# Industrial Systems Modeling and Optimization

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## Assignment II
## Capacitated Lot-Sizing Problem with Setup Times (*CLSP-ST*)

**Instructions:**

- Submit your group's assignment report (**Group-xx_Assignment-II_report.zip**) as a single zipped file, which must contain a **self-contained Jupyter Notebook** (readable with visual studio code) as well as any file necessary to run the code, and a **report** answering thoroughly the assignment questions and discussing your results.

- Starts: 24 March, 2025 at 8:00 – **Due: 18 May, 2025 at 23:59**.

The *Multi-item Capacitated Lot-Sizing Problem with Setup Times* (*MiCLSP-ST*) aims at determining an optimal production plan of a set of items $i \in P = \{1, \dots, N\}$, produced on a shared single capacity constrained resource, over a finite planning horizon $H = \{1, \dots, T\}$ of $T$ discrete time periods (also called time buckets). The optimal production plan must specify for each item $i \in P$, the production quantities, $x_{it}$, produced in periods $t \in H$, to meet the items' known dynamic demands over the planning horizon $H$. The production costs vary over the planning horizon $H$ and consist of setup, production and holding costs. Each unit of item $i \in P$ produced in period $t \in H$ costs $c_{it}$, and consumes $r_i$ units of the resource's capacity $\kappa_t$. If production of an item $i \in P$ is initiated in period $t \in H$, a setup cost $f_{it}$ is incurred in that period, and an amount of time of the resource's capacity $\kappa_t$, the setup time $\tau_i$, is used for setting up the system to produce the item. In the case discussed here, it is assumed that the setup times depend only the product being produced, and are not sequence dependent. Finally, the inventory, $s_{it}$, of item $i \in P$, at the end of period $t \in H$, costs $h_{it}$ per-unit.

The constraints of the problem include material balance equations (2), where $d_{it}$ is the demand of item $i \in P$ in period $t \in H$; the capacity constraints (3), where $\kappa_t$ is the capacity limit of the resource in period $t \in H$; the production setup constraints (4), requiring setups in periods in which production occurs (4); and constraints (5) requiring the setup variables, $y_{it}$, to be binary variable assuming 1 if item $i \in P$ is produced in period $t \in H$. Backorders are not allowed (6). Initial inventories are assumed to be zero (7). Overtime is not allowed. Note that it would be easy to extend the model to allow $r_i$ and $\tau_i$ to be period dependent. The (*MiCLSP-ST*) problem can be formulated as follows:

(*MiCLSP-ST*$_{\text{IP}}$): $Z_{IP}$ = Minimize $\sum_{t \in H} \sum_{i \in P} (f_{it} y_{it} + c_{it} x_{it} + h_{it} s_{it})$

Subject to:

$$s_{i,t-1} + x_{it} - s_{it} = d_{it}, \quad \text{for all } i \in P, t \in H \quad (2)$$

$$\sum_{i \in P} (r_i x_{it} + \tau_i y_{it}) \leq \kappa_t, \quad \text{for all } t \in H \quad (3)$$

$$x_{it} - \left(\sum_{q=t}^{T} d_{iq}\right) y_{it} \leq 0, \quad \text{for all } i \in P, t \in H \quad (4)$$

$$y_{it} \in \{0,1\}, \quad \text{for all } i \in P, t \in H \quad (5)$$

$$x_{it} \geq 0, s_{it} \geq 0, \quad \text{for all } i \in P, t \in H \quad (6)$$

$$s_{i0} = 0, \quad \text{for all } i \in P \quad (7)$$

This problem is known to be a hard optimization problem to solve (see Bitran and Yanasse, 1982). Observe, however, that if constraints (3) are relaxed, the resulting sub-problems are single-item uncapacitated lot-sizing problems, studied in assignment I. We would like to take advantage of our knowledge on the single-item uncapacitated lot-sizing problem and propose good solutions to the (*MiCLSP-ST*) problem.

1. Solve the LP-relaxation (*MiCLSP-ST*$_{\text{LP}}$) of the above (*MiCLSP-ST*$_{\text{IP}}$) model and the (*MiCLSP-ST*$_{\text{IP}}$) model itself for the instances provided, and then report the solution values ($Z_{LP}$ and $Z_{IP}$) together with the computational times of both (*MiCLSP-ST*$_{\text{LP}}$) and (*MiCLSP-ST*$_{\text{IP}}$) for each instance.

2. As for the single-item lot-sizing problem, an alternative formulation for the current problem, also called plant location reformulation (*PLRF*$_{\text{IP}}$), is obtained by replacing in the model (*MiCLSP-ST*$_{\text{IP}}$) the production variables $x_{it}$ by the variables $w_{st}^{i}$ given by:

$$x_{is} = \sum_{t=s}^{T} d_{it} w_{st}^{i}; \quad \text{for all } i \in P, t \in H$$

   Where $w_{st}^{i}$ can be interpreted as the portion of demand of item $i \in P$ in period $t \in H$ fulfilled by production in period $s \in H, s \leq t$. Write down this reformulation and check the validity of your proposed model.

3. Solve the LP-relaxation (*PLRF*$_{\text{LP}}$) of the (*PLRF*$_{\text{IP}}$) model and the (*PLRF*$_{\text{IP}}$) model itself for the instances provided, and then report the solution values ($RFZ_{LP}$ and $RFZ_{IP}$) together with the computational times of both (*PLRF*$_{\text{LP}}$) and (*PLRF*$_{\text{IP}}$) for each instance. How does $RFZ_{LP}$ compare to $Z_{LP}$? Compare and discuss the computational times of these four models.

4. As mentioned above, observe that the only constraints in (*MiCLSP-ST*$_{\text{IP}}$) which tie the items together are the resource capacity constraints (3). Consider applying Lagrangian relaxation to the (*MiCLSP-ST*$_{\text{IP}}$), where constraints (3) are relaxed and dualized into the objective function with Lagrange multipliers ($\lambda_t$). Discuss the properties of the relaxed problem (*MiCLSP-ST*$_{\text{LR}}$) with value $Z_{LR}(\boldsymbol{\lambda})$. Propose a Lagrangian relaxation approach to determine the Lagrangian Dual value $Z_{LD}$, and then suggest a Lagrangian heuristic to obtain approximate solutions and values to the (*MiCLSP-ST*$_{\text{IP}}$).

5. Apply your approach to the instances provided and report the solution values $Z_{LD}$ and approximate values of $Z_{IP}$, as well as the computational times. Provide also the *LP* and *LD* gaps.

6. Consider now applying Lagrangian relaxation to the plant location reformulation (*PLRF*$_{\text{IP}}$), where here also the capacity constraints (3) are relaxed and dualized into the objective function with Lagrange multipliers ($\mu_t$). Discuss the properties of the relaxed problem (*PLRF*$_{\text{LR}}$) with value $RFZ_{LR}(\boldsymbol{\mu})$. Determine the Lagrangian Dual values $RFZ_{LD}$ for the instances provided and compare these values to $RFZ_{LP}$.

7. Chapter 6 (slide 6-12 to 6-15) discussed a possible reformulation of the multi-item production planning problem that is prone to the application of Dantzig-Wolfe decomposition technique. Extend that formulation to the (*MiCLSP-ST*$_{\text{IP}}$) problem. Define clearly the master problem as well as the sub-problems and discuss their related properties.

8. Apply the Dantzig-Wolfe decomposition approach to the instances provided and report the solution values ($Z_{DW}$) and computational times. Provide here also the gaps and discuss any issue you came across when you were implementing Dantzig-Wolfe decomposition.

9. Now, notice that if for each item the periods at which the setups take place are given in advance, the resulting problem become an LP problem to solve for the optimal production quantities. Of course the resulting LP-problem might be infeasible due to capacities and setup-times. Consider implementing Benders Decomposition where the complicating variables are the setup variables, $y_{it}$. Discuss the properties of the resulting master problem and the sub-problems in this case.

10. Apply Benders Decomposition approach to the instances provided and report the solution values ($Z_{BD}$) and computational times. Provide also the gaps and discuss any issue in implementing Benders Decomposition you came across. Compare all results and comment on your favorite approach if any!

**Challenge:** – (A bonus will be added to the final grade if this question is answered completely and correctly. This implies submitting your code of the L-shaped algorithm!)

11. Reconsider the 6-periods **capacitated** single-item uncapacitated lot-sizing problem, and assume now that the demand in each period is normally distributed and follows the distribution $N(\mu = 100, \sigma = 20)$. To approximately solve the recourse problem, we first start by discretizing the distribution of the random demand $\boldsymbol{d_t}$ at the end of each period $t$. We assume that $\boldsymbol{d_t}$ takes the realizations $(\boldsymbol{\mu} \pm \boldsymbol{k\sigma})$ for $k = 0, 1.5, 2.5$! Approximate the probability corresponding to each realization.

    11.1 Solve the expected value problem ($P^{EV}$) and determine the expected value solution. Then determine the expect result *EEV* of using this expected value solution, approximately and then exactly using the normal distribution.

    11.2 Using the above discretization of the distributions of the demands, obtain an approximation of the *wait and see* bound *WS* of the recourse problem and compare it to *EEV*.

    11.3 Using the above same discretization of the distributions of the demands and the L-shaped method, solve the recourse problem. Determine *EVPI* and *VSS*.

**Note**: Instances are given in the excel.

**References:**

William W. Trigeiro, L. Joseph Thomas, John O. McClain, (1989) Capacitated Lot Sizing with Setup Times. Management Science 35(3):353-366. http://dx.doi.org/10.1287/mnsc.35.3.353

Bitran, G. R., H. H. Yanasse. 1982. Computational complexity of the capacitated lot size problem. Management Sci. 28 1174