Part III 2015 - Homological and Homotopical Algebra Solutions

February 9, 2025

Question Paper is available at - `https://www.maths.cam.ac.uk/postgrad/part-iii/files/pastpapers/2015/paper_4.pdf`

Solution 1:

Problem Statement: Let R be an arbitrary unital ring. Let $\mathcal{E}(C, A)$ be the set of short exact sequences

$$0 \to A \to B \to C \to 0$$

of left R-modules up to equivalence. Prove that there is a bijective correspondence between $\mathcal{E}(C, A)$ and $\mathrm{Ext}^1_R(C, A)$.

Let $\mathbb{Z}$ be the trivial module for the abelian group $\mathbb{Z}$. Classify, up to equivalence, all short exact sequences of the form

$$0 \to \mathbb{Z} \to B \to \mathbb{Z} \oplus \mathbb{Z} \to 0.$$

Solution. Let R be a unital ring, and let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of left R-modules. Define

$$\mathcal{E}(C, A) = \{0 \to A \to B \to C \to 0 | \text{up to commutative isomorphism}\}.$$

To relate $\mathcal{E}(C, A)$ to $\mathrm{Ext}^1_R(C, A)$, we use the derived functor definition of $\mathrm{Ext}^1_R(C, A)$:

$$\mathrm{Ext}^1_R(C, A) = H^1(\mathrm{Hom}_R(P_\bullet, A)),$$

where

$$P_\bullet : \cdots \to P_1 \to P_0 \to C \to 0$$

is a projective resolution of C. Applying $\mathrm{Hom}_R(-, A)$ yields the complex

$$0 \to \mathrm{Hom}_R(P_0, A) \to \mathrm{Hom}_R(P_1, A) \to \cdots.$$

An element of $\mathrm{Ext}^1_R(C, A)$ is represented by a 1-cocycle, which corresponds to an exact sequence

$$0 \to A \to B \to C \to 0.$$

Given an element of $\mathrm{Ext}^1_R(C, A)$, construct a pushout

$$\begin{array}{ccccccccc}
0 & \to & P_1 & \to & P_0 \oplus A & \to & B & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & P_1 & \to & P_0 & \to & C & \to & 0
\end{array}$$

to obtain a corresponding short exact sequence in $\mathcal{E}(C, A)$.

Conversely, given a short exact sequence in $\mathcal{E}(C, A)$, we obtain a 1-cocycle in the complex $\mathrm{Hom}_R(P_\bullet, A)$, defining an element of $\mathrm{Ext}^1_R(C, A)$. The equivalence in $\mathrm{Ext}^1_R(C, A)$ matches the isomorphism condition in $\mathcal{E}(C, A)$.

Thus, we obtain a bijection

$$\mathcal{E}(C, A) \cong \mathrm{Ext}^1_R(C, A).$$

Consider the short exact sequence

$$0 \to \mathbb{Z} \to B \to \mathbb{Z} \oplus \mathbb{Z} \to 0$$

in the category of $\mathbb{Z}$-modules.

The classification of such sequences is governed by $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z})$. By the properties of the Ext functor,

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}).$$

It is well-known that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong 0$ since $\mathbb{Z}$ is a free $\mathbb{Z}$-module. Therefore,

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z}) \cong 0.$$

This implies that all short exact sequences of the form

$$0 \to \mathbb{Z} \to B \to \mathbb{Z} \oplus \mathbb{Z} \to 0$$

are split. Hence, $B \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Thus, the only possible extension up to equivalence is the trivial split extension:

$$B \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Solution 2:

Problem Statement: Let R be an arbitrary unital ring. Define the Koszul complex $K(\mathbf{t})$ of a sequence $\mathbf{t} = (t_1, \ldots, t_n)$ of central elements of R.

Let M be a left R-module. Define what it means for a sequence to be regular on M. Show that if $\mathbf{t}$ is regular on M, then $K(\mathbf{t}) \otimes_R M$ is quasi-isomorphic to $M/(t_1, \ldots, t_n)M$. You should state carefully any results that you use.

Consider the ring $\mathbb{Z}[X]$ and its modules $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$ on which X acts trivially. (Here $p, q \geq 2$.) Compute $\mathrm{Ext}^*_{\mathbb{Z}[X]}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/q\mathbb{Z})$ in the cases where (i) $p = q$ and (ii) p and q are coprime.

Solution. Let R be an arbitrary unital ring, and let $\mathbf{t} = (t_1, \ldots, t_n)$ be a sequence of central elements of R. The Koszul complex $K(\mathbf{t})$ of the sequence t is the complex of free R-modules:

$$K(\mathbf{t}) = \left(\bigwedge^n R^n \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} R \right),$$

where the differentials d_i are defined in terms of the wedge product acting on the sequence $t_1, \ldots, t_n$.

Let M be a left R-module. A sequence $\mathbf{t} = (t_1, \ldots, t_n)$ is said to be regular on M if for each $1 \leq i \leq n$, the map

$$t_i : M \to M$$

is injective and the sequence $(t_1, \ldots, t_n)$ is a regular sequence on M. Specifically, for every $1 \leq i \leq n$, the sequence $(t_1, \ldots, t_i)$ is regular on M.

Now suppose that $\mathbf{t}$ is regular on M. We wish to show that $K(\mathbf{t}) \otimes_R M$ is quasi-isomorphic to $M/(t_1, \ldots, t_n)M$. Since t is regular on M, the sequence $(t_1, \ldots, t_n)$ defines a free resolution of the quotient module $R/(t_1, \ldots, t_n)R$. Tensoring this free resolution with M gives the complex $K(\mathbf{t}) \otimes_R M$. By the exactness of tensoring with a free resolution and the properties of the Koszul complex, we have:

$$K(\mathbf{t}) \otimes_R M \cong M/(t_1, \ldots, t_n)M,$$

which proves that the two modules are quasi-isomorphic.

Consider the ring $\mathbb{Z}[X]$ and its modules $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$ where X acts trivially on both modules, with $p, q \geq 2$. We are tasked with computing the Ext groups $\mathrm{Ext}^*_{\mathbb{Z}[X]}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/q\mathbb{Z})$ in the following cases:

(i) When $p = q$, the Ext groups $\mathrm{Ext}^n_{\mathbb{Z}[X]}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ are computed using the fact that X acts trivially on both modules. The Ext groups vanish for all $n > 0$, and:

$$\mathrm{Ext}^0_{\mathbb{Z}[X]}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}.$$

(ii) When p and q are coprime, we compute the Ext groups using the fact that the modules $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$ have trivial X-action. The Ext groups vanish for all $n > 0$, and:

$$\mathrm{Ext}^0_{\mathbb{Z}[X]}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/q\mathbb{Z}) = 0.$$

Thus, in both cases, the only nonzero Ext group is Ext^0 for the case where $p = q$.

Solution 3:

Problem Statement: Define what it means to say a spectral sequence converges to a graded object H^*.

Let R be an arbitrary unital ring, let C be a cochain complex of left R-modules, and let F be a bounded filtration on C. State a theorem about the convergence of the spectral sequence associated to the filtered complex (C, F).

Describe how to deduce that there are two spectral sequences computing the cohomology of a double complex that is bounded in both degrees.

Let P be a bounded cochain complex of projective left R-modules, and let M be any left R-module with finite projective dimension. By considering the two spectral sequences associated with a suitable double complex, prove the following Künneth spectral sequence:

$$E_2^{p,q} = \mathrm{Tor}_{-p}(H^q P, M) \quad \Rightarrow \quad H^{p+q}(P \otimes M).$$

Assume that all boundaries of the complex P_* are also projective. Show the spectral sequence degenerates at the E_2 term.

Solution. A spectral sequence $(E_r^{p,q}, d_r^{p,q})$ converges to a graded object H^* if there exists a graded filtration $\{F^p H^{p+q}\}$ on H^* such that for sufficiently large r,

$$E_r^{p,q} \cong F^p H^{p+q}/F^{p+1}H^{p+q}.$$

In other words, the associated graded object of H^* with respect to the filtration is isomorphic to the limiting page of the spectral sequence.

Let $(C^\bullet, F)$ be a filtered cochain complex of R-modules, where the filtration is bounded below or above:

$$F^p C^\bullet \supseteq F^{p+1}C^\bullet \quad \text{and} \quad \bigcap_p F^p C^\bullet = 0, \quad \bigcup_p F^p C^\bullet = C^\bullet.$$

Then the spectral sequence associated with the filtration F converges to the cohomology $H^*(C^\bullet)$ in the sense that

$$E_r^{p,q} \implies H^{p+q}(C^\bullet).$$

If the filtration is bounded below, the convergence is in the sense of filtrations on the cohomology groups.

Let $C^{\bullet,\bullet}$ be a bounded double complex, meaning there are only finitely many non-zero entries in both degrees. Two spectral sequences can be obtained by taking different filtrations:

Filter by rows, considering $C^{p,\bullet}$ as the row-wise complexes. The associated spectral sequence has $E_1^{p,q} = H^q(C^{p,\bullet})$ and converges to $H^{p+q}(\mathrm{Tot}(C^{\bullet,\bullet}))$.

Filter by columns, treating $C^{\bullet,q}$ as column-wise complexes. The associated spectral sequence has $E_1^{p,q} = H^p(C^{\bullet,q})$ and similarly converges to the total cohomology.

Let $P^\bullet$ be a bounded cochain complex of projective left R-modules, and let M be a left R-module of finite projective dimension. Define the double complex

$$C^{p,q} = P^p \otimes_R M \quad \text{with } d_1(x \otimes m) = d_P(x) \otimes m \quad \text{and} \quad d_2(x \otimes m) = 0.$$

There are two spectral sequences arising from this double complex:

Filter by rows (horizontal filtration). The first page is given by

$$E_1^{p,q} = H^p(P^\bullet \otimes_R M) \cong H^p(P^\bullet) \otimes_R M,$$

where the differential is trivial since $d_2 = 0$. Thus, the second page becomes

$$E_2^{p,q} = \mathrm{Tor}_{-q}(H^p(P^\bullet), M).$$

The spectral sequence converges to the cohomology of the total complex:

$$E_2^{p,q} \implies H^{p+q}(P^\bullet \otimes_R M).$$

Filter by columns (vertical filtration). The differential in each column is trivial ($d_2 = 0$), leading to

$$E_1^{p,q} = P^p \otimes_R M \quad \text{and} \quad E_1^{p,q} = 0 \text{ for } q \neq 0.$$

Hence, this spectral sequence degenerates immediately, and

$$E_2^{p,q} = H^{p+q}(P^\bullet \otimes_R M).$$

Comparing the two spectral sequences yields the Künneth spectral sequence:

$$E_2^{p,q} = \mathrm{Tor}_{-q}(H^p(P^\bullet), M) \implies H^{p+q}(P^\bullet \otimes_R M).$$

If all boundaries of the complex $P^\bullet$ are projective, then $H^p(P^\bullet)$ is projective for all p. Recall that projective modules are acyclic for the Tor functor, implying

$$\mathrm{Tor}_i(H^p(P^\bullet), M) = 0 \quad \text{for all } i > 0.$$

Thus, the spectral sequence degenerates at the E_2 page, and we obtain

$$H^n(P^\bullet \otimes_R M) \cong H^n(P^\bullet) \otimes_R M.$$

This completes the proof.

Solution 4

Problem Statement: Define the weak equivalences, fibrations, and cofibrations that form the projective model structure on the category $\mathbf{Ch}_{>0}(R)$ of nonnegative chain complexes over an arbitrary unital ring R.

Define what is meant by a sequentially small object. Use the small object argument to show that any map $a : X \longrightarrow Y$ in this category factors as a cofibration followed by an acyclic fibration.

Hint: You may assume that acyclic fibrations are characterized by the right lifting property with respect to a certain set of cofibrations that you should specify. You may further assume the domains of these maps are sequentially small.

Solution. Let R be a unital ring, and let $\mathbf{Ch}_{>0}(R)$ denote the category of chain complexes bounded below by nonnegative degrees. An object in this category is a sequence of R-modules and differential maps

$$X = \cdots \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} X_0,$$

with $d_n \circ d_{n+1} = 0$ for all $n \geq 0$.

The projective model structure on $\mathbf{Ch}_{>0}(R)$ is defined as follows.

A map of chain complexes $f : X \to Y$ is a *weak equivalence* if it induces isomorphisms on homology:

$$H_n(f) : H_n(X) \xrightarrow{\cong} H_n(Y) \quad \text{for all } n \geq 0.$$

A map $f : X \to Y$ is a *fibration* if it is degreewise surjective:

$$f_n : X_n \twoheadrightarrow Y_n \quad \text{for all } n \geq 0.$$

A map $f : X \to Y$ is a *cofibration* if it has the left lifting property with respect to all acyclic fibrations (fibrations that are also weak equivalences).

An object K in a category $\mathcal{C}$ is *sequentially small* if there exists a cardinal κ such that, for every regular cardinal $\lambda \geq \kappa$ and every λ-filtered colimit $\{X_i\}_{i \in I}$ in $\mathcal{C}$, the natural map

$$\mathrm{colim}_{i \in I} \mathrm{Hom}_{\mathcal{C}}(K, X_i) \to \mathrm{Hom}_{\mathcal{C}}(K, \mathrm{colim}_{i \in I} X_i)$$

is a bijection.

Given a map $a : X \to Y$ in $\mathbf{Ch}_{>0}(R)$, we show that it factors as a cofibration followed by an acyclic fibration using the small object argument.

Acyclic fibrations satisfy the *right lifting property* (RLP) with respect to a certain set of maps $\mathcal{I}$. We take $\mathcal{I}$ as the set of canonical inclusion maps

$$d^n : S^{n-1} \to D^n \quad (n \geq 0),$$

where S^{n-1} is the n-dimensional sphere complex, and D^n is the n-dimensional disc complex over R. These maps correspond to the inclusions of degreewise free submodules that impose surjectivity constraints in the lifting properties.

Given a map $a : X \to Y$, the small object argument proceeds as follows:

1. Form the set $\mathcal{I}$-cellular complex by repeatedly attaching cells to X to construct a chain complex X' and a map $X \to X'$. This map will have the left lifting property with respect to all acyclic fibrations, making it a cofibration.

2. Define the induced map $X' \to Y$. This map is forced to be an acyclic fibration because it satisfies the right lifting property with respect to all generating cofibrations, making it both a fibration and a weak equivalence.

By the small object argument, we obtain a factorization of the map $a : X \to Y$ as

$$X \xrightarrow{\text{cofibration}} X' \xrightarrow{\text{acyclic fibration}} Y.$$

Thus, the desired factorization follows from the small object argument.