negative integers j.
The action of controlled version of the U operator on an arbitrary state, is given by,

$$\alpha|0\rangle|u\rangle + \beta|1\rangle|u\rangle \xrightarrow{CU} \alpha|0\rangle|u\rangle + e^{2\pi i\phi}\beta|1\rangle|u\rangle$$

QPE requires two registers where first register contains t qubits which are in state $|0\rangle$. t depends on the precision required in the estimated phase. Suppose that the qubits in the first register are numbered from 1 to t. Second register stores $|u\rangle$ and has as many qubits necessary to store $|u\rangle$. We are given controlled U^{2^j} operators as black-box functions.

1) Apply Hadamard to first register which has t qubits, all initialized to $|0\rangle$. We get

$$\frac{1}{2^{t/2}}\big(|0\rangle + |1\rangle\big)\otimes\cdots\otimes\big(|0\rangle + |1\rangle\big)|u\rangle .$$

2) Apply CU^{2^j} gate where qubit $t-j$ is the control for $j = 0,\ldots,t-1$ and the qubits representing the state $|u\rangle$ being the target. We get,

$$\frac{1}{2^{t/2}}\big(|0\rangle + e^{2\pi i\phi 2^{t-1}}|1\rangle\big)\otimes\cdots \otimes\big(|0\rangle + e^{2\pi i\phi 2^{1}}|1\rangle\big)\otimes\big(|0\rangle + e^{2\pi i\phi 2^{0}}|1\rangle\big)|u\rangle \tag{1}$$

After some simplification this reduces to,

$$\frac{1}{2^{t/2}}\sum_{k=0}^{2^t-1} e^{2\pi i k\phi}|k\rangle|u\rangle \tag{2}$$

We express ϕ exactly in t bits. That is,

$$\phi = 0.\phi_1\ldots\phi_t = \frac{\phi_1\ldots\phi_t}{N} = \frac{x}{2^t},$$

for an *integer* x. The resulting state now is,

$$\frac{1}{2^{t/2}}\sum_{k=0}^{2^t-1} e^{\frac{2\pi i k x}{N}}|k\rangle|u\rangle .$$

3) The state of the first register is clearly of the form of Fourier transform (QFT) of state $|x\rangle$, hence applying $QFT^\dagger$ to the first register, we exactly measure $|x\rangle|u\rangle = |\phi_1\ldots\phi_t\rangle|u\rangle$.

$$\frac{1}{2^{t/2}}\sum_{k=0}^{2^t-1} e^{\frac{2\pi i k x}{N}}|k\rangle|u\rangle \xrightarrow{QFT^\dagger\otimes I} |x\rangle|u\rangle .$$

Finally the required phase is just $\phi = \frac{x}{2^t}$.

For the case where ϕ cannot be written exactly with a t bit binary expression (i.e $e^{i2\pi\phi} \neq e^{i2\pi(\frac{x}{2^t})}$ for an integer x) the QPE algorithm would still produce a very good *approximation* of ϕ. In this case, $QFT^\dagger$ on 2 followed by simplification of the summation yields the state

$$\frac{1}{2^t}\sum_{k'}\left[\frac{1-\left[e^{\left(\frac{-2\pi i}{2^t}(k'-\phi 2^t)\right)}\right]^{2^t}}{1-\left[e^{\left(\frac{-2\pi i}{2^t}(k'-\phi 2^t)\right)}\right]}\right]|k'\rangle|u\rangle \tag{3}$$

Thus, probability of observing the state k' is,

$$P(k') = \frac{1}{2^{2t}}\left|\left[\frac{1-\left[e^{\left(\frac{-2\pi i}{2^t}(k'-\phi 2^t)\right)}\right]^{2^t}}{1-\left[e^{\left(-\frac{2\pi i}{2^t}(k'-\phi 2^t)\right)}\right]}\right]\right|^2 \tag{4}$$

The probability peaks when $k' \approx \phi 2^t$. Hence we have an approximate estimate of ϕ.

QAE uses QPE to estimate the amplitude of a desired quantum state. Let's say we have a unitary operator A acting on a register of $(n+1)$ qubits. $|\psi_0\rangle_n$ and $|\psi_1\rangle_n$ are some n-qubit normalized states, $a \in [0,1]$ is unknown and QAE estimates a.

$$A|0\rangle_{n+1} = |\Psi\rangle = \sqrt{1-a}\,|\psi_0\rangle_n|0\rangle + \sqrt{a}\,|\psi_1\rangle_n|1\rangle \tag{5}$$

Technique from Amplitude Amplification:
We use the Grover Operator defined by $Q = AS_0A^{\dagger}S_{|\psi_0\rangle}$, where $S_0 = 1 - 2|0\rangle\langle 0|$ and $S_{|\psi_0\rangle} = 1 - 2|\psi_0\rangle|0\rangle\langle\psi_0|\langle 0|$, represent the reflection operators about the state $|0\rangle$ and the good state $|\psi_0\rangle$ respectively. The full action of the operator Q is to rotate the state $A|0\rangle_{n+1}$ by an angle 2θ in the $2D$ space spanned by $|\psi_0\rangle_n|0\rangle$ and $|\psi_1\rangle_n|1\rangle$, where $a = \sin^2(\theta)$.
The matrix representation of the operator Q is

$$Q = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} = (\cos 2\theta)1 + i(\sin 2\theta)\sigma_y \tag{6}$$

From this, the eigenvalues of the Q operator can be evaluated and are found to be $\exp(+i2\theta)$ and $\exp(-i2\theta)$ with the eigenvectors say, $|\psi_+\rangle$ and $|\psi_-\rangle$ respectively. In the eigenbasis of Q we can express our state Ψ as,

$$\Psi = a_1|\psi_+\rangle + a_2|\psi_-\rangle$$

Also, the action of Q on $|\Psi\rangle$ is,

$$Q|\Psi\rangle = a_1e^{+i2\theta}|\psi_+\rangle + a_2e^{-i2\theta}|\psi_-\rangle \tag{7}$$

Technique from Phase Estimation:
Denote $2\theta = \frac{2\pi q}{2^m}$, where m is the size of the estimation register in the QPE algorithm.
Now applying QPE,

$$QPE(|0\rangle|\Psi\rangle) = a_1|q\rangle|\psi_+\rangle + a_2|q + 2^{m-1}\rangle|\psi_-\rangle$$

Note that the phases corresponding to the eigenvectors $|\psi_+\rangle$ and $|\psi_-\rangle$ in 7 differ by angle π; hence the term $|q + 2^{m-1}\rangle$ in the above expression makes sense as,

$$\frac{2\pi(q + 2^{m-1})}{2^m} = \frac{2\pi q}{2^m} + \pi$$

If upon measurement of the estimation register we obtain q' and $q' > 2^{m-1}$ then we get the phase corresponding to $|\psi_-\rangle$ branch, hence our estimated q (say q_p) from the QPE algorithm is,

$$q_p = q' - 2^{m-1}$$

Thus the value of 2θ is $\frac{2\pi q_p}{2^m}$ and hence the the value of a is simply,

$$a = \sin^2\theta = \sin^2(\frac{\pi q_p}{2^m})$$

This result is exact, for he case where q_p is an integer. For the case where q_p is not an integer, QPE will yield an estimate for θ.

Option Pricing Methodology

The article closely follows the option pricing methodology described in [1].
For an option with payoff f, let's suppose the $\mathcal{A}$ operator creates the following state:

$$\sum_{i=0}^{2^n-1}\sqrt{1-f(S_i)}\sqrt{p_i}\,|S_i\rangle|0\rangle + \sum_{i=0}^{2^n-1}\sqrt{f(S_i)}\sqrt{p_i}\,|S_i\rangle|1\rangle.$$

where S_i represent the possible values the underlying asset can take and the corresponding probabilities are p_i. Here we have assumed that the prices can take 2^n possible values on expiration and on a quantum circuit n qubits can be used to represent all these price paths.

Comparing with 5, we find that

$$a = \sum_{i=0}^{2^n-1} f(S_i)\,p_i = E[f(S)]$$

The value of a which can be estimated using QAE.

These are the following critical steps involved:

1) Load the probability distribution P of the random variables $\mathbf{X}$
The price of a path-independent option depends only on the distribution of the underlying asset price S_t at the option maturity t and the payoff function f of the option. The distribution of S_t is truncated to the range $[S_{t,\min}, S_{t,\max}]$ and this interval is discretized to $\{0,\ldots,2^n-1\}$ to encode 2^n potential price values the asset can take upon expiration, utilizing n qubits. The distribution loading operator $\mathcal{P}$ creates a state

$$|0\rangle_n \xrightarrow{\mathcal{P}} |\psi\rangle_n = \sum_{i=0}^{2^n-1}\sqrt{p_i}|i\rangle_n \tag{8}$$

with $i \in \{0,\ldots,2^n-1\}$ to represent S_t. The n-qubit state $|i\rangle_n = |i_{n-1}\ldots i_0\rangle$ encodes the integer $i = 2^{n-1}i_{n-1} + \ldots + 2i_1 + i_0 \in \{0,\ldots,2^n-1\}$ with $i_k \in \{0,1\}$ and $k = 0,\ldots,n-1$. We note that the outcomes of a random variable X can be mapped to the integer set $\{0,\ldots,2^n-1\}$ using an affine mapping.

2) Construct a quantum circuit which computes the payoff $f(\mathbf{X})$ An example for a call option payoff is the European Call Option $f\left(S_t\right)$ given as

$$f\left(S_t\right)=\max\left(0, S_t-K\right)$$

To load this pay-off function into the quantum circuit and compute the expectation value of the pay-off, the state in 8 along with an ancilla qubit the following procedure as described in **reference** is followed:

The payoff function of European call-option is piece-wise linear and hence for $f : \{0, \ldots, 2^n-1\} \rightarrow [0,1]$ we write $f(i)=f_1 i+f_0$. We create a circuit that performs

$$|i\rangle_n|0\rangle \rightarrow |i\rangle_n(\cos[f(i)]|0\rangle+\sin[f(i)]|1\rangle)$$

for all i using controlled Y-rotations implemented with CNOT and single-qubit gates.

Using this technique, the following transformation can be achieved on the state $|\psi\rangle_n|0\rangle$

$$\begin{aligned}&|\psi\rangle_n|0\rangle \rightarrow \sum_{i=0}^{2^n-1}\sqrt{p_i}|i\rangle_n\left[\cos\left(c\tilde{f}(i)+\frac{\pi}{4}\right)\right.\\&\left.|0\rangle+\sin\left(c\tilde{f}(i)+\frac{\pi}{4}\right)|1\rangle\right.\end{aligned} \tag{9}$$

Here $\tilde{f}(i)$ is a scaled version of $f(i)$ given by

$$\tilde{f}(i)=2\frac{f(i)-f_{\min}}{f_{\max}-f_{\min}}-1$$

with $f_{\min}=\min_i f(i)$ and $f_{\max}=\max_i f(i)$, and $c \in [0,1]$ is an additional scaling parameter. The relation is chosen so that $\tilde{f}(i) \in [-1,1]$.

3) Calculate the expectation value of the payoff $E_P[f(\mathbf{X})]$.

The probability to find the ancilla qubit in state $|1\rangle$, is given by

$$P_1=\sum_{i=0}^{2^n-1} p_i \sin^2\left(c\tilde{f}(i)+\frac{\pi}{4}\right)$$

Using the approximation $\sin^2\left(c\tilde{f}(i)+\frac{\pi}{4}\right) \approx c\tilde{f}(i)+\frac{1}{2}$, leads to

$$\begin{aligned}P_1 &\approx \sum_{i=0}^{2^n-1} p_i\left(c\tilde{f}(i)+\frac{1}{2}\right)\\&=c\frac{2E[f(X)]-f_{\min}}{f_{\max}-f_{\min}}-c+\frac{1}{2}\end{aligned} \tag{10}$$

Thus we could recover $E[\max(0, i-K)]$ from P_1 up to a scaling factor and a constant.

Classiq Demo: Amplitude Estimation and PostProcessing

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Option Pricing using Classiq SDK

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CONCLUSION

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