Assignment 6: Solutions

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Solution 1:

Problem Statement: Let $f : X \to Y$ be a continuous map of topological spaces, and $\mathscr{G} \in \mathcal{S}h_Y$. Then the inverse image $f^{-1}\mathscr{G}$ of $\mathscr{G}$ is the sheaf on X obtained from the fibre-product $X \times_Y \mathscr{E}(\mathscr{G})$ consisting of points $(x, e) \in X \times \mathscr{E}(\mathscr{G})$ such that $f(x) = \pi_{\mathscr{G}}(e)$, where $\pi = \pi_{\mathscr{G}} : \mathscr{E}(\mathscr{G}) \to Y$ is the standard projection map. The space $X \times \mathscr{E}(\mathscr{G})$ is given the product topology, and $X \times_Y \mathscr{E}(\mathscr{G})$ the subspace topology. It is easy to see that the natural map $X \times_Y \mathscr{E}(\mathscr{G}) \to X$ makes $X \times_Y \mathscr{E}(\mathscr{G})$ into an étale space over X and hence gives a sheaf, which we denote $f^{-1}\mathscr{G}$. Equivalently, consider the presheaf $f^{\#}\mathscr{G}$ given by

$$U \mapsto \varinjlim_{V \supseteq f(U)} \mathscr{G}(V),$$

where the direct limit is taken over open subsets V of Y containing $f(U)$. Then $f^{-1}\mathscr{G}$ is the sheafification of $f^{\#}\mathscr{G}$. It is not hard to show that f^{-1} is a left adjoint to f_*, i.e. one has a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathcal{S}h_X}(f^{-1}\mathscr{G}, \mathscr{H}) \cong \mathrm{Hom}_{\mathcal{S}h_Y}(\mathscr{G}, f_*\mathscr{H}).$$

The (easy) details may be found in
`https://www.cmi.ac.in/~pramath/random_notes/upper-lower-star.`

1. Let $f : X \to Y$ be a continuous map of topological spaces.
   (a) Show that f^{-1} is exact. Using this, show that $f_*\mathscr{E}$ is an injective sheaf on Y if $\mathscr{E}$ is injective on X.
   (b) Given $x \in X$ and $y = f(x)$, show that for any sheaf $\mathscr{G}$ on Y, there is a functorial map $\mathscr{G}_y \to (f^{-1}\mathscr{G})_x$.
   (c) Show that a map of ringed spaces $(f, f^{\#}) : (X, \mathscr{A}) \to (Y, \mathscr{B})$ is equivalent to the data $(f, f^{\flat})$ with $f^{\flat} : f^{-1}\mathscr{B} \to \mathscr{A}$ a map of sheaves of rings. Show further that if $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are locally ringed spaces, then $(f, f^{\#})$ is a map of locally ringed spaces if and only if for every $x \in X$, with $y = f(x)$, the composite

$$\mathscr{B}_y \to (f^{-1}\mathscr{B})_x \xrightarrow{f_x^{\flat}} \mathscr{A}_x$$

   is a local homomorphism.

Solution: (a) The functor f^{-1} is defined as the composition of:

- $f^{\#}$: A presheaf functor given by

$$f^{\#}\mathscr{G}(U) = \varinjlim_{V \supseteq f(U)} \mathscr{G}(V),$$

  where $U \subset X$ is open and V ranges over open neighborhoods of $f(U)$ in Y,

- Sheafification of the resulting presheaf, denoted as $f^{-1}\mathscr{G}$.

To show that f^{-1} is exact, we use the characterization of exactness in $\mathcal{Sh}_X$. A complex

$$\mathscr{F} \to \mathscr{G} \to \mathscr{H}$$

is exact at $\mathscr{G}$ if and only if for all $x \in X$, the complex of stalks

$$\mathscr{F}_x \to \mathscr{G}_x \to \mathscr{H}_x$$

is exact at $\mathscr{G}_x$.

The stalks of $f^{-1}\mathscr{G}$ at $x \in X$ are given by

$$(f^{-1}\mathscr{G})_x = \varinjlim_{V \ni f(x)} \mathscr{G}(V),$$

where the direct limit is taken over open neighborhoods V of $f(x)$. Thus, for a complex of sheaves $0 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}'' \to 0$ on Y, the exactness of the stalks

$$\mathscr{G}'_y \to \mathscr{G}_y \to \mathscr{G}''_y$$

at $y = f(x)$ implies the exactness of the complex

$$(f^{-1}\mathscr{G}')_x \to (f^{-1}\mathscr{G})_x \to (f^{-1}\mathscr{G}'')_x$$

at x.

Since f^{-1} preserves the stalkwise exactness of the complex, f^{-1} is exact.

Suppose $\mathscr{E}$ is an injective sheaf on X. We want to show that $f_*\mathscr{E}$ is injective on Y.

Let $0 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}'' \to 0$ be a short exact sequence on Y. To prove the injectivity of $f_*\mathscr{E}$, we need to show that:

$$0 \to \operatorname{Hom}_{\mathcal{Sh}_Y}(\mathscr{G}', f_*\mathscr{E}) \to \operatorname{Hom}_{\mathcal{Sh}_Y}(\mathscr{G}, f_*\mathscr{E}) \to \operatorname{Hom}_{\mathcal{Sh}_Y}(\mathscr{G}'', f_*\mathscr{E}) \to 0$$

is exact.

By the adjunction property of f^{-1} and f_*, we have:

$$\operatorname{Hom}_{\mathcal{Sh}_Y}(\mathscr{G}, f_*\mathscr{E}) \cong \operatorname{Hom}_{\mathcal{Sh}_X}(f^{-1}\mathscr{G}, \mathscr{E}).$$

Since $\mathscr{E}$ is injective, the sequence:

$$0 \to \operatorname{Hom}_{\mathcal{Sh}_X}(f^{-1}\mathscr{G}', \mathscr{E}) \to \operatorname{Hom}_{\mathcal{Sh}_X}(f^{-1}\mathscr{G}, \mathscr{E}) \to \operatorname{Hom}_{\mathcal{Sh}_X}(f^{-1}\mathscr{G}'', \mathscr{E}) \to 0$$

is exact.

Therefore, the adjunction ensures that:

$$0 \to \mathrm{Hom}_{\mathcal{S}h_Y}(\mathscr{G}', f_*\mathscr{E}) \to \mathrm{Hom}_{\mathcal{S}h_Y}(\mathscr{G}, f_*\mathscr{E}) \to \mathrm{Hom}_{\mathcal{S}h_Y}(\mathscr{G}'', f_*\mathscr{E}) \to 0$$

is also exact, proving that $f_*\mathscr{E}$ is injective.

(b) Let $x \in X$ and $y = f(x)$. The stalk of a sheaf $\mathscr{G}$ on Y at y is given by

$$\mathscr{G}_y = \varinjlim_{y \in V} \mathscr{G}(V),$$

where V ranges over all open neighborhoods of y, and $\mathscr{G}(V)$ is the set of sections of $\mathscr{G}$ over V. An element $g \in \mathscr{G}_y$ is represented by a family of sections $g_V \in \mathscr{G}(V)$, with g_V restricting to $g_{V'}$ whenever $V' \subseteq V$.

The stalk of the pullback sheaf $f^{-1}\mathscr{G}$ at $x \in X$ is given by

$$(f^{-1}\mathscr{G})_x = \varinjlim_{x \in U} f^{-1}\mathscr{G}(U),$$

where U ranges over all open neighborhoods of x. Since $f^{-1}\mathscr{G}$ is defined by

$$f^{-1}\mathscr{G}(U) = \varinjlim_{V \supseteq f(U)} \mathscr{G}(V),$$

we have

$$(f^{-1}\mathscr{G})_x = \varinjlim_{V \ni f(x)} \mathscr{G}(V).$$

Given $g \in \mathscr{G}_y$, represented by sections $g_V \in \mathscr{G}(V)$, there exists a natural map to $(f^{-1}\mathscr{G})_x$. For any open set U containing x, where $f(U) \subseteq V$, the section g_V determines a corresponding element in $f^{-1}\mathscr{G}(U)$. Taking the direct limit over all $U \ni x$, this defines an element in $(f^{-1}\mathscr{G})_x$.

This process defines a map

$$\mathscr{G}_y \to (f^{-1}\mathscr{G})_x,$$

where each $g \in \mathscr{G}_y$ is sent to its corresponding element in $(f^{-1}\mathscr{G})_x$. The map is functorial because it respects the restriction of sections. If $g_V \in \mathscr{G}(V)$ restricts to $g_{V'} \in \mathscr{G}(V')$ for $V' \subseteq V$, the corresponding elements in $(f^{-1}\mathscr{G})_x$ are consistent. Moreover, the map respects morphisms of sheaves: for any morphism $\phi : \mathscr{G} \to \mathscr{H}$, the diagram

$$\begin{tikzcd}
\mathscr{G}_y \arrow[r] \arrow[d, "\phi_y"] & (f^{-1}\mathscr{G})_x \arrow[d, "f^{-1}\phi_x"] \\
\mathscr{H}_y \arrow[r] & (f^{-1}\mathscr{H})_x
\end{tikzcd}$$

commutes.

Thus, for any sheaf $\mathscr{G}$ on Y, there exists a natural, functorial map

$$\mathscr{G}_y \to (f^{-1}\mathscr{G})_x.$$

(c)

A ringed space $(X, \mathscr{A})$ consists of a topological space X equipped with a sheaf of rings $\mathscr{A}$. A morphism of ringed spaces $(f, f^\#) : (X, \mathscr{A}) \to (Y, \mathscr{B})$ consists of:

1. A continuous map $f : X \to Y$,
2. A morphism of sheaves of rings $f^\# : \mathscr{B} \to f_*\mathscr{A}$,

where $f_*\mathscr{A}$ is the pushforward sheaf on Y.

Using the adjunction between f^{-1} and f_*, a morphism of sheaves $\mathscr{B} \to f_*\mathscr{A}$ is equivalent to a morphism $f^{-1}\mathscr{B} \to \mathscr{A}$ on X. Concretely, the adjunction provides a bijection:

$$\mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathscr{B}, f_*\mathscr{A}) \cong \mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{B}, \mathscr{A}).$$

Thus, given $f^\# : \mathscr{B} \to f_*\mathscr{A}$, we obtain a unique map $f^\flat : f^{-1}\mathscr{B} \to \mathscr{A}$, and conversely, given $f^\flat$, we can construct $f^\#$.

The data of $(f, f^\#)$ is therefore equivalent to the data of $(f, f^\flat)$, where $f^\flat : f^{-1}\mathscr{B} \to \mathscr{A}$ is a morphism of sheaves of rings.

For locally ringed spaces, recall that $(X, \mathscr{A})$ is a ringed space such that at every point $x \in X$, the stalk $\mathscr{A}_x$ is a local ring. A morphism of locally ringed spaces $(f, f^\#) : (X, \mathscr{A}) \to (Y, \mathscr{B})$ must satisfy the condition that the induced map on stalks:

$$f_x^\# : \mathscr{B}_{f(x)} \to \mathscr{A}_x$$

is a local homomorphism of rings. To verify this condition, consider the composite map:

$$\mathscr{B}_y \to (f^{-1}\mathscr{B})_x \to \mathscr{A}_x,$$

where $y = f(x)$, $\mathscr{B}_y$ is the stalk of $\mathscr{B}$ at y, and $(f^{-1}\mathscr{B})_x$ is the stalk of $f^{-1}\mathscr{B}$ at x.

The first map $\mathscr{B}_y \to (f^{-1}\mathscr{B})_x$ is the natural map from the stalk of $\mathscr{B}$ at y to the stalk of the pullback sheaf $f^{-1}\mathscr{B}$ at x. The second map $(f^{-1}\mathscr{B})_x \to \mathscr{A}_x$ is induced by $f^\flat : f^{-1}\mathscr{B} \to \mathscr{A}$ on stalks. The composite:

$$\mathscr{B}_y \to (f^{-1}\mathscr{B})_x \to \mathscr{A}_x$$

must be a local homomorphism, meaning: 1. $f_x^\#$ maps the maximal ideal $m_{\mathscr{B}_y}$ of $\mathscr{B}_y$ into the maximal ideal $m_{\mathscr{A}_x}$ of $\mathscr{A}_x$, 2. The induced map on residue fields is well-defined.

Since the composite $\mathscr{B}_y \to \mathscr{A}_x$ factors through $(f^{-1}\mathscr{B})_x$, the condition that $f_x^\#$ is a local homomorphism is equivalent to the map:

$$\mathscr{B}_y \to (f^{-1}\mathscr{B})_x \to \mathscr{A}_x$$

being a local homomorphism.

A morphism of ringed spaces $(f, f^\#)$ is therefore equivalent to the data $(f, f^\flat)$, where $f^\flat : f^{-1}\mathscr{B} \to \mathscr{A}$ is a morphism of sheaves of rings. For locally ringed spaces, the condition that $(f, f^\#)$ is a morphism of locally ringed spaces is equivalent to requiring that for every $x \in X$, with $y = f(x)$, the composite:

$$\mathscr{B}_y \to (f^{-1}\mathscr{B})_x \to \mathscr{A}_x$$

is a local homomorphism of rings.

Solution 2:

The tensor product of sheaves of modules and the upper-star functor. Let $(X, \mathscr{O}_X)$ be a ringed space, and let $\mathscr{F}, \mathscr{G} \in \mathrm{Mod}_{\mathscr{O}_X}$. We have a presheaf $\mathscr{F}^P \otimes_{\mathscr{O}_X} \mathscr{G}$ given by:

$$U \rightsquigarrow \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U),$$

where U is an open subset of X. The tensor product $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is defined as the sheafification of $\mathscr{F}^P \otimes_{\mathscr{O}_X} \mathscr{G}$. Note that $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \in \mathrm{Mod}_{\mathscr{O}_X}$.

Combining the universal property of sheafifications and the universal property of tensor products, we see that there is a universal $\mathscr{O}_X$-bilinear map:

$$\mathscr{F} \times \mathscr{G} \to \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G},$$

which is $\mathscr{O}_X(U)$-bilinear over each open set $U \subseteq X$. This satisfies the following universal property: if $\mathscr{F} \times \mathscr{G} \to \mathscr{H}$ is a bilinear map of $\mathscr{O}_X$-modules, then there is a unique map:

$$\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \to \mathscr{H},$$

in $\mathrm{Mod}_{\mathscr{O}_X}$, such that the bilinear map factors through:

$$\mathscr{F} \times \mathscr{G} \xrightarrow{\text{universal}} \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \to \mathscr{H}.$$

If $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a map of ringed spaces, and $\mathscr{G} \in \mathrm{Mod}_{\mathscr{O}_Y}$, we define:

$$f^*\mathscr{G} = f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X,$$

where $\mathscr{O}_X$ is an $f^{-1}\mathscr{O}_Y$-algebra via $f^\#$. Note that f^* is right exact but need not be exact. It is easy to see that:

$$\mathrm{Hom}_X(f^*\mathscr{G}, \mathscr{H}) \cong \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{H}),$$

where $\mathrm{Hom}_X(-,-)$ is the Hom functor in $\mathrm{Mod}_{\mathscr{O}_X}$.

Problem Statement 2: Let $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, $f : X \to Y$ a map of schemes, and M an A-module. Show that $f^*\widetilde{M}$ is the sheafification of $M \otimes_A B$.

Solution: Let $f \colon X \to Y$ be a morphism of schemes, where $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$, and let M be an A-module. We aim to show that $f^*\widetilde{M}$ is the sheafification of $M \otimes_A B$.

Recall that for any morphism of schemes $f \colon X \to Y$, the pullback of a quasi-coherent sheaf $\mathscr{F}$ on Y is defined as:

$$f^*\mathscr{F} = f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X.$$

In our case, $\mathscr{F} = \widetilde{M}$, where $\widetilde{M}$ is the quasi-coherent sheaf on $Y = \operatorname{Spec}(A)$ associated to the A-module M. Thus, we have:

$$f^*\widetilde{M} = f^{-1}\widetilde{M} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X.$$

The structure sheaf $\mathscr{O}_Y$ on Y corresponds to the ring A, and $\mathscr{O}_X$ on X corresponds to B. Under the equivalence of categories $\mathsf{Qcoh}(\operatorname{Spec}(A)) \simeq \mathsf{Mod}(A)$, the sheaf $\widetilde{M}$ corresponds to M, and for affine schemes, we identify:

$$\Gamma(U, \widetilde{M}) = M \otimes_A \Gamma(U, \mathscr{O}_Y),$$

where $U \subseteq Y$ is an open affine subset. Pulling back to X, we have:

$$f^*\widetilde{M} = \widetilde{M \otimes_A B}.$$

To justify this isomorphism rigorously, note that f^* satisfies the adjunction:

$$\mathrm{Hom}_X(f^*\widetilde{M}, \mathscr{G}) \cong \mathrm{Hom}_Y(\widetilde{M}, f_*\mathscr{G}),$$

where $\mathscr{G}$ is any $\mathscr{O}_X$-module. Under the equivalence $\mathsf{Qcoh}(\mathrm{Spec}(A)) \simeq \mathsf{Mod}(A)$ and $\mathsf{Qcoh}(\mathrm{Spec}(B)) \simeq \mathsf{Mod}(B)$, this adjunction corresponds to the usual adjunction between scalar restriction and scalar extension of modules:

$$\mathrm{Hom}_B(M \otimes_A B, N) \cong \mathrm{Hom}_A(M, N),$$

where N is a B-module.

Therefore, $f^*\widetilde{M} \cong \widetilde{M \otimes_A B}$ as desired.

Solution 3:

Problem Statement 3: Suppose X is a scheme and $\mathscr{A}$ a sheaf of $\mathscr{O}_X$-algebras on X such that, as an $\mathscr{O}_X$-module, $\mathscr{A}$ is quasi-coherent. Show that there is an X-scheme $\mathbf{Spec}(\mathscr{A})$ such that the structure morphism $\Phi : \mathbf{Spec}(\mathscr{A}) \to X$ is an affine map, and such that $\Phi_*\mathscr{O}_{\mathbf{Spec}(\mathscr{A})}$ is canonically isomorphic to $\mathscr{A}$. Show further that the canonical map $\Phi^\# : \mathscr{O}_X \to v_*\mathscr{O}_{\mathbf{Spec}(\mathscr{A})}$ is, under this identification, the algebra map $\mathscr{O}_X \to \mathscr{A}$. Conversely, if $\Psi : S \to X$ is an affine map of schemes, show that $\mathbf{Spec}(\Psi_*\mathscr{O}_S)$ is canonically isomorphic to S and that under this identification, the structure map $\mathbf{Spec}(\Psi_*\mathscr{O}_S) \to X$ agrees with Ψ.

Solution: On any affine open subset $U \subseteq X$, we know that $\mathscr{A}|_U$ corresponds to a quasi-coherent $\mathscr{O}_X(U)$-algebra $A = \mathscr{A}(U)$. The existence of $\mathrm{Spec}\, A$ as a scheme over U follows from the affine case. Using the uniqueness of gluing for quasi-coherent sheaves, we can construct a scheme $\mathbf{Spec}(\mathscr{A})$ that locally corresponds to $\mathrm{Spec}\, A$. Thus, we obtain a scheme $\mathbf{Spec}(\mathscr{A})$ equipped with a canonical structure morphism

$$\Phi : \mathbf{Spec}(\mathscr{A}) \to X,$$

where Φ is affine because it is locally the spectrum of a ringed space over $\mathrm{Spec}\, A$.

For the first part, the pushforward of the structure sheaf satisfies

$$\Phi_*\mathscr{O}_{\mathbf{Spec}(\mathscr{A})}(U) = \mathscr{O}_{\mathbf{Spec}(\mathscr{A})}(\Phi^{-1}(U)) = A = \mathscr{A}(U).$$

Hence, $\Phi_*\mathscr{O}_{\mathbf{Spec}(\mathscr{A})} \cong \mathscr{A}$. The canonical map $\Phi^\# : \mathscr{O}_X \to \Phi_*\mathscr{O}_{\mathbf{Spec}(\mathscr{A})}$, under this identification, is the natural algebra homomorphism $\mathscr{O}_X \to \mathscr{A}$, which satisfies the required compatibility conditions.

For the converse, let $\Psi : S \to X$ be an affine morphism of schemes. Then, the pushforward sheaf $\Psi_*\mathscr{O}_S$ is a quasi-coherent $\mathscr{O}_X$-algebra. Locally, on an affine subset $U = \mathrm{Spec}\, B \subseteq X$, we have $S|_U \cong \mathrm{Spec}\, A$ for $A = \Psi_*\mathscr{O}_S(U)$. By the uniqueness of affine schemes associated with quasi-coherent algebras, we obtain a canonical isomorphism

$$\mathbf{Spec}(\Psi_*\mathscr{O}_S) \cong S.$$

Under this identification, the structure map $\mathbf{Spec}(\Psi_*\mathscr{O}_S) \to X$ agrees with Ψ, as both maps locally correspond to the ring homomorphisms induced by the algebra structure of $\Psi_*\mathscr{O}_S$ over $\mathscr{O}_X$.