Gradient Descent

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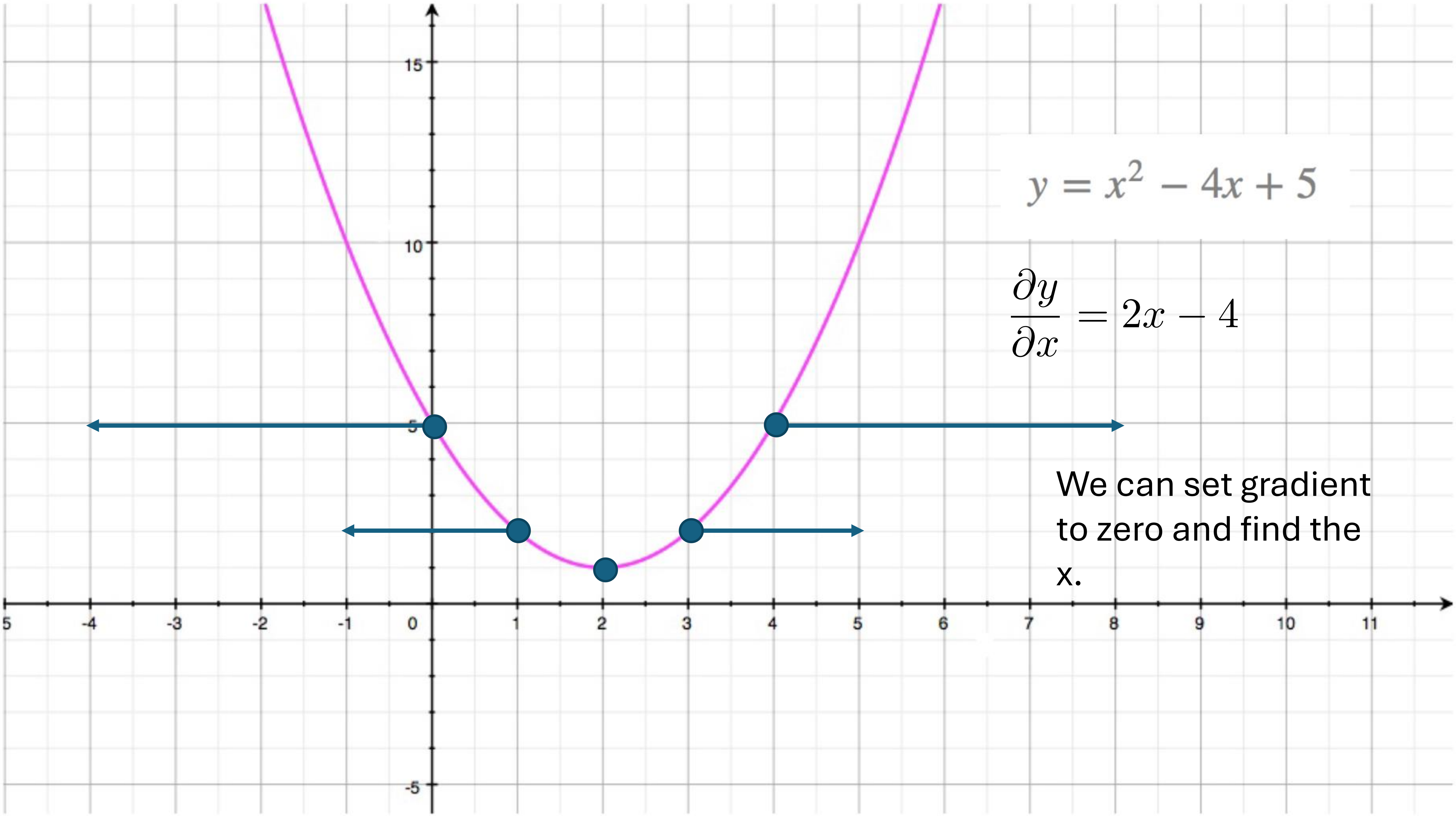
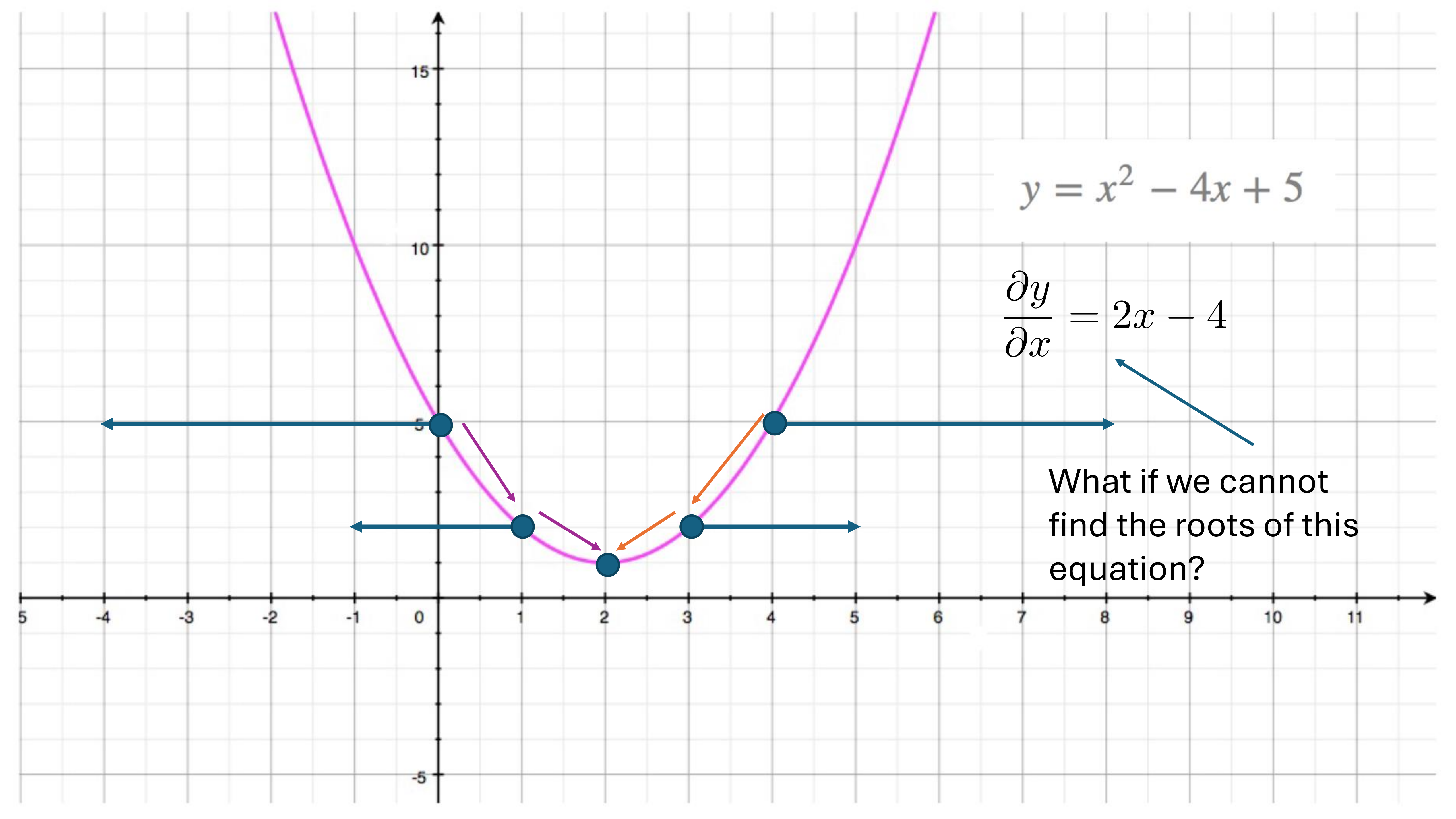
STUDY LOCALLY. LIVE GLOBALLY.



$$y = x^2 - 4x + 5$$

$$\frac{\partial y}{\partial x} = 2x - 4$$

We can set gradient to zero and find the x.



$$y = x^2 - 4x + 5$$

$$\frac{\partial y}{\partial x} = 2x - 4$$

What if we cannot find the roots of this equation?

Training

- Loss function:

$$L[\phi]$$

Returns a scalar that is smaller when model maps inputs to outputs better

- Find the parameters that minimize the loss:

$$\hat{\phi} = \underset{\phi}{\operatorname{argmin}} \left[L[\phi] \right]$$

Example: 1D Linear regression loss function

$$\hat{\phi} = \underset{\phi}{\operatorname{argmin}} \left[L[\phi] \right]$$

$\text{minimize } \|y - X\phi\|_2^2$

$$\frac{\partial L}{\partial \varphi} = 0 \quad \text{solve for } \hat{\varphi}$$

$$\phi = (X^T X)^{-1} X^T y \longrightarrow \text{Closed-form solution}$$

Where:

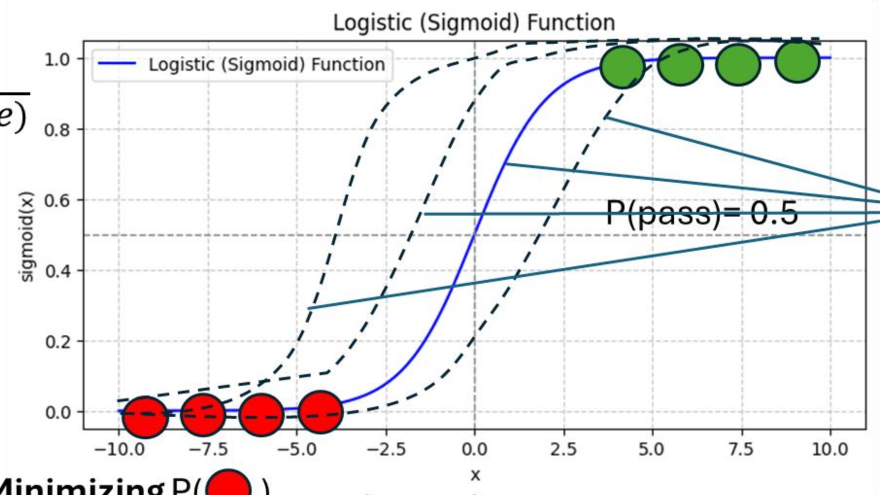
- $(X^T X)^{-1}$ is the inverse of the matrix product of the transpose of the design matrix X^T and X.
- X^T is the transpose of the design matrix.
- y is the vector of observed target values.

Example: Logistic Regression

Maximizing P( )

$$\frac{1}{1+e^{-(\beta_0+\beta_1 score)}}$$

P(pass)



Parameterized by

β_0, β_1

Minimizing P( )

$\beta_0 + \beta_1 score$

Or **maximizing** 1 - P( )

From maximization to minimization.

$$L(\beta) = -\prod_{s\ in\ y_i=1} p(x_i) \ * \prod_{s\ in\ y_i=0} (1-p(x_i))$$

$\frac{\partial L}{\partial \beta} = 0$ solve for $\hat{\beta}$?

Gradient descent algorithm

Step 1. Compute the derivatives of the loss with respect to the parameters:

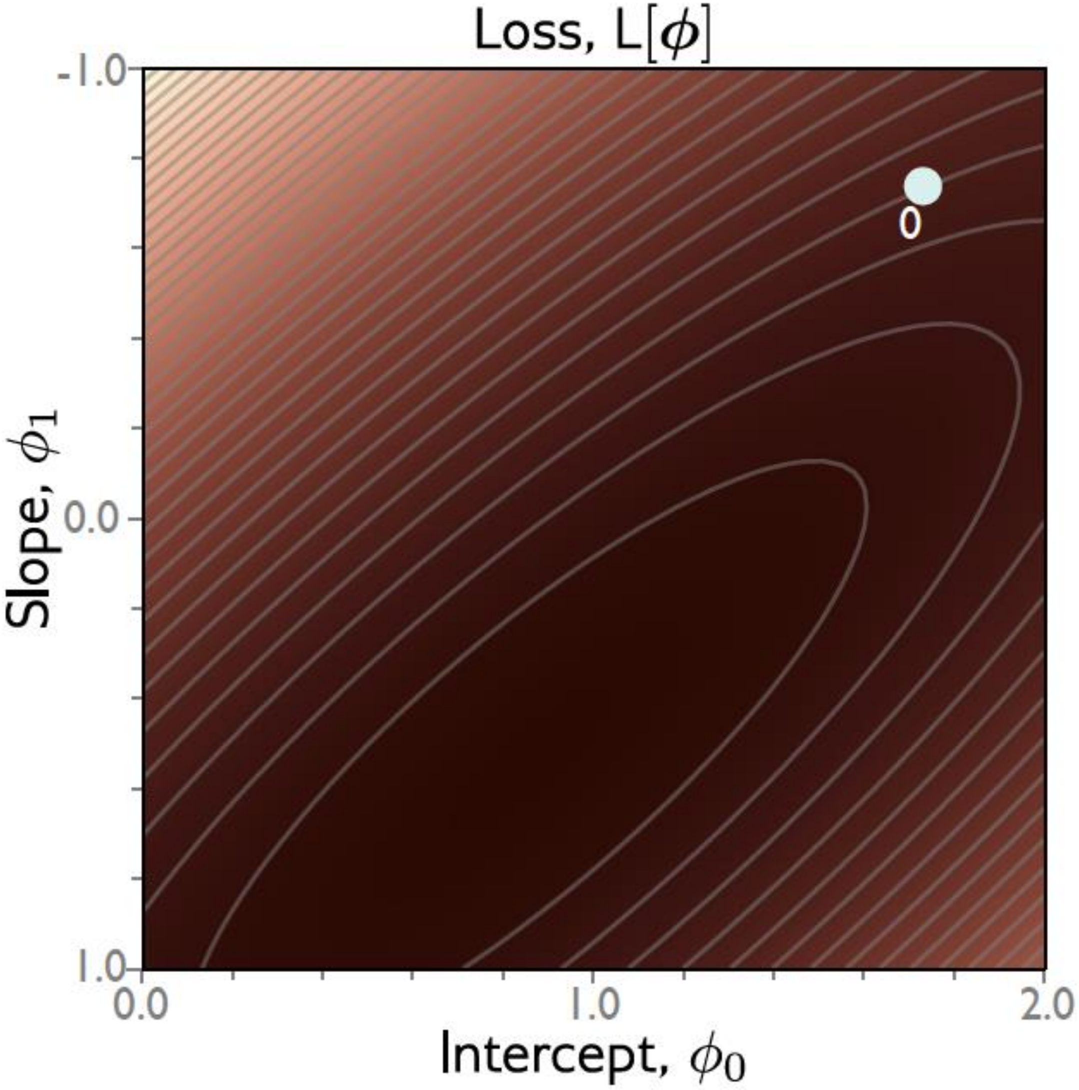
$$\frac{\partial L}{\partial \phi} = \begin{bmatrix} \frac{\partial L}{\partial \phi_0} \\ \frac{\partial L}{\partial \phi_1} \\ \vdots \\ \frac{\partial L}{\partial \phi_N} \end{bmatrix}.$$

Step 2. Update the parameters according to the rule:

$$\phi \longleftarrow \phi - \alpha \frac{\partial L}{\partial \phi},$$

where the positive scalar α determines the magnitude of the change.

Gradient descent for linear regression



Step 1: Compute derivatives (slopes of function) with
Respect to the parameters

$$
\begin{aligned}
L[\boldsymbol{\phi}] &= \sum_{i=1}^{I} \ell_i = \sum_{i=1}^{I} (\mathrm{f}[x_i, \boldsymbol{\phi}] - y_i)^2 \\
&= \sum_{i=1}^{I} (\phi_0 + \phi_1 x_i - y_i)^2
\end{aligned}
$$

$$
\frac{\partial L}{\partial \boldsymbol{\phi}} = \frac{\partial}{\partial \boldsymbol{\phi}} \sum_{i=1}^{I} \ell_i = \sum_{i=1}^{I} \frac{\partial \ell_i}{\partial \boldsymbol{\phi}}
$$

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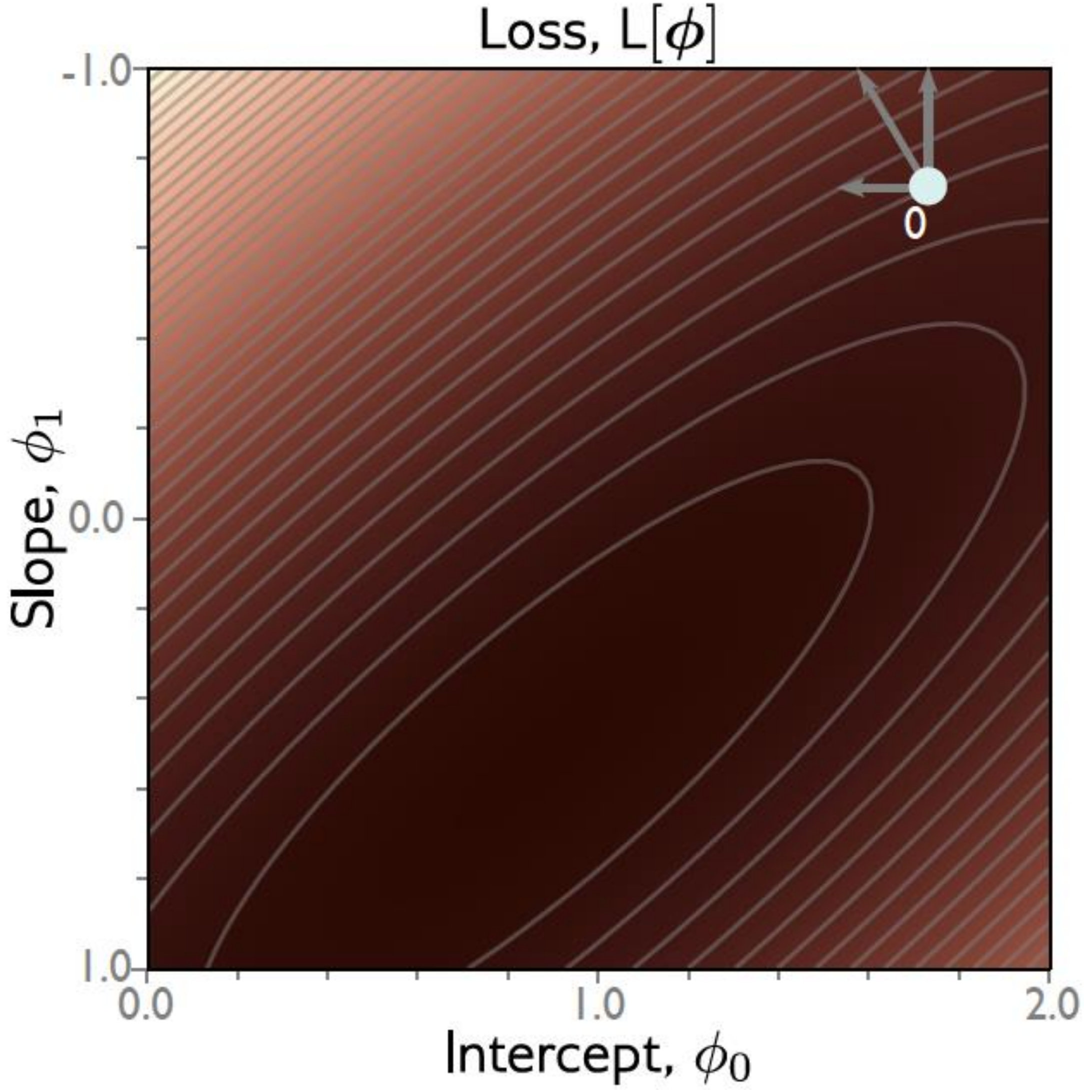
\frac{\partial \ell_i}{\partial \boldsymbol{\phi}} = \begin{bmatrix} \frac{\partial \ell_i}{\partial \phi_0} \\ \frac{\partial \ell_i}{\partial \phi_1} \end{bmatrix} = \begin{bmatrix} 2(\phi_0 + \phi_1 x_i - y_i) \\ 2x_i(\phi_0 + \phi_1 x_i - y_i) \end{bmatrix}
$$

Gradient descent for linear regression



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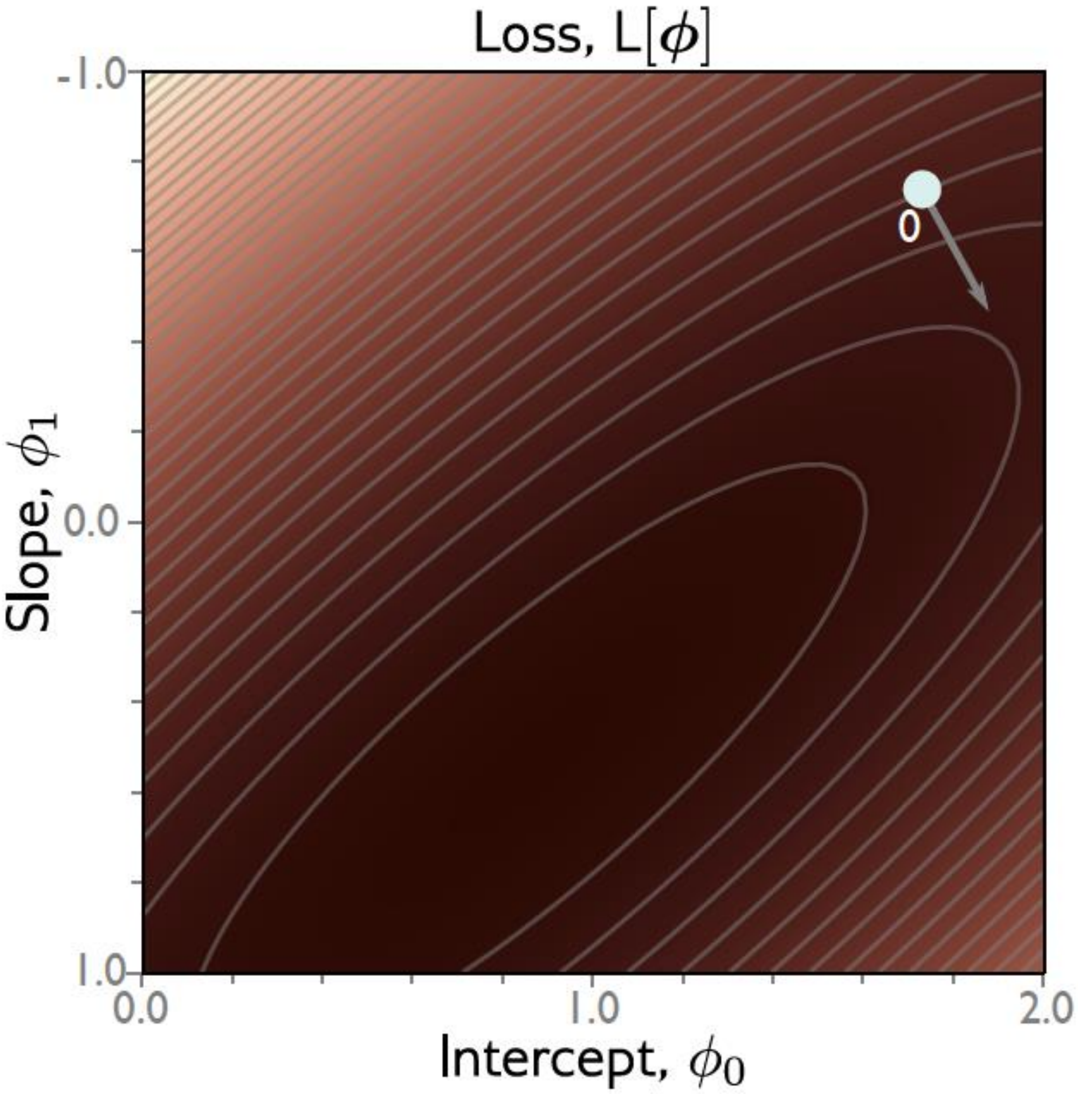

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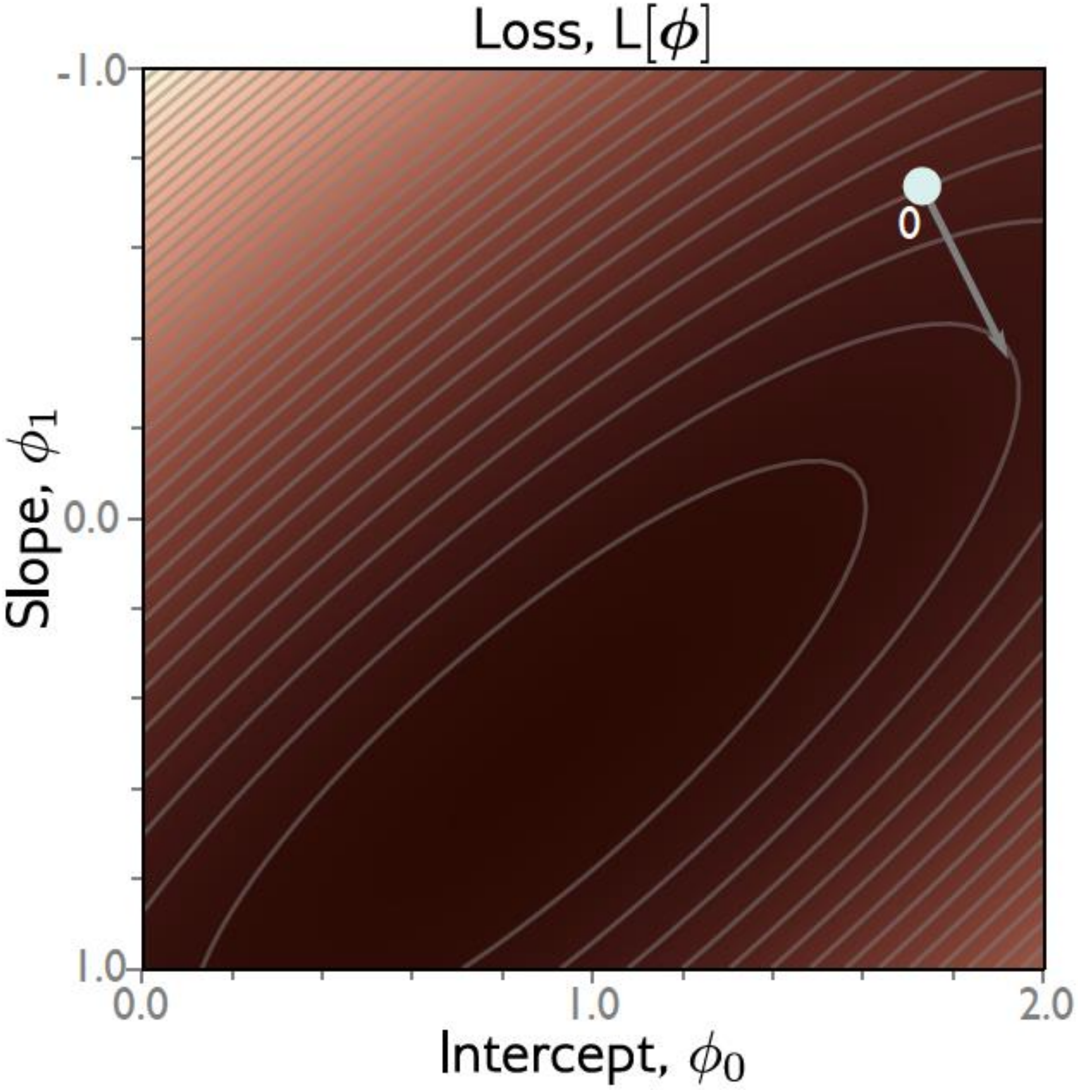
Step 2: Update parameters according to rule

$$\phi \longleftarrow \phi - \alpha \frac{\partial L}{\partial \phi}$$

α = step size or learning rate if fixed

Gradient descent for linear regression



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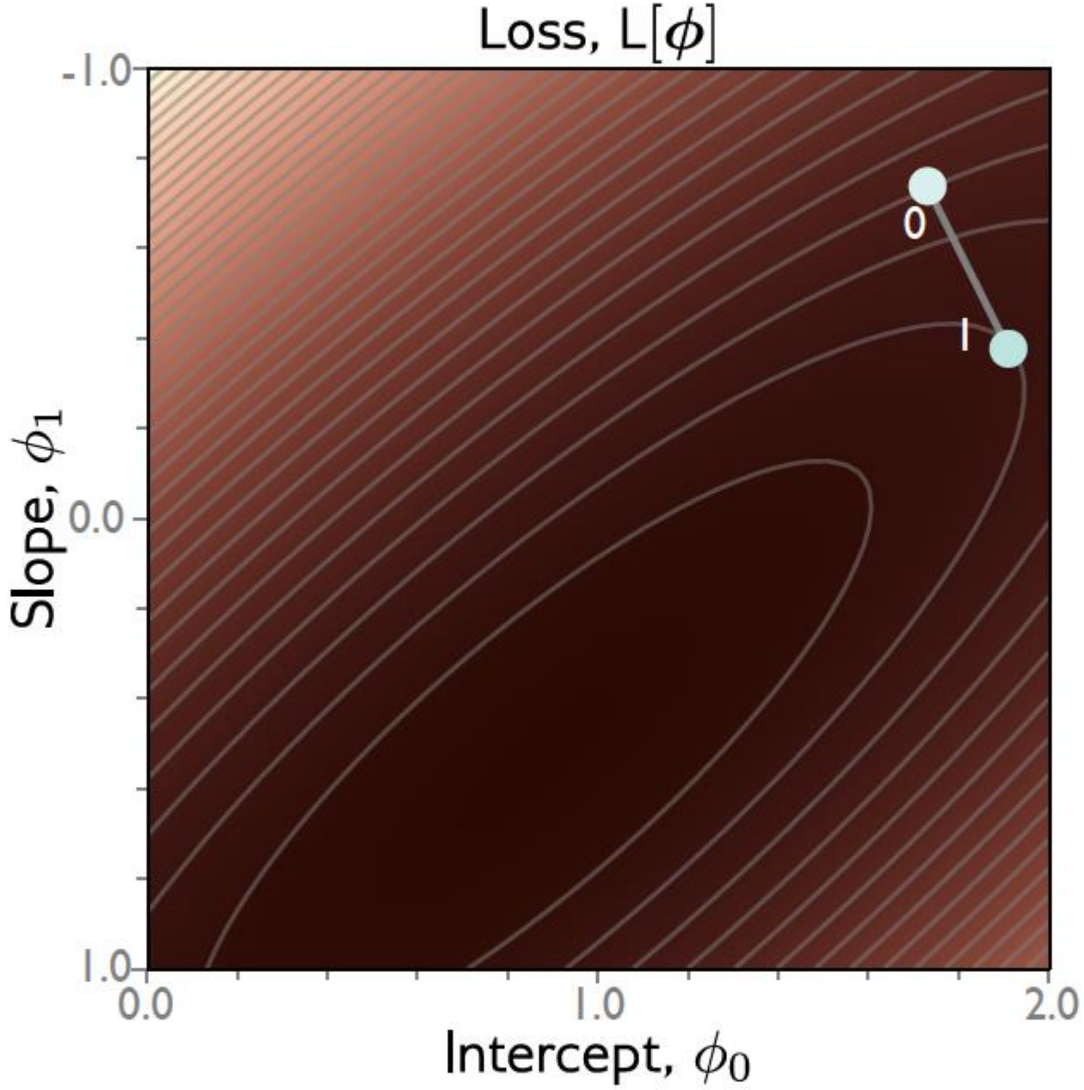

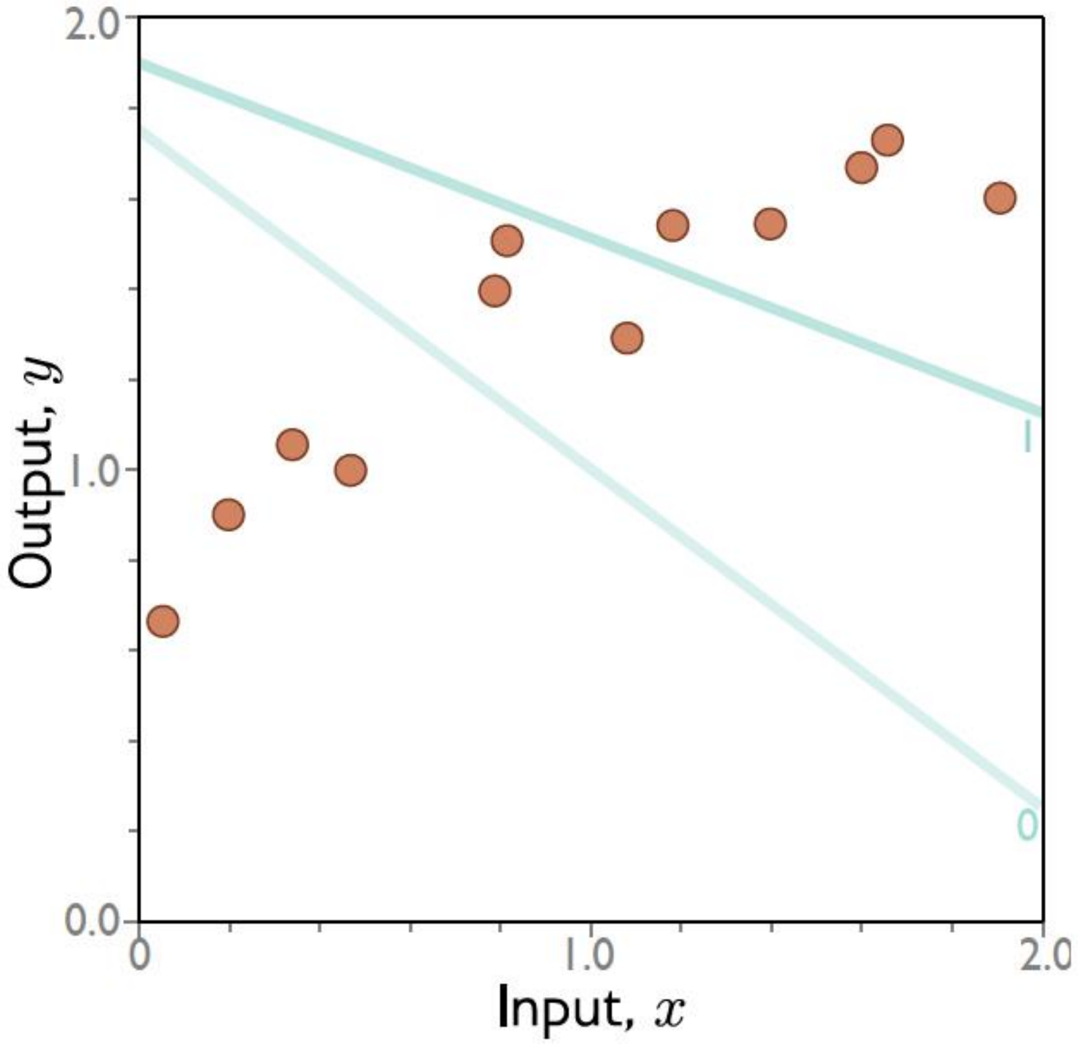

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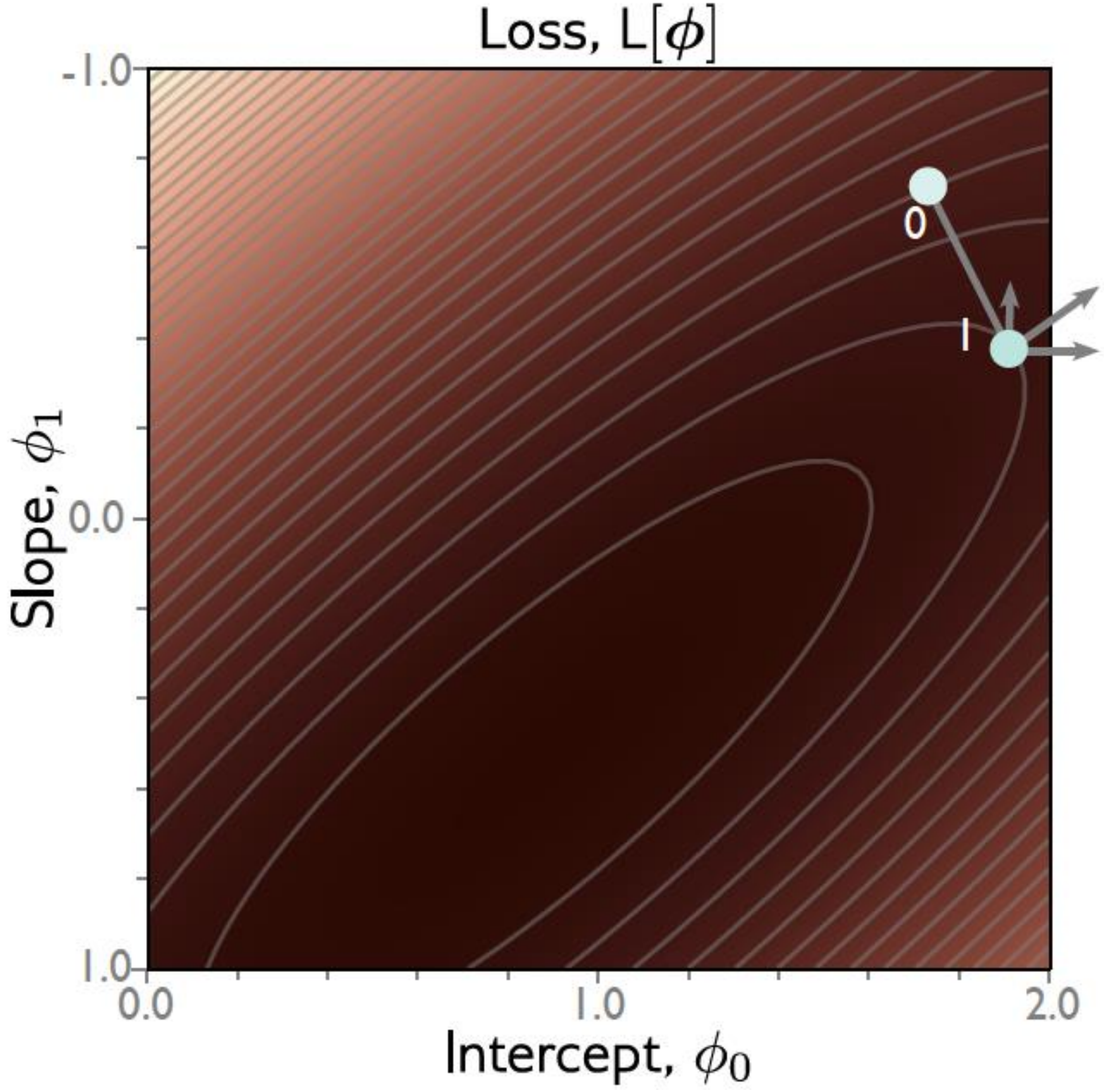

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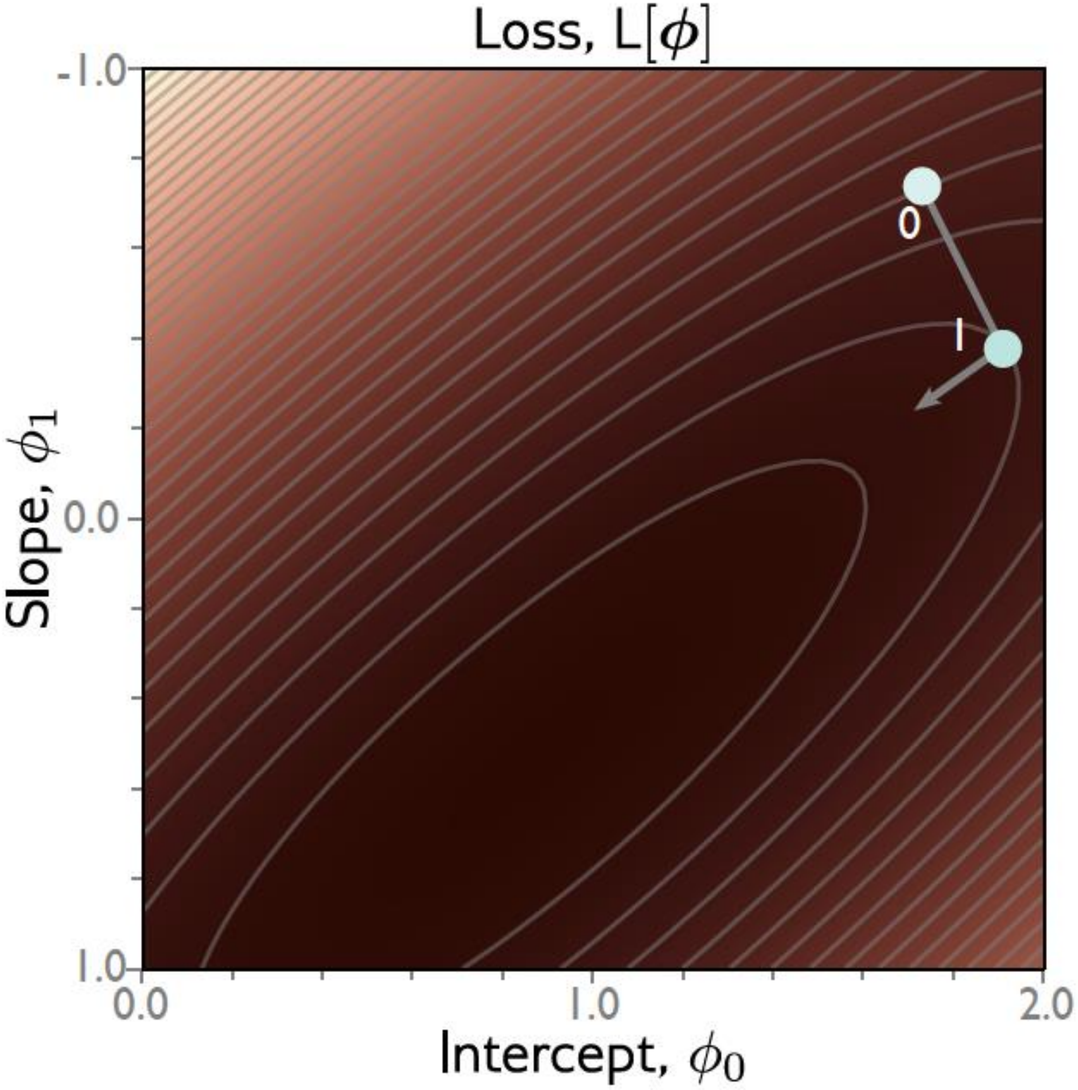

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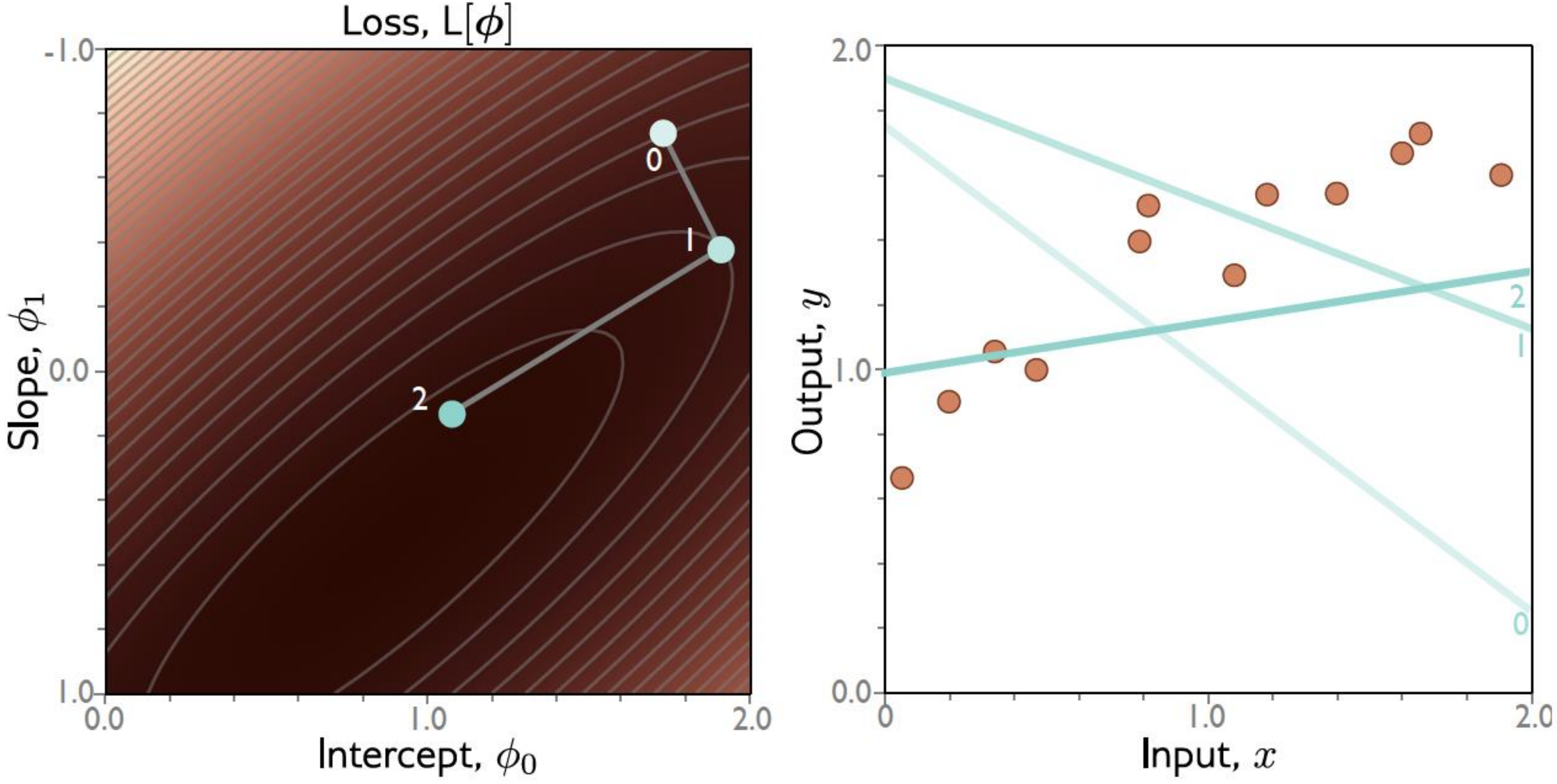

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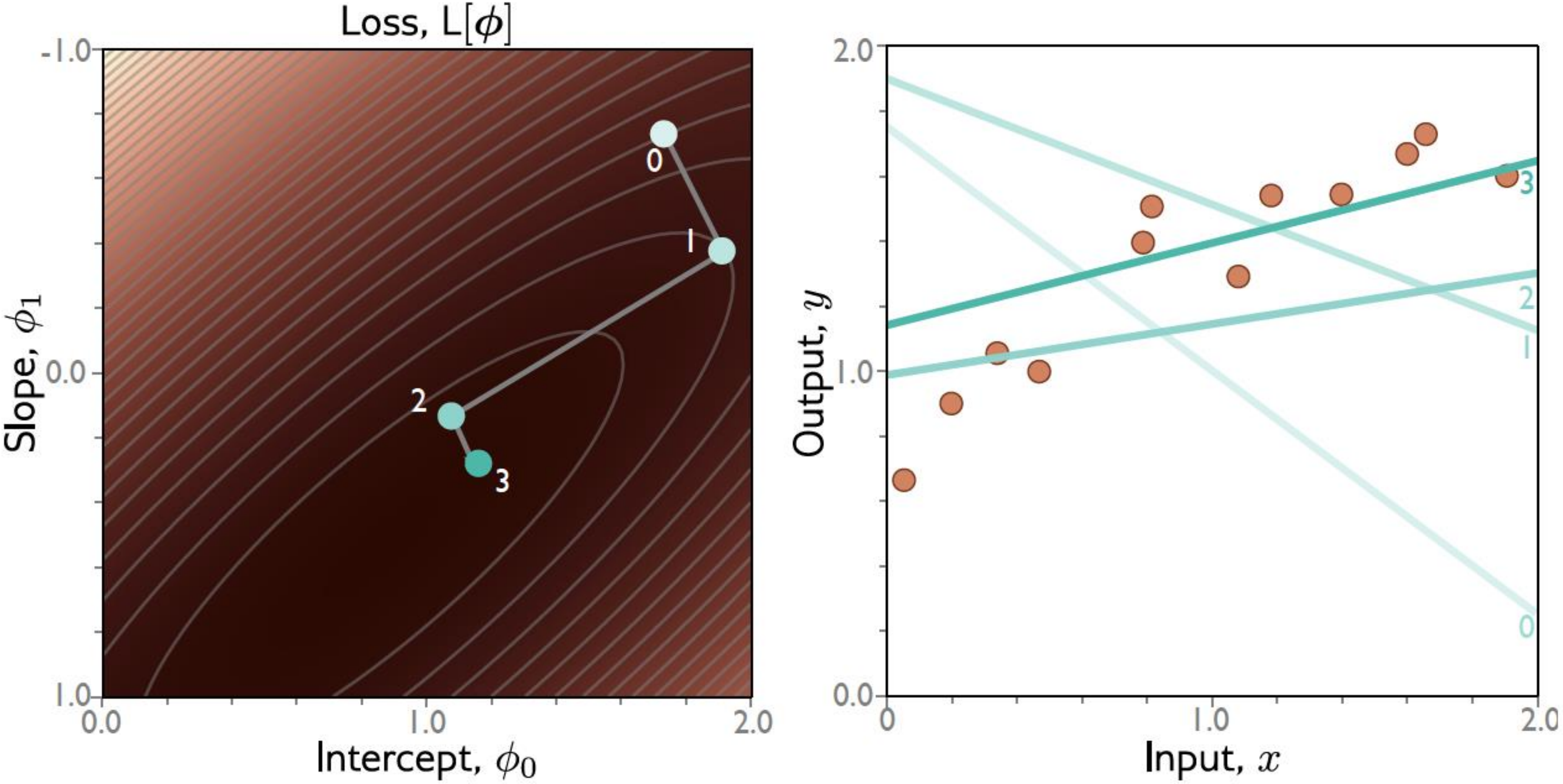

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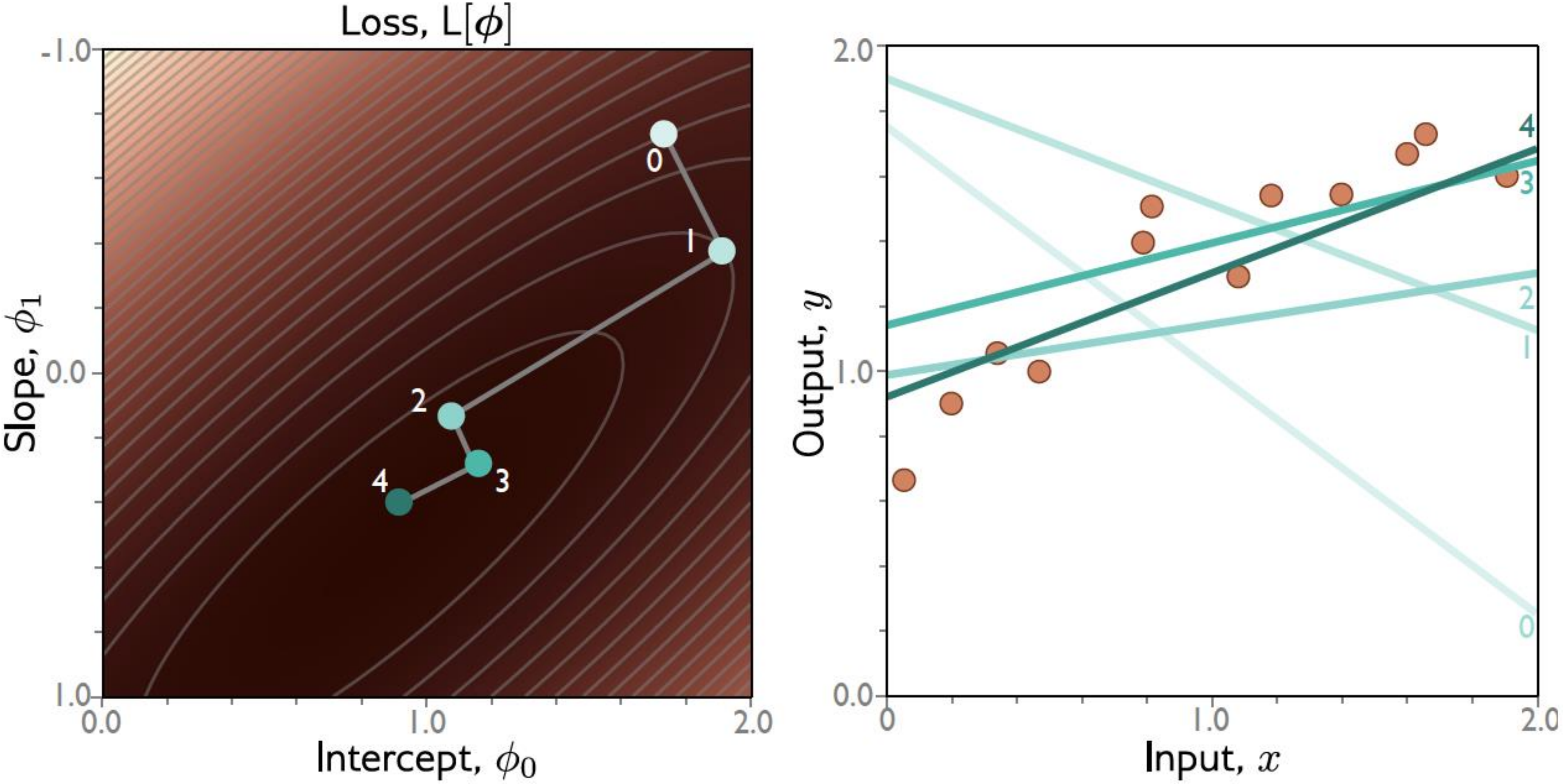

Closed Form Solution vs. Gradient Descent

$\phi = (X^T X)^{-1} X^T y$ → Closed-form solution

Where:

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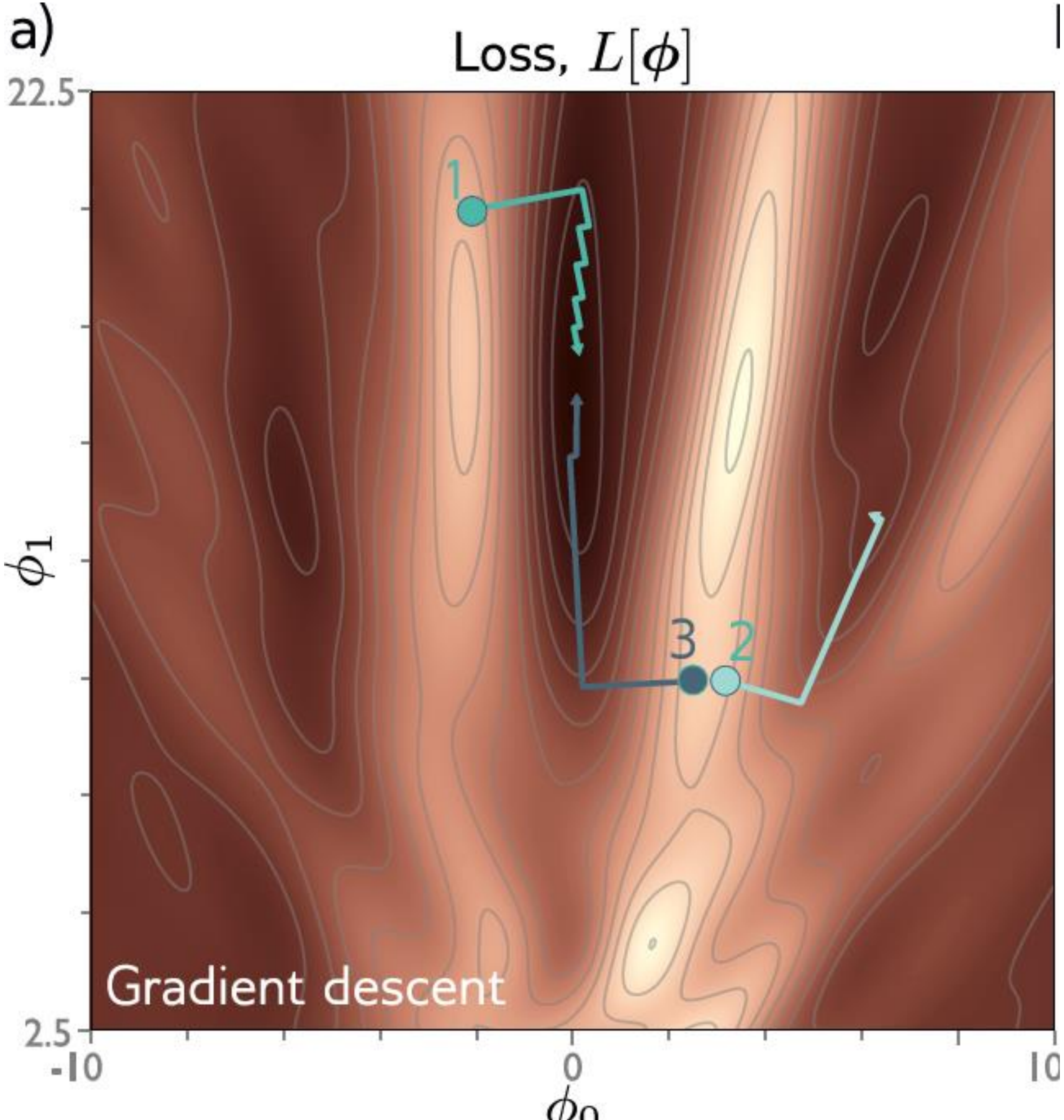

Batch Gradient Descent

$$\phi_{t+1} \longleftarrow \phi_t - \alpha \sum_{i=1}^{I} \frac{\partial \ell_i[\phi_t]}{\partial \phi},$$

Mini-Batch Gradient Descent

$$\phi_{t+1} \longleftarrow \phi_t - \alpha \sum_{i \in \mathcal{B}_t} \frac{\partial \ell_i[\phi_t]}{\partial \phi},$$

Stochastic Gradient Descent

Batch size is 1.

Fixed learning rate α