AUTOMATIC CONTROL
KTH
Applied Estimation EL2320
Exam 8:00-12:00 January 18, 2016

Aids: None, no books, no notes, nor calculators

Observe:

- Name and person number on every page
- Answers should be in English (or Swedish)
- Only write on one side of the sheets
- Specify the total number of handed in pages on the cover
- Be careful to label each answer with the question number and letter
- Be careful to only state things you know are true as one incorrect statement can negate an otherwise correct or partially correct answer.
- That said also be sure to answer the question asked.
- Motivate answer and clearly state any additional assumptions you may need to make.

Results: The results will be available on 'mina sidor', in a few weeks
Any problems with the recording of grades, STEX Studerandexpeditionen, Osquldasv. 10.

Responsible: John Folkesson, 08-790-6201

Good to know (what all the symbols mean you should be able to figure, this is just in case you forgot how to put them together):

$$G(\mathbf{x}, \mu, \Sigma) = \frac{1}{\sqrt{|2\pi\Sigma|}} \exp \frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \tag{1}$$

Typical Non-linear system equations:

$\mathbf{x}_t = \mathbf{g}(\mathbf{u}_t, \mathbf{x}_{t-1}) + \varepsilon_t$

$\mathbf{z}_t = \mathbf{h}(\mathbf{x}_t) + \delta_t$

'Kalman Gain'

$K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$

Predict phase:

$\bar{\Sigma}_t = R_t + G_t \Sigma_{t-1} G_t^T$

$\bar{\mu}_{\mathbf{t}} = \mathbf{g}(\mathbf{u}_t, \mu_{t-1})$

Update:

$\Sigma_t = \bar{\Sigma}_t - K_t H_t \bar{\Sigma}_t$

$\mu_t = \bar{\mu}_{\mathbf{t}} + K_t \eta_t$

Questions

1. (5p) You flip a coin three times and count the number of heads. This number corresponds to a discrete random variable X that can take on the values {0,1,2,3}:

   a) What is the probability of each of these four values? (hint, list the possible outcomes as 'triples' such as head, head, tail) (1p)
   ans.$P(0) = \frac{1}{8}$, $P(1) = \frac{3}{8}$, $P(2) = \frac{3}{8}$, $P(1) = \frac{1}{8}$

   b) What is the expectation value of X? (1 p)

   ans. $E[X] = 1.5$

   c) What is the expectation value of X^2? (1 p)

   ans. $E[X^2] = \frac{1}{8}[1 * 0 + 3 * 1 + 3 * 4 + 1 * 9] = 3.0$

   d) What is the variance of X? (1p)
   ans. $E[X^2] - E[X]^2 = 3 - 2.25 = 0.75$

   e) You now throw a die and get a number from 1 to 6 which corresponds to a second discrete random variable Y. What is the probability that X=2 and Y=2? (1p)
   Since the tosses are independent, $P(X = 2, Y = 2) = P(X = 2)P(Y = 2) = \frac{3}{8*6} = \frac{1}{16}$

2. (7p) This question is on Bayesian inference. Given a prior distribution over a random variables X and Y :
$P(Y=1)=\frac{10}{16}$; $P(Y=2)=\frac{6}{16}$; $P(X=1)=\frac{3}{4}$; $P(X=2)=\frac{1}{4}$,
and the following conditional probabilities:
$P(Y=1|X=1)=\frac{3}{4}$; $P(Y=1|X=2)=\frac{1}{4}$;
$P(Y=2|X=1)=\frac{1}{4}$; $P(Y=2|X=2)=\frac{3}{4}$,
a) are X and Y independent? (1p)
ans. no
b) You now measure Y and find $Y=1$. What is your belief of X now? (2p)
ans. $P(X=1|Y=1)=\frac{P(Y=1|X=1)P(X=1)}{P(Y=1)} \propto P(Y=1|X=1)P(X=1)=\frac{3}{4}\frac{3}{4}$
$P(X=2|Y=1)=\frac{P(Y=1|X=2)P(X=2)}{P(Y=1)} \propto P(Y=1|X=2)P(X=2)=\frac{1}{4}\frac{1}{4}$
$P(X=1|Y=1)=\frac{9}{10}$, $P(X=2|Y=1)=\frac{1}{10}$

c) Now you measure another random variable Z with
$P(Z=1|X=1,Y=1)=\frac{3}{4}$, $P(Z=1|X=2,Y=1)=\frac{1}{4}$,
$P(Z=2|X=1,Y=1)=\frac{1}{4}$, $P(Z=2|X=2,Y=1)=\frac{3}{4}$,
and find $Z=1$. You do not know how Z and Y are related but you do know that Z can only take on values 1 or 2 and that it is not completely determined by Y. That is, knowing Y might give some indication of Z but does not tell me with certainty what Z is.
What is your belief of X now? (2p)
ans. $P(X=1|Z=1,Y=1) \propto P(Z=1|X=1,Y=1)P(X=1|Y=1)=\frac{3}{4}\frac{9}{10}$
$P(X=2|Z=1,Y=1) \propto P(Z=1|X=2,Y=1)P(X=2|Y=1)=\frac{1}{4}\frac{1}{10}$
$P(X=1|Z=1,Y=1)=\frac{27}{28}$, $P(X=2|Z=1,Y=1)=\frac{1}{28}$

d) Now you measure yet another random variable W with
$P(W=1|X=1,Y=1)=\frac{3}{4}$, $P(W=1|X=2,Y=1)=\frac{1}{4}$,
$P(W=2|X=1,Y=1)=\frac{1}{4}$, $P(W=2|X=2,Y=1)=\frac{3}{4}$,
and find $W=1$. You do not know how W and Y are related but you do know that W can only take on values 1 or 2 and that it is not completely determined by Y. You also know that Z determines the value of W with certainty. So the value of Z says what the value of W is.
What is your belief of X now? (2p)
ans. $P(X=1|W=1,Z=1,Y=1)=\frac{27}{28}$, $P(X=2|W=1,Z=1,Y=1)=\frac{1}{28}$

3. (8p) This question is on the Kalman Filter. You are filling your car's 20 liter gas tank at an average rate of 0.1 $\frac{l}{s}$ (liters per second). That rate is not constant and you estimate that the variance over one second is 0.01 $(l)^2$. The pump does not have the normal sensor to stop the flow when the tank is fulls so you need to estimate the level in the tank to avoid spilling gasoline on the ground. Let x_i be the number of liters of gas in the tank after i seconds. You think that you started with $x_0 = 1.0$ with a standard deviation of 1 liter. You decide to estimate using a Kalman filter with intervals of one second between predictions.

   a) Set up the prediction equations, define all parameters and give the values for them. (1p)

   $\bar{\mu}_i = \mu_{i-1} + 0.1$; $\mu_0 = 1$; $\bar{\Sigma}_i = \Sigma_{i-1} + R_i$; $\Sigma_0 = 1$; $R_i = 0.01$;

   b) What is the posteriori distribution over x after 1 second of filling the tank? What is it after 5 seconds? (2p)

   ans. after 1 second it is $G(x_1, 1.1, 1.01)$
   after 5 seconds it is $G(x_5, 1.5, 1.05)$

   c) Your partner is in the car and is reading out the gas level on the fuel gauge every 5 seconds. These readings, while having no delay, are uncertain with Gaussian noise of variance 0.01. The first reading is 1.6 liters. What are the update equations with all parameters specified? (2p)

   ans. For i a multiple of 5 and the reading of z_i:
   $K_i = \bar{\Sigma}_i(\bar{\Sigma}_i + Q_i)^{-1}$; $Q_i = 0.01$;
   $\mu_i = \bar{\mu}_i + K_i(z_i - \bar{\mu}_i)$; $z_5 = 1.6$
   $\Sigma_i = (1 - K_i)\bar{\Sigma}_i$
   For other i the updated mean and variance are equal to the predicted ('bar') values.

   d) What is posteriori distribution now? You do not need to carry out all the multiplications and divisions. Just an expression for each parameter with all the numbers replacing the symbols. (1p) (1p)

   $K_5 = 1.05/(1.06)$; $\mu_5 = 1.5{+}(1.05/1.06)(0.1) = (1.590{+}0.105)/1.06 = 1.695/1.06$;
   $\Sigma_5 = (0.01/1.06)1.05 = 0.0105/1.06$;
   $G(x_5, 1.695/1.06, 1.05/106)$

   e) The probability of being within 2 standard deviations of the mean of a Gaussian is about 95%. You decide to stop pumping when there is more than a 2.5%

chance that the level is at the top of the tank. What does this criteria mean in terms of your estimated parameters after i seconds? (2p)

ans. stop when $\mu_i + 2 * \sqrt(\Sigma_i) > 20$.

4. (12 p) These questions are on the particle filter, PF, used for state estimation.

a) You decide to swap the Kalman Filter for a PF for the previous problem on filling the gas tank. You have a very long list of random numbers drawn from a standard normal distribution (ie. mean of 0 and variance of 1) which we will call 'L_1' and another drawn from a uniform distribution between 0 and 1 called 'L_2'. How do you generate the initial set of particles using these lists? What do you store for each particle? What do each of these particles represent? (2p)

ans. You need to draw samples for x_0 from $G(x_0, 1, 1)$. Since L_1 is a standard Normal it has the right variance, 1.0 but its mean is 0 so we add 1.0 to the first M numbers from L_1 for M particles. These M numbers then represent possible values for the number of liters in the tank at the start.

$x_0^j = r_1^j + 1$ where j is the particle index from 1, M and r_1^j is the jth number on L_1.

b) How will you generate the particle set after 1 second of pumping? Again be specific and use your random numbers. (2p)

ans. Since the motion equation has Gaussian noise we again use L_1 and use the next M numbers on that list. These we need to multiply by the standard deviation, 0.1, and add the mean 0.1 plus the value for each particle.

$x_1^j = x_0^j + 0.1 + 0.1 r_1^{j+M}$.

c) How and when will you first weight each particle? The measurement was as before 1.6 with variance 0.01. (2p)

You will weight the particles after the 5th diffusion step. The weight will be given by:

ans. $w_5^j \propto G(x_5^j, 1.6, 0.01)$; $\sum_i w_5^j = 1$.

d) If you have M particles how will you do the re-sampling step? Explain completely and use the random numbers. (2p)

For example one could use the first M numbers of L_2 and for each number chose the smallest k such that:

$\sum_{i=1}^k w_5^j <= r_2^j$. Then re-sample the kth particle.

e) Why do we re-sample? (2p)

ans. To avoid having samples with near zero weights. To focus the particles in the more likely regions. ...

f) If we wanted to compute a criteria for stopping based on the probability of overflow, how might we do that? (2p)

ans. If p was the threshold probability of overflow for stopping then we could stop when p*M of the particles were bigger than 20.

5. (7p) This problem is about the Extended Kalman Filter, EKF and refers to the formulas at the start of the exam.

a) If the prior distribution (starting belief), is in fact Gaussian what problems might it still problems cause for the early state estimation using the EKF? Explain in terms of the parameters of the prior distribution and the motion and measurement models. (2p)

ans. If the variance of the initial belief is large relative to the non-linearity in the measurement model then the linear approximation will not be accurate in some regions where there was significant probability. This will lead to inconsistency.

b) In the formulas at the start of the exam, how are the components $G_{t,ij}$ of G_t computed? (1p)

ans. $G_{t,ij} = \frac{\partial \mathbf{g}_i}{\partial x_{t-1,j}}$

c) How are the $H_{t,ij}$ of H_t computed? (1p)

ans. $H_{t,ij} = \frac{\partial \mathbf{h}_i}{\partial x_{t,j}}$

d) What is the assumed pdf of δ_t (1p)

ans. $G(\delta_t, 0, Q_t)$

e) When tuning our filter why might we decrease R_t? What effect does that have on the estimated state trajectory? (2p)

We might decreasing R_t to increase our confidence in the motion model. It will have the effect of smoothing the trajectory more.

6. (11p)

a) When is the particle filter a better choice of estimator than the extended Kalman filter? (2 pt.)

Ans. When the distribution is non-Gaussian.

b) What does it mean when we say the estimator is inconsistent? (1p)

ans. This means that the posteriori distribution is not the best fit to the true posteriori distribution. For example the mean or the variance might be wrong.

c) What distinguishes an information filter from the Kalman filter? (2 pt.)

ans. The information filter does all the calculation using the information matrix rather than the covariance matrix. The information matrix is the inverse of the covariance matrix. It also does not directly compute the mean.

d) How does the iterated Extended Kalman Filter, IEKF improve consistency over the EKF? (2p)

ans. By iterating the update of the mean it is able to find a better place to linearize the measurement function.

e) Why might the Unscented Kalman Filter improve estimates over an EKF? (2p)

ans. It does not rely on a linearization at a single point but rather numerically computes the integrals and expectations using several points. Thus it can better handle non-linearities.

f) When using FastSLAM, why do we not need to compute correlations between landmarks in the Gaussian over the map coordinates? (2p)

ans. Because the landmark positions are uncorrelated given the path of the robot and each particle represents a given path.