AUTOMATIC CONTROL
KTH
Applied Estimation EL2320/EL3320
Exam 8:00-12:00 March 14, 2016

Aids: None, no books, no notes, nor calculators

Observe:

- Name and person number on every page
- Answers should be in English (or Swedish)
- Only write on one side of the sheets
- Specify the total number of handed in pages on the cover
- Be careful to label each answer with the question number and letter
- Be careful to only state things you know are true as one incorrect statement can negate an otherwise correct or partially correct answer.
- That said also be sure to answer the question asked.
- Motivate answer and clearly state any additional assumptions you may need to make.

Results: The results will be available on 'mina sidor', in a few weeks
Any problems with the recording of grades, STEX Studerandexpeditionen, Osquldasv. 10.

Responsible: John Folkesson, 08-790-6201

Good to know (what all the symbols mean you should be able to figure, this is just in case you forgot how to put them together):

$$G(\mathbf{x}, \mu, \Sigma) = \frac{1}{\sqrt{|2\pi\Sigma|}} \exp \frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \tag{1}$$

Typical Non-linear system equations:

$\mathbf{x}_t = \mathbf{g}(\mathbf{u}_t, \mathbf{x}_{t-1}) + \varepsilon_t$

$\mathbf{z}_t = \mathbf{h}(\mathbf{x}_t) + \delta_t$

'Kalman Gain'

$K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$

Predict phase:

$\bar{\Sigma}_t = R_t + G_t \Sigma_{t-1} G_t^T$

$\bar{\mu}_{\mathbf{t}} = \mathbf{g}(\mathbf{u}_t, \mu_{t-1})$

Update:

$\Sigma_t = \bar{\Sigma}_t - K_t H_t \bar{\Sigma}_t$

$\mu_t = \bar{\mu}_{\mathbf{t}} + K_t \eta_t$

Questions

1. (6p) You have 10 balls in a bag, 5 white and 5 black. You draw balls from the bag without replacement. You associate a discrete binary random variable X_i with the selected ball on the ith draw where $x_i = 0$ if the ith draw is white and $X_i = 1$ if it is black.

   NOTICE WITHOUT REPLACEMENT!

   a) What is the probability distribution for the first draw? $P(X_1) =$ (1p)?
   ans. $P(X_1 = 0) = 0.5$, $P(X_1 = 1) = 0.5$

   b) What is the expectation value of X_1? (1p)
   ans. $E(X_1) = 0.5$

   c) What is the variance of X_1? (1p)
   ans. $E(X_1^2) - E(X_1)^2 = 0.25$

   d) What is the probability distribution for the second draw? $P(X_2) =$? (1p)
   ans. $P(X_2 = 0) = 0.5$, $P(X_2 = 1) = 0.5$

   e) What is the probability of drawing a white then a black ball on the first two draws (1p)?
   ans. $P(X_1 = 0, X_2 = 1) = \frac{25}{90}$

   f) Are X_1 and X_2 independent (1p)?
   ans. No

2. (6p) Given three binary random variables A, B, and C, with:

$P(A,B)$	$B=0$	$B=1$
$A=0$	0.12	0.08
$A=1$	0.48	0.32

$P(C=0\|A,B)$	$B=0$	$B=1$
$A=0$	1.0	0.5
$A=1$	0.0	0.5

a) What is $P(A,B,C)$? (1p) ans.

$P(A,B,C)$	$B=0$	$B=1$
$A=0,\ C=0$	0.12	0.04
$A=1,\ C=0$	0.0	0.16
$A=0,\ C=1$	0.0	0.04
$A=1,\ C=1$	0.48	0.16

b) What are $P(A)$ and $P(B)$ (1p)? ans. $P(A=0)=0.2$, $P(B=0)=0.6$

c) Are A and B independent? (2p) ans. yes

d) What is $P(A,B|C)$? (1p) ans.

$P(A,B\|C)$	$B=0$	$B=1$
$A=0,\ C=0$	$\frac{12}{32}$	$\frac{4}{32}$
$A=1,\ C=0$	0.0	$\frac{16}{32}$
$A=0,\ C=1$	0.0	$\frac{4}{68}$
$A=1,\ C=1$	$\frac{48}{68}$	$\frac{16}{68}$

e) What is $P(A|C)$? (1p) ans.

$P(A\|C)$	$A=0$	$A=1$
$C=0$	0.5	0.5
$C=1$	$\frac{1}{17}$	$\frac{16}{17}$

3. (9p) This question is on the Kalman Filter. An elevator is traveling up in a tall building. The indicator light tells the floor number and changes as the elevator floor is at the same height as the building floor. This gives a measurements of the height of the elevator that are independent and unbiased but with a variance of $2.0m^2$. The floors are exactly 5.0 meters apart.

   The elevator is in motion upwards and assumed to be traveling at a nearly constant but unknown speed. At time $t_2 = 0$ the indicator changed to 2 and at $t_3 = 5.0$ seconds it changed to 3. This gives an initial estimate of the speed along with its variance at t_3.

   Let x_k be the height above floor 3 in meters and v_k be the speed at the moment when the indicator changes to value k. The v_k is subject to random fluctuations due to various disturbances but these are not correlated over time so that the average velocities over between passing two floors are independent random variables with variance of $0.01(m/s)^2$.

   Assume that the only sources of uncertainty are the two given above. The times t_k can be considered accurate.

   a) What is the estimated initial state (both x_3 and v_3 and the covariance matrix), at t_3. (1p)
   ans. $\mu_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\Sigma_3 = \begin{pmatrix} 2.0 & 0.4 \\ 0.4 & 0.16 \end{pmatrix}$

   b) Set up the prediction equations, define all parameters and give the values for them. (2p)
   ans. $\bar{\mu}_k = \begin{pmatrix} 1 & (t_k - t_{k-1}) \\ 0 & 1 \end{pmatrix} \mu_{k-1}$;
   $\bar{\Sigma}_k = + \begin{pmatrix} 0 & 0 \\ 0 & 0.01 \end{pmatrix} + \begin{pmatrix} 1 & (t_k - t_{k-1}) \\ 0 & 1 \end{pmatrix} \Sigma_{k-1} \begin{pmatrix} 1 & 0 \\ (t_k - t_{k-1}) & 1 \end{pmatrix}$

   c) The indicator changes to 4 at $t_4 = 9.0$ seconds. What is the belief after the predict step at k=4? (2p)
   ans. $\bar{\mu}_4 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$;
   $\bar{\Sigma}_4 = \begin{pmatrix} 7.76 & 1.04 \\ 1.04 & 0.17 \end{pmatrix}$

   d) Use the indicator as a direct measurement of the height, x_k. What are the update equations and the parameters? (2p)
   $C = (1, 0)$
   $K_k = \bar{\Sigma}_k C^T (C \Sigma C^T + 2)^{-1}$

$\mu_k = \bar{\mu}_k + K_k(z_k - \bar{\mu}_k)$, where $z_k = (k-3) * 5$
$\Sigma_k = (1 - K_k C)\bar{\Sigma}_k$

e) What is posteriori distribution after the first update at $t_4 = 9.0$? You do not need to carry out all the multiplications and divisions. Just an expression for each parameter with all the numbers replacing the symbols. (2p)
ans. A Gaussian with mean and Covariances of:
$\mu_4 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + K_4(1);$
$K_4 = \frac{1}{9.76}\begin{pmatrix} 7.76 \\ 1.04 \end{pmatrix};$
$\Sigma_4 = \begin{pmatrix} 7.76 & 1.04 \\ 1.04 & 0.17 \end{pmatrix} - \frac{1}{9.76}\begin{pmatrix} 7.76(7.76) & 1.04(7.76) \\ 1.04(7.76) & 1.04(1.04) \end{pmatrix};$

4. (12 p) These questions are on the particle filter, PF, used for state estimation. We decide to swap the Kalman Filter for a PF for the previous problem. Now instead of estimating the initial state we start by assume we do not know anything about the initial state other than that x_3 is between -1 and +1 meters and that the speed v_3 is between -2 and 2 meters per second.

a) How do you generate the initial set of particles, at t_3? (2p)
ans. Select M random numbers each from uniform distributions on (-1,1) and (-2,2). Then form M pairs from these arranged as M vectors, $\mathbf{x}_i$, i=1,2,...M.

b) Describe one full iteration of the particle filter taking the estimate from the initial set of particles at t_3 through all steps of the filter once and ending with a set of particles at t_4. Explain what will happen to each of the particles at each step that you describe and how the overall distribution will change. (7p)
ans. 1. transform all the particles with the equations of motion:
$\mathbf{x}_i = \begin{pmatrix} 1 & (t_4 - t_3) \\ 0 & 1 \end{pmatrix} \mathbf{x}_i$ where $t_4 - t_3 = 4$ in this case;
2. add to each a separate random number drawn from N(0,0.01).
3. compute the importance weight for each particle:
$w_i = N(5 - x_i, 2)$ where x_i is the first component of $\mathbf{x}_i$.
4. Normalize the weights by dividing by $\sum_{i=1}^{M} w_i$
5. Re-Sample by drawing M numbers, r_k, from uniform (0,1). Then chose for each the particle n given by the smallest n such that $\sum_{i=1}^{n} w_i > r_k$

The effect on the particle distribution is:

1. All particles x values are shifted either up or down randomly since the velocities are random. The partcle set will be more spread our in x. The v values are unchanged.
2. This step adds additional random shift to x and now also v. spreading things out even more.
3. N/A
4. N/A
5. The particles that better explained the measurements will have a higher weight so they will servive better. Probably no particles with negative velocities and negative x values will survive. The particles will cluster around x= 5 and v=1.25.

c) Explain in terms of this example how using too few particles could cause problems. Be specific so that I know that you understand the terms you use. That is, do not just say too few particles could cause X. Explain what happens

is to someone that has not taken the course and does not know the terminology (such as X). (3p)
ans. If we were to have too few particles then we may find that none of the particles describe the measurements well. Every particle will have an x_i more than 1.41 m from the floor height above floor 3. This is what is know as particle depletion. Since the motion model is relatively tight with a standard deviation in the noise of on 0.1 m this condition will persist and there is little hope that the random noise added to the particles in the diffusion step (2 above) will correct this.

5. (10p) These questions are on the Extended Kalman Filter. The formulas for the Extended Kalman Filter appear at the start of the exam. Use that notation when answering these questions.

a) What is the form of the pdf for the EKF state estimator? (1p)
ans. It is a Gaussian.

b) If the state is one dimensional and the motion model is non-linear, draw figures to illustrate and explain in words two situations: One in which the predicted mean and covariance will be reasonably accurate and one where there will be considerable distortion of the distribution during the predict phase of the EKF. (3p)
ans. See fig 3.5 in the book.

c) Explain how the error in the predicted mean state can cause problems when computing H_t (2p)
The update computes the Jacobian of the measurement model and evaluates it at the predicted mean. If this mean is wrong then the filter could diverge due to the incorrect value for H. In fact if we could only linearize at the true state we could hold the EKF consistent. It is even possible that if the linearization is wrong it is possible for the updated state to worse explain the measurement than th state before the update.

d) Explain the effect of the measurements on the distribution's mean and covariance? (2p)
The measurement will be used in the update step to move the mean to a position that (usually) better explains the measurement and the covariance will decrease in some components.

e) Explain how the mean of an EKF estimate can be different from the true state and the estimator still be 'consistent'? (2 pt.)
The mean will normally not equal the true state but if it is consistent with the uncertainty estimate then the estimateor is said to be consistent. By this we often mean that the mean seems to stay mostly within the '2 sigma bounds' that is within 2 standard devaitions of the true state.

Or one could say that the estimate is consistent if it agrees statistically with the measurements. The true state is not relevent since the estimate is only based on the measurements.

6. (7p)

a) Give the names of 4 estimators that use a Gaussian distribution to represent the belief? (2 pt.)
ans. UKF, IEKF, EKF, Information Filter,

b) Give three methods of density extraction from a set of particles? (2p)
ans. Gaussian Kernals on each particle, Take the mean and covariance of the particle set and use them to form a single Gaussian. Create a histogram over a partition of the state space.

c) What is data association? (1 pt.)
ans. Data association is assigning each measurement to a measurent model. The idea is to find the correct cause of the measurement.

d) How might we do data association? (1p)
ans. We can compute the likelihood of each measurement model and chose the most likely.

e) What is a MAP estimator? (1p)
ans. The maximum a posteriori estimator is the state given by $argmax_{x_t} p(x_t|z_{1...t})$ when we have an a prior probability and a model of the measurements.