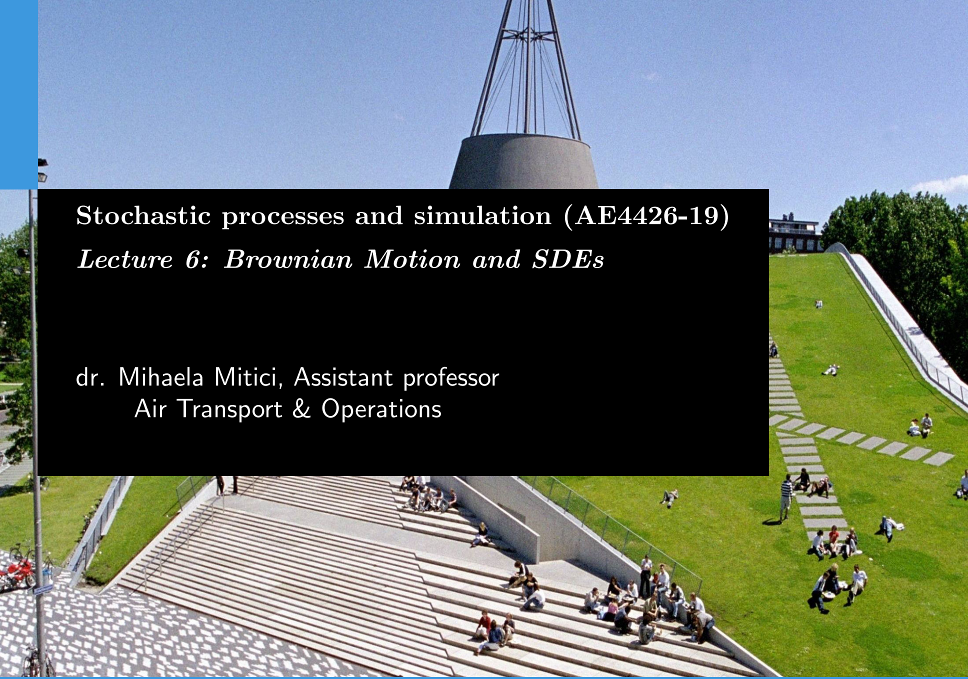

Stochastic processes and simulation (AE4426-19)

Lecture 6: Brownian Motion and SDEs

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Outline

If you want to read more

M. Taylor & S. Karlin. *An introduction to stochastic models*. Third Edition. Academic Press. 1998

Chapter VIII, Sections 1.1, 1.2 (Introduction to Brownian Motion)

René L. Schilling et. al., *Brownian Motion : An Introduction to Stochastic Processes*, De Gruyter, Inc., 2012 (**e-book** available at TUD Library)

Chapter 18: SDE
Chapter 19: Simulation of Brownian Motion

Next Subsection

Continuous-time continuous-state stochastic processes

Definition

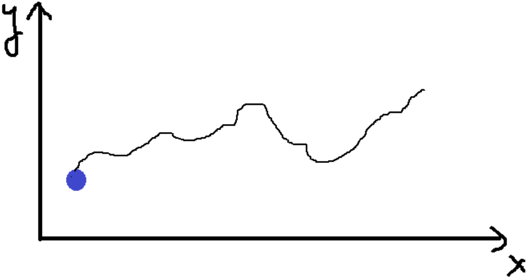
A continuous-time continuous-state stochastic process $\{X_t\}$ with $t \in T$, $T = [0, \infty)$ is a process that assumes values in $\mathbb{R}^n$ (instead of a discrete set).

Example: Brownian Motion

Brownian Motion - the beginnings

R. Brown - random movement of pollen particles in water (1827)

https://www.youtube.com/watch?v=R5t-oA796to



Next Subsection

Examples of applications

- Aircraft trajectory under wind uncertainty.

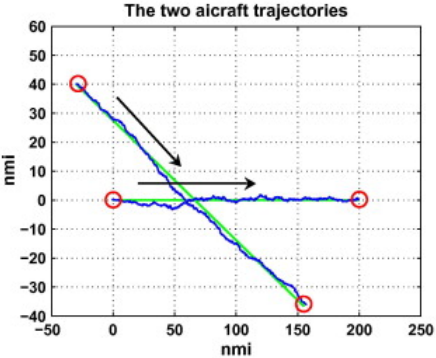


Figure: A top view of the flight plans of two aircraft and the actual aircraft trajectories("Conflict probability estimation between aircraft with dynamic importance splitting", D. Jacquemart and J. Morio, Safety Science, 2013.)

- Estimating the probability of aircraft conflict/collision.

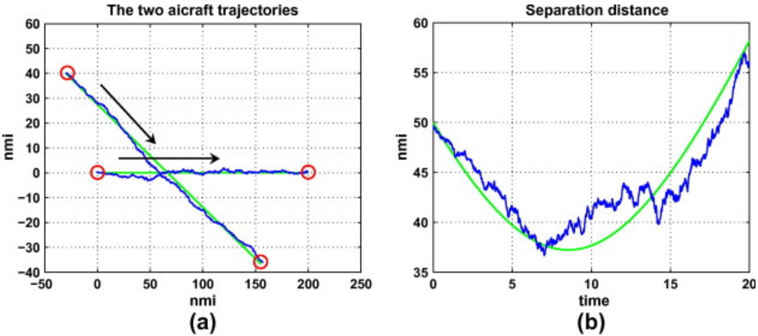
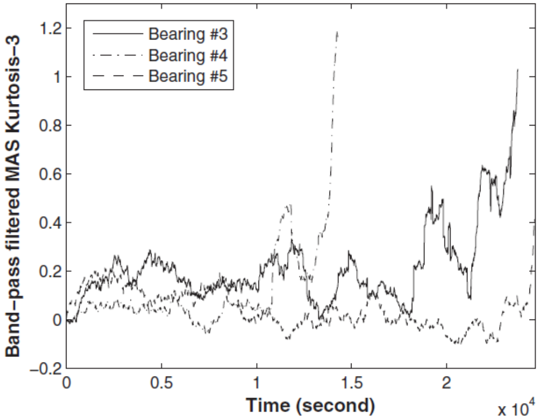


Figure: "Conflict probability estimation between aircraft with dynamic importance splitting", D. Jacquemart and J. Morio, Safety Science, 2013.

Example of applications (2)

- Degradation/wear of aircraft components



The degradation paths of ball bearings #3, #4, and #5 under operating condition 1.

Figure: Zhang, Hanwen, et al. "Predicting remaining useful life based on a generalized degradation with fractional Brownian motion." Mechanical Systems and Signal Processing 115 (2019): 736-752.

Next Subsection

Standard Brownian Motion

Definition

Standard Brownian motion is an $\mathbb{R}$ valued stochastic process $\{B_t\}_{t\geq 0}$ with the following properties:

i) $P(B_0 = 0) = 1$.
ii) Every increment $B_t - B_s, t > s$, is **normally distributed** with mean 0 and variance $\sigma^2(t-s)$, with $\sigma^2 = 1$.
iii) For any pair of disjoint (non-overlapping) time intervals $[t_1, t_2]$ and $[t_3, t_4]$, say $t_1 < t_2 \leq t_3 < t_4$, the increments $B_{t_4} - B_{t_3}$ and $B_{t_2} - B_{t_1}$ are **independent and stationary** with distribution given in ii).
iv) $\{B_t\}$ has **continuous** sample paths.

Stationary increments

Definition

A process $\{X_t\}$ is said to have stationary independent increments if

$$X_{t+h} - X_{s+h}$$

has the same distribution as

$$X_t - X_s$$

for every $t > s \in T$ and every $h > 0$.

- **Observation 1: (Standard) Brownian motion $\{B_t\}$ has stationary, independent increments**.
- Observation 2: The variance of the increments is proportional to the length of the time difference and invariant to the location of the interval on the time line.

Next Subsection

Mean and (Co)variance

- Mean of B_t for $t \geq 0$.

$$\begin{aligned}\mathbb{E}[B_t|B_s] =& \mathbb{E}[B_t - B_s + B_s|B_s] \\ =& \mathbb{E}[B_t - B_s|B_s] + \mathbb{E}[B_s|B_s] \\ =& \mathbb{E}[B_t - B_s|B_s - B_0] + B_s \\ =& \mathbb{E}[B_t - B_s] + B_s \\ =& 0 + B_s.\end{aligned}$$

- If conditioning on $B_0 = 0$:

$$\mathbb{E}[B_t|B_0] = B_0 = 0. \tag{1}$$

From assumption $P(B_0 = 0) = 1$ and (1) it follows:

$$E[B_t] = 0.$$

Mean and Covariance

- Covariance of B_s and B_t, for $0 \leq s < t$:

$$\begin{aligned}
\mathrm{Cov}[B_t, B_s] &= \mathbb{E}[B_t B_s] - \mathbb{E}[B_t]\mathbb{E}[B_s] \\
&= \mathbb{E}[(B_t - B_s + B_s)B_s] - 0 \\
&= \mathbb{E}[(B_t - B_s)B_s] + \mathbb{E}[B_s^2] \\
&= \mathbb{E}[B_t - B_s]\mathbb{E}[B_s] + \mathbb{E}[B_s^2] \\
&= 0 + \mathbb{E}[B_s^2] \\
&= s.
\end{aligned}$$

Next Subsection

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Transition probability function for SBM

Reminder: Probability density function of a normally distributed random variable $Y \sim N(\mu, \sigma^2)$:

$$N(y; \mu, \sigma^2) = \frac{1}{\sqrt{2\sigma^2\pi}} exp(-\frac{(y-\mu)^2}{2\sigma^2}), y \in \mathbb{R}.$$

- Transition probability for SBM, $t_1 > t_0$:

$$p_{B_{t_1}|B_{t_0}}(y|x) = \frac{1}{\sqrt{2\pi(t_1 - t_0)}} exp(-\frac{(y-x)^2}{2(t_1 - t_0)})$$
$$\sim N(x, t_1 - t_0).$$

Mean and Variance

$$\begin{aligned}
\mathbb{E}[B_{t_1}|B_{t_0}=x] &= \int_{-\infty}^{\infty} y p_{B_{t_1}|B_{t_0}}(y|x) dy \\
&= \int_{-\infty}^{\infty} y\ N(y;x,t_1-t_0) dy \\
&= \int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi(t_1-t_0)}} exp(-\frac{(y-x)^2}{2(t_1-t_0)}) dy \\
&= x.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[B_{t_1}^2|B_{t_0}=x] &= \int_{-\infty}^{\infty} y^2 p_{B_{t_1}|B_{t_0}}(y|x) dy \\
&= \int_{-\infty}^{\infty} y^2\ N(y;x,t_1-t_0) dy \\
&= \int_{-\infty}^{\infty} y^2 \frac{1}{\sqrt{2\pi(t_1-t_0)}} exp(-\frac{(y-x)^2}{2(t_1-t_0)}) dy \\
&= t_1-t_0+x^2.
\end{aligned}$$

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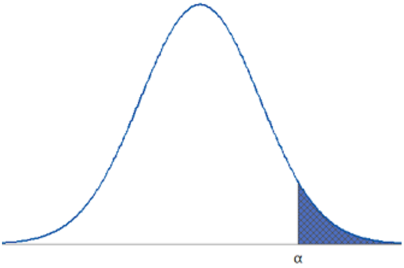
How to determine $P(B_t > \alpha)$?

$$P(B_t > \alpha) = \int_\alpha^\infty p_{B_t}(y)dy = \int_\alpha^\infty N(y; 0, t)dy$$
$$= \frac{1}{\sqrt{2\pi t}} \int_\alpha^\infty exp(\frac{-y^2}{2t})dy.$$



In general,

$$\begin{aligned} P(B_{t_1} > \alpha | B_{t_0} = x_0) &= \int_{\alpha}^{\infty} N(y; x_0, t_1 - t_0) dy \\ &= \frac{1}{\sqrt{2\pi(t_1 - t_0)}} \int_{\alpha}^{\infty} exp(\frac{-(y - x_0)^2}{2(t_1 - t_0)}) dy. \end{aligned}$$

Next Subsection

Markov property

From the property of independent increments of a BM, if we know that $B_s = x_s$, then knowing the values $B_\tau, \tau < s$ (the past) do not affect our knowledge of $B_t, t > s$. Formally, for $t_0 < t_1 < \ldots < t_n < t$,

$$p_{B_{t_n}|B_{t_{n-1}},\ldots,B_{t_1}}(x_n|x_{n-1},\ldots,x_1) = p_{B_{t_n}|B_{t_{n-1}}}(x_n|x_{n-1}).$$

Monte Carlo simulation of Brownian Motion

1. Divide simulation horizon t into k equal time intervals of length Δ as follows $t_1 = t_0 + \Delta, \ldots, t_k = \Delta + t_{k-1}$.
2. Generate increments:

$$
\begin{aligned}
B_{t_0} &= 0. \\
B_{t_1} - B_{t_0} &\sim N(0, t_1 - t_0) \\
B_{t_2} - B_{t_1} &\sim N(0, t_2 - t_1) \\
&\ldots
\end{aligned}
$$

Reminder: To simulate a random variable v from $N(0, t_1 - t_0)$:
Step 1: Generate $u \sim N(0, 1)$.
Step 2: $v := u \cdot \sqrt{t_1 - t_0}$.

Next Subsection

Ordinary differential equations (ODEs)

Ordinary Differential Equation (ODE):

$$\frac{dX(t)}{dt} = f(t, X)$$

So,

$$dX(t) = f(t, X)dt.$$

With initial conditions $X(0) = x_0$, we can write in integral form

$$X(t) = x_0 + \int_0^t f(s, X(s))ds,$$

where $X(t) = X(t, x_0, 0)$ is the solution with initial conditions $X(0) = x_0$.

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Stochastic differential equations (SDEs)

An SDE is a differential equation in which one or more of the terms is a stochastic process. The solution of an SDE is also a stochastic process.

$$\frac{dX_t}{dt} = f(X_t, \omega_t, t),$$

where ω_t is the stochastic term.
So,

$$dX(t) = f(t, X_t, \omega_t)dt.$$

Stochastic differential equations (SDEs)

Definition

A typical stochastic differential equation (SDE) has the form:

$$dX_t = a(t, X_t)dt + b(t, X_t)dB_t, \tag{2}$$

where B_t is standard Brownian motion, $a(.)$ and $b(.)$ are given functions of time t and the current state.

Next Subsection

Solution of the SDE in integral form

A solution of the SDE in (2) is a continuous stochastic process which satisfies the integral equation:

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s, t \geq 0.$$

- The integral $\int_0^t a(s, X_s)ds$ is the usual Riemann integral
  But $\int_0^t b(s, X_s)dBs$ is a stochastic (Itô) integral!

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Itô integral

The Itô integral can be defined as the limit:

$$\lim_{N\to\infty} \sum_{i=1}^{N} b(t_{i-1},\omega)(B(t_i,\omega) - (B(t_{i-1},\omega)),$$

for each sequence of partitions $(t_0, t_1, ..., t_N)$ of the interval $[0, t]$ such that $\max_i(t_i - t_{i-1}) \to 0$.

This stochastic integral is a random variable, the samples of which depend on the individual realizations of the paths $B(., \omega)$.

Example SDE

$$dX_t = \mu dt + \sigma B_t.$$

Solving the SDE (integrating):

$$\begin{aligned} \int_0^T dX_t &= \int_0^T \mu dt + \int_0^T \sigma dB_t \\ X_T - X_0 &= \mu T + \sigma B_T \\ X_T &= X_0 + \mu T + \sigma B_T \end{aligned}$$

The distribution of X_T is $N(X_0 + \mu T, \sigma^2 T)$ since:

$$\begin{aligned} \mathbb{E}[X_T] &= X_0 + \mu T \\ Var[X_T] &= \sigma^2 T \\ Cov[X_T, X_S] &= \sigma^2 S, \text{ for } S < T. \end{aligned}$$

Monte Carlo simulation of SDEs

Euler method (discretization of time)

Consider the following SDE:

$$dX_t = a(t, X_t)dt + b(t, X_t)dB_t, \text{ with } X_0 = x_0.$$

1. Let $[0, T]$ be the time horizon over which we simulate. Let $\Delta = T/N$, N large.
2. Set $Y_0 = x_0$.
3. Recursively define Y_n for $1 \leq n \leq N$ as

$$Y_{n+1} = Y_n + a(Y_n)\Delta + b(Y_n)\sqrt{\Delta}U, \text{ where } U \sim N(0, 1).$$