## CATEGORY THEORY: ASSIGNMENT 8

**Instructions:**

(1) Only the key ideas count for scores, so write precise and concise answers.
(2) Handwritten as well as typed up solutions are allowed for submission.
(3) Answer ALL questions. **Max. marks:** 10

A Let the functor $T : \mathbf{Set} \to \mathbf{Set}$ be defined on objects by $T(X) := \mathbb{Z} \times X$, and on morphisms by $T(f) := 1_{\mathbb{Z}} \times f$. Also define a natural transformations $\mu : TT \Rightarrow T$ whose component $\mu_X : TTX \to TX$ is given by $\mu_X(m, n, x) := (m + n, x)$, and yet another natural transformation $\eta : 1_{\mathbf{Set}} \Rightarrow T$ whose component $\eta_X : X \to TX$ is given by $\eta_X(x) := (0, x)$.

(a) Show that the triple $\mathbb{T} := (T, \mu, \eta)$ is a monad on **Set**. [3]

(b) Find an adjoint pair of functors $F \dashv G$, $F : \mathbf{Set} \leftrightarrows \mathcal{C} : G$, that induces this monad on **Set**. [2]

B Without using any version of Beck's monadicity theorem, show that the inclusion of a reflective subcategory is a monadic functor. [5]

**Assignment- 8**

**A (a)** Let $T\colon \mathbf{Set} \to \mathbf{Set}$ be defined by $T(X) = \mathbb{Z} \times X$. For a function $f\colon X \to Y$, $T(f) = 1_{\mathbb{Z}} \times f$. The natural transformations associated with $T$ are $\mu\colon TT \Rightarrow T$ and $\eta\colon 1_{\mathbf{Set}} \Rightarrow T$, where:

$$\mu_X\colon T(T(X)) \to T(X)$$

is given by:

$$\mu_X(m, n, (p, q, x)) = (m + n + p, q, x),$$

and:

$$\eta_X\colon X \to T(X)$$

is given by:

$$\eta_X(x) = (0, x).$$

To verify that $(T, \mu, \eta)$ is a monad, we check the associativity and unit laws.
Consider the associativity law:

$$\mu_X \circ T(\mu_X) = \mu_X \circ \mu_{T(X)}.$$

First, compute $T(\mu_X)$:

$$T(\mu_X)(m, n, (p, q, x)) = \mu_X(m, n, (p, q, x)) = (m + n + p, q, x).$$

Then:

$$\mu_X \circ T(\mu_X)(m, n, (p, q, x)) = \mu_X(m + n + p, q, x) = (m + n + p + q, x).$$

Next, compute $\mu_X \circ \mu_{T(X)}$:

$$\mu_{T(X)}\colon T(T(T(X))) \to T(T(X))$$

is given by:

$$\mu_{T(X)}(m, n, (p, q, (r, s, x))) = (m + n + p + r, q + s, x).$$

$$\mu_X \circ \mu_{T(X)}(m, n, (p, q, x)) = \mu_X(m + n + p, q, x) = (m + n + p + q, x).$$

Thus:

$$\mu_X \circ T(\mu_X) = \mu_X \circ \mu_{T(X)},$$

demonstrating associativity.
For the unit law:

$$\mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)}.$$

Compute $T(\eta_X)$:

$$T(\eta_X)(x) = (0, x).$$

$$\mu_X \circ T(\eta_X)(x) = \mu_X(0, x) = (0, x).$$

Thus:

$$\mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)},$$

verifying the unit law.
Therefore, $(T, \mu, \eta)$ is a monad on **Set**.

**(b)** Let $T : \mathbf{Set} \to \mathbf{Set}$ be the functor defined as $T(X) = \mathbb{Z} \times X$ for objects $X$ and $T(f) = 1_{\mathbb{Z}} \times f$ for morphisms $f : X \to Y$. The natural transformations are $\eta_X : X \to \mathbb{Z} \times X$, where $\eta_X(x) = (0, x)$, and $\mu_X : \mathbb{Z} \times (\mathbb{Z} \times X) \to \mathbb{Z} \times X$, where $\mu_X(m, n, x) = (m + n, x)$.

The Eilenberg-Moore category $\mathbf{Set}^T$ consists of objects $(X, \alpha)$, where $X \in \mathbf{Set}$ and $\alpha : \mathbb{Z} \times X \to X$ satisfies

$$\alpha \circ \eta_X = \mathrm{id}_X \quad \text{(i.e., } \alpha(0, x) = x\text{)},$$

$$\alpha \circ (1_{\mathbb{Z}} \times \alpha) = \alpha \circ \mu_X \quad \text{(i.e., } \alpha(m, \alpha(n, x)) = \alpha(m + n, x)\text{)}.$$

A morphism $f : (X, \alpha) \to (Y, \beta)$ satisfies $f \circ \alpha = \beta \circ (1_{\mathbb{Z}} \times f)$, meaning the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z} \times X & \xrightarrow{\alpha} & X \\ \downarrow 1_{\mathbb{Z}} \times f & & \downarrow f \\ \mathbb{Z} \times Y & \xrightarrow{\beta} & Y \end{array}$$

Define the left adjoint functor $F : \mathbf{Set} \to \mathbf{Set}^T$ by $F(X) = (\mathbb{Z} \times X, \mu_X)$, where $\mu_X(m, n, x) = (m + n, x)$. For a morphism $f : X \to Y$, define $F(f) = 1_{\mathbb{Z}} \times f : \mathbb{Z} \times X \to \mathbb{Z} \times Y$.

The right adjoint functor $G : \mathbf{Set}^T \to \mathbf{Set}$ maps a $T$-algebra $(X, \alpha)$ to its underlying set $X$, and for a morphism $f : (X, \alpha) \to (Y, \beta)$, define $G(f) = f$.

The monad $T$ on $\mathbf{Set}$ is induced by the adjunction $F \dashv G$, where:

$$T(X) = G(F(X)) = G(\mathbb{Z} \times X, \mu_X) = \mathbb{Z} \times X,$$

$$\eta_X(x) = (0, x),$$

$$\mu_X(m, n, x) = (m + n, x).$$

To verify that $F \dashv G$ induces the monad $T$, we check the triangular identities. For the left triangular identity $G\epsilon F = 1_F$, where $\epsilon : FG \Rightarrow 1_{\mathbf{Set}^T}$ is the counit of the adjunction, we verify that applying $F$ followed by $G$ gives back the original algebra structure. Similarly, the right triangular identity $F\eta G = 1_G$, where $\eta : 1_{\mathbf{Set}} \Rightarrow GF$ is the unit of the adjunction, holds because applying $G$ followed by $F$ yields the identity.

Thus, the adjoint pair $F \dashv G$ induces the monad $T = (\mathbb{Z} \times (-), \mu, \eta)$.

**Sol B** An adjunction $F \dashv G : \mathcal{C} \leftrightarrows \mathcal{D}$ is called monadic if the Eilenberg–Moore comparison functor

$$K : \mathcal{D} \to \mathcal{C}^T$$

is part of an equivalence of categories, where $T = GF$ is the monad on $\mathcal{C}$ induced by the adjunction, and $\mathcal{C}^T$ is the Eilenberg–Moore category of $T$-algebras. The functor $K$ sends an object $d \in \mathcal{D}$ to the $T$-algebra in $\mathcal{C}^T$ associated to $G(d)$, i.e.,

$$K(d) = (G(d), G(\epsilon_d) : T(G(d)) \to G(d)),$$

where $\epsilon_d : FG(d) \to d$ is the counit of the adjunction.

Now, consider the reflective subcategory situation. Let $\mathcal{A} \subseteq \mathcal{B}$ be a reflective subcategory, which means that there is an adjunction

$$L \dashv i : \mathcal{B} \leftrightarrows \mathcal{A},$$

where $L : \mathcal{B} \to \mathcal{A}$ is the left adjoint (the reflector), and $i : \mathcal{A} \to \mathcal{B}$ is the right adjoint (the inclusion functor). The goal is to show that this adjunction is monadic. For this, we need to show that the comparison functor

$$K : \mathcal{A} \to \mathcal{B}^T$$

is an equivalence of categories, where $T = iL$ is the monad on $\mathcal{B}$ induced by the adjunction $L \dashv i$.

The monad $T : \mathcal{B} \to \mathcal{B}$ is given by $T = iL$. The unit of the monad is the unit of the adjunction $L \dashv i$, denoted by $\eta_B : B \to iL(B)$ for each $B \in \mathcal{B}$, and the multiplication of the monad $T$ comes from the counit of the adjunction, $\mu_B : iL(iL(B)) \to iL(B)$.

The Eilenberg–Moore comparison functor $K : \mathcal{A} \to \mathcal{B}^T$ sends each object $A \in \mathcal{A}$ to a $T$-algebra in $\mathcal{B}^T$. Specifically, for each $A \in \mathcal{A}$, $K(A) = (A, \theta_A)$, where the structure map $\theta_A : T(A) = iL(A) \to A$ is an isomorphism because $A$ is in the reflective subcategory $\mathcal{A}$, meaning that $L(A) = A$ and $iL(A) \cong A$.

To show that the adjunction is monadic, we need to show that $K : \mathcal{A} \to \mathcal{B}^T$ is an equivalence of categories. This can be done by verifying two things: (1) The functor $K$ is fully faithful because for any two objects $A, A' \in \mathcal{A}$, there is a natural isomorphism $\mathcal{A}(A, A') \cong \mathcal{B}(i(A), i(A'))$. Since $\mathcal{A} \subseteq \mathcal{B}$, this is a natural identification of hom-sets. (2) Every $T$-algebra $(B, \theta_B) \in \mathcal{B}^T$ is isomorphic to some object in $\mathcal{A}$. Specifically, for each $T$-algebra $(B, \theta_B)$, the object $L(B) \in \mathcal{A}$ is such that $iL(B) \cong B$ as $T$-algebras. Thus, every object in $\mathcal{B}^T$ arises from some object in $\mathcal{A}$.

Since $K$ is fully faithful and essentially surjective, it follows that $K$ is an equivalence of categories. Therefore, the adjunction $L \dashv i$ is monadic.

More generally, a functor $G : \mathcal{D} \to \mathcal{C}$ is called monadic if it has a left adjoint $F : \mathcal{C} \to \mathcal{D}$ such that the adjunction $F \dashv G$ is monadic. That is, the comparison functor

$$K : \mathcal{D} \to \mathcal{C}^T$$

is an equivalence of categories, where $T = GF$ is the monad induced by the adjunction.

Using this definition of monadicity, the inclusion functor $i : \mathcal{A} \to \mathcal{B}$ in a reflective subcategory is monadic because the comparison functor $K : \mathcal{A} \to \mathcal{B}^T$ is an equivalence of categories. Thus, the adjunction $L \dashv i$ is monadic.

## CATEGORY THEORY: ASSIGNMENT 9

**Instructions:**

(1) Only the key ideas count for scores, so write precise and concise answers.
(2) Handwritten as well as typed up solutions are allowed for submission.
(3) Answer ALL questions. **Max. marks:** 10

A Suppose $(L, \leq)$ is the linear poset with 3 elements, where $L := \{0, 1, 2\}$ with $0 < 1 < 2$, through of as a category. Define a bifunctor $\otimes : L \times L \to L$ by $a \otimes b := \min\{a, b\}$. Show that $((L, \leq), \otimes, 2)$ is a monoidal category. [3]
   Is this monoidal structure symmetric? [1]
   Is this monoidal structure closed? [1]

B An abelian group $A$ is said to be a torsion abelian group if for any $a \in A$ there is a positive integer $n$ such that $na = 0$. Show that the full subcategory **TorAb** of **Ab** consisting of torsion abelian groups is an abelian category. You may use that **Ab** is an abelian category. [3]

C Are the following categories abelian? Justify briefly. [1 each]
   (a) **Gr**–the category of groups and group homomorphisms
   (b) **Ring**–the category of unital rings and unit-preserving morphisms

**Sol A**

We are given the poset $L = \{0, 1, 2\}$ with $0 < 1 < 2$, viewed as a category, and a bifunctor $\otimes$ : $L \times L \to L$ defined by $a \otimes b = \min\{a, b\}$. We aim to show that $((L, \leq), \otimes, 2)$ forms a monoidal category.

A monoidal category consists of:

- a category $\mathcal{C}$,
- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- an object $I$ (the unit object),
- a natural isomorphism $\alpha$ (the associator) with components $\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$,
- a natural isomorphism $\lambda$ (the left unitor) with components $\lambda_A : I \otimes A \to A$,
- a natural isomorphism $\rho$ (the right unitor) with components $\rho_A : A \otimes I \to A$.

First, we verify that $L = \{0, 1, 2\}$ forms a category. The objects of $L$ are $\{0, 1, 2\}$, and there is a morphism $a \to b$ if and only if $a \leq b$. Each element has an identity morphism, and morphisms compose correctly: if $a \leq b$ and $b \leq c$, then $a \leq c$. Thus, $L$ is a category.

The bifunctor $\otimes : L \times L \to L$ is defined by $a \otimes b = \min\{a, b\}$. It respects morphisms: if $a \leq a'$ and $b \leq b'$, then $\min(a, b) \leq \min(a', b')$, which means $\otimes$ is a valid bifunctor.

Next, we show that $2 \in L$ acts as the unit object. For any $a \in L$, we have:

$$a \otimes 2 = \min(a, 2) = a \quad \text{and} \quad 2 \otimes a = \min(2, a) = a,$$

so 2 behaves as the unit object with respect to $\otimes$.

Now, we define the associator $\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$. Using the definition of $\otimes$ as min, we compute:

$$A \otimes (B \otimes C) = \min(A, \min(B, C)) = \min(A, B, C),$$

and

$$(A \otimes B) \otimes C = \min(\min(A, B), C) = \min(A, B, C).$$

Thus, $\alpha_{A,B,C}$ is the identity isomorphism for all $A, B, C \in L$.

Similarly, we define the left unitor $\lambda_A : 2 \otimes A \to A$ and the right unitor $\rho_A : A \otimes 2 \to A$. Both $\lambda_A$ and $\rho_A$ are identity isomorphisms since:

$$2 \otimes A = \min(2, A) = A \quad \text{and} \quad A \otimes 2 = \min(A, 2) = A.$$

Thus, the unit object 2, the bifunctor $\otimes$, and the associator $\alpha$, left unitor $\lambda$, and right unitor $\rho$ satisfy the conditions required for a monoidal category. Therefore, $((L, \leq), \otimes, 2)$ is a monoidal category.

First, we check whether the monoidal structure $((L, \leq), \otimes, 2)$ is symmetric. In a symmetric monoidal category, there must exist a natural isomorphism $\gamma_{A,B} : A \otimes B \to B \otimes A$ for every pair of objects $A, B$ such that $\gamma_{A,B} \circ \gamma_{B,A} = \mathrm{id}_{A \otimes B}$.

In the case of $((L, \leq), \otimes, 2)$, the bifunctor $\otimes$ is defined by $a \otimes b = \min(a, b)$. Since $\min(a, b) = \min(b, a)$, it follows that:

$$a \otimes b = b \otimes a$$

for all $a, b \in L$. Therefore, the braiding $\gamma_{A,B}$ is simply the identity map. Clearly, $\gamma_{A,B} \circ \gamma_{B,A} = \mathrm{id}_{A \otimes B}$, as both maps are the identity. Thus, the monoidal structure is symmetric.

Next, we check whether the monoidal structure is closed. In a closed monoidal category, for every object $A$, the functor $A \otimes - : \mathcal{C} \to \mathcal{C}$ has a right adjoint, called the internal hom or exponential, denoted $[A, -]$. This means that for all objects $A, B, C \in \mathcal{C}$, there must be a natural isomorphism:

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(B, [A, C]).$$

For the poset $((L, \leq), \otimes, 2)$, where $\otimes$ is the minimum function $\min(a, b)$, the hom-sets correspond to the inequality $a \leq b$. Thus:

$$\mathrm{Hom}(A \otimes B, C) = \mathrm{Hom}(\min(A, B), C) = (\min(A, B) \leq C).$$

For $A \otimes -$ to have a right adjoint, there must exist an object $[A, C]$ such that:

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(B, [A, C]).$$

This implies that $[A, C]$ must satisfy:

$$(\min(A, B) \leq C) \cong (B \leq [A, C]).$$

However, in the poset $L = \{0, 1, 2\}$, there is no object $[A, C]$ that consistently satisfies this adjunction property for all $A, B, C$. For example, if $A = 1$ and $C = 1$, no single object $[A, C]$ can satisfy the required isomorphism for all $B$. Thus, $((L, \leq), \otimes, 2)$ does not have internal hom objects, meaning the monoidal structure is not closed.

**Sol B**

Let **TorAb** be the full subcategory of **Ab**, the category of abelian groups, consisting of torsion abelian groups. A group $G$ is called torsion if for every element $g \in G$, there exists a positive integer $n$ such that $n \cdot g = 0$. We will show that **TorAb** satisfies the four axioms of an abelian category:

A zero object in a category is both initial and terminal. In **Ab**, the zero abelian group 0 is both the initial and terminal object. Since 0 is a torsion group (as every element satisfies $n \cdot 0 = 0$), it belongs to **TorAb**. Hence, **TorAb** has a zero object.

Let $f : G \to H$ be a morphism in **TorAb**, where $G$ and $H$ are torsion abelian groups. The kernel of $f$ in **Ab** is the subgroup

$$\ker(f) = \{g \in G \mid f(g) = 0\}.$$

Since $G$ is a torsion group, every element of $\ker(f) \subseteq G$ is also torsion, meaning $\ker(f)$ is a torsion group. Hence, $\ker(f) \in$ **TorAb**.

The cokernel of $f$ in **Ab** is the quotient group

$$\mathrm{coker}(f) = H/f(G),$$

where $f(G)$ is the image of $f$ in $H$. Since $H$ is a torsion group, every element of $H$ has finite order, and the quotient $H/f(G)$ consists of elements with finite order as well. Thus, $\mathrm{coker}(f) \in$ **TorAb**.

A sequence of morphisms $G \xrightarrow{f} H \xrightarrow{g} K$ in a category is exact at $H$ if $\ker(g) = \mathrm{im}(f)$. Since **TorAb** is a full subcategory of **Ab**, it inherits exact sequences from **Ab**. That is, if $G \to H \to K$ is an exact sequence in **Ab** and $G, H, K \in$ **TorAb**, then the sequence is exact in **TorAb**. Therefore, **TorAb** is an exact subcategory of **Ab**.

In an abelian category, the product and coproduct of any two objects must coincide, i.e., it must have biproducts. In **Ab**, the biproduct of two abelian groups $G$ and $H$ is their direct sum $G \oplus H$. Since the direct sum of two torsion abelian groups is again a torsion abelian group (i.e., every element of $G \oplus H$ has finite order), **TorAb** is closed under direct sums. Hence, **TorAb** has biproducts, as required in an abelian category.

**Sol C**

**(a)** The category **Gr** is **not abelian**. Although it has a zero object (the trivial group $\{e\}$) and kernels, it lacks cokernels and biproducts, as products and coproducts do not coincide.

**(b)** The category **Ring** is **not abelian**. There is no zero object since the terminal object is 0 and the initial object is $\mathbb{Z}$. Additionally, products and coproducts do not coincide, and the category is not additive.

## CATEGORY THEORY: ASSIGNMENT 10

**Instructions:**

(1) Only the key ideas count for scores, so write precise and concise answers.
(2) Handwritten as well as typed up solutions are allowed for submission.
(3) Answer ALL questions. **Max. marks:** 10

A Given an abelian category $\mathcal{A}$ and $n \in \mathbb{Z}$, show with justification that
  (a) $H_n : \mathrm{Ch}_\bullet(\mathcal{A}) \to \mathcal{A}$ is a functor; [4]
  (b) $H_n : \mathrm{Ch}_\bullet(\mathcal{A}) \to \mathcal{A}$ is essentially surjective. [1]

B Show exactness of the sequence $K_1 \to K_2 \to K_3 \to Q_1 \to Q_2 \to Q_3$ obtained from the Snake lemma at $K_3$. You may freely use diagram chasing, thanks to the Freyd-Mitchell embedding theorem. [5]

**Sol A** Let $\mathcal{A}$ be an abelian category, and consider the homology functor

$$H_n : \mathrm{Ch}_\bullet(\mathcal{A}) \to \mathcal{A}$$

for each $n \in \mathbb{Z}$.

Given a chain map $f_\bullet : (C_\bullet, d_\bullet^C) \to (D_\bullet, d_\bullet^D)$, the map induced on homology is defined as:

$$H_n(f_\bullet) : H_n(C_\bullet) \to H_n(D_\bullet),$$

where $H_n(f_\bullet)(\mathrm{cls}(z_n)) = \mathrm{cls}(f_n(z_n))$ for an $n$-cycle $z_n \in Z_n(C_\bullet)$, and $\mathrm{cls}(z_n)$ is the equivalence class of $z_n$ in homology.

Let $z_n$ be an $n$-cycle, so $d_n^C z_n = 0$. Since $f_\bullet$ is a chain map, we have:

$$d_n^D f_n(z_n) = f_{n-1}(d_n^C z_n) = f_{n-1}(0) = 0,$$

which shows that $f_n(z_n)$ is also an $n$-cycle in $D_\bullet$. Hence, $H_n(f_\bullet)$ maps cycles to cycles.

Now, suppose that $z_n$ and $z_n'$ are homologous, i.e., $z_n - z_n' \in \mathrm{im}(d_{n+1}^C)$, so $z_n - z_n' = d_{n+1}^C(c_{n+1})$ for some $c_{n+1} \in C_{n+1}$. Applying $f_\bullet$, we get:

$$f_n(z_n) - f_n(z_n') = f_n(d_{n+1}^C(c_{n+1})) = d_{n+1}^D(f_{n+1}(c_{n+1})),$$

which shows that $f_n(z_n)$ and $f_n(z_n')$ are homologous in $D_\bullet$. Therefore, $H_n(f_\bullet)(\mathrm{cls}(z_n)) = H_n(f_\bullet)(\mathrm{cls}(z_n'))$, and $H_n(f_\bullet)$ is well-defined.

1. Identity morphism: For the identity chain map $\mathrm{id}_{C_\bullet}$, we have:

$$H_n(\mathrm{id}_{C_\bullet})(\mathrm{cls}(z_n)) = \mathrm{cls}(\mathrm{id}_{C_n}(z_n)) = \mathrm{cls}(z_n),$$

showing that $H_n(\mathrm{id}_{C_\bullet}) = \mathrm{id}_{H_n(C_\bullet)}$.

2. Composition of chain maps: Let $f_\bullet : C_\bullet \to D_\bullet$ and $g_\bullet : D_\bullet \to E_\bullet$ be two chain maps. The composition of these maps is $(g_\bullet \circ f_\bullet)$, and for any $n$-cycle $z_n \in Z_n(C_\bullet)$, we have:

$$H_n(g_\bullet \circ f_\bullet)(\mathrm{cls}(z_n)) = \mathrm{cls}((g_n \circ f_n)(z_n)) = \mathrm{cls}(g_n(f_n(z_n))).$$

This is equal to:

$$H_n(g_\bullet)(\mathrm{cls}(f_n(z_n))) = H_n(g_\bullet)(H_n(f_\bullet)(\mathrm{cls}(z_n))),$$

which shows that:

$$H_n(g_\bullet \circ f_\bullet) = H_n(g_\bullet) \circ H_n(f_\bullet).$$

Thus, $H_n$ is a functor from $\mathrm{Ch}_\bullet(\mathcal{A})$ to $\mathcal{A}$.

Let $\mathcal{A}$ be an abelian category, and consider the category of chain complexes over $\mathcal{A}$, denoted by $\mathsf{Ch}_\bullet(\mathcal{A})$. The $n$-th homology functor is defined as

$$H_n : \mathrm{Ch}_\bullet(\mathcal{A}) \to \mathcal{A},$$

where for any chain complex $C_\bullet$, the $n$-th homology is given by

$$H_n(C_\bullet) = \frac{\ker(d_n : C_n \to C_{n-1})}{\mathrm{im}(d_{n+1} : C_{n+1} \to C_n)}.$$

We seek to prove that the homology functor $H_n$ is essentially surjective, meaning that for any object $A \in \mathcal{A}$, there exists a chain complex $C_\bullet \in \mathrm{Ch}_\bullet(\mathcal{A})$ such that $H_n(C_\bullet) \cong A$.

Given an object $A \in \mathcal{A}$, construct a chain complex $C_\bullet$ as follows:

$$C_m = \begin{cases} A & \text{if } m = n, \\ 0 & \text{if } m \neq n, n-1. \end{cases}$$

Define the differentials $d_n : C_n \to C_{n-1}$ and $d_{n+1} : C_{n+1} \to C_n$ to be zero. Explicitly, the complex is:

$$C_\bullet = \cdots \to 0 \to A \xrightarrow{0} 0 \to \cdots$$

where $A$ is concentrated in degree $n$.

Now, compute the homology of this chain complex at each degree. For degree $n$,

$$H_n(C_\bullet) = \frac{\ker(d_n : C_n \to C_{n-1})}{\operatorname{im}(d_{n+1} : C_{n+1} \to C_n)} = \frac{\ker(0 : A \to 0)}{\operatorname{im}(0 : 0 \to A)} = \frac{A}{0} \cong A.$$

For $m \neq n$, either $C_m = 0$ or the differentials involved are zero, so the homology is

$$H_m(C_\bullet) = 0.$$

Thus, for any object $A \in \mathcal{A}$, we have constructed a chain complex $C_\bullet$ such that $H_n(C_\bullet) \cong A$ and $H_m(C_\bullet) = 0$ for $m \neq n$. Therefore, the homology functor $H_n : \mathrm{Ch}_\bullet(\mathcal{A}) \to \mathcal{A}$ is essentially surjective.

**Sol B**

$$\begin{tikzcd}
 & \ker(u_1) \arrow[r, "\overline{f_1}"] \arrow[d, hook, "i_1"] & \ker(u_2) \arrow[r, "\overline{f_2}"] \arrow[d, hook, "i_2"] & \ker(u_3) \arrow[d, hook, "i_3"] \arrow[dddll, rounded corners, "\delta"] & \\
 & M_1 \arrow[r, "f_1"] \arrow[d, "u_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "u_2"] & M_3 \arrow[r] \arrow[d, "u_3"] & 0 \\
0 \arrow[r] & N_1 \arrow[r, "g_1"] \arrow[d, two heads, "\pi_1"] & N_2 \arrow[r, "g_2"] \arrow[d, two heads, "\pi_2"] & N_3 \arrow[d, two heads, "\pi_3"] & \\
 & \operatorname{coker}(u_1) \arrow[r, "\overline{g_1}"] & \operatorname{coker}(u_2) \arrow[r, "\overline{g_2}"] & \operatorname{coker}(u_3) &
\end{tikzcd}$$

We aim to show the exactness of the sequence:

$$\ker(u_1) \to \ker(u_2) \to \ker(u_3) \to \operatorname{coker}(u_1) \to \operatorname{coker}(u_2) \to \operatorname{coker}(u_3)$$

obtained from the Snake Lemma at $\ker(u_3)$. We proceed with the proof.

We first consider the part of the sequence:

$$\ker(u_2) \xrightarrow{\overline{f_2}} \ker(u_3) \xrightarrow{\delta} \operatorname{coker}(u_1).$$

Clearly, $\operatorname{im}(\overline{f_2}) \subseteq \ker(\delta)$. For any $x_3 \in \operatorname{im}(\overline{f_2})$, there exists a preimage $x_2 \in \ker(u_2)$. Since $u_2(x_2) = 0$, we have $\delta(x_3) = 0$. Therefore, $\operatorname{im}(\overline{f_2}) \subseteq \ker(\delta)$.

Conversely, suppose $x_3 \in \ker(\delta)$. Let $x_2 \in K_2$ be a preimage of $x_3$, and let $y_1 \in Q_1$ be a preimage of $u_2(x_2)$. Since $x_3 \in \ker(\delta)$, we have $y_1 \in \operatorname{im}(u_1)$, meaning there exists a preimage $x_1 \in K_1$ such that $u_1(x_1) = y_1$. Writing

$$u_2(f_1(x_1)) = u_1(x_1) = y_1 = u_2(x_2),$$

we obtain $x_2 - f_1(x_1) \in \ker(u_2)$. Therefore,

$$x_3 = f_2(x_2) = f_2(x_2 - f_1(x_1)) \in \operatorname{im}(\overline{f_2}),$$

and hence $\ker(\delta) \subseteq \operatorname{im}(\overline{f_2})$.

Thus, the sequence is exact at $\ker(u_3)$:

$$\ker(u_2) \xrightarrow{\overline{f_2}} \ker(u_3) \xrightarrow{\delta} \operatorname{coker}(u_1).$$

Now, consider the sequence:

$$\ker(u_3) \xrightarrow{\delta} \operatorname{coker}(u_1) \xrightarrow{\overline{g_1}} \operatorname{coker}(u_2).$$

We first show that $\operatorname{im}(\delta) \subseteq \ker(\overline{g_1})$. Take an element $x_3 \in \ker(u_3)$, and let $x_2 \in K_2$ be a preimage of $x_3$, and $y_1 \in Q_1$ be a preimage of $u_2(x_2)$ under $\overline{g_1}$. Then $y_1$ is a representative of $\delta(x_3)$, and since $\overline{g_1}(y_1) = u_2(x_2) \in \operatorname{im}(u_2)$, we conclude that

$$\overline{g_1}(\delta(x_3)) = 0,$$

so $\mathrm{im}(\delta) \subseteq \ker(\overline{g_1})$.

Conversely, take an element $y_1 \in \ker(\overline{g_1})$. Then $\overline{g_1}(y_1) \in \mathrm{im}(u_2)$, meaning there is a preimage $x_2 \in K_2$ of $\overline{g_1}(y_1)$. Using the equation $u_3(f_2(x_2)) = g_2(u_2(x_2)) = g_2(\overline{g_1}(y_1)) = 0$ (due to the exactness of the original sequence), we conclude that $x_3 := f_2(x_2) \in \ker(u_3)$. Thus, from the construction of $x_3$, we have $\delta(x_3) = y_1$, and therefore $\ker(\overline{g_1}) \subseteq \mathrm{im}(\delta)$.

Hence, the sequence is exact at $\mathrm{coker}(u_1)$:

$$\ker(u_3) \xrightarrow{\delta} \mathrm{coker}(u_1) \xrightarrow{\overline{g_1}} \mathrm{coker}(u_2).$$

In conclusion, the sequence

$$\ker(u_1) \to \ker(u_2) \to \ker(u_3) \to \mathrm{coker}(u_1) \to \mathrm{coker}(u_2) \to \mathrm{coker}(u_3)$$

is exact at both $\ker(u_3)$ and $\mathrm{coker}(u_1)$, as required by the Snake Lemma.

# CATEGORY THEORY: ASSIGNMENT 11

**Instructions:**

(1) Only the key ideas count for scores, so write precise and concise answers.
(2) Handwritten as well as typed up solutions are allowed for submission.
(3) Answer ALL questions. **Max. marks:** 10

A Suppose $P, Q, R$ are objects of an abelian category satisfying $P \oplus Q \cong R$. Show that $R$ is projective if and only if $P$ and $Q$ are projective. [2]

B Show that the pullback of an epimorphism along any morphism in **Sets** is again an epimorphism. [2]
Say that an object $P$ of a (not-necessarily-abelian) category $\mathcal{C}$ is projective if $\mathcal{C}(P, -)$ preserves epimorphisms. Show that every object of the category **Sets** is projective if and only if the axiom of choice holds. You may use question B above. [1]

D Verify in **Set** that a monomorphism $m$ has the left lifting property with respect to any epimorphism $e$. [2]
If $f$ is a morphism in **Sets** that has the left lifting property with respect to all epimorphisms, then show that $f$ is a monomorphism. [2]
Show that any morphism $f$ in **Sets** can be factorised as $f = em$, where $m$ is a monomorphism and $e$ is an epimorphism. [1]

**Sol A**

Let $P$, $Q$, and $R$ be objects of an abelian category $\mathcal{A}$ such that $P \oplus Q \cong R$. We aim to show that $R$ is projective if and only if both $P$ and $Q$ are projective.

Assume first that $R$ is projective. By definition, this means that the functor $\mathrm{Hom}_{\mathcal{A}}(R, -)$ is exact. We consider the natural isomorphism:

$$\mathrm{Hom}_{\mathcal{A}}(A, R) \cong \mathrm{Hom}_{\mathcal{A}}(A, P \oplus Q) \cong \mathrm{Hom}_{\mathcal{A}}(A, P) \oplus \mathrm{Hom}_{\mathcal{A}}(A, Q)$$

for any object $A$ in $\mathcal{A}$. Since $R$ is projective, $\mathrm{Hom}_{\mathcal{A}}(R, -)$ being exact implies that the induced sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(A, P) \to \mathrm{Hom}_{\mathcal{A}}(A, R) \to \mathrm{Hom}_{\mathcal{A}}(A, Q)$$

is exact. Consequently, $\mathrm{Hom}_{\mathcal{A}}(P, -)$ and $\mathrm{Hom}_{\mathcal{A}}(Q, -)$ must also be exact, which shows that $P$ and $Q$ are projective.

Now assume that both $P$ and $Q$ are projective. By the definition of projectivity, $\mathrm{Hom}_{\mathcal{A}}(P, -)$ and $\mathrm{Hom}_{\mathcal{A}}(Q, -)$ are exact functors. We again utilize the natural isomorphism:

$$\mathrm{Hom}_{\mathcal{A}}(A, R) \cong \mathrm{Hom}_{\mathcal{A}}(A, P \oplus Q) \cong \mathrm{Hom}_{\mathcal{A}}(A, P) \oplus \mathrm{Hom}_{\mathcal{A}}(A, Q)$$

The exactness of $\mathrm{Hom}_{\mathcal{A}}(P, -)$ and $\mathrm{Hom}_{\mathcal{A}}(Q, -)$ implies that the functor $\mathrm{Hom}_{\mathcal{A}}(R, -)$ is exact as well, given that the direct sum of exact functors is exact.

Thus, $R$ is projective.

We have shown that $R$ is projective if and only if both $P$ and $Q$ are projective. This completes the proof.

**Sol B**

Let $f : A \to B$ be a regular epimorphism in **Set**, which means that $f$ is surjective. Let $g : C \to B$ be any morphism in **Set**. Consider the pullback diagram:

$$\begin{array}{ccc} P & \xrightarrow{\pi_1} & A \\ \downarrow \pi_2 & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

where $P = \{(a, c) \in A \times C \mid f(a) = g(c)\}$, and the projection maps are $\pi_1 : P \to A$ and $\pi_2 : P \to C$.

We want to show that the map $\pi_1 : P \to A$ is a regular epimorphism, i.e., that $\pi_1$ is surjective. Take any $a \in A$. Since $f$ is surjective, there exists $b \in B$ such that $f(a) = b$.

Now, consider the function $g : C \to B$. By the definition of $g$, there exists some $c \in C$ such that $g(c) = b = f(a)$. Therefore, the pair $(a, c) \in P$ satisfies $f(a) = g(c)$, and hence $\pi_1(a, c) = a$.

Thus, for every $a \in A$, there exists a pair $(a, c) \in P$ such that $\pi_1(a, c) = a$. This shows that $\pi_1$ is surjective, and since in **Set**, regular epimorphisms are precisely the surjections, $\pi_1$ is a regular epimorphism.

**Sol C**

**Axiom 1.** *Let $R$ be an equivalence relation on a set $X$. Then there is a set $X/R$, and a function $q : X \to X/R$ with the following properties:*

*1.* $(x, y) \in R \iff q(x) = q(y)$,

*2. For any set $Y$ and function $f : X \to Y$ that is constant on equivalence classes (i.e., $f(x) = f(y)$ whenever $(x, y) \in R$), there exists a unique function $g : X/R \to Y$ such that $f = g \circ q$.*

*The function $q$ is called the coequalizer of the projections $p_0, p_1 : R \to X$, where $p_0(x, y) = x$ and $p_1(x, y) = y$.*

Now we will show *"Every object in* **Sets** *is projective if and only if the Axiom of Choice holds"* using the axiom above.

*Proof.* Let $f : X \to Y$ be an epimorphism, and let $g : S \to Y$ be any function.

Suppose every set is projective. Let $f : X \to Y$ be a surjection, and consider the kernel pair of $f$, $p_0, p_1 : X \times_Y X \to X$, where:

$$p_0(x_1, x_2) = x_1, \quad p_1(x_1, x_2) = x_2,$$

with $X \times_Y X = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$. This is an equivalence relation on $X$.

By Axiom 1, the coequalizer of $p_0$ and $p_1$ exists. Let $q : X \to X/R$ be this coequalizer, so:

$$q \circ p_0 = q \circ p_1.$$

Since $f$ is an epimorphism, $f$ coequalizes $p_0$ and $p_1$, which implies there is a unique map $m : X/R \to Y$ such that:

$$f = m \circ q.$$

Now, suppose every object in **Sets** is projective. In particular, $S$ is projective. Since $m : X/R \to Y$ is a surjection and $g : S \to Y$ is a function, the projectivity of $S$ ensures there exists a map $h' : S \to X/R$ such that:

$$m \circ h' = g.$$

Define $h = q \circ h'$. Then:

$$f \circ h = m \circ q \circ h' = g.$$

Thus, we have lifted $g$ to a function $h : S \to X$, showing that $S$ is projective. This holds for any set $S$.

Now, we show that the Axiom of Choice holds. Consider a family of non-empty sets $\{A_i\}_{i \in I}$. Let $X = \coprod_{i \in I} A_i$, and define a surjection $f : X \to I$ by sending each element of $A_i$ to $i$. Since $I$ is projective by assumption, there exists a map $h : I \to X$ such that:

$$f \circ h = \mathrm{id}_I.$$

This map $h$ selects an element from each set $A_i$, giving a choice function, and thus, the Axiom of Choice holds.

**Conversely**, suppose the Axiom of Choice holds. We now show that every set is projective.

Let $S$ be any set, and let $f : X \to Y$ be a surjection. We are given a map $g : S \to Y$ and aim to find a map $h : S \to X$ such that:

$$f \circ h = g.$$

Define the kernel pair of $f$, $p_0, p_1 : X \times_Y X \to X$, as before. By Axiom 1, there exists a coequalizer $q : X \to X/R$ that coequalizes $p_0$ and $p_1$. Since $f$ coequalizes $p_0$ and $p_1$, there is a unique function $m : X/R \to Y$ such that:

$$f = m \circ q.$$

Now, by the Axiom of Choice, for each $s \in S$, there exists an element $x_s \in X$ such that $f(x_s) = g(s)$. Define a function $h : S \to X$ by selecting such an $x_s$ for each $s$. Then:

$$f \circ h(s) = g(s),$$

so $S$ is projective.

Thus, every set in **Sets** is projective if and only if the Axiom of Choice holds. □

**Sol D**

Let **Set** denote the category of sets and functions. We aim to show that any monomorphism $m : A \to B$ in **Set** has the left lifting property with respect to any epimorphism $e : X \to Y$.

First, recall that a weak factorization system $(E, M)$ on a category $C$ consists of two classes of morphisms, $E$ and $M$, such that: 1. Any morphism in $E$ has the left lifting property with respect to any morphism in $M$. 2. Every morphism $f : C \to D$ in $C$ can be factored as $f = m \circ e$, where $e \in E$ and $m \in M$.

In **Set**, there is a standard weak factorization system $(E, M)$, where: - $E$ is the class of epimorphisms (surjections), - $M$ is the class of monomorphisms (injections).

For any function $f : X \to Y$ in **Set**, we can always factor $f$ as:

$$f = m \circ e$$

where $e : X \to \mathrm{Im}(f)$ is a surjection (epimorphism) onto the image of $f$, and $m : \mathrm{Im}(f) \to Y$ is an injection (monomorphism).

Next, we show that every monomorphism $m \in M$ has the left lifting property with respect to any epimorphism $e \in E$. This means that given any commutative square:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & X \arrow[d, "e"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

where $m \in M$ is a monomorphism and $e \in E$ is an epimorphism, there must exist a function $h : B \to X$ such that:

$$h \circ m = f \quad \text{and} \quad e \circ h = g.$$

We now proceed to prove this property in detail.

Since $e : X \to Y$ is surjective, for each $b \in B$, we know that there exists at least one $x_b \in X$ such that $e(x_b) = g(b)$. Therefore, we can define a function $h : B \to X$ by choosing for each $b \in B$ a corresponding element $x_b \in X$ such that $e(x_b) = g(b)$.

This choice ensures that $e \circ h = g$, so the right side of the lifting condition is satisfied. It remains to check that $h \circ m = f$.

Since $m : A \to B$ is injective, for any element $a \in A$, the element $m(a) \in B$ has a unique corresponding $x_{m(a)} \in X$ such that $e(x_{m(a)}) = g(m(a))$. However, by commutativity of the original square, we have:

$$g \circ m = e \circ f.$$

Thus, for each $a \in A$, we get:

$$e(x_{m(a)}) = g(m(a)) = e(f(a)).$$

Since $e$ is an epimorphism (surjective), this implies $x_{m(a)} = f(a)$. Therefore, $h(m(a)) = f(a)$, which gives $h \circ m = f$, completing the left side of the lifting condition.

Hence, there exists a function $h : B \to X$ such that:

$$h \circ m = f \quad \text{and} \quad e \circ h = g,$$

which verifies that $m$ has the left lifting property with respect to $e$.

Thus, we have shown that in the category **Set**, every monomorphism has the left lifting property with respect to every epimorphism, as required.

Let $f : A \to B$ be a morphism in the category **Set** that has the *left lifting property* (LLP) with respect to all epimorphisms. We aim to show that $f$ is a monomorphism, i.e., for any pair of functions $g_1, g_2 : X \to A$, if $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$.

Given $f : A \to B$, suppose there are two functions $g_1, g_2 : X \to A$ such that:

$$f \circ g_1 = f \circ g_2.$$

We want to show that $g_1 = g_2$.

Consider the function $h : X \to X$ defined by:

$$h(x) = \begin{cases} g_1(x) & \text{if } g_1(x) = g_2(x), \\ a & \text{if } g_1(x) \neq g_2(x), \end{cases}$$

where $a \in A$ is some arbitrary element. Define the function $e : X \amalg \{*\} \to X$, which is the projection map that maps the disjoint union $X \amalg \{*\}$ onto $X$, i.e.,

$$e(x) = x \quad \text{for all } x \in X, \quad \text{and} \quad e(*) = x_0 \quad \text{for some } x_0 \in X.$$

This map $e$ is clearly an epimorphism (surjective). Consider the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{g_1} & A \\ {\scriptstyle \mathrm{id}_X}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow{f \circ g_1} & B \end{array}$$

Since $f$ has the left lifting property with respect to all epimorphisms, there exists a map $k : X \amalg \{*\} \to A$ such that the following diagram commutes:

$$\begin{array}{ccc} X \amalg \{*\} & \xrightarrow{k} & A \\ {\scriptstyle e}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow{f \circ g_1} & B \end{array}$$

Thus, we have $f(k(x)) = f(g_1(x))$ for all $x \in X$. Since $f$ is a function, and we assumed that $f \circ g_1 = f \circ g_2$, this forces $k(x) = g_1(x) = g_2(x)$. Hence, $g_1 = g_2$, showing that $f$ is injective.

Thus, $f$ is a monomorphism.

Let $f : A \to B$ be a morphism in the category **Set**. We aim to factor $f$ as $f = e \circ m$, where $m$ is a monomorphism (injection) and $e$ is an epimorphism (surjection).

First, we define the image of $f$, denoted as $\mathrm{Im}(f)$. The image of $f$ is the set:

$$\mathrm{Im}(f) = \{f(a) \mid a \in A\} \subseteq B.$$

Since $f$ maps elements from $A$ into $B$, the image is a subset of $B$.

Next, we define two functions: 1. The epimorphism $e : A \to \mathrm{Im}(f)$ given by $e(a) = f(a)$ for all $a \in A$. 2. The monomorphism $m : \mathrm{Im}(f) \to B$ which is the inclusion map, defined by:

$$m(b) = b \quad \text{for all } b \in \mathrm{Im}(f).$$

Now, we need to check the properties of $e$ and $m$: - The map $e$ is surjective because for every element $b \in \mathrm{Im}(f)$, there exists an element $a \in A$ such that $e(a) = f(a) = b$. Therefore, $e$ is an epimorphism.

- The map $m$ is injective because if $m(b_1) = m(b_2)$ for some $b_1, b_2 \in \mathrm{Im}(f)$, then $b_1 = b_2$. Therefore, $m$ is a monomorphism.

Now we can express $f$ as follows:

$$f(a) = m(e(a)) \quad \text{for all } a \in A.$$

This shows that:

$$f = m \circ e.$$

Thus, we have factored $f$ as $f = e \circ m$, where $m$ is a monomorphism and $e$ is an epimorphism.

# CATEGORY THEORY: ASSIGNMENT 12

**Instructions:**

(1) Only the key ideas count for scores, so write precise and concise answers.
(2) Handwritten as well as typed up solutions are allowed for submission.
(3) Answer ALL questions. **Max. marks:** 10

A Suppose $\mathcal{E}$ is an elementary topos and $A$ is an object of $\mathcal{E}$. Show the following:
  (a) $\mathcal{E}$ is a well-powered category. [1]
  (b) $A^0 \cong 1$. [1]

B Show the following for sets $A, B_1, B_2$:
  (a) $A^{B_1 \times B_2} \cong (A^{B_1})^{B_2}$. [2]
  (b) $A^{B_1 + B_2} \cong A^{B_1} \times A^{B_2}$. [2]

C For an object $F$ of $\mathcal{E} := [\mathbf{2}, \mathbf{Sets}]$, where $\mathbf{2}$ is the discrete category with only two objects, say $a, b$, show that $\mathbf{Sub}_{\mathcal{E}}(F)$ is in bijective correspondence with $P(F(a)) \times P(F(b))$, where $P(X)$ is the power set of $X$. Hence find a subobject classifier for $\mathcal{E}$ with justification. [3]

D Suppose $S_1, S_2$ are sieves on an object $C$ of a small category $\mathcal{C}$. Show that $S_1 \cap S_2$ is again a sieve on $C$. [1]

**Sol A**

**(a)**

A category $C$ is well-powered if for every object $X \in C$, the collection of subobjects of $X$, denoted $\mathrm{Sub}(X)$, forms a set. In an elementary topos $E$, there exists a *subobject classifier* $\Omega \in E$, meaning that for every monomorphism $f : A \to X$, there is a unique characteristic morphism $\chi_f : X \to \Omega$ that makes the following diagram commute:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & 1 \arrow[d] \\ X \arrow[r, "\chi_f"] & \Omega \end{tikzcd}$$

Subobjects of $X$ correspond to certain morphisms from $X$ to $\Omega$. Since the hom-set $\mathrm{Hom}_E(X, \Omega)$ is a set, this implies that the collection of subobjects of $X$ is also a set, meaning $E$ is well-powered.

**(b)**

The exponential object $A^0$ represents the set of morphisms from the terminal object 1 to $A$. In other words, $A^0 = \mathrm{Hom}_E(1, A)$, where 1 is the terminal object in $E$.

For any object $A$, the set $\mathrm{Hom}_E(1, A)$ consists of the *global elements* of $A$, i.e., elements of $A$ without dependence on other objects. Since $A^0 \cong \mathrm{Hom}_E(1, A)$, and for the terminal object we have $1^0 \cong 1$, it follows that $A^0 \cong 1$.

**Sol B**

**(a)** To prove the isomorphism $A^{(B_1 \times B_2)} \cong (A^{B_1})^{B_2}$, we will use the structure of a cartesian closed category. This isomorphism illustrates how exponentiation distributes over products in such categories.

First, we understand the object $A^{(B_1 \times B_2)}$, which represents the set of all functions from $B_1 \times B_2$ to $A$:

$$A^{(B_1 \times B_2)} = \{f : B_1 \times B_2 \to A\}.$$

Next, we consider the object $(A^{B_1})^{B_2}$, which represents the set of all functions from $B_2$ to $A^{B_1}$:

$$(A^{B_1})^{B_2} = \{g : B_2 \to (B_1 \to A)\}.$$

Now, we define a bijection between these two sets of functions.

From $A^{(B_1 \times B_2)}$ to $(A^{B_1})^{B_2}$: Given a function $f : B_1 \times B_2 \to A$, we construct a corresponding function $g : B_2 \to A^{B_1}$ by defining:

$$g(b_2)(b_1) = f(b_1, b_2).$$

From $(A^{B_1})^{B_2}$ to $A^{(B_1 \times B_2)}$: Given a function $g : B_2 \to A^{B_1}$, we construct a corresponding function $f : B_1 \times B_2 \to A$ by defining:

$$f(b_1, b_2) = g(b_2)(b_1).$$

Now we need to check that these constructions are inverses of each other. Starting with $f \in A^{(B_1 \times B_2)}$: - We construct $g : B_2 \to A^{B_1}$ by setting $g(b_2)(b_1) = f(b_1, b_2)$. - Then we reconstruct $f$ from $g$ by setting $f'(b_1, b_2) = g(b_2)(b_1) = f(b_1, b_2)$.

Thus, $f' = f$.

Starting with $g \in (A^{B_1})^{B_2}$: - We construct $f : B_1 \times B_2 \to A$ by setting $f(b_1, b_2) = g(b_2)(b_1)$. - Then we reconstruct $g$ from $f$ by setting $g'(b_2)(b_1) = f(b_1, b_2) = g(b_2)(b_1)$.

Thus, $g' = g$.

Using the adjunction properties of exponentiation in a cartesian closed category, we have shown that there is a natural isomorphism between the two objects $A^{(B_1 \times B_2)}$ and $(A^{B_1})^{B_2}$. Therefore, the isomorphism holds:

$$A^{(B_1 \times B_2)} \cong (A^{B_1})^{B_2}.$$

**(b)** To prove the isomorphism $A^{B_1+B_2} \cong A^{B_1} \times A^{B_2}$, we will again use the structure of a cartesian closed category.

1. Coproduct: The coproduct $B_1 + B_2$ represents the disjoint union of the objects $B_1$ and $B_2$ with canonical injection morphisms $i_1 : B_1 \to B_1 + B_2$ and $i_2 : B_2 \to B_1 + B_2$.

2. Exponentiation: The object $A^B$ denotes the object of morphisms from $B$ to $A$. So $A^{B_1}$ and $A^{B_2}$ represent morphisms from $B_1$ and $B_2$ to $A$.

Now, for a fixed object $C$, consider morphisms from $C \times (B_1 + B_2)$ to $A$:

$$\mathcal{C}(C \times (B_1 + B_2), A).$$

Using the universal property of coproducts, we have:

$$\mathcal{C}(C \times (B_1 + B_2), A) \cong \mathcal{C}(C \times B_1, A) \times \mathcal{C}(C \times B_2, A).$$

By the exponentiation adjunction, we have:

$$\mathcal{C}(C \times B_1, A) \cong \mathcal{C}(C, A^{B_1}),$$

and

$$\mathcal{C}(C \times B_2, A) \cong \mathcal{C}(C, A^{B_2}).$$

Thus, we can combine these results:

$$\mathcal{C}(C \times (B_1 + B_2), A) \cong \mathcal{C}(C, A^{B_1}) \times \mathcal{C}(C, A^{B_2}).$$

This implies:

$$\mathcal{C}(C \times (B_1 + B_2), A) \cong \mathcal{C}(C, A^{B_1} \times A^{B_2}).$$

By the properties of exponentiation in a cartesian closed category, we conclude:

$$A^{B_1+B_2} \cong A^{B_1} \times A^{B_2}.$$

**Sol C** Let $[2, \mathbf{Sets}]$ be the category of functors from the discrete category with two objects $\{a, b\}$ to the category of sets. A functor $F : [2] \to \mathbf{Sets}$ assigns:

$$F(a) \quad \text{to the object} \quad a, \quad F(b) \quad \text{to the object} \quad b,$$

with no morphisms between $a$ and $b$. Thus, each functor $F$ is a pair of sets $(F(a), F(b))$.

Next, we describe $\mathrm{Sub}_{[2,\mathbf{Sets}]}(F)$, the subobjects of $F$. A subobject corresponds to a subfunctor $G \subseteq F$, which assigns:

$$G(a) \subseteq F(a), \quad G(b) \subseteq F(b).$$

Hence, the subobjects of $F$ are in bijection with pairs of subsets of $F(a)$ and $F(b)$, meaning:

$$\mathrm{Sub}_{[2,\mathbf{Sets}]}(F) \cong P(F(a)) \times P(F(b)),$$

where $P(X)$ denotes the power set of a set $X$.

In a topos, a subobject classifier is an object $\Omega$ such that for any object $F$, there is a bijection between subobjects of $F$ and morphisms $F \to \Omega$.

The subobject classifier in **Sets** is $\Omega_{\mathbf{Sets}} = \{0, 1\}$. In $[2, \mathbf{Sets}]$, the subobject classifier should assign to each object $a$ and $b$ a copy of $\{0, 1\}$, so the subobject classifier $\Omega$ is the functor:

$$\Omega(a) = \{0, 1\}, \quad \Omega(b) = \{0, 1\},$$

which is $\Omega = (\{0, 1\}, \{0, 1\})$.

Given any object $F = (F(a), F(b))$, the subobjects of $F$ are classified by a morphism $F \to \Omega$. This morphism corresponds to characteristic functions:

$$\chi_a : F(a) \to \{0, 1\}, \quad \chi_b : F(b) \to \{0, 1\},$$

indicating whether each element of $F(a)$ and $F(b)$ belongs to the subsets $G(a) \subseteq F(a)$ and $G(b) \subseteq F(b)$.

Thus, the subobject classifier for $[2, \mathbf{Sets}]$ is the functor $\Omega = (\{0, 1\}, \{0, 1\})$.

**Sol D** Let $C$ be a category, and let $\mathrm{Sieve}(C)$ denote the set of sieves on an object $C$ in $C$. The set $\mathrm{Sieve}(C)$ is partially ordered by inclusion, i.e., for two sieves $S_1$ and $S_2$ on $C$, we say $S_1 \leq S_2$ if $S_1 \subseteq S_2$. In any poset, the intersection of any collection of elements is still an element of the poset. Thus, the intersection of sieves is well-defined. Our goal is to prove that the intersection of any family of sieves on $C$ is also a sieve on $C$.

Let $\{S_i\}_{i \in I}$ be a family of sieves on $C$. The intersection is defined as

$$S = \bigcap_{i \in I} S_i = \{f : D \to C \mid f \in S_i \text{ for all } i \in I\}.$$

We need to show that $S$ is a sieve, i.e., that it is closed under precomposition.

Take any morphism $f : D \to C$ in $S$, so that $f \in S_i$ for all $i \in I$. Since each $S_i$ is a sieve, it is closed under precomposition. Therefore, for any morphism $g : E \to D$, we have $f \circ g \in S_i$ for each $i \in I$. Thus,

$$f \circ g \in \bigcap_{i \in I} S_i = S.$$

Since $S$ is closed under precomposition, it satisfies the definition of a sieve. Therefore, $\bigcap_{i \in I} S_i$ is a sieve on $C$.