AUTOMATIC CONTROL
KTH
Applied Estimation EL2320/EL3320
Exam 14:00-18:00 April 10, 2017

Aids: None, no books, no notes, nor calculators

Observe:

- Name and person number on every page
- Answers should be in English (or Swedish)
- Only write on one side of the sheets
- Specify the total number of handed in pages on the cover
- Be careful to label each answer with the question number and letter
- Be careful to only state things you know are true as one incorrect statement can negate an otherwise correct or partially correct answer.
- That said also be sure to answer the question asked.
- Motivate answer and clearly state any additional assumptions you may need to make.

Results: The results will be available on 'mina sidor', in a few weeks
Any problems with the recording of grades, STEX Studerandexpeditionen, Osquldasv. 10.

Responsible: John Folkesson, 08-790-6201

Good to know (what all the symbols mean you should be able to figure, this is just in case you forgot how to put them together):

$$G(\mathbf{x}, \mu, \Sigma) = \frac{1}{\sqrt{|2\pi\Sigma|}} \exp \frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \tag{1}$$

$\mathbf{x} \sim N(\mu, \Sigma)$ means $\mathbf{x}$ has the pdf of $G(\mathbf{x}, \mu, \Sigma)$

Typical Non-linear system equations:

$\mathbf{x}_t = \mathbf{g}(\mathbf{u}_t, \mathbf{x}_{t-1}) + \varepsilon_t$

$\mathbf{z}_t = \mathbf{h}(\mathbf{x}_t) + \delta_t$

'Kalman Gain'

$K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$

Predict phase:

$\bar{\Sigma}_t = R_t + G_t \Sigma_{t-1} G_t^T$

$\bar{\mu}_\mathbf{t} = \mathbf{g}(\mathbf{u}_t, \mu_{t-1})$

Update:

$\Sigma_t = \bar{\Sigma}_t - K_t H_t \bar{\Sigma}_t$

$\mu_t = \bar{\mu}_\mathbf{t} + K_t \eta_t$

Questions

1. (6p) A joint probability over random variables x and y is given by:

$P(x,y)$	$y=0$	$y=1$
$x=0$	0.01	0.09
$x=1$	0.09	0.81

a) What is the probability for x? $P(x) =?$ (1p)
ans. P(x=0)=.1, P(x=1)=.9

b) What is the expectation value of x? $E(x) =?$ (1p)
ans. .9

c) What is the variance of x? (1p)
ans. .09

d) What is the conditional probability, $P(x|y) =?$ (1p)
ans. P(x=0—y=0)=P(x=0—y=1)=.1, P(x=1—y=0)=P(x=1—y=1)=.9

e) What is the covariance of x and y, C_{xy} (2p)? ans. 0

2. (5p) A Histogram filter in a one dimensional space has bins of unit width centered on each integer from -10 to 10. The probability of bin i is notated as p_i. These bins start with $p_{-5} = .25$, $p_0 = .5$, $p_5 = .25$ and all other bins with 0 probability.

A measurement is taken with measurement model:

$P(z\|i)$	z =0	z =1
i = -10	0.1	0.9
i = -9	0.2	0.8
i = -8	0.3	0.7
i = -7	0.4	0.6
i = -6	0.5	0.5
i = -5	0.6	0.4
i = -4	0.7	0.3
i = -3	0.8	0.2
i = -2	0.9	0.1
i = -1	1	0
$i = 0$	1	0
$i = 1$	1	0
$i = 2$	0.8	0.2
$i = 3$	0.7	0.3
$i = 4$	0.6	0.4
$i = 5$	0.5	0.5
$i = 6$	0.3	0.7
$i = 7$	0.2	0.8
$i = 8$	0.1	0.9
$i = 9$	0	1

a) What is the posteriori if the measurement is z=1? (3p)
P(x=-5—z=1)=5/9, P(x=5—z=1)=4/9, all others are 0.

b) The transition model from state i to state j is

$P(j = i - 1|i) = .25$
$P(j = i|i) = .5$
$P(j = i + 1|i) = .25$

for i between -9 and 9.

If we disregard the measurement in (a) and apply this transition model to the original priori, what is the posteriori? (2p) ans. P(x=-6)=P(x=-4)=P(x=4)=P(x=6)=1/16, P(x=-5)=P(x=-1)=P(x=1)=P(x=5)=1/8, P(x=0)=1/4, rest are 0

3. (6p) This question is on the Kalman Filter. A signal is being estimated. The initial state at k=0 is $x_0 = 0$, with a variance or .01. The motion model is given by:

   $x_{k+1} = x_k + \sin(k\pi/2) + \eta$
   where $\eta \sim N(0, 0.15)$

   a) What is the estimated state and the covariance matrix, at k=1. (3p)
   ans. $\bar{\mu}_1 = 0$, $\bar{\Sigma}_1 = .16$

   b) The measurement model is

   $p(z_k|x_k) = x_k + \epsilon$
   where $\epsilon \sim N(0, 0.09)$

   A measurement of $z_1 = .2$ is made. What is the posteriori distribution? (3p)
   Gaussian with mean $\mu_1 = .2(16/25)$, $\Sigma_1 = .16(9/25)$

4. (12 p) These questions are on the particle filter, PF, used for state estimation.

   a) During the re-sample step using systematic re-sampling a random number of 0.2 is chosen from an uniform distribution over the interval (0,1). The number of particles is 10. The particle weights are

   {0.01, 0.1, 0.1, 0.1, 0.3, 0.02, 0.07, 0.1, 0.1, 0.1 },

   corresponding to particle 1 to 10. How many of each particle remain after re-sampling? (3p)
   ans. 0, 1, 1, 1, 3, 1, 0, 1, 1, 1

   b) If instead of systematic re-sampling in (a) you used the 0.2 as the first random number using vanilla re-sampling, which particle would be sampled. (3p)
   ans. 3

   c) Describe how the weights are computed for each particle? (3p)
   $w_i = p(z|x_i)$

   d) We have a spread particle set and take a very accurate measurement which actually can only be explained by states in a very small region of state space. What problem will the standard PF have? What might be a way to modify it to fix the problem? (3p) ans. It may have no particles in the small region. This could be fixed by changing the proposal distribution to be one that samples states that explain the measurements. Then using the prior particle set to deine a density that can be used to set importance weights.

5. (6p) These questions are on the Extended Kalman Filter or EKF. The state vector is given by $(x_t, y_t)^T$.

$z_t = x_t^2 + \epsilon$
where $\epsilon \sim N(0,1)$

$x_t = x_{t-1} + y_{t-1} + \eta_x$
$y_t = y_{t-1} - .1y_{t-1}^3 + \eta_y$
where $\eta_x \sim N(0,1)$ and $\eta_y \sim N(0,1)$

$\bar{\mu}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\bar{\Sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$z_0 = 3$

The formulas for the Extended Kalman Filter appear at the start of the exam. Use that notation when answering these questions. Explictily write the values for all the matricies and vectors used. When giving your answers complete all the arithmatic.

a) What is the belief after the update at t=0? (3p) ans. $H_0 = (2,0)$, $K = .4(1,0)^T$, $\mu_0 = (1.8,1)^T$ $\Sigma_0 = \begin{pmatrix} .2 & 0 \\ 0 & 1 \end{pmatrix}$

b) What is the belief after the prediction to t=1? (3p) $\mu_1 = (.28, 0.9)^T$, $G_1 = \begin{pmatrix} 1 & 1 \\ 0 & .7 \end{pmatrix}$, $\bar{\Sigma}_1 = \begin{pmatrix} 2.2 & .7 \\ 0.7 & 1.49 \end{pmatrix}$

6. (15p)

a) If the data association for a series of measurements $\{z_1, z_2, ...z_n\}$ is given by $\{c_1, c_2, ...c_n\}$ write an expression for the maximum likelihood data association for the last measurement $\hat{c}_n =$?. (2p)
ans. $\hat{c}_n = argmax_{c_n} p(z_n|c_1, ...c_n, z_1, ...z_{n-1})$

b) What are proposal and target distribution in the particle filter? (1p)
ans. The samples are drawn from the proposal distribution. The desired (true) distribution is the target.

c) What is the factorization of the posteriori that reduces the number of particles needed in the Rao-Blackwellized Particle filter? (hint, it is uded in FASTSLAM) Explain your notation. (2p)
ans. $p(x_1, ...x_n) = p(x_1, ..., x_m|x_{m+1}, ...x_n)p(x_{m+1}, ...x_n)$

d) Write out the form of the probability density when using a Gaussian kernel density extraction on a set of particles. Explain all the terms and notation. (2p)
ans. $p(\mathbf{x}) = (1/M)\sum_i G(\mathbf{x}, \mathbf{x}^i, \sigma)$

e) What is the trade off when choosing the number of particles in the particle filter? (2p)
ans. computations vs accuracy

f) Conceptually how does the Kullback-Leibler divergence, KLD, sampling scheme work? (That is, what is it that leads to a bigger or small M.) (2p)
ans. It measures the spread of the distribution, more spread gives higher M.

g) For the robot localization problem using maximum likelihood data association and a mahalanobis test for outlier detection, how might an overconfident estimate create a problem? (2p)
ans. Too small a covariance will lead to too high a mahalanobis distance which will lead to too many measurements being discarded as outliers.

h) How do the iterative Extended Kalman Filter, iEKF, and Unscented Kalman Filter UKF differ? (2p)
ans. The iEKF finds a good point to linearize around by iteratively repeating the update step. The UKF estimates the non-linarity by passing several separated points through the non-linear function and measuring the spead of the resulting states.