Stochastic processes and simulation (AE4426-19)

Lecture 4: Gaussian processes & difference equations

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Outline

Learning Goals

At the end of this lecture, you will be able to:

1. Define a discrete-state continuous-time Gauss- Markov process.
2. Define a stochastic difference equation.
3. Check if a stochastic process is Gauss-Markov.
4. Monte Carlo simulate a discrete-state continuous-time Markov process.

Discrete-Time Continuous-State Markov Process

- Discrete time: $T \in \{1, 2, \ldots\}$.
- State space $\mathbb{R}^n, n \in \mathbb{N}^+$ (continuous state space!).
- Similar to the finite state case, $p_{X_{t+1}|X_t}(x|y)$ is to be understood as the transition probability density function at $X_{t+1} = x$ given $X_t = y$, with

$$p_{X_{t+1}|X_t}(x|y) \geq 0, \forall i, j \in \mathbb{R}^n$$
$$\int p_{X_{t+1}|X_t}(x|y)dx = 1.$$

- Markov property is satisfied.

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Stochastic Difference Equations

A general stochastic difference equation has the form

$$X_t = f(t, X_{t-1}, \omega_t), t \geq 1,$$

where $\{X_t\}$ takes values from $\mathbb{R}^n$ and $\omega_0, \omega_1, \ldots, \omega_t$ i.i.d. drawings from $[0, 1]$.

Example:

$$X_t = 2X_{t-1} + 3W_t,$$

$\{X_t\}$ an $\mathbb{R}$-valued stochastic process, X_0 is Gaussian and independent of W_t, with W_t Gaussian.

Application - an example

Flight path under wind uncertainty, where the wind is normally distributed.

Example*: s_t is the 3D position of an aircraft at time t, having velocity v_t at time t, heading angle ψ, flight path angle γ, bank angle ϕ and i.i.d. normally distributed W_t^x, W_t^y, W_t^z variation in the position, $\Delta \to 0$.

$$
\begin{aligned}
s^x_{t+\Delta} &= s^x_t + \Delta v_t \cos\psi \cos\gamma + \Delta W^x_t,\\
s^y_{t+\Delta} &= s^y_t + \Delta v_t \sin\psi \cos\gamma + \Delta W^y_t,\\
s^z_{t+\Delta} &= s^z_t + \Delta v_t \sin\gamma + \Delta W^z_t,\\
v_{t+\Delta} &= v_t + \Delta \frac{g \tan\phi}{v_t}.
\end{aligned}
$$

*A multi-Aircraft model for conflict detection and resolution algorithm evaluation", W. Glover and J. Lygeros, 2004, Section 3: Aircraft dynamics

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1. Discrete-Time Continuous-State Markov Process
   - Stochastic Difference Equations
   - Gauss-Markov process
   - Difference equations of Gauss-Markov process
   - Gaussian process - Mean
   - Gaussian process - Covariance
2. Monte Carlo simulation of a discrete-time continuous-state Markov process
   - Simulation

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Gauss-Markov process

Gauss-Markov process: an example of a discrete-time continuous-state Markov process.

1. A Gaussian process is a process with the property that for any $t_1, \ldots, t_n$, $(X_{t_1}, \ldots, X_{t_n})$ have a joint Gaussian distribution.
2. A Markov process is said to satisfy the **Markov property** when its future and past states are conditionally independent given its present state, i.e.,

$$p_{X_t|X_{t-1},X_{t-2},\ldots,X_0}(x|x', x'', \ldots, x^*) = p_{X_t|X_{t-1}}(x|x').$$

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Difference equations of Gauss-Markov process

$$X_0 = C_0 V_0 \tag{1}$$
$$X_t = X_{t-1} + BU(t) + CV_t, t \geq 1, \tag{2}$$

$X_t \in \mathbb{R}^n$, $U(t)$ a deterministic, scalar-valued sequence, $\{V_t\}$ a sequence of i.i.d. standard Gaussian random variables, i.e., $V_t \sim N(0,1)$, B and C are n-size vectors.

1. Gaussian process

$$X_{t+1} = (X_{t+1} - X_t) + (X_t - X_{t-1}) + \ldots + X_1.$$

   X_{t+1} is a linear combination of $X_{t+1}, X_t, \ldots, X_1$, which is Gaussian. Thus, $X_{t+1}, X_t, \ldots, X_1$ are themselves jointly normal.

2. Markov property

   The sequence generated by (2) is Markov since, given X_{t-1}, X_t depends only on ω_t, which is independent of $X_{t-2}, \ldots, X_0$.

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Gaussian process - Mean

- Reminder: Linear functions of Gaussian random variables are also Gaussian.

  From (1), X_0 follows a normal distribution with mean $\bar{X}_0 = 0$ and covariance $\bar{R}_0 = C_0 C_0^T$, where C_0^T is the transpose of C_0. Then, from (1) and (2), X_t follows a normal distribution with mean $\bar{X}_t$ and covariance $\bar{R}_t$.

$$
\begin{aligned}
\bar{X}_{t+1} &= \mathbb{E}[X_{t+1}] \\
&= \mathbb{E}[X_t + BU_{t+1} + CV_{t+1}] \\
&= \mathbb{E}[X_t] + B\mathbb{E}[U(t+1)] + C\mathbb{E}[V_{t+1}] \\
&= \mathbb{E}[X_t] + BU(t+1) \\
&= \bar{X}_t + BU(t+1).
\end{aligned}
$$

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Gaussian process - Covariance

$$
\begin{aligned}
\bar{R}_{t+1} &= \mathbb{E}[(X_{t+1} - \bar{X}_{t+1})(X_{t+1} - \bar{X}_{t+1})^T] \\
&= \mathbb{E}[(X_t + BU_{t+1} + CV_{t+1} - \bar{X}_t - BU_{t+1}) \\
&\cdot (X_t + BU_{t+1} + CV_{t+1} - \bar{X}_t - BU_{t+1})^T] \\
&= \mathbb{E}[(X_t - \bar{X}_t)(X_t - \bar{X}_t)^T] + \mathbb{E}[(X_t - \bar{X}_t)(CV_{t+1})^T] \\
&+ \mathbb{E}[(CV_{t+1})(X_t - \bar{X}_t)^T] + \mathbb{E}[CV_{t+1}(CV_{t+1})^T] \\
&= \mathbb{E}[(X_t - \bar{X}_t)(X_t - \bar{X}_t)^T] + C\mathbb{E}[V_{t+1}(V_{t+1})^T]C^T \\
&= \bar{R}_t + CC^T.
\end{aligned}
$$

Note: $\bar{R}_0 = C_0 C_0^T$.

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Monte Carlo simulation of a discrete-time continuous-state Markov process

1. Initial condition:

$$X_0 = f(\omega_0), \text{ where } \omega_0 \sim N(\mu, \sigma^2)$$

2. The outcome of X_t is defined by X_{t-1} and ω_t, where $\omega_0, \omega_1, \ldots, \omega_t$ are independent drawings from $\mathit{Uniform}(0, 1)$. Thus,

$$X_t = f(t, X_{t-1}, \omega_t).$$

Note: In Matlab, the function **randn** *generates a random variable* $\sim N(0, 1)$*, while the function* **rand** *generates a random variable* $\sim U(0, 1)$.

Simulating $Y \sim N(\mu_Y, \sigma_Y^2)$

Let $Y \sim N(\mu_Y, \sigma_Y^2)$ and $X \sim N(0,1)$.
Question: How to simulate Y using $N(0,1)$?

Reminder: A linear combination of normally distributed random variable is again a normally distributed random variable.

$Y = \alpha X + \beta$ is normally distributed. But what are the values for α and β such that $Y \sim N(\mu_Y, \sigma_Y^2)$?

$$\mathbb{E}[Y] = \mathbb{E}[\alpha X + \beta] = \beta$$
$$Var[Y] = Var[\alpha X + \beta] = \alpha^2 Var[X] = \alpha^2$$

So, to generate $Y \sim N(\mu_Y, \sigma_Y^2)$, we use $\sigma_Y \cdot randn + \mu_Y$.