Deriving the Iterated Extended Kalman Filter From Bayes' Rule

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Assume that we have some estimate of the state variable x, denoted by $\hat{x}$. Further, assume that the probability distribution of the state vector is normally distributed with mean equal to our estimate, and with covariance P. That is

$$\mathcal{P}(x) \sim \mathcal{N}(\hat{x}, P). \tag{1}$$

After acquiring a new measurement z, we would like to update our estimate of x. Assume that the measurement is given by some function of the current state plus some observation noise

$$z = h(x) + \nu \tag{2}$$

where the observation noise ν is zero mean with covariance R so that

$$\mathcal{P}(\nu) \sim \mathcal{N}(0, R). \tag{3}$$

The estimation problem is to determine the value of the state vector x^* which maximizes $\mathcal{P}(x|z)$.

$$\begin{aligned} x^* &= Argmax_x\{\mathcal{P}(x|z)\} \\ &= Argmax_x\{k\mathcal{P}(z|x)\mathcal{P}(x)\} \\ &= Argmin_x\{-Log(\mathcal{P}(z|x)\mathcal{P}(x))\} \\ &= Argmin_x\{\frac{1}{2}(z - h(x))^T R^{-1}(z - h(x) + \frac{1}{2}(x - \hat{x})P^{-1}(x - \hat{x})\} \end{aligned}$$

Finding the estimate x^* which maximizes the posterior probability then becomes equivalent to finding the minimum of the quadratic surface given by

$$f(x) = \frac{1}{2}(z - h(x))^T R^{-1}(z - h(x) + \frac{1}{2}(x - \hat{x})P^{-1}(x - \hat{x}) \tag{4}$$

We can find the minimum of f using Newton-Raphson, by computing the first and second derivatives of f with respect to x,

$$\begin{aligned}
\nabla_x f(x) &= -H_x^T R^{-1}(z - h(x)) + P^{-1}(x - \hat{x}) && (5)\\
\nabla_x^2 f(x) &= H_x^T R^{-1} H_x + P^{-1} && (6)
\end{aligned}$$

where H_x is the Jacobian of the measurement function $h(x)$ and we have made the assumption that $H_{xx} = \nabla_x^2 h(x)$ is small. The first observation we make is that the second derivative of f gives the covariance of the updated estimate, so

$$(P')^{-1} = H_x^T R^{-1} H_x + P^{-1} \tag{7}$$

which is the well known Kalman update equation for covariance in the Kalman filter.

Newton-Raphson minimization of (4) requires iteratively updating x by an amount d given by

$$\nabla_x^2 f d = -\nabla_x f \tag{8}$$

or

$$d = -(\nabla_x^2 f)^{-1} \nabla_x f. \tag{9}$$

Using (4), we find

$$d = P' H_x^T R^{-1}(z - h(x)) - P' P^{-1}(x - \hat{x}). \tag{10}$$

The updated state estimate $\hat{x}'$ then becomes

$$\hat{x}' = \hat{x} + P' H_x^T R^{-1}(z - h(x)) - P' P^{-1}(x - \hat{x}). \tag{11}$$

This is the update equation for the Iterated Extended Kalman Filter.