Serre and Grothendieck Duality

Serre Duality is one of the cornerstones of classical algebraic geometry, providing a deep link between the cohomology groups of a coherent sheaf and its dual on a smooth projective variety. Let X be a smooth projective variety of dimension n over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. Then Serre Duality states:

$$H^i(X, \mathscr{F}) \cong H^{n-i}(X, \mathscr{F}^\vee \otimes \omega_X)^*,$$

where $\omega_X = \det(\Omega^1_X)$ is the canonical sheaf (or dualizing sheaf), $\mathscr{F}^\vee = \mathcal{H}om(\mathscr{F}, \mathcal{O}_X)$ is the dual sheaf, and the isomorphism is realized through a perfect pairing:

$$H^i(X, \mathscr{F}) \times H^{n-i}(X, \mathscr{F}^\vee \otimes \omega_X) \to k.$$

Grothendieck Duality generalizes this framework to proper morphisms of schemes. Let $f : X \to Y$ be a proper morphism of noetherian schemes. The key is the construction of a functor $f^!$, which assigns to a complex $\mathscr{G}^\bullet \in D^b_c(Y)$ a complex $f^!\mathscr{G}^\bullet \in D^b_c(X)$. This functor satisfies a trace isomorphism:

$$\mathrm{Tr}_f : Rf_* f^! \mathscr{G}^\bullet \to \mathscr{G}^\bullet,$$

enabling the definition of a dualizing complex $\omega_{X/Y}$ for f. For smooth morphisms of relative dimension n, $f^!\mathscr{G}^\bullet \cong \omega_{X/Y}[n] \otimes^{\mathbb{L}} \mathscr{G}^\bullet$, where $\omega_{X/Y}$ is the relative dualizing sheaf.

Using Tr_f, one defines a duality morphism:

$$\theta_f : Rf_* R\mathcal{H}om_X(\mathscr{F}^\bullet, f^!\mathscr{G}^\bullet) \to R\mathcal{H}om_Y(Rf_*\mathscr{F}^\bullet, \mathscr{G}^\bullet),$$

where $\mathscr{F}^\bullet \in D^-_{\mathrm{qc}}(X)$ and $\mathscr{G}^\bullet \in D^b_c(Y)$. This isomorphism extends Serre Duality to more general contexts, where f need not be smooth or X projective.

A refinement of θ_f involves the δ-functorial trace map:

$$\theta'_f : Rf_* R\mathcal{H}om_X(\mathscr{F}^\bullet, f^\#\mathscr{G}^\bullet) \to R\mathcal{H}om_Y(Rf_*\mathscr{F}^\bullet, \mathscr{G}^\bullet),$$

where $f^\#$ is a modification of $f^!$, incorporating pullback adjustments to $\mathscr{G}^\bullet$. When $Y = \mathrm{Spec}(k)$, the θ'_f morphism specializes to the classical pairing in Serre Duality:

$$\mathrm{H}^{-i}(\theta_f) : \mathrm{H}^{n-i}(X, \mathscr{F}^\vee \otimes \omega_X) \to \mathrm{Hom}_k(H^i(X, \mathscr{F}), k).$$

The projection formula:

$$Rf_*(f^!\mathscr{G}^\bullet) \cong Rf_*(\omega_{X/Y}[n] \otimes^{\mathbb{L}} \mathscr{G}^\bullet)$$

connects the relative dualizing sheaf $\omega_{X/Y}$ with $f^!$. This formula facilitates the explicit computation of duality maps.