AUTOMATIC CONTROL
KTH
Applied Estimation EL2320/EL3320
Exam 9:00-13:00 Jan 13, 2018

Aids: None, no books, no notes, nor calculators

Observe:

- Name and person number on every page
- Answers should be in English (or Swedish)
- Only write on one side of the sheets
- Be careful to label each answer with the question number and letter
- Be careful to only state things you know are true as one incorrect statement can negate an otherwise correct or partially correct answer.
- That said also be sure to answer the question asked.
- Motivate answer and clearly state any additional assumptions you may need to make.

Results: The results will be available on 'mina sidor', in a few weeks
Any problems with the recording of grades, STEX Studerandexpeditionen, Osquldasv. 10.

Responsible: John Folkesson, 08-790-6201

Good to know (what all the symbols mean you should be able to figure, this is just in case you forgot how to put them together):

$$G(\mathbf{x}, \mu, \Sigma) = \frac{1}{\sqrt{|2\pi\Sigma|}} \exp \frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \tag{1}$$

$\mathbf{x} \sim N(\mu, \Sigma)$ means $\mathbf{x}$ has the pdf of $G(\mathbf{x}, \mu, \Sigma)$

Typical Non-linear system equations:

$\mathbf{x}_t = \mathbf{g}(\mathbf{u}_t, \mathbf{x}_{t-1}) + \varepsilon_t$

$\mathbf{z}_t = \mathbf{h}(\mathbf{x}_t) + \delta_t$

'Kalman Gain'

$K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$

Predict phase:

$\bar{\Sigma}_t = R_t + G_t \Sigma_{t-1} G_t^T$

$\bar{\mu}_{\mathbf{t}} = \mathbf{g}(\mathbf{u}_t, \mu_{t-1})$

Update:

$\Sigma_t = \bar{\Sigma}_t - K_t H_t \bar{\Sigma}_t$

$\mu_t = \bar{\mu}_{\mathbf{t}} + K_t \eta_t$

Questions

1. (6p) A conditional probability over x given y is given by:

$P(x\|y)$	y=-1	y=0	y=1
x=-1	.1	.8	?
x=0	.1	?	.15
x=1	?	.15	.75

The probability of y, P(y), is uniform over the three values -1, 0, and 1.

a) Complete the table by filling in the ? with values? (1p)

$P(x\|y)$	y=-1	y=0	y=1
x=-1	.1	.8	.1
x=0	.1	.05	.15
x=1	.8	.15	.75

b) What is the joint probability over x and y, $P(x, y)$? (1p)

$60 \cdot P(x, y)$	-1	y=0	y=1
x=-1	2	16	2
x=0	2	1	3
x=1	16	3	15

c) If we know that y is not -1 what is the probability that x is 0,
ie. $P(x = 0|y \neq -1)$? (1p)

	$P(x\|y \neq -1)$
x=-1	.45
x=0	.1
x=1	.45

d) What is the expectation value of x if $y \neq -1$? (1p)
0

e) What is the variance of x if $y \neq -1$? (1p)
0.9

f) Are x and y independent (1p)?
No

2. (5p) You are on vacation. The place you are staying has a known weather pattern where:

   - 8 of 10 days are nice (ie. sunny).
   - If the previous day was nice then the pattern is 9 of 10.
   - If the day is going to be nice there is a 50% chance that there are some clouds at sunrise regardless of the weather the day before.
   - If the day is not going to be nice then there are always morning clouds.

   You can let N_0 and N_1 be the symbols for nice yesterday and nice today respectively. Let C indicate that there are some clouds in the morning today. $\neg N_0$ indicates not nice yesterday and so on. So we do not consider morning clouds as 'the day is ruined' but rather it still might be a nice day. You can assume that the chance of clouds in the morning does not depend on the weather the day before given the weather for the rest of the day is either nice or not, (ie. morning clouds are a markov process with respect to the day's weather state).

   You just woke up at sunrise the morning after a nice day.

   a) What is the chance that today will be nice, $P(N_1|N_0) =$? (1p)
   0.9

   b) You look outside and see that there are some clouds. What is the probability of a nice day today, $P(N_1|C, N_0) =$? (2p)
   $\frac{9}{11} = .818188182$.
   1 point for correct Bayes rule but not understanding to normalize to find the denominator

   c) What is the probability of a nice day after a not nice day $P(N_1|\neg N_0) =$? (2p)

   0.4
   1 point for setting up the total probability equation correct but making a numerical error or pluging in wrong numbers. 2 points for pluging in the right numbers but not bothering to solve the linear equation.

3. (6p) This question is on the Kalman Filter or KF. A very deep vertical hole is drilled into a uniformly dense spherical asteriod and a ball is dropped into it. The ball experiences an acceleration due to gravity at time t of:

$a_t = \frac{4\pi}{3} G\rho r_t$

Where G is the graitational constant, ρ is the density of the asteriod and r_t is the distance of the ball to the center or the asteriod at time t. Then it turns out that:

$a_t = 10(1 - \frac{x_t}{R})$

in mks units where $R = 1000$ (meter) is the radius of the asteriod and $x_t = R - r_t$ is the depth below the surface that the ball has reached. (The numbers end up with the asteriod having an impossible high density but nevermind that.) The ball was released from rest at $x_0 = 0$ and allowed to fall freely. We will use intervals of one second to numerically estimate the position of the ball using the simplified formula:

$x_{t+1} = x_t + v_t + 0.5a_t + \epsilon_0$
$v_{t+1} = v_t + a_t + \epsilon_1$
$a_{t+1} = 10 - \frac{x_t}{100} + \epsilon_2$

where $\epsilon_i \sim N(0, 1.0)$ for $i = 0, 1, 2$. Assume no uncertainly in the depth or acceleration at $t = 0$ but an uncertainty of 1 in the initial velocity.

a) What is the estimated state and the covariance matrix, at t=1. (3p)

$$\mu_0 = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix}, \Sigma_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A = \begin{pmatrix} 1 & 1 & .5 \\ 0 & 1 & 1 \\ -0.01 & 0 & 0 \end{pmatrix},$$
$$\bar{\mu}_1 = \begin{pmatrix} 5 \\ 10 \\ 10 \end{pmatrix}, \bar{\Sigma}_1 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1 point for using formulas right but having wrong values. 2 points for right values but numerical error.

b) A laser measures the depth of 5.3 at $t = 1$ with a variance of 1.0. What is the posteriori distribution? (3p)

$$K = \begin{pmatrix} .67 \\ .33 \\ 0 \end{pmatrix}, \mu_1 = \begin{pmatrix} 5.2 \\ 10.1 \\ 10 \end{pmatrix}, \Sigma_1 = \begin{pmatrix} .67 & .33 & 0 \\ .33 & 1.67 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. (6 p) These questions are on the Extended Kalman filter, EKF, used for state estimation for robot localization in 2 dimensions with point features.
The prediction formula is:

$x_{t+1} = x_t + v\Delta t cos(\theta_t) + \epsilon_x$
$y_{t+1} = y_t + v\Delta t sin(\theta_t) + \epsilon_y$
$\theta_{t+1} = \theta_t + \omega\Delta t + \epsilon_\theta$

where the constants are: $v\Delta t = 1m$
$\omega\Delta t = \pi/6$ rads. (note: $cos(\pi/6) = \frac{\sqrt{3}}{2}$, and $sin(\pi/6) = \frac{1}{2}$)

and where $\epsilon_x \sim N(0, 0.2)$, $\epsilon_y \sim N(0, 0.2)$, and $\epsilon_\theta \sim N(0, 0.04)$.

The intial state is at the origin with no correlations between the three state components. The inital variances in x and y are both .05 m^2. The initial variance in θ is .01 rad^2.

The formulas for the Extended Kalman Filter appear at the start of the exam. Use that notation when answering these questions.

a) What is the belief after the prediction phase of the EKF? (3p)

$$\mu_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma_0 = \begin{pmatrix} 0.05 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}, G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\bar{\mu}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{\pi}{6} \end{pmatrix}, \bar{\Sigma}_1 = \begin{pmatrix} .25 & 0 & 0 \\ 0 & .26 & 0.01 \\ 0 & 0.01 & 0.05 \end{pmatrix}$$

There is a point feature at (2, 0) in the x y plane. A distance to it is measured and the measurement is 0.75 m. The variance in the measurement is .25 m^2

b) What is the belief after the update phase of the EKF? (3p)

$$K = \begin{pmatrix} -0.5 \\ 0 \\ 0 \end{pmatrix}, \mu_1 = \begin{pmatrix} 1.125 \\ 0 \\ \frac{\pi}{6} \end{pmatrix}, \Sigma_1 = \begin{pmatrix} .125 & 0 & 0 \\ 0 & 0.26 & 0.01 \\ 0 & 0.01 & 0.05 \end{pmatrix}$$

5. (12p) These questions are on the Particle Filter, PF, used to estimate the one dimensional state, x, of a system, where x can be any real number. The inital state is uniformly distributed with x somewhere between -1 and +1 inclusive, $x \in [-1, 1]$. You are using three paricles. -You draw 9 random numbers between 0 and 100 and get 10, 50, 85, 20, 5, 90, 40, 30, 60.

   The motion of the system is given by:

   $p(x_k|x_{k-1}) = 0.5$ for $|x_k - x_{k-1}| < 1$ and 0 otherwise.

   Measurements, y_k, are taken after the motion according to the measurement model:

   $p(y_k|x_k) = 1$ for $|y_k - x_k| < 0.25$;
   $p(y_k|x_k) = .5$ for $0.75 > |y_k - x_k| \geq 0.25$;
   (and 0 otherwise).

   a) What is your initial particle set if you use the first three radom numbers to draw from the inital distribution? (3p)
   $\{\frac{10-50}{50}, \frac{50-50}{50}, \frac{85-50}{50}\} = \{-0.8, 0, 0.7\}$
   b) Use the next three random numbers to carry out the diffusion step. (3p)
   $x_1^{(1)} = -0.8 + \frac{20-50}{50} = -1.4$
   $x_1^{(2)} = 0 + \frac{5-50}{50} = -0.9$
   $x_1^{(3)} = 0.7 + \frac{90-50}{50} = 1.5$

   c) The measurement is now -1.0. Use the next three random numbers to resample the particle set. (Be very clear on how you are doing the resampling) (6p)
   $w^{(i)} \propto p(y_1 = -1|x_1^{(i)})$
   $w^{(1)} = \frac{.5}{1.5} = \frac{1}{3}$, $w^{(2)} = \frac{1}{1.5} = \frac{2}{3}$, $w^{(3)} = 0$
   $\frac{1}{3} < 0.4 \rightarrow$ particle 2
   $\frac{1}{3} > 0.3 \rightarrow$ paricle 1
   $\frac{1}{3} < 0.6 \rightarrow$ particle 2
   $\{-0.9, -1.4, -0.9\}$

6. (15p)

a) Give three ways to reduce particle deprivationin a Particle Filter estimate? (3p)
Increase M, Inject random particles, delay resampling, stratified resampling, systematic (Low Variance) resampling.

b) Give an example where a Extended Kalman filter would need to be used instead of a Kalman Filter? (1p)
Any system with non-linear dynamics or measurement model. for ex. Robot Localization.

c) Give an example where a Kalman filter would be a better choice than a Particle filter? (1p)
When the dynamics and measurements are linear with mono modal noise and the belief is mono modal. For ex. a random walk with linear Gaussian models.

d) Give an example where a particle filter would be better than an Extended Kalman Filter. (1p)
When there are multiple modes in the belief such as localization with several separted areas of significant probability.

e) Give two variations on the Extended Kalman Filter, EKF, and describe how each might work better than the statndard EKF? (4p)
Iterative EKF finds a better linearization point for the update by iterating the update several times. The Unscented KF passes several sigma points thru the none linearities in order to be able to apprximate the higher order terms than linear.

f). If one skips the resampling steps of the Particle Filter what effect would that have on the distribution of particles after several iterations? (2p)
The paricle set would be more spread than if we did resampling. The measurements would not have any effect on the particle distribution.

g) Give two ways to convert a set of particles to a continous probability density function. (2p)
Guassian kernals on each particle, Gaussian fit to entire particle set means and covariance, histogram over the state space.

h) Given a system with a series of states $\{\mathbf{x}_k\}$ and a series of measurements $\{\mathbf{z}_k\}$, for k= 1,2,3,...n, write a mathematical formula for the maximum a posteriori, MAP, state estimate? (1p)
$argmax_{\{\mathbf{x}_k\}} p(\{\mathbf{x}_k\}|\{\mathbf{z}_k\})$