An Expository Note on the Grothendieck-Riemann-Roch Theorem, Part-I

Virat Chauhan

Contents

1 Introduction

In this note, we provide an expository discussion of the Grothendieck-Riemann-Roch (GRR) theorem, a far-reaching generalization of the classical Riemann-Roch theorem, formulated by Alexander Grothendieck using the tools of sheaf cohomology, K-theory, and intersection theory.

1.1 Classical Riemann-Roch Theorem

Let C be a smooth projective curve over an algebraically closed field, and let D be a divisor on C. A divisor D is a formal sum of points on C with integer coefficients:

$$D = \sum_{P \in C} n_P \cdot P, \quad n_P \in \mathbb{Z}.$$

The degree of the divisor D is defined as the sum of the coefficients:

$$\deg(D) = \sum_{P \in C} n_P.$$

Associated with a divisor D is the vector space $L(D)$, the space of global sections of the line bundle $\mathcal{O}_C(D)$, defined as

$$L(D) = H^0(C, \mathcal{O}_C(D)),$$

where $\mathcal{O}_C(D)$ is the sheaf of meromorphic functions on C whose poles and zeros are prescribed by D. The dimension of $L(D)$, denoted by $\ell(D)$, is

$$\ell(D) = \dim L(D).$$

The classical Riemann-Roch theorem relates the dimension $\ell(D)$ of the space of global sections of D to the degree of D and the genus g of the curve C. Specifically, for any divisor D on C, we have

$$\ell(D) - \ell(K_C - D) = \deg(D) + 1 - g,$$

where K_C is the canonical divisor associated with the sheaf of differentials Ω^1_C, and g is the genus of C.

The Euler characteristic $\chi(D)$ of the divisor D is given by

$$\chi(D) = \ell(D) - \ell(K_C - D) = \deg(D) + 1 - g.$$

This formula connects the geometric properties of the curve, such as the degree of a divisor and the genus, with the algebraic properties of the space of global sections.

For example, on a curve of genus $g = 0$, such as $\mathbb{P}^1$, the Riemann-Roch theorem simplifies to:

$$\ell(D) = \deg(D) + 1 \quad \text{for} \quad \deg(D) \geq 0.$$

In the case of genus $g = 1$, the theorem shows that divisors of degree zero correspond to the points of the Jacobian variety of the curve.

In the case where $D = K_C$ (i.e., the canonical divisor), the Riemann-Roch theorem gives

$$\ell(K_C) = g,$$

which counts the number of independent holomorphic differentials on the curve.

Thus, the classical **Riemann-Roch theorem** provides a powerful tool for studying the relationship between the geometry of a curve and the algebraic properties of divisors on the curve.

2 Background on Grothendieck's Language

2.1 Sheaf Cohomology

A review of the necessary tools from sheaf theory and cohomology. Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups on X. The global sections functor $\Gamma(X, -)$ is defined as

$$\Gamma(X, \mathcal{F}) = \mathcal{F}(X).$$

Since $\Gamma(X, -)$ is a left exact functor, its higher right-derived functors are defined as the cohomology groups $H^i(X, \mathcal{F})$ for $i \geq 0$. These are the sheaf cohomology groups of $\mathcal{F}$ on X:

$$H^i(X, \mathcal{F}) = R^i\Gamma(X, \mathcal{F}).$$

If we have a short exact sequence of sheaves on X,

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

the functors $H^i(X, -)$ yield a long exact sequence in cohomology:

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}') \to \cdots$$

One way to compute sheaf cohomology is through Čech cohomology. Let $\mathcal{U} = \{U_i\}$ be an open cover of X. The Čech cochain groups are defined as

$$C^k(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < i_1 < \cdots < i_k} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_k}),$$

with the coboundary maps $\partial^k : C^k(\mathcal{U}, \mathcal{F}) \to C^{k+1}(\mathcal{U}, \mathcal{F})$ given by

$$(\partial^k \sigma)_{i_0 i_1 \ldots i_{k+1}} = \sum_{j=0}^{k+1} (-1)^j \sigma_{i_0 \ldots \hat{i}_j \ldots i_{k+1}}.$$

The Čech cohomology groups are defined as the cohomology of the Čech complex:

$$\check{H}^k(\mathcal{U}, \mathcal{F}) = \frac{\ker(\partial^k)}{\operatorname{im}(\partial^{k-1})}.$$

For a sufficiently fine open cover $\mathcal{U}$, the Čech cohomology groups $\check{H}^i(\mathcal{U}, \mathcal{F})$ coincide with the sheaf cohomology groups $H^i(X, \mathcal{F})$, by Leray's theorem:

$$H^i(X, \mathcal{F}) \cong \lim_{\mathcal{U}} \check{H}^i(\mathcal{U}, \mathcal{F}).$$

A sheaf $\mathcal{F}$ is called flasque if for every open subset $U \subset X$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. Flasque sheaves are important because they are acyclic, meaning

$$H^i(X, \mathcal{F}) = 0 \quad \text{for all } i > 0.$$

In particular, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of sheaves, and $\mathcal{F}$ is flasque, then $\Gamma(X, -)$ gives rise to a short exact sequence of global sections:

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0.$$

Riemann-Roch Theorem Suppose X is a smooth projective variety of dimension 1 (i.e., a smooth projective curve). For every Cartier divisor D on X, the Riemann-Roch theorem asserts that:

$$\chi(X, \mathcal{O}_X(D)) = \dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^1(X, \mathcal{O}_X(D)).$$

By the duality theorem, we also have:

$$\chi(X, \mathcal{O}_X(D)) = \dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \mathcal{O}_X(K_X - D)),$$

where K_X is the canonical divisor on X, and H^0 refers to the space of global sections.

Therefore, the Riemann-Roch theorem for a smooth projective curve simplifies to:

$$\chi(X, \mathcal{O}_X(D)) = \deg(D) + 1 - \dim_k H^0(X, \mathcal{O}_X(K_X)),$$

where $\deg(D)$ is the degree of the divisor D, and the genus g of the curve is defined by:

$$g = \dim_k H^0(X, \mathcal{O}_X(K_X)).$$

Thus, the Euler characteristic of the divisor D on X is given by:

$$\chi(X, \mathcal{O}_X(D)) = \deg(D) + 1 - g.$$

This formula shows that the Euler characteristic of a line bundle on a smooth projective curve depends only on the degree of the corresponding divisor and the genus of the curve.

In the context of algebraic geometry, for a projective variety X, the cohomology groups $H^i(X, \mathcal{O}_X(n))$ of the twisted sheaves $\mathcal{O}_X(n)$ play a crucial role. For instance, by Serre's Vanishing Theorem, there exists an integer n_0 such that for all $n \geq n_0$,

$$H^i(X, \mathcal{O}_X(n)) = 0 \quad \text{for all } i > 0.$$

The Grothendieck-Riemann-Roch theorem relates the pushforward of a sheaf along a proper morphism $f : X \to Y$ to the Todd class of the relative tangent bundle and the Chern character of the sheaf. Let $\mathcal{F}$ be a coherent sheaf on X. The theorem states that in $A^*(Y)$,

$$\mathrm{ch}(f_*\mathcal{F}) \cdot \mathrm{Td}(T_Y) = f_*(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{Td}(T_X)).$$

Here, ch denotes the Chern character, and Td denotes the Todd class, both of which can be computed using sheaf cohomology and intersection theory.

2.2 Chow Groups and Chern Classes

We introduce the concepts of Chow groups, intersection theory, and Chern classes, which play a key role in the formulation of GRR. Let X be a smooth projective variety over a field k. The Chow group $\mathrm{CH}_p(X)$ of codimension-p cycles modulo rational equivalence is a central object in the study of intersection theory on X. A cycle of codimension p on X is a formal finite linear combination of irreducible subvarieties of codimension p:

$$Z = \sum_i n_i [V_i],$$

where $n_i \in \mathbb{Z}$ and $V_i \subset X$ are irreducible subvarieties of codimension p. Rational equivalence is defined by considering divisors of rational functions on subvarieties. Two cycles Z and Z' are rationally equivalent if their difference $Z - Z'$ is the divisor of a rational function. The Chow group $\mathrm{CH}_p(X)$ is then defined as the group of cycles modulo rational equivalence:

$$\mathrm{CH}_p(X) = Z_p(X)/\sim .$$

For a smooth projective variety X, the Chow groups are graded by codimension, and the Chow ring $\mathrm{CH}^*(X)$ is defined as:

$$\mathrm{CH}^*(X) = \bigoplus_{p=0}^{\dim X} \mathrm{CH}^p(X),$$

where $\mathrm{CH}^p(X) = \mathrm{CH}_{\dim X - p}(X)$. The ring structure on $\mathrm{CH}^*(X)$ is given by the intersection product, which reflects the geometric intersection of cycles:

$$\mathrm{CH}^p(X) \times \mathrm{CH}^q(X) \to \mathrm{CH}^{p+q}(X).$$

This product is bilinear, associative, and commutative. It turns $\mathrm{CH}^*(X)$ into a graded commutative ring with identity, where the identity element corresponds to the fundamental class of the variety.

Let E be a vector bundle of rank r over X. The Chern classes $c_i(E) \in \mathrm{CH}^i(X)$ are algebraic invariants associated with E that encode its geometry. The total Chern class of E is defined as:

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E),$$

where $c_i(E) \in \mathrm{CH}^i(X)$ for $i = 1, 2, \ldots, r$, and $c_r(E)$ is the top Chern class. These classes satisfy the following properties: - For a line bundle L, $c_1(L)$ is the divisor class associated with L. - Chern classes are natural, meaning that for any morphism $f : Y \to X$, the Chern classes pull back: $f^*c_i(E) = c_i(f^*E)$. - For a short exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$, the total Chern class satisfies the Whitney sum formula:

$$c(E) = c(E') \cdot c(E'').$$

- The top Chern class $c_r(E)$ represents the zero-dimensional cycle associated with the Euler characteristic of E.

The splitting principle states that for any vector bundle E of rank r, there exists a smooth proper morphism $\pi : X' \to X$ such that $\pi^* E$ splits as a direct sum of line bundles:

$$\pi^* E = L_1 \oplus L_2 \oplus \cdots \oplus L_r.$$

Using the splitting principle, the total Chern class $c(E)$ can be computed as:

$$c(E) = \prod_{i=1}^{r} (1 + c_1(L_i)),$$

where L_i are line bundles, and $c_1(L_i)$ are their first Chern classes. This allows us to reduce the computation of Chern classes of a general vector bundle to that of line bundles.

The Chern character $\mathrm{ch}(E)$ is a homomorphism from the K-theory of X to its Chow ring $\mathrm{CH}^*(X) \otimes \mathbb{Q}$. For a vector bundle E of rank r, the Chern character is defined as:

$$\mathrm{ch}(E) = r + c_1(E) + \frac{1}{2} c_1(E)^2 - c_2(E) + \cdots,$$

where the higher terms involve polynomials in the Chern classes. The Chern character satisfies:

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$$

and

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F).$$

The Todd class $\mathrm{td}(E)$ of a vector bundle E of rank r is defined as:

$$\mathrm{td}(E) = \prod_{i=1}^{r} \frac{\xi_i}{1 - e^{-\xi_i}},$$

where ξ_i are the formal Chern roots of E, satisfying $c(E) = \prod_{i=1}^{r} (1 + \xi_i)$. This can be expanded in terms of the Chern classes of E:

$$\mathrm{td}(E) = 1 + \frac{1}{2} c_1(E) + \frac{1}{12} (c_1(E)^2 + c_2(E)) + \cdots.$$

The Chern classes, Chern character, and Todd class are crucial for understanding the Grothendieck-Riemann-Roch theorem, which will be discussed next. This theorem provides a relationship between the pushforward of a class in K-theory and its image in the Chow ring via the Chern character and the Todd class. The pushforward of a vector bundle E under a proper morphism $f : X \to Y$ satisfies:

$$\mathrm{ch}(f_*(E)) = f_* \left(\mathrm{td}(T_f) \cdot \mathrm{ch}(E) \right),$$

where $\mathrm{td}(T_f)$ is the Todd class of the relative tangent bundle T_f of the morphism.

In summary, the Chow ring $\mathrm{CH}^*(X)$ and the Chern classes of vector bundles provide a powerful algebraic framework for studying the intersection theory on a variety X. The interplay between these objects forms the foundation for deeper results in algebraic geometry, including the Grothendieck-Riemann-Roch theorem.

3 The Grothendieck-Riemann-Roch Theorem

3.1 Statement of the Theorem

We present the theorem in its general form for a proper morphism $f : X \to Y$ between smooth varieties. Let $f : X \to Y$ be a proper morphism of smooth quasiprojective varieties. For any $\alpha \in K(X)$, the following identity holds in the cohomology ring $H^*(Y, \mathbb{Q})$:

$$ch(f_*\alpha) \cdot td(T_Y) = f_* \left(ch(\alpha) \cdot td(T_X) \right),$$

where ch denotes the Chern character, $td(T_X)$ and $td(T_Y)$ are the Todd classes of the tangent bundles of X and Y, and f_* on the right-hand side refers to the pushforward in cohomology.

The proof is based on the construction of the pushforward in both K-theory and cohomology and verifying the behavior of Chern characters and Todd classes under this operation. Consider the following commutative diagram of K-theory and cohomology:

$$\begin{tikzcd} K(X) \arrow[r, "f_*"] \arrow[d, "ch\cdot td(T_X)"'] & K(Y) \arrow[d, "ch\cdot td(T_Y)"] \\ H^*(X, \mathbb{Q}) \arrow[r, "f_*"] & H^*(Y, \mathbb{Q}) \end{tikzcd}$$

For a proper morphism $f : X \to Y$, the pushforward in K-theory is given by:

$$f_*([F]) = \sum_i (-1)^i [R^i f_* F],$$

where F is a coherent sheaf on X, and $R^i f_* F$ denotes the higher direct images.

The Chern character is a homomorphism from K-theory to cohomology, and the Todd class is defined using the Chern roots of the tangent bundle. For any vector bundle E on X, the Todd class is:

$$td(E) = \prod_{i=1}^{r} \frac{c_1(L_i)}{1 - e^{-c_1(L_i)}},$$

where L_i are the line bundle components from a filtration of E.

A key component of the proof is the deformation to the normal cone technique, which allows reduction of the general case of a proper morphism to the case of a closed embedding and a projection.

By the multiplicativity of the Todd class and the additivity of the Chern character, we compute:

$$ch(f_*\alpha) \cdot td(T_Y) = f_*(ch(\alpha) \cdot td(T_X)),$$

which completes the proof.

Oriented Cohomology Theory

Oriented cohomology theories provide a general framework for extending classical cohomology theories, such as Chow rings and K-theory, to a broader class of algebraic varieties and morphisms. These theories are distinguished by their association with a formal group law, which governs how characteristic classes, such as Chern classes, interact with tensor products of line bundles.

Let A^* be an oriented cohomology theory on the category of smooth quasi-projective varieties over a field k. This is a contravariant functor from the category of smooth quasi-projective varieties over k to graded rings, satisfying the following key properties:

For any disjoint union $X_1 \sqcup X_2$ of smooth varieties, we have an isomorphism:

$$A(X_1 \sqcup X_2) \cong A(X_1) \oplus A(X_2).$$

This property reflects the fact that cohomology classes can be computed separately for disjoint components.

For any vector bundle $E \to X$ over a smooth variety X, the pullback map in cohomology is an isomorphism:

$$A(X) \xrightarrow{\cong} A(E).$$

This is analogous to the homotopy invariance property in classical cohomology, which ensures that vector bundles do not alter the cohomology of the base space.

For any projective morphism $f : X \to Y$ between smooth quasi-projective varieties, there exists a pushforward map, also known as the Gysin map:

$$f_* : A(X) \to A(Y),$$

which satisfies functoriality and commutes with pullback maps. This pushforward generalizes the integration of cohomology classes along the fibers of a projective morphism and is crucial for defining the Grothendieck-Riemann-Roch theorem in this context.

A key feature of oriented cohomology theories is their Euler structure, which associates to each line bundle L over a smooth variety X an element called the Euler class $e(L) \in A(X)$. The Euler class generalizes the first Chern class in ordinary cohomology and satisfies the following fundamental property:

$$e(L_1 \otimes L_2) = F_A(e(L_1), e(L_2)),$$

where $F_A(u, v)$ is the formal group law associated with the oriented cohomology theory A^*. The formal group law governs how the Chern classes of tensor products of line bundles interact, and it distinguishes oriented cohomology theories from classical theories.

For example, in algebraic cobordism, the formal group law is given by:

$$F_A(u, v) = u + v + \beta uv,$$

where β is a characteristic element depending on the cohomology theory. This formal group law encodes the behavior of characteristic classes in the oriented cohomology theory and plays a central role in the generalization of the Grothendieck-Riemann-Roch theorem.

One of the key applications of Euler classes in oriented cohomology is the projective bundle formula. Let $E \to X$ be a vector bundle of rank n, and consider the projective bundle $P(E)$ of lines in E. Then the cohomology ring of $P(E)$ in an oriented cohomology theory is given by:

$$A(P(E)) \cong A(X)[t]/(t^n - e(E)),$$

where t is the Euler class of the tautological line bundle over $P(E)$, and $e(E)$ is the Euler class of the vector bundle E. This formula is analogous to the projective bundle theorem in classical cohomology but incorporates the more general structure of Euler classes and the formal group law.

A particularly important example of an oriented cohomology theory is algebraic cobordism, denoted by $\Omega^*(X)$. Algebraic cobordism is the universal oriented cohomology theory, meaning that any other oriented cohomology theory can be obtained as a homomorphism from algebraic cobordism. The formal group law for algebraic cobordism is universal, making this theory a powerful tool for understanding more specific cohomology theories like Chow rings or K-theory.

Grothendieck-Riemann-Roch Theorem in Algebraic Cobordism

The Grothendieck-Riemann-Roch theorem (GRR) can be extended to the framework of algebraic cobordism, which serves as a universal oriented cohomology theory. This generalization incorporates the formal group law of algebraic cobordism into the definition of characteristic classes, such as the Chern character and the Todd genus.

Let $f : X \to Y$ be a projective morphism between smooth quasi-projective varieties, and let $\Omega^*(X)$ and $\Omega^*(Y)$ denote the algebraic cobordism rings of X and Y, respectively. The algebraic cobordism version of the Grothendieck-Riemann-Roch theorem states:

$$f_*(\alpha) = \mathcal{T}_\Omega(X/Y) \cdot \mathrm{Ch}_\Omega(\alpha),$$

where: - $f_* : \Omega^*(X) \to \Omega^*(Y)$ is the pushforward in algebraic cobordism, - $\mathcal{T}_\Omega(X/Y)$ is the Todd class defined using the formal group law of algebraic cobordism, - $\mathrm{Ch}_\Omega(\alpha)$ is the Chern character in algebraic cobordism, which takes values in the cobordism theory.

In algebraic cobordism, the characteristic classes are governed by a formal group law. For two line bundles L_1 and L_2 over a smooth variety, the formal group law $F_\Omega(u, v)$ describes how the Chern classes of their tensor product interact:

$$F_\Omega(u, v) = u + v + \beta uv,$$

where β is a parameter depending on the cobordism theory. The Todd class $\mathcal{T}_\Omega(X/Y)$ is constructed using this formal group law, reflecting the interaction between Chern classes of vector bundles in cobordism.

This generalization can be represented in the following commutative diagram, where the pushforward in cobordism is compatible with the Todd class and the Chern character:

$$\begin{tikzcd}
\Omega^*(X) \arrow[r, "f_*"] \arrow[d, "\mathrm{Ch}_\Omega \cdot \mathcal{T}_\Omega(X)"] & \Omega^*(Y) \arrow[d, "\mathrm{Ch}_\Omega \cdot \mathcal{T}_\Omega(Y)"] \\
A^*(X) \arrow[r, "f_*"] & A^*(Y)
\end{tikzcd}$$

The Grothendieck-Riemann-Roch (GRR) theorem classically relates the Chern character and Todd class to the pushforward in K-theory. In the context of *Algebraic Cobordism*, introduced by Levine and Morel, this result can be extended to a more general framework where cobordism groups $\Omega^*(X)$ replace the classical cohomology groups and K-theory.

Let $f : X \to Y$ be a projective morphism between smooth quasi-projective varieties over a field k, and let E be a vector bundle on X. The classical Grothendieck-Riemann-Roch theorem states:

$$\mathrm{ch}(f_*(E)) = f_*(\mathrm{ch}(E) \cdot \mathrm{Td}(X)),$$

where $\mathrm{ch}(E)$ is the Chern character of E, $\mathrm{Td}(X)$ is the Todd class of T_X, and f_* denotes the pushforward in K-theory.

In algebraic cobordism, the pushforward f_* is defined analogously, but the Chern character and Todd class are modified to reflect the cobordism formalism. Specifically, the cobordism groups $\Omega^*(X)$ form a universal oriented cohomology theory, which means that any other oriented theory (such as Chow groups or K-theory) factors through $\Omega^*(X)$. The GRR theorem in algebraic cobordism is formulated as follows.

Let $\Omega^*(X)$ be the algebraic cobordism ring of a smooth quasi-projective variety X. The first Chern class $c_1(L)$ of a line bundle L on X is governed by a *formal group law* F_Ω in cobordism. For any two line bundles L and M over X, we have

$$c_1(L \otimes M) = F_\Omega(c_1(L), c_1(M)),$$

where $F_\Omega(u, v)$ is a power series

$$F_\Omega(u, v) = u + v + \sum_{i,j\geq 1} a_{i,j} u^i v^j,$$

with coefficients $a_{i,j} \in \Omega^*(\mathrm{Spec}(k))$. This generalizes the classical additive and multiplicative formal group laws found in Chow groups and K-theory, respectively. The power series F_Ω reflects the universal nature of algebraic cobordism and governs the interaction between first Chern classes in this theory.

Let A_* be an oriented Borel-Moore homology theory that is compatible with algebraic cobordism. Then there exists a natural transformation from algebraic cobordism Ω^* to A_*, and the Grothendieck-Riemann-Roch theorem takes the form:

$$\mathrm{ch}_A(f_*(E)) = f_*\left(\mathrm{ch}_A(E) \cdot \mathrm{Td}_A(T_X)\right),$$

where $\mathrm{ch}_A(E)$ is the Chern character in A_*, $\mathrm{Td}_A(T_X)$ is the Todd class associated with the tangent bundle T_X, and f_* is the pushforward in $\Omega^*(X)$. The Todd class in cobordism is constructed via the formal group law F_Ω, and the Chern character $\mathrm{ch}_\Omega(E)$ is a formal power series in the Chern classes $c_i(E)$:

$$\mathrm{ch}_\Omega(E) = \sum_{i\geq 0} \frac{c_i(E)}{i!}.$$

The Todd class $\mathrm{Td}_\Omega(T_X)$ is similarly defined by the Chern roots of T_X in cobordism.

Algebraic cobordism is universal in the sense that any other oriented cohomology theory factors through it. That is, for any oriented cohomology theory A_*, there exists a unique natural transformation $\Omega^*(X) \to A^*(X)$ that respects the operations of the cohomology theory. In particular, for Chow groups $\mathrm{CH}^*(X)$ and algebraic K-theory $K_0(X)$, there are natural transformations:

$$\Omega^*(X) \to \mathrm{CH}^*(X) \quad \text{and} \quad \Omega^*(X) \to K_0(X),$$

which allow us to recover the classical Grothendieck-Riemann-Roch theorem for Chow groups and K-theory as special cases of the cobordism GRR theorem.

In particular, for a smooth quasi-projective variety X over a field k that admits resolution of singularities, there is an isomorphism $\Omega^*(X) \cong \mathrm{CH}^*(X)$. Thus, the Grothendieck-Riemann-Roch theorem in cobordism recovers the classical GRR theorem in Chow theory:

$$\mathrm{ch}(\mathrm{CH}_*(f_*(E))) = f_*(\mathrm{ch}(\mathrm{CH}_*(E)) \cdot \mathrm{Td}(\mathrm{CH}_*(T_X))).$$

Similarly, by applying the natural transformation from algebraic cobordism to K-theory, we recover the classical GRR theorem for $K_0(X)$:

$$\mathrm{ch}(K_0(f_*(E))) = f_*(\mathrm{ch}(K_0(E)) \cdot \mathrm{Td}(K_0(T_X))).$$

This demonstrates the compatibility of algebraic cobordism with other cohomology theories, emphasizing its role as the universal theory that unifies and generalizes classical cohomology and K-theory.

Moreover, the pushforward in algebraic cobordism respects the formal group law, and hence, the Grothendieck-Riemann-Roch theorem in cobordism provides a more refined version of the classical result that incorporates the richer structure of cobordism Chern classes and Todd classes, governed by the universal formal group law F_Ω.

4 Categorified GRR

The categorified Grothendieck-Riemann-Roch theorem (GRR) generalizes the classical Grothendieck-Riemann-Roch theorem by categorifying the notions involved, particularly by lifting the concept of the Chern character and its compatibility with pushforward to a higher categorical setting. The classical GRR theorem involves a relationship between the pushforward of perfect complexes and the corresponding Chern character of these complexes in the setting of derived algebraic geometry. The GRR theorem extends this to an ∞-categorical framework where both the Chern character and the pushforward are now ∞-categorical objects.

We start with a symmetric monoidal functor $f : D \to C$ between stable, presentable symmetric monoidal ∞-categories. In this setting, we consider dualizable modules over these categories, denoted as $\mathrm{Mod}_C^{\mathrm{dual}}$ and $\mathrm{Mod}_D^{\mathrm{dual}}$, respectively. These are categories of dualizable objects that are equipped with a symmetric monoidal structure.

The Chern character in the classical sense takes values in Hochschild homology. For a symmetric monoidal ∞-category C, the categorified Chern character is a functor Ch^{S^1} from $\mathrm{Mod}_C^{\mathrm{dual}}$ to the S^1-equivariant Hochschild homology, denoted $(LC)^{S^1}$, where LC represents the loop space of C.

Let $f : D \to C$ be a symmetric monoidal functor. The pushforward functor $f_* : \mathrm{Mod}_C^{\mathrm{dual}} \to \mathrm{Mod}_D^{\mathrm{dual}}$ preserves the dualizable objects and induces a map between the corresponding Hochschild homologies $Lf_R : (LC)^{S^1} \to (LD)^{S^1}$, where Lf_R is the right adjoint of the induced functor on loop spaces $Lf : LD \to LC$.

The gRR theorem asserts the existence of a commutative diagram:

$$\begin{tikzcd}
\mathrm{Mod}_C^{\mathrm{dual}} \arrow[r, "\mathrm{Ch}^{S^1}"] \arrow[d, "f_*"'] & (LC)^{S^1} \arrow[d, "Lf_R"] \\
\mathrm{Mod}_D^{\mathrm{dual}} \arrow[r, "\mathrm{Ch}^{S^1}"'] & (LD)^{S^1}
\end{tikzcd}$$

This diagram expresses that the pushforward f_* is compatible with the categorified Chern character Ch^{S^1}. The functor Ch^{S^1} encodes a higher-categorical analog of the Chern character, landing in the S^1-equivariant Hochschild homology of the corresponding category.

In the classical Grothendieck-Riemann-Roch theorem, we consider a proper map $f : X \to Y$ between smooth, quasi-projective schemes, and the diagram expressing the compatibility of the pushforward f_* with the Chern character ch is:

$$\begin{tikzcd}
\iota_0\mathrm{Perf}(X) \arrow[r, "\mathrm{ch}"] \arrow[d, "f_*"'] & HH(\mathrm{Perf}(X)) \arrow[d, "f_*"] \\
\iota_0\mathrm{Perf}(Y) \arrow[r, "\mathrm{ch}"'] & HH(\mathrm{Perf}(Y))
\end{tikzcd}$$

Here, $\iota_0\mathrm{Perf}(X)$ is the groupoid of perfect complexes on X, and $HH(\mathrm{Perf}(X))$ is the Hochschild homology of $\mathrm{Perf}(X)$. The commutativity of this diagram implies that the Chern character of the pushforward $f_*(E)$ coincides with the pushforward of the Chern character $f_*(\mathrm{ch}(E))$, up to a Todd class correction.

In the categorified setting, we extend this to a functor between ∞-categories of dualizable modules. The map f_* between $\mathrm{Mod}_C^{\mathrm{dual}}$ and $\mathrm{Mod}_D^{\mathrm{dual}}$ respects the dualizability and the monoidal structure, and thus preserves the structure needed for the categorified Chern character. The functor Ch^{S^1}, which assigns to every dualizable module its Hochschild homology equipped with an S^1-action, is functorial with respect to f_*.

The classical Grothendieck-Riemann-Roch theorem (GRR) asserts that the Chern character commutes with pushforwards in the category of perfect complexes on smooth quasi-projective schemes, up to a correction factor involving the Todd class. More precisely, given a proper map $f : X \to Y$ between smooth schemes, the GRR theorem states that the Chern character $\mathrm{ch}(f_*(E))$ of the pushforward of a perfect complex E on X is related to the pushforward of the Chern character of E on X, twisted by the Todd class $\mathrm{Td}(X/Y)$:

$$\mathrm{ch}(f_*(E)) = f_* \left(\mathrm{ch}(E) \cup \mathrm{Td}(X/Y) \right).$$

The categorified Grothendieck-Riemann-Roch theorem (GRR) generalizes this result in the context of higher categories. In the GRR, the role of the Chern character is played by the categorified Chern character, and the pushforward is considered in the setting of dualizable modules over symmetric monoidal ∞-categories.

To move from the categorified GRR to the classical GRR, we follow a process of decategorification. The categorified GRR applies to higher-categorical objects (e.g., dualizable modules), while the classical GRR applies to more familiar algebraic objects such as vector bundles or perfect complexes.

Let $f : D \to C$ be a rigid symmetric monoidal functor between stable symmetric monoidal ∞-categories, and let T be a dualizable C-module category. Similarly, let T' be a dualizable D-module category, and suppose we have a right adjointable morphism $g : f_*(T) \to T'$. In this setup, we define the categorified Chern characters:

$$\mathrm{Ch} : \mathrm{Hom}_{\mathrm{Mod}_C^{\mathrm{dual}}}(C, T) \to \mathrm{Hom}_{(LC)^{S^1}}(1_{LC}, \mathrm{Ch}(T)),$$

$$\mathrm{Ch} : \mathrm{Hom}_{\mathrm{Mod}_D^{\mathrm{dual}}}(D, T') \to \mathrm{Hom}_{(LD)^{S^1}}(1_{LD}, \mathrm{Ch}(T')).$$

The pushforward functor between the hom-spaces is defined as follows: given a morphism $x : C \to T$, the pushforward map sends it to the composite:

$$D \to f_* C \xrightarrow{f_*(x)} f_* T \xrightarrow{g} T'.$$

On the level of Chern characters, the pushforward map for the categorified Chern character is defined similarly, sending a morphism $1_{LC} \to \mathrm{Ch}(T)$ to the composite:

$$1_{LD} \to (Lf)_R 1_{LC} \to (Lf)_R \mathrm{Ch}(T) \to \mathrm{Ch}(T').$$

This construction leads to the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Mod}_C^{\mathrm{dual}}}(C, T) \arrow[r, "\mathrm{Ch}"] \arrow[d, "f_*"] & \mathrm{Hom}_{(LC)^{S^1}}(1_{LC}, \mathrm{Ch}(T)) \arrow[d, "(Lf)_R"] \\
\mathrm{Hom}_{\mathrm{Mod}_D^{\mathrm{dual}}}(D, T') \arrow[r, "\mathrm{Ch}"] & \mathrm{Hom}_{(LD)^{S^1}}(1_{LD}, \mathrm{Ch}(T')).
\end{tikzcd}$$

In the classical limit, the rigid symmetric monoidal functor f corresponds to a proper morphism between smooth quasi-projective schemes, and the Chern characters become the classical Chern characters of perfect complexes. The commutative diagram at the level of dualizable modules corresponds to the following classical GRR diagram:

$$\begin{tikzcd}
\iota_0 \mathrm{Perf}(X) \arrow[r, "\mathrm{ch}"] \arrow[d, "f_*"] & HH(\mathrm{Perf}(X)) \arrow[d, "f_*"] \\
\iota_0 \mathrm{Perf}(Y) \arrow[r, "\mathrm{ch}"] & HH(\mathrm{Perf}(Y)).
\end{tikzcd}$$

The HKR (Hochschild-Kostant-Rosenberg) isomorphism allows us to express this commutative diagram in terms of differential forms. Using the HKR isomorphism, we have:

$$HH(\mathrm{Perf}(X)) \cong \bigoplus_{i\geq 0} \Omega^i_X,$$

and similarly for Y. The second half of the classical GRR theorem involves a correction term given by the Todd class:

$$\mathrm{ch}(f_*(E)) = f_*\left(\mathrm{ch}(E) \cup \mathrm{Td}(X/Y)\right).$$

Thus, the classical GRR theorem follows from the decategorification of the categorified GRR by applying the HKR isomorphism and introducing the Todd class to account for the noncommutativity of the pushforward with the Chern character at the level of differential forms.

Rough Outline of Topics to Be Covered in Part 2

- Relation Between K-Theory and Cohomology,
- Applications:
  - Examples,
  - Intersection Theory and Enumerative Geometry,
  - Moduli Spaces,
- Some More Generalizations:
  - Derived Categories,
  - RR for Deligne-Mumford Stacks,

Acknowledgments

I would like to thank Dr. Gopal Datt who inspired me for writing this expository note.

References

[1] Hartshorne, R. *Algebraic Geometry*. GTM, Springer, 1977.

[2] Fulton, W. *Intersection Theory*. Springer, 1998.

[3] Weibel, C. *The K-book: An Introduction to Algebraic K-theory*. AMS, 2013.

[4] Ulrich Görtz, Torsten Wedhorn, *Algebraic Geometry II: Cohomology of Schemes* With Examples and Exercises, Springer Studium Mathematik - Master, Springer Nature, 2023.

[5] Marc Levine, Fabien Morel, *Algebraic Cobordism*, Springer Monographs in Mathematics, Springer, 2007.

[6] Marc Hoyois, Pavel Safronov, Sarah Scherotzke, Nicolò Sibilla, The categorified Grothendieck-Riemann-Roch theorem, `https://arxiv.org/abs/1804.00879v2`