Advanced Problems

Exercises Collected by Virat Chauhan

1: Enriched Kan Extensions via Coends

Let R be a commutative ring and consider categories enriched in R-modules (so that Hom-sets carry an R-module structure and composition is R-bilinear). Define an R-linear category $\mathcal{A}$ with two objects a, b, where

$$\mathrm{Hom}_{\mathcal{A}}(a,a) = R, \quad \mathrm{Hom}_{\mathcal{A}}(b,b) = R, \quad \mathrm{Hom}_{\mathcal{A}}(a,b) = R, \quad \mathrm{Hom}_{\mathcal{A}}(b,a) = 0,$$

and composition is given by R-bilinear maps (so there is one non-identity arrow $\alpha\colon a \to b$).

Let $\mathcal{B}$ be another R-linear category with two objects c, d and one non-identity arrow $f\colon c \to d$. Define an enriched functor $F\colon \mathcal{A} \to \mathcal{B}$ by $F(a) = c$, $F(b) = d$, and $F(\alpha) = f$.

Let $\mathcal{M}$ be the R-linear category of R-modules. Define $G\colon \mathcal{A} \to \mathcal{M}$ as the enriched functor determined by modules M_a, M_b and an R-linear map $g\colon M_a \to M_b$ corresponding to α.

Compute the *enriched left Kan extension* $\mathrm{Lan}_F G\colon \mathcal{B} \to \mathcal{M}$ using the coend formula:

$$(\mathrm{Lan}_F G)(x) = \int^{y \in \mathcal{A}} \mathcal{B}(F(y), x) \otimes_R G(y).$$

Compute $(\mathrm{Lan}_F G)(c)$ and $(\mathrm{Lan}_F G)(d)$ explicitly in terms of M_a, M_b, and g. Identify the resulting coequalizers or pushouts.

2: Homotopy Kan Extensions in $(\infty, 1)$-Categories

Kan extensions in ordinary category theory become **homotopy Kan extensions** in a homotopical setting. For example, in the model category of topological spaces (or simplicial sets), the *left homotopy Kan extension* along a functor p is computed by the homotopy colimit of the diagram. Consider the following specific diagram $X\colon C \to \mathbf{Top}$ in the category of spaces:

$$C: \quad 0 \longrightarrow 1,$$

and let $p\colon C \to *$ be the functor to the terminal category. Define $X(0) = S^0$ (the discrete space with two points) and $X(1) = *$ (a singleton), and let the map $X(0) \to X(1)$ collapse both points to the basepoint. The **homotopy left Kan extension** $\mathrm{L}_p(X)$ at the point $*$ is the homotopy colimit of the diagram $S^0 \rightrightarrows *$.

(a) Describe how to compute this homotopy colimit as a homotopy pushout (or suspension) of the diagram. Argue why ordinary colimit (which would just yield a point) is not the correct answer, and explain how one obtains the homotopy-correct result by considering a cofibrant replacement of the diagram or by using mapping cylinders.

(b) Show that the homotopy colimit (hence $\mathrm{L}_p(X)(*)$) is homeomorphic to S^1. (*Hint: one can view two maps from a point together with a homotopy between their restrictions to* S^0 *as describing a circle.*) In fact, as observed by Carlson on Math StackExchange, the "two null-homotopies" on S^0 force the result to be S^1. Conclude that in the $(\infty, 1)$-category of spaces this Kan extension produces the circle.

Context hint: In general, Lurie shows that pointwise left Kan extensions in $(\infty, 1)$*-categories exist whenever the target admits the required homotopy colimits (see* MathOverflow *). This example exhibits the construction explicitly.*

3: Kan Extension and Induction of Group Actions

Let $\varphi : H \to G$ be a homomorphism of groups, and let BH, BG denote the one-object categories with endomorphism monoids H, G respectively (the *classifying categories* of the groups). A functor $X : BH \to \mathbf{Set}$ is equivalently a (left) H-set. The restriction functor $\varphi^* : \mathrm{Fun}(BG, \mathbf{Set}) \to \mathrm{Fun}(BH, \mathbf{Set})$ sends a G-set to the same set with H-action via φ.

By general theory (using the Grothendieck construction), φ^* has a left adjoint given by the left Kan extension along $B\varphi : BH \to BG$,
see math.stackexchange.com.

In concrete terms:

(a) Show that the left Kan extension $\mathrm{Lan}_{B\varphi}(X)$ of an H-set X along $B\varphi : BH \to BG$ is a G-set whose underlying set can be identified with

$$\int^{h \in BH} G(\varphi(h), *) \times X(h) \cong G \times_H X,$$

the *coend* or quotient set $G \times X/\sim$ with $(g\varphi(h), x) \sim (g, hx)$. (This describes the usual "induced G-set" or *induction of X from H to G*.)

(b) Verify explicitly (by unraveling the Kan extension coend or by a universal property argument) that $\mathrm{Lan}_{B\varphi}(X)$ is isomorphic to the usual induced set $G \underset{H}{\times} X$. In particular, describe the G-action on $G \underset{H}{\times} X$ and show it satisfies the universal property of the Kan extension.

Hint: The general fact cited above (see math.stackexchange.com *) implies the existence of this Kan extension as a left adjoint. The computation via coends gives a constructive formula:*

$$(\mathrm{Lan}_{B\varphi} X)(*) = \int^{h \in BH} G(*, \varphi(h)) \times X(h).$$

4: Kan Extensions in ∞-Categories

Let $f\colon \mathcal{C}' \to \mathcal{C}$ be a functor of small ∞-categories, and let $\mathcal{D}$ be a presentable ∞-category (so $\mathcal{D}$ has all small colimits math.ias.edu). Consider the precomposition functor

$$f^* : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}', \mathcal{D}), \quad F \mapsto F \circ f.$$

1. **Left Kan extension.** For a given functor $F\colon \mathcal{C}' \to \mathcal{D}$, define the *pointwise left Kan extension* $f_!F\colon \mathcal{C} \to \mathcal{D}$ by the formula

   $$(f_!F)(c) = \operatorname*{colim}_{(c' \to c) \in \mathcal{C}'_{/c}} F(c').$$

   Show that this construction indeed produces a functor $f_!F$ and that $f_!\colon \mathrm{Fun}(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is left adjoint to f^*. In particular, verify the universal property

   $$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(f_!F, G) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}',\mathcal{D})}(F, G \circ f)$$

   by constructing the unit and counit maps and checking the triangle identities. (Here $\mathcal{C}'_{/c}$ denotes the ∞-category of objects of $\mathcal{C}'$ over c.) Use the fact that $\mathcal{D}$ has all colimits math.ias.edu to ensure these Kan extensions exist.

2. **Right Kan extension.** Assume further that $\mathcal{D}$ admits all small limits (for instance $\mathcal{D}$ could be presentable and complete). Define the *pointwise right Kan extension* $f_*F\colon \mathcal{C} \to \mathcal{D}$ by

   $$(f_*F)(c) = \lim_{(c \to c') \in (\mathcal{C}'_{c/})} F(c').$$

   Show that this formula yields a functor $f_*\colon \mathrm{Fun}(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *which is right adjoint to* f^*. Again verify the adjunction homotopy equivalence

   $$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(G, f_*F) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}',\mathcal{D})}(G \circ f, F).$$

   Hints: In each case, you should identify the comma (over-)categories $\mathcal{C}'_{/c}$ and $(c \downarrow \mathcal{C}')$ and use the universal property of colimits/limits in $\mathcal{D}$. You may cite that in HA Lurie shows colimit-preserving functors between presentable ∞-categories are left adjoints math.ias.edu.

5: The Cofiber Functor in a Stable ∞-Category

Let $\mathcal{C}$ be a stable ∞-category (so $\mathcal{C}$ is pointed and every morphism admits both a fiber and a cofiber, see HA, Definition 1.1.1.9).

Consider the ∞-category of arrows $\mathrm{Fun}(\Delta^1, \mathcal{C})$, whose objects are morphisms $f\colon X \to Y$ in $\mathcal{C}$. There is a functor

$$\mathrm{cofib}\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$$

which sends a morphism $f\colon X \to Y$ to its cofiber, i.e., the pushout of the span $X \leftarrow X \to Y$. There is also a functor

$$Z\colon \mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C}),$$

which sends an object X to the zero morphism $0 \to X$ (where 0 denotes the zero object in $\mathcal{C}$, see HA, Remark 1.1.1.8).

1. **Adjunction.** Show that cofib is left adjoint to Z. That is, construct natural transformations

$$\eta\colon \mathrm{id}_{\mathcal{C}} \to Z \circ \mathrm{cofib}, \quad \varepsilon\colon \mathrm{cofib} \circ Z \to \mathrm{id}_{\mathcal{C}}$$

exhibiting the unit and counit of the adjunction, and verify the triangular identities. Intuitively, this expresses that a morphism $f : X \to Y$ is equivalent to specifying an object $\mathrm{cofib}(f)$ together with a map $0 \to \mathrm{cofib}(f)$ (cf. HA, Remark 1.1.1.8).

2. **Kan Extension Perspective.** Deduce from HA, Remark 1.1.1.8 that cofib arises as the left Kan extension of the inclusion of the fiber category. Consequently, cofib preserves all colimits which exist in $\mathrm{Fun}(\Delta^1, \mathcal{C})$. In particular, explain why cofib preserves small colimits as a left adjoint.

Remark. According to Lurie, every morphism in a pointed ∞-category admits a cofiber section up to contractible choice. Consequently, the forgetful functor

$$\theta\colon \{\textit{cofiber squares in } \mathcal{C}\} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$$

is a trivial fibration and thus admits a homotopy section. This provides a functorial cofiber construction up to equivalence, identifying cofib *as a left adjoint to the embedding $Z\colon \mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ (cf. HA, Remark 1.1.1.8).*

6: Extension of Scalars for Module ∞-Categories

Let $(\mathcal{C}^{\otimes}, \otimes, \mathbf{1})$ be a presentable symmetric monoidal ∞-category which is also stable — for example, Sp, the ∞-category of spectra, or the derived ∞-category of chain complexes. Suppose we are given a morphism $f : A \to B$ of E_1-algebra objects in $\mathcal{C}$ (that is, an associative algebra map). This induces a restriction (or forgetful) functor

$$f^* : \mathrm{Mod}_B(\mathcal{C}) \longrightarrow \mathrm{Mod}_A(\mathcal{C}),$$

which sends a B-module N to its underlying A-module via f. Here, $\mathrm{Mod}_A(\mathcal{C})$ and $\mathrm{Mod}_B(\mathcal{C})$ denote the ∞-categories of left modules over A and B, respectively. These are known to be presentable and stable (cf. *Higher Algebra*, Chapter 4).

- **Left adjoint (Extension of Scalars).** Define the extension-of-scalars functor

$$f_! : \mathrm{Mod}_A(\mathcal{C}) \longrightarrow \mathrm{Mod}_B(\mathcal{C}), \quad M \mapsto B \otimes_A M,$$

where $B \otimes_A M$ denotes the relative tensor product in $\mathcal{C}$. Show that $f_!$ is left adjoint to f^*, i.e., for every $M \in \mathrm{Mod}_A(\mathcal{C})$ and $N \in \mathrm{Mod}_B(\mathcal{C})$, there is a natural equivalence

$$\mathrm{Map}_{\mathrm{Mod}_B(\mathcal{C})}(B \otimes_A M, N) \simeq \mathrm{Map}_{\mathrm{Mod}_A(\mathcal{C})}(M, f^*N).$$

Use the fact that $\mathcal{C}$ is symmetric monoidal and that the relative tensor product satisfies the expected universal property. You may assume standard results from *Higher Algebra*, such as the presentability and stability of $\mathrm{Mod}_A(\mathcal{C})$, and the colimit-preserving nature of the tensor product.

- **Right adjoint (Coextension of Scalars).** Suppose that B is dualizable as an A-module object in $\mathcal{C}$. Then show that f^* also admits a right adjoint given by

$$f_* : \mathrm{Mod}_A(\mathcal{C}) \longrightarrow \mathrm{Mod}_B(\mathcal{C}), \quad M \mapsto \mathrm{Hom}_A(B, M),$$

where $\mathrm{Hom}_A(B, M)$ denotes the internal Hom in $\mathrm{Mod}_A(\mathcal{C})$, assuming that $\mathcal{C}$ is closed symmetric monoidal. Verify formally that this is right adjoint to f^*, i.e., for every $M \in \mathrm{Mod}_A(\mathcal{C})$ and $N \in \mathrm{Mod}_B(\mathcal{C})$, there is a natural equivalence

$$\mathrm{Map}_{\mathrm{Mod}_A(\mathcal{C})}(f^*N, M) \simeq \mathrm{Map}_{\mathrm{Mod}_B(\mathcal{C})}(N, \mathrm{Hom}_A(B, M)).$$

Remark. The existence of both the left adjoint $f_!$ and the right adjoint f_* can be understood as instances of (left and right) Kan extensions in the ∞-categorical setting. The foundational theory developed in *Higher Algebra* ensures the existence of such adjoints under the stated conditions, particularly that $\mathrm{Mod}_A(\mathcal{C})$ is presentable and that the tensor product preserves colimits in each variable.

7. ∞-Operads and Monoidal ∞-Categories

An **∞-operad** is defined by Lurie (HA Definition 2.1.1.10) as a coCartesian fibration

$$p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$$

satisfying certain inert-lifting and Segal conditions. Equivalently, a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is an ∞-operad with essentially surjective map to $\mathrm{N}(\mathrm{Fin}_*)$, and a *monoidal functor* corresponds to a map of ∞-operads over $\mathrm{N}(\mathrm{Fin}_*)$.

Using this perspective, prove the following:

- Show that if $i : \mathcal{A}^{\otimes} \to \mathcal{B}^{\otimes}$ is a map of ∞-operads over $\mathcal{O}^{\otimes}$, then the induced forgetful functor

$$\theta : \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C})$$

(between algebras in a fixed coCartesian fibration $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$) **admits a left adjoint** (a "free $\mathcal{B}$-algebra" functor) whenever certain operadic colimits exist. Specifically, combine HA Proposition 3.1.3.3 and Corollary 3.1.3.4 to formulate the necessary criterion (involving the existence of operadic colimit diagrams) and prove that this criterion is sufficient.

- Apply this to the inclusion of ∞-operads $\mathsf{Assoc} \hookrightarrow \mathsf{Comm}$ (the associative into the commutative operad). Show that if $\mathcal{C}^{\otimes}$ admits sifted colimits (e.g. $\mathcal{C}$ presentable), then the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$ has a left adjoint — the **free commutative algebra** on a given associative algebra. Describe (at least informally) how this free commutative algebra can be constructed by forming the relevant colimits in $\mathcal{C}$.

8. Derived ∞-Categories and Sheaf Theory

Let $\mathcal{A}$ be a Grothendieck abelian category (for instance, sheaves of abelian groups on a space). Its derived ∞-category $D(\mathcal{A})$ can be constructed by inverting quasi-isomorphisms in chain complexes. Prove that:

- $D(\mathcal{A})$ is a **presentable stable ∞-category** and admits a natural t-structure whose heart is equivalent to $\mathcal{A}$ (HA Proposition 1.3.5.21).
  (Hint: Use the model of bounded-above complexes of injectives or projectives.)

- Deduce that the homotopy category $hD(\mathcal{A})$ is a **triangulated category** (with translation given by the suspension in $D(\mathcal{A})$) by applying HA Theorem 1.1.2.14.
  Verify that the distinguished triangles in $hD(\mathcal{A})$ correspond exactly to the usual exact triangles of complexes.

- As an illustration, let X be a topological space. Show that $D(\mathrm{Shv}(X))$ (the derived ∞-category of sheaves of abelian groups on X) **satisfies descent**: for any open cover $\{U_i\}$, the global sections functor can be computed as the limit of the Čech nerve of the cover.
  (Equivalently, prove that sheaf cohomology can be computed via the ∞-categorical Čech complex, using left Kan extensions and the Beck–Chevalley condition.)

9. Stable ∞-Categories and Triangulated Structures

Recall Lurie's definition: an ∞-category $\mathcal{C}$ is **stable** if it has a zero object and every map admits a fiber and a cofiber, with fibers and cofibers agreeing (Def. 1.1.1.9). Prove:

- $\mathcal{C}$ has all finite limits and colimits, and a square in $\mathcal{C}$ is a pushout if and only if it is a pullback (HA Prop. 1.1.3.4). Conclude that the suspension functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ (taking cofibers of identity maps) is an equivalence of $\mathcal{C}$.

- Using HA Theorem 1.1.2.14, show that $h\mathcal{C}$ carries a natural triangulated structure. Namely, verify that given any morphism $f\colon X \to Y$ in $\mathcal{C}$ one can form a fiber-cofiber sequence

  $$X \to Y \to Z \to X[1]$$

  whose image in $h\mathcal{C}$ is a distinguished triangle. Check that Verdier's axioms (TR1–TR4) hold.

- Finally, let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between stable ∞-categories. Prove that the induced functor

  $$hF\colon h\mathcal{C} \to h\mathcal{D}$$

  is an exact functor of triangulated categories (it commutes with the shift and sends distinguished triangles to distinguished triangles). *Example:* verify these statements for $\mathcal{C} = \mathrm{Mod}_R$ where R is an E_∞-ring and for $\mathcal{C} = \mathrm{Perf}(A)$ for a ring A.

10. Spectral Algebra and E_n-Ring Structures

Let $E_n^{\otimes}$ denote the little n-cubes ∞-operad (so E_1 is the associative operad and E_∞ is the commutative one). Using Lurie's Dunn Additivity Theorem (HA 5.1.2.2: mathias.edu), show that:

- For nonnegative k, ℓ, there is an equivalence of ∞-operads $E_k \otimes E_\ell \simeq E_{k+\ell}$.
  Conclude that equipping an ∞-category with an $E_{k+\ell}$-monoidal structure is equivalent to giving it compatible E_k- and E_ℓ-monoidal structures. In particular, deduce that an E_n-algebra in a symmetric monoidal ∞-category $\mathcal{C}$ can be viewed as an associative algebra object in the $(n-1)$-fold monoidal ∞-category of E_{n-1}-algebras.

- Let R be an E_n-ring spectrum (an E_n-algebra in $\mathcal{S}p$).
  Prove that the ∞-category Mod_R of left R-modules is naturally an E_{n-1}-monoidal stable ∞-category. Describe this tensor product (often given by the relative tensor over R) and check that for $n = 2$ it is (braided) monoidal, while for $n = \infty$ it becomes symmetric monoidal.
  Show that Mod_R is presentable and stable, and that restriction-of-scalars along a map of E_n-rings has both adjoints (extension and coextension of scalars).

  Bonus: Analyze the homotopy groups $\pi_* R$ of an E_n-ring; for instance, explain why for $n \geq 2$ the graded ring $\pi_*(R)$ is graded-commutative.

11. ∞-Topoi and Descent

Consider a flat morphism of commutative rings $f : A \to B$. Let $B^\bullet = B^{\otimes_A \bullet}$ denote the Čech nerve (the cosimplicial ring with $B^{\otimes_A n}$ in degree n). Using the Barr–Beck argument as in HA §4.7.5 (mathias.edu), prove the following descent statement:

- Define the ∞-category of **descent data** $\mathrm{Desc}(f)$ as the limit of the cosimplicial ∞-category
  $$\cdots \rightrightarrows \mathrm{Mod}_{B \otimes_A B} \to \mathrm{Mod}_B \to \mathrm{Mod}_A.$$
  Show that $\mathrm{Desc}(f) \simeq \varprojlim \mathrm{Mod}_{B^{\otimes_A \bullet}}$.

- Prove that if $f : A \to B$ is faithfully flat, then the canonical functor $\mathrm{Mod}_A \to \mathrm{Desc}(f)$ is an equivalence of ∞-categories (generalizing Grothendieck's theorem). Conclude that quasi-coherent sheaves (or vector bundles) satisfy faithfully flat descent.

- **(Related)** More generally, let $\mathcal{X}$ be an ∞-topos. Prove that for any hypercover $U_\bullet \to *_{\mathcal{X}}$, the limit of the diagram of ∞-categories $\mathrm{Shv}(X_{/U_n})$ recovers $\mathrm{Shv}(\mathcal{X})$, giving descent for sheaves of spaces or spectra on $\mathcal{X}$.

12. Presentability, Adjunctions, and Kan Extensions

Let $\mathcal{C}$ be a small ∞-category and $\mathcal{D}$ a presentable ∞-category.

(a) Show that any functor $F : \mathcal{C} \to \mathcal{D}$ admits a **left Kan extension** $\widetilde{F} : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ along the Yoneda embedding $j : \mathcal{C} \to \mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, which preserves all (small) colimits. Deduce the universal property $\mathrm{Fun}^{\mathcal{L}}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, and hence that colimit-preserving functors out of a presheaf category are determined by their restriction to the dense subcategory $\mathcal{C}$.
mathias.edu

(b) Using Lurie's adjoint functor theorem for ∞-categories, prove that any colimit-preserving functor between presentable ∞-categories admits a right adjoint (this follows since in the presentable setting "having a right adjoint" is equivalent to preserving colimits).

Example: Consider the inclusion of the subcategory of compact/projective generators $i : \mathcal{C}_0 \to \mathcal{C}$ in a presentable ∞-category $\mathcal{C}$; then any functor $F : \mathcal{C}_0 \to \mathcal{D}$ to a cocomplete ∞-category $\mathcal{D}$ extends uniquely (as a left Kan extension) to a colimit-preserving functor $\mathcal{C} \to \mathcal{D}$.

(c) Finally, analyze left Kan extensions along fully faithful functors: if $j : \mathcal{C}_0 \to \mathcal{C}$ has a left adjoint, show $\mathrm{Lan}_j(F)$ exists for every $F : \mathcal{C}_0 \to \mathcal{E}$ and can be computed by composing F with that left adjoint. In general, formulate necessary and sufficient conditions for a left Kan extension along an inclusion j to exist in terms of (co)limits in the target.

13: Tensor Products of ∞-Operads

The ∞-category of ∞-operads is known to be presentable mathoverflow.net.

Let $\mathcal{O}_1\otimes, \mathcal{O}_2\otimes$ be two small ∞-operads. *Using the existence of (small) colimits in* Op_∞, show that there is a Boardman–Vogt tensor product operad

$$\mathcal{O}_1^{\otimes} \otimes \mathcal{O}_2^{\otimes},$$

characterized by the universal property that for any symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, algebras over $\mathcal{O}_1^{\otimes} \otimes \mathcal{O}_2^{\otimes}$ in $\mathcal{C}$ are the same as pairs of an $\mathcal{O}_1^{\otimes}$-algebra and an $\mathcal{O}_2^{\otimes}$-algebra in $\mathcal{C}$.

In particular, describe the set of colors (objects over $\langle 1 \rangle$) of $\mathcal{O}_1^{\otimes} \otimes \mathcal{O}_2^{\otimes}$ in terms of those of $\mathcal{O}_1^{\otimes}$ and $\mathcal{O}_2^{\otimes}$, and verify explicitly that when one of the operads is the commutative operad $\mathrm{Comm}^{\otimes}$, the tensor product yields the other operad (i.e., $\mathrm{Comm}^{\otimes} \otimes$ = the unit) mathoverflow.net.

14: Monoidal ∞-Categories and CoCartesian Fibrations

Recall that a symmetric monoidal ∞-category can be defined as a coCartesian fibration

$$p : \mathcal{C}^{\otimes} \to \Gamma^{\mathrm{op}}$$

(over the category of pointed finite sets) satisfying two axioms (M1) and (M2) idrissi.eu.

For instance, if $\mathcal{C}$ is an ordinary category with finite products, one constructs $\mathcal{C}^{\otimes}$ so that the fiber over $\langle n \rangle_+$ is $\mathcal{C}^n$, and p is coCartesian.

(a) Prove in detail that in this construction p indeed is coCartesian and that the fiber over $\langle n \rangle_+$ is (equivalent to) $\mathcal{C}^n$ (properties (M1) and (M2) in Lurie's language) idrissi.eu.

(b) Deduce from this that an algebra over the associative operad (an E_1-algebra) in $\mathcal{C}^{\otimes}$ is the same as a monoid object in $\mathcal{C}$, and an algebra over $\mathrm{Comm}^{\otimes}$ is the same as a commutative monoid object in $\mathcal{C}$. (In other words, recover classical notions of algebra objects via the fibration language.)

15: Dunn's Additivity Theorem for Little Cubes

The Little Cubes operads E_k satisfy a fundamental additivity property: the tensor product $E_m \otimes E_n$ is equivalent to E_{m+n}. State and prove Dunn's Additivity Theorem (cf. HA §5.1.2), which asserts that the functor

$$E_m \otimes E_n \longrightarrow E_{m+n}$$

constructed by sending a pair of configurations of m-cubes and n-cubes into a configuration of $(m+n)$-cubes is an equivalence of ∞-operads (math.ias.edu).

Use the operadic tensor product and the little cubes model to construct this equivalence. As an application, show that giving an E_{m+n}-algebra in a symmetric monoidal ∞-category is equivalent to giving an E_n-algebra in the category of E_m-algebras (and vice versa).

16: Iterated Loop Spaces as E_n-Algebras

Let X be a pointed topological space and let $\Omega^k X$ be its k-fold loop space. Using the little k-cubes operad model, show that $\Omega^k X$ carries a canonical E_k-algebra structure. More precisely, describe the action of the topological little k-cubes operad on $\Omega^k X$ by sending a collection of rectilinear embeddings of k-cubes with disjoint images to the operation that "concatenates" loops (see HA 5.2.6, `math.ias.edu`).

Prove that this indeed defines an E_k-monoid structure on $\Omega^k X$. Conversely, show that any *grouplike* E_k-space is (up to homotopy) an iterated loop space of some connected space (May's Recognition Principle in the ∞-categorical context).

17: Faithfully Flat Descent via Barr–Beck

In *Higher Algebra* §4.7.5,[1] Lurie sets up an ∞-categorical version of descent theory for ring spectra. Let

$$f^* : A \to B$$

[1] HA §4.7.5 reference link

be a map of E_∞-rings which is *faithfully flat* (i.e., B is a flat A-module and $\pi_0(f)$ is faithfully flat). Consider the extension-of-scalars functor

$$f^* : \mathrm{Mod}_A \to \mathrm{Mod}_B$$

and its left adjoint

$$f_* : \mathrm{Mod}_B \to \mathrm{Mod}_A$$

(i.e., the restriction-of-scalars functor).

Use the ∞-categorical Barr–Beck theorem to show that f^* exhibits Mod_A as the limit (in Cat_∞) of the simplicial diagram:

$$\mathrm{Mod}_B \rightrightarrows\!\!\!\!\rightarrow \mathrm{Mod}_{B\otimes_A B} \rightrightarrows\!\!\!\!\rightarrow \mathrm{Mod}_{B\otimes_A B\otimes_A B} \dashrightarrow \cdots$$

i.e.,

$$\mathrm{Mod}_A \simeq \mathrm{Tot}\left(\mathrm{Mod}_{B^{\otimes_A(\bullet+1)}}\right).$$

In particular, prove that the forgetful functor with descent data

$$\mathrm{Desc}(f) \simeq \lim\left(\mathrm{Mod}_B \rightrightarrows\!\!\!\!\rightarrow \mathrm{Mod}_{B\otimes_A B} \rightrightarrows\!\!\!\!\rightarrow \cdots\right)$$

is equivalent to Mod_A.

(You may assume or prove the key hypotheses of the Barr–Beck theorem in this context.)

18: Adjoint Functor Theorem for ∞-Categories

State and prove the Adjoint Functor Theorem in the setting of presentable ∞-categories (as sketched in SAG, Remark 2.6; see rezk.web.illinois.edu).

In particular, let

$$F\colon \mathcal{A} \to \mathcal{B}$$

be a functor between presentable ∞-categories.

- Prove that F admits a *right adjoint* if and only if it preserves all small colimits.
- Prove that F admits a *left adjoint* if and only if it preserves all small limits and is accessible.

As an application, use this to characterize when a symmetric monoidal functor between presentable symmetric monoidal ∞-categories admits a (lax symmetric monoidal) right adjoint.

19: Sheafification and ∞-Topoi

Let X be a topological space (or more generally a small ∞-site). Show that the ∞-category $\mathrm{Shv}(X)$ of space-valued sheaves on X is an ∞-topos. Equivalently, prove that the inclusion

$$\mathrm{Shv}(X) \hookrightarrow \mathrm{PSh}(X)$$

of sheaves into all presheaves admits a left exact left adjoint (the sheafification functor). In other words, the localization $\mathrm{PSh}(X) \to \mathrm{Shv}(X)$ is left exact.

(For guidance, recall that for a cover $\{U_i \to U\}$ in X, sheafification enforces the usual Čech descent limit condition, and one shows this localization preserves finite limits — see rezk.web.illinois.edu.)

Conclude that $\mathrm{Shv}(X)$ satisfies Lurie's axioms for an ∞-topos (SAG Def. 2.4).

[Hint: In the case of an ordinary topological space this is classical; you may use the fact that the sheafification of presheaves of spaces preserves finite limits.]

20: Slices of ∞-Topoi

Let $\mathcal{X}$ be an ∞-topos and let $U \in \mathcal{X}$ be any object. Prove that the slice ∞-category $\mathcal{X}_{/U}$ is again an ∞-topos.

(For example, one can use the fact that if $\mathcal{X} = \mathrm{Shv}(T)$ is presented by a site T, then $\mathcal{X}_{/U} \simeq \mathrm{Shv}(T_{/U})$, where $T_{/U}$ is the "slice site" of objects over U.)

As part of the solution, verify Example 2.9 of SAG: when $\mathcal{X} = \mathcal{S}$ (the ∞-category of spaces) and U is a space, the slice $\mathcal{S}_{/U}$ is an ∞-topos whose underlying 1-topos is $\mathrm{Fun}(\pi_1 U, \mathrm{Set})$ — see rezk.web.illinois.edu.

Deduce in general that the underlying 1-topos of $\mathcal{X}_{/U}$ depends only on the fundamental groupoid of U.

21: Stabilization and Spectrum Objects

Let $\mathcal{C}$ be a pointed presentable ∞-category with all finite limits and colimits. Define its stabilization $\mathrm{Sp}(\mathcal{C})$ as the ∞-category of spectrum objects in $\mathcal{C}$ (as in HA Ch. 6). Show that $\mathrm{Sp}(\mathcal{C})$ is a stable presentable ∞-category and that the canonical functor

$$\Sigma^\infty : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$$

exhibits $\mathrm{Sp}(\mathcal{C})$ as the *stable envelope* of $\mathcal{C}$.

That is, prove that for any stable presentable ∞-category $\mathcal{D}$, functors $\mathrm{Sp}(\mathcal{C}) \to \mathcal{D}$ are equivalent to functors $\mathcal{C} \to \mathcal{D}$ that take finite colimits in $\mathcal{C}$ to colimits in $\mathcal{D}$. Equivalently, show that $\mathrm{Sp}(\mathcal{C})$ is initial among stable presentable ∞-categories receiving a colimit-preserving functor from $\mathcal{C}$.

(In *Higher Algebra* 7.3.1.4, Lurie proves precisely that Σ^∞ exhibits $\mathrm{Sp}(\mathcal{C})$ as the stable envelope — see math.ias.edu.)

22: Localizations of E_∞-Rings

Let R be a (connective) E_∞-ring and let $x \in \pi_0(R)$ be an element. Define the localization $R[x^{-1}]$ by freely inverting x in the category of E_∞-rings (see section 7.2.3 in *Higher Algebra*, Lurie 2017).

Prove that this localization exists and is characterized by the usual universal property: any map $R \to R'$ of E_∞-rings in which x becomes invertible factors uniquely through $R[x^{-1}]$.

Show that on homotopy groups this agrees with the classical formula

$$\pi_*(R[x^{-1}]) \cong \pi_*(R)[x^{-1}],$$

by verifying that $\pi_0(R[x^{-1}]) \cong \pi_0(R)[x^{-1}]$ and that the localization behaves correctly on higher homotopy groups.

As an example, compute the localization of the p-local sphere spectrum at p, and compare it to the Eilenberg–MacLane spectrum of the localized ring.

23: Cotangent Complex of E_∞-Rings

In *Higher Algebra* §7.3, Lurie defines the cotangent complex $L_{A/R}$ for a map $R \to A$ of E_∞-rings. Give this definition—for example, via the stabilization of the slice category $\mathrm{Alg}_{R/}$—and prove its basic properties.

Show in particular that if A is an ordinary discrete commutative ring (viewed as an E_∞-ring), then

$$L_A \simeq \Omega^1_{A/\mathbb{Z}}$$

in degree 0, where $\Omega^1_{A/\mathbb{Z}}$ is the classical module of Kähler differentials, and that L_A vanishes in higher homotopy groups.

More generally, prove that

$$L_{B/R} \simeq 0$$

precisely when B is étale over R (assuming connective, finitely presented hypotheses).

Use these computations to derive obstruction-theoretic consequences for maps of rings (e.g. formal smoothness and unramifiedness in the spectral sense).

24: Algebras and Modules in Presentable Categories

Let $\mathcal{C}^\otimes$ be a presentable symmetric monoidal ∞-category whose tensor product preserves colimits separately in each variable. Show that for any small ∞-operad $\mathcal{O}$, the ∞-category $\mathrm{Alg}_\mathcal{O}(\mathcal{C})$ of $\mathcal{O}$-algebra objects in $\mathcal{C}$ is itself presentable.

Concretely, construct colimits in $\mathrm{Alg}_\mathcal{O}(\mathcal{C})$ by taking suitable operadic (lax) colimit diagrams in $\mathcal{C}$, and show that the forgetful functor

$$\mathrm{Alg}_\mathcal{O}(\mathcal{C}) \to \mathcal{C}$$

creates these colimits.

Deduce in particular that $\mathrm{Alg}_{E_1}(\mathcal{C})$ and $\mathrm{Alg}_{E_\infty}(\mathcal{C})$ admit all small colimits, and that the free algebra functors preserve colimits.

Finally, analyze the case of module categories: if A is an algebra in $\mathcal{C}$, prove that the ∞-category of A-modules in $\mathcal{C}$ is also presentable, and that base-change along maps of algebras satisfies the expected adjointability properties (as discussed in *Higher Algebra*, Ch. 4).

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