Industrial Systems Modeling and Optimization

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Assignment I
The Single-Item Uncapacitated Lot-Sizing Problem and Extensions (*SI-ULSP*)

Instructions:

- Submit your group's assignment report (**Group-xx_Assignment-I_report.zip**) as a single zipped file, which must contain a **self-contained Jupyter Notebook** (readable with visual studio code) as well as any file necessary to run the code, and a **report** answering thoroughly the assignment questions and discussing your results.

- Starts: February 24, 2025 at 01:00, **Due: March 23, 2025 at 23:59**!

The Single-Item Uncapacitated Lot-Sizing Problem:

The single-item uncapacitated lot-sizing problem with or without backlogging (*SI-ULSP*) considers a single production facility (plant), which must produce to fully satisfy the time-varying demand of a single-item over a finite planning horizon $H = \{1, \ldots, T\}$ of T periods. It is assumed that the facility has enough capacity to produce the item in any quantity. A fixed production (setup) cost f_t and a unit production cost c_t are incurred if production take place in a period t. A unit holding cost h_t is incurred for each unit held in inventory at the end of period t. If backlogging is allowed, that is, the demand d_t in period t may also be partially or fully satisfied from productions that take place in periods $\{t + 1, t + 2, \ldots, T\}$, a unit backlogging cost b_t is incurred for each unit backlogged at the end of period t. The purpose is to determine a production plan that specifies when and how much of the item to produce in each period, so that the demand in each period is fully satisfied and at a minimum total cost.

Part I: Uncapacitated Lot-Sizing Problem <u>without</u> Backlogging

In the single-item uncapacitated lot-sizing problem without backlogging (*SI-ULSP*), backlogging is not allowed, that is, the demand d_t in each period t must be fully satisfied from productions that took place in periods $\{1, 2, \ldots, t\}$.

1. Using only the variables defined below, propose a mixed-integer programming model that solves this production planning problem:

    - x_t : Amount of item produced in period t,
    - s_t : Amount of item held in inventory at the end of period t,
    - y_t : A 0-1 setup variable which assumes value 1 if $x_t > 0$, and 0 otherwise,

You may assume that no initial inventory is available in the beginning of the planning horizon H, that is (s_0 = 0). Unless otherwise indicated, you may also assume no inventory is left at the end of the planning horizon, that is (s_T = 0).

2. Solve the LP-relaxation of the model for the instances provided using Gurobi, then solve it as a MILP, and report the solution values together with the computational times for each instance of both the LP-relaxation and the MILP models.

3. Analyze the optimal MILP-solutions you obtained for each instance (draw them for small instances). Do you observe any pattern in these solutions? Show that that there exists always an optimal MILP-solution that displays this *pattern*. Using this MILP-solutions' pattern, propose a polynomial dynamic programming algorithm to solve the (*SI-ULSP*) problem (see Wagner-Whitin, 1958).

4. Define the following (l, S)-inequalities as follows: $\sum_{t\in S} x_t \leq \sum_{t\in S} D_{tl} y_t + s_l$, where $D_{tl} = \sum_{i=t}^{l} d_i$; for any $l \in H = \{1, \ldots, T\}$ and any subset $S \subseteq \{1,2,\ldots,l\}$. If X denotes the set of feasible solutions of the uncapacitated lot-sizing problem, show that these inequalities are valid for X.

5. For each instances provided pick the LP-relaxation solution and use the following separation procedure to find the most violated (l, S)-inequality, add it to the model and resolve again to find the newest LP-relaxation solution.

   (l, S) inequality separation procedure for an LP-relaxation solution (x^k, y^k, s^k):
   - For $l = 1, \ldots, T$; let $S_l = \{j, 1 \leq j \leq l$: $(x_j^k - D_{jl} y_j^k) > 0\}$, and
     Compute $v_l^k = \sum_{t\in S_l}(x_j^k - D_{jl} y_j^k) - s_l^k$;
   - If $v_l^k > 0$, then the (l, S_l) is violated by (x^k, y^k, s^k), and the inequality with the largest $v_{l^*}^k = max_{l=1,\ldots,T}\{v_l^k: v_l^k > 0\}$ is the most violated $(l_{l^*}^k, S_{l^*}^k)$ inequality.

6. Using Gurobi and this procedure for each instance and keep iterating until no (l, S)-inequalities can be generated anymore. Examine carefully the resulting solution (x^*, y^*, s^*). Normally this should be IP-feasible, i.e., y^* should be binary.

7. Now, define the variable w_{qt} to be the amount produced in period q ($q \leq t$) and used to meet the demand of the item in period t; that is the fraction of the demand of the item in period t that is produced in period q. Reformulate the single-item uncapacitated lot-sizing problem using only w_{qt} and y_t variables. This reformulation is called the facility location reformulation (*FLPR*).

8. Solve this facility location reformulation (*FLPR*) for the instances provided again, but this time by dropping any integrality constraints. Discuss your findings. What can you conclude from the obtained results, what can be said about this reformulation of *SI-ULSP*.

Part II: Uncapacitated Lot-Sizing Problem <u>with</u> Backlogging

When backlogging is allowed, that is, the demand d_t in period t may also be partially or fully satisfied from productions that take place in periods $\{t + 1, t + 2, \ldots, T\}$, a unit backlogging cost b_t is incurred for each unit backlogged at the end of period t.

9. Using the variables defined below, propose a mixed-integer programming model (*SI-ULSPwB*) that solves this production planning problem:

    - x_t : Amount of item produced in period t,
    - s_t : Amount of item held in inventory at the end of period t,
    - r_t : Amount of item backlogged at the end of period t,
    - y_t : A 0-1 set-up variable which assumes value 1 if $x_t > 0$,

    You may assume that no initial inventory and backlog is available in the beginning of the planning horizon H, that is ($s_0 = r_0 = 0$). Unless otherwise indicated, you may also assume no inventory is left at the end of the planning horizon and no backlog, that is ($s_T = r_T = 0$).

10. Analyze the optimal MILP-solutions you obtained for each instance (draw them for small instances). Do you observe any pattern in these solutions? Show that that there exists always an optimal MILP-solution that displays this *pattern*. Using this MILP-solutions' pattern, propose a polynomial dynamic programming algorithm to solve the (*SI-ULSPwB*) problem (see see Zangwill 1969).

11. The facility location reformulation for the *SI-ULSP* extends also to the *SI-ULSPwB*. There is however another reformulation, called the *Shortest Path Reformulation* (*SPR*).of (*SI-ULSPwB*). Write down this reformation using the variables y and the following:

    w_t = 1 if the demand for period t is produced in period t, and 0 otherwise;
    φ_{ut} = 1 if production in u includes the future demand precisely up to period $t \geq u$, and 0 otherwise;
    ψ_{ut} = 1 if production in u includes backlogged demand precisely from period $t \leq u$, and 0 otherwise;

12. Solve this reformation (*SPR*) for the instances provided, here also by dropping any integrality constraints. Discuss your findings. What can you conclude from the obtained results, what can be said about this reformulation of (*SI-ULSPwB*).

References:

1. Harvey M. Wagner and Thomson M. Whitin; 'Dynamic Version of the Economic Lot Size Model'; Management Science , 1958, vol. 5, no. 1, pp. 89-96.
2. Williard I. Zangwill; 'A Backlogging Model and a Multi-Echelon Model of a Dynamic Economic Lot Size Production System - A Network Approach'; Management Science , 1969, vol. 15, no. 9, pp. 506-527.