Assignment 2

AE4426-19 Stochastic processes and simulation, 2020-2021

Deadline: 5th Jan 2021, 21:00

Note: For each additional day of delay, 5pcts (out of 100) are substracted.

1 Security check at an airport (46 pcts)

Consider a security check lane where passengers arrive according to a Poisson process $\{N_t\}$ with rate λ passengers per minute. The passengers wait in a queue to be checked. Each passenger is checked, one by one, for an exponential amount of time with mean $1/\mu$ minutes.

Using analytical derivations:

a) Determine the probability that at least 5 passengers arrive at the security lane in the period 10:00-10:25.

b) Knowing that at 10:00 there are no passengers at the security lane, determine the probability that the first passenger arrives some time after 10:03 minutes, but not later than 10:09.

c) We start with an empty passenger queue. Let $T_1, T_2, T_3, ...$ denote the successive times at which passengers arrive at the security lane. Determine $\mathbb{E}[T_4|N_8 = 2]$ and the variance of the time of the 4th passenger arrival $\text{Var}[T_4]$.

d) Determine $\mathbb{E}[N_5|N_t = 7], t < 3$.

e) You have just arrived at the security check and you are now the first passenger in the queue for this check. In front of you, a passenger has been undergoing security check already for 15min now. If, in the next 5min, the passenger in front of you does not complete his check, then a new security check point is opened and you immediately start your check at this new point. Determine what is the expected time you have to wait from now until you start your check. Assume $\lambda = 0.2$ and $\mu = 0.3$.

Monte Carlo simulation (please submit also the code)

f) Suppose the queue is empty at time 0. Passengers start arriving at a rate of $\lambda = 0.3$. Customer service surveys have shown that it is desired that the passenger waiting time in the queue stays below 8 minutes. If there is only one queue, at what rate must passengers be checked to ensure that at least 95% of passengers has a waiting time below 8 minutes? Determine an appropriate value of μ by simulating the situation for a duration of 1 hour. (Hint: compute the standard deviation)

g) Due to social distancing constraints, a queue only has enough space to accommodate a maximum of 7 passengers. When a queue reaches full capacity, a second queue must be opened. Assume passengers are checked at a rate of $\mu = 0.4$. Plot the empirical distribution of the number of passengers in the queue after 1 hour (starting from an empty queue). What is the probability that a second queue is required?

Hint for Python users:

- numpy.cumsum(x) is a useful function to plot the empirical CDF

Hint for MATLAB users:

- cdf(x) automatically generates an empirical CDF

2 Degradation of aircraft component(24pcts)

The dynamics of damage evolution in various rotational machinery can be modeled by a nonlinear dynamical system with two hidden states and one measured output (representing for, example, the measured dimension of the wear). Let the degradation condition of a rotational machinery be represented as a stochastic process with the following difference equations:

$$\mathbf{X}_{t+1} = f(\mathbf{X}_t, \mathbf{V}_t, \mathbf{A}, \mathbf{B}),$$
$$Y_t = g(\mathbf{X}_t, e_t, \mathbf{C}),$$

where $\mathbf{X}_{t+1}$ is the true condition of the machinery at time $t+1$; Y_t is the measured degradation condition of the machinery at time t, $\mathbf{V}_t$ is a sequence of i.i.d. Gaussian random variables, $V_t \sim N(0,1)$, with $\mathbf{V}_t$ independent of $\mathbf{X}_t, \mathbf{X}_{t-1}, ..., \mathbf{X}_0$; $\mathbf{C}$ is the set of model parameters; e_t is the measurement error, with $e_t \sim N(0, 0.1)$ and e_t independent of $e_{t-1}, ..., e_0$. Formally, we consider:

$$\mathbf{X}_{t+1} = \mathbf{A}\mathbf{X}_t + \mathbf{B} + \mathbf{V}_t,$$
$$Y_t = \mathbf{C}\mathbf{X}_t + e_t,$$

where $\mathbf{X}_t = \begin{bmatrix} X_t^{(1)} \\ X_t^{(2)} \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix} \mathbf{C} = [1 \quad 0]$, and $\mathbf{V}_t = \begin{bmatrix} V_t^{(1)} \\ V_t^{(2)} \end{bmatrix}$, $\mathbf{X}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

a) Determine analytically $\mathbb{E}[\mathbf{X}_3]$ and $Cov[\mathbf{X}_3, \mathbf{X}_3]$, i.e., characterize the degradation level of the machinery after being used 3 time steps.

Monte Carlo simulation: (please submit also the code)

b) Estimate the first time when the measured level of the degradation exceeds a level $L = 1000$, i.e., determine $\min\{t : Y_t \geq 1000\}$.

3 Brownian motion (30 pcts)

The degradation over time of a component (wearing) is given by:

$$X_t = \beta t + \sigma B_t,$$

where β is a constant for degradation drift, σ is a diffusion coefficient, and B_t is standard Brownian motion. Consider $X_0 = 0$.

Using analytical derivations:

a) Determine $\mathbb{E}[X_5^2]$.

b) Determine $\text{Var}(3X_5 - X_2)$.

c) Determine $\mathbb{E}[B_5 | B_2 = 2]$.

Monte Carlo simulation: (please submit also the code)

d) Using $\beta = 1$ and $\sigma = 2$, estimate $\mathbb{E}[X_5]$ and $\text{Var}[X_5]$. Consider 10, 100, 10.000 simulation runs. What do you observe?

e) Using $\beta = 1$ and $\sigma = 2$, plot the empirical distribution of X_5 using both a histogram and a Dirac function.

NOTE: For each simulation exercise, the grading is:

30% - Declaration of the variables used with explanation of their meaning; Clarity of the code and proper use of comments.

20% - Final numerical results are correct.

50% - Correct logic of the simulation. The simulation answers the question posed.