AUTOMATIC CONTROL
KTH
Applied Estimation EL2320/EL3320
Exam 14:00-18:00 Jan 9, 2017

Aids: None, no books, no notes, nor calculators

Observe:

- Name and person number on every page
- Answers should be in English (or Swedish)
- Only write on one side of the sheets
- Specify the total number of handed in pages on the cover
- Be careful to label each answer with the question number and letter
- Be careful to only state things you know are true as one incorrect statement can negate an otherwise correct or partially correct answer.
- That said also be sure to answer the question asked.
- Motivate answer and clearly state any additional assumptions you may need to make.

Results: The results will be available on 'mina sidor', in a few weeks
Any problems with the recording of grades, STEX Studerandexpeditionen, Osquldasv. 10.

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Good to know (what all the symbols mean you should be able to figure, this is just in case you forgot how to put them together):

$$G(\mathbf{x}, \mu, \Sigma) = \frac{1}{\sqrt{|2\pi\Sigma|}} \exp \frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \tag{1}$$

$\mathbf{x} \sim N(\mu, \Sigma)$ means $\mathbf{x}$ has the pdf of $G(\mathbf{x}, \mu, \Sigma)$

Typical Non-linear system equations:

$\mathbf{x}_t = \mathbf{g}(\mathbf{u}_t, \mathbf{x}_{t-1}) + \varepsilon_t$

$\mathbf{z}_t = \mathbf{h}(\mathbf{x}_t) + \delta_t$

'Kalman Gain'

$K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$

Predict phase:

$\bar{\Sigma}_t = R_t + G_t \Sigma_{t-1} G_t^T$

$\bar{\mu}_{\mathbf{t}} = \mathbf{g}(\mathbf{u}_t, \mu_{t-1})$

Update:

$\Sigma_t = \bar{\Sigma}_t - K_t H_t \bar{\Sigma}_t$

$\mu_t = \bar{\mu}_{\mathbf{t}} + K_t \eta_t$

Questions

1. (6p) A joint probability density function is given by:

   $p(x, y) = A(x^2y)$ for x and y in the interval [0,1]

   a) What is the probability density for x? $p(x) =?$ (1p) ans. $3x^2$

   b) What is the expectation value of x? $E(x) =?$ (1p) ans. $\frac{3}{4}$

   c) What is the variance of x? (1p) ans. $\frac{3}{80}$

   d) What is the conditional probability, $p(x|y) =?$ (1p) ans. $3x^2$

   e) Are x and y independent (1p)? ans. yes

   f) What is the covariance of x and y, C_{xy} (1p) ans. 0?

2. (5p) The prior probability density of a random variable x is uniform for $x \in (-2, 2)$ and zero otherwise. A binary measurement z takes on the two values 0 or 1 and depends on x according to the measurement model:

   $P(z = 1|x < 0) = 0.75$ $P(z = 1|x \geq 0) = 0.5$

   a) What is $P(z = 0|x < 0)$? (1p) ans. 0.25

   b) A measurement gives z=0. What is the probability that $(x < 0)$ after that measurement? (2p) ans. $\frac{1}{3}$

   c) What is the new belief of x given the measurement? (2p)
   ans. $p(x|z = 0) = \frac{1}{6} : x \in (-2, 0)$ and $\frac{1}{3} : x \in [0, 2)$

3. (6p) This question is on the Kalman Filter or KF. A wagon is rolling down and incline on rails. Its state is given by a distance along the track, its speed, and its acceleration, (x, v, a). The initial state at k=0 is (0, 0, 1). The uncertainty of this state is not significant. The motion model is given by:

$$x_{k+1} = x_k + v_k + 0.5a_k$$

$$v_{k+1} = v_k + a_k$$
$$a_{k+1} = a_k + \eta$$

where $\eta \sim N(0, 0.25)$

a) What is the estimated state and the covariance matrix, at k=1. (3p)
ans. $\bar{\mu}_1 = (0.5, 1, 1)^T$

$$\bar{\Sigma}_1 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b) A direct measurement $z = 1.1$ is made of the speed at k=1 with a variance of 0.01. What is the posteriori distribution? You do not need to do the final arithmatic but all symbols should have been replaced by numbers and not too messy. (3p)
ans. $\mu_1 = (0.5, 1, 1)^T$

$$\Sigma_1 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. (12 p) These questions are on the particle filter, PF, used for state estimation. We decide to swap the Kalman Filter for a PF for the previous problem. We draw 3 particles from the initial distribution at k=0.

a) What are the states for each of these 3 particles at k=0? (2p)
ans. all 3 are the same and equal to (0, 0, 1).

b) Describe the first step of the particle filter taking the estimate from the initial set of 3 particles at $k = 0$ through to a set of 3 particles at $k = 1$. You will need the following 3 samples (imagined to be) taken from the standard normal distribution, $N(0, 1)$: $\{0, 0.1, 1\}$. (3p)
ans. the 3 particles have states:
$\mathbf{x}^1 = (0.5, 1, 1)$, $\mathbf{x}^2 = (0.5, 1, 1.05)$, and $\mathbf{x}^3 = (0.5, 1, 1.5)$.

c) Describe how do you now use the measurement? (3p)
ans. The gaussian measurement model will be used to find the weight:
$w_i \propto P(z = 1.1|\mathbf{x}^i) = N(1.1 - x_2^i, .01)$ which are then normalized to sum to one and used in the resampling step. In this case since the measurement depends only on the speed and all three particles have the same speed all weights will be equal to one third.

d) Assume the weights of the three particles look like $w_1 = 3, w_2 = 4, w_3 = 1$. (They really do not but use these weights here.) You are given the following three numbers drawn from a uniform distribution over $(0, 1)$: (.5, .1, .4). What is the particle set after resampling both with the 'vanilla' (or multinomial) method and the systematic (or low variance) method? (4p) ans vanilla: x^2, x^1, x^2; systematic: x^1, x^2, x^2

5. (6p) These questions are on the Extended Kalman Filter or EKF. The formulas for the Extended Kalman Filter appear at the start of the exam. Use that notation when answering these questions.

We now change the motion model of the previous two questions by:

$a_{k+1} = a_k - v_k^2 + \eta$

a) What is the probability density function for the EKF state estimator after the predict step at k=1? Show your work using the notation from page 2. (3p)

ans. $G = \begin{pmatrix} 1 & 1 & 0.5 \\ 0 & 1 & 1 \\ 0 & -2v_k & 1 \end{pmatrix}$ For k=0 $v_k = 0$ so this is exatly the same linear matrix used in question 3. The noise model is also the same so the answer is identical.

b) Now the measurement of the velocity is also non-linear:

$z_k = v_k^2 + \epsilon$

where $\epsilon \sim N(0, 0.01)$

The measurement is $z_1 = 1.21$. What is the probability density function for the EKF state estimator after the update step at k=1? (3p)

ans. $H_1 = (0, 2v_k, 0) = (0, 2, 0)$ This is 2 times the linear matrix from question 3. So the answer is found the same way and it turns out to be the same since the Kalman Gain is agian 0.

6. (15p)

a) What is particle deprivation? (1p)
ans. There are no particles in regions with significant probality of containing the state.

b) What can cause the EKF to become inconsistent? (1p)
ans. The non-linearities are large over regions within the uncertainty. We linearise at a state far from the MAP state.

c) What is maximum likelihood data association? (1p)
ans. We chose the data association, c_i that maximizes $p(z|x, c_i)$.

d) How would you find the MAP estimate if you are using a particle filter? (2p)
ans. First do some sort of density extration such as approximating the particle set with a Guassian or a set of Gaussian Kernals. Then find the state that maximizes that density.

e) What advantage does an Unscented Kalman filter, UKF have over the extended Kalman filter, EKF? (1p)
ans. The EKF bases the estimate on a linear approximation at a single point in state space while the UKF passes several points through the non-linearity and thus samples the nonlinearity over a wider domain.

f) Describe the use of importance weights in terms of the proposal and target distributions. (2p)
ans. The weights are propotional to the ratio of the target to the proposal disribution evaluated at the particle's state. $w^i = \frac{target}{proposal}_{\mathbf{x}=\mathbf{x}^i}$

g) What does the Iterative Extended Kalman filter iEKF try to do better than the EKF and how does it do that? (2p)
ans. It trys to find a better state at which to linearize the measurement model. It does this by repeating the update where each time it uses the previous updated state as the new linearization point.

h) A mixture of Gaussians has a weight for each Gaussian component. How are these weights set? (1p)
ans. By the likelihoods of the observations conditional on the Gaussian component.

i) How can ambiguous data association in robot localization lead to distributions with two peaks (modes)? (2p)
ans. Two features that are separated by a distance greater than the measurement noise but less than the uncertainty in the prior state both explain the measurement about equally well. Each of these two data associations give a

tight distribution that does not overlap the other hypothesis much. Thus not knowing which is correct we have two peaks.

j) What is Gaussian Kernal density extraction? (1p)
ans. Each particle is associated with a Gaussian component in a mixure of Gauassians where all components have equal weight. The mean of this Gaussian component is the particle state and the covariance is chosen to give a smooth sum for the mixture.

k) How do the particle filter and histogram filter differ in representing the probability density of the belief. (1p) ans. The histogram filter subdivides the state space into a finite number of regions and computes the probability that the state lies in each region. The particle filter draws a set of particles from the belief and thus has many densely packed particles in regions of high probability and none where probabiliy is low.