

Advanced Problems

Exercises Collected by Virat Chauhan

1. Enriched Kan Extensions via Coends

Let R be a commutative ring and consider categories enriched in R -modules (so that Hom-sets carry an R -module structure and composition is R -bilinear). Define an R -linear category \mathcal{A} with two objects a, b , where

$$\mathrm{Hom}_{\mathcal{A}}(a, a) = R, \quad \mathrm{Hom}_{\mathcal{A}}(b, b) = R, \quad \mathrm{Hom}_{\mathcal{A}}(a, b) = R, \quad \mathrm{Hom}_{\mathcal{A}}(b, a) = 0,$$

and composition is given by R -bilinear maps (so there is one non-identity arrow $\alpha: a \rightarrow b$).

Let \mathcal{B} be another R -linear category with two objects c, d and one non-identity arrow $f: c \rightarrow d$. Define an enriched functor $F: \mathcal{A} \rightarrow \mathcal{B}$ by $F(a) = c$, $F(b) = d$, and $F(\alpha) = f$.

Let \mathcal{M} be the R -linear category of R -modules. Define $G: \mathcal{A} \rightarrow \mathcal{M}$ as the enriched functor determined by modules M_a, M_b and an R -linear map $g: M_a \rightarrow M_b$ corresponding to α .

Compute the *enriched left Kan extension* $\mathrm{Lan}_F G: \mathcal{B} \rightarrow \mathcal{M}$ using the coend formula:

$$(\mathrm{Lan}_F G)(x) = \int^{y \in \mathcal{A}} \mathcal{B}(F(y), x) \otimes_R G(y).$$

Compute $(\mathrm{Lan}_F G)(c)$ and $(\mathrm{Lan}_F G)(d)$ explicitly in terms of M_a, M_b , and g . Identify the resulting coequalizers or pushouts.

2. Homotopy Kan Extensions in $(\infty, 1)$ -Categories

Kan extensions in ordinary category theory become **homotopy Kan extensions** in a homotopical setting. For example, in the model category of topological spaces (or simplicial sets), the *left homotopy Kan extension* along a functor p is computed by the homotopy colimit of the diagram. Consider the following specific diagram $X: C \rightarrow \mathbf{Top}$ in the category of spaces:

$$C: \quad 0 \longrightarrow 1,$$

and let $p: C \rightarrow *$ be the functor to the terminal category. Define $X(0) = S^0$ (the discrete space with two points) and $X(1) = *$ (a singleton), and let the map $X(0) \rightarrow X(1)$ collapse both points to the basepoint. The **homotopy left Kan extension** $L_p(X)$ at the point $*$ is the homotopy colimit of the diagram $S^0 \rightrightarrows *$.

- (a) Describe how to compute this homotopy colimit as a homotopy pushout (or suspension) of the diagram. Argue why ordinary colimit (which would just yield a point) is not the correct answer, and explain how one obtains the homotopy-correct result by considering a cofibrant replacement of the diagram or by using mapping cylinders.
- (b) Show that the homotopy colimit (hence $L_p(X)(*)$) is homeomorphic to S^1 . (*Hint: one can view two maps from a point together with a homotopy between their restrictions to S^0 as describing a circle.*) In fact, as observed by Carlson on [Math StackExchange](#), the “two null-homotopies” on S^0 force the result to be S^1 . Conclude that in the $(\infty, 1)$ -category of spaces this Kan extension produces the circle.

Context hint: In general, Lurie shows that pointwise left Kan extensions in $(\infty, 1)$ -categories exist whenever the target admits the required homotopy colimits (see [MathOverflow](#)). This example exhibits the construction explicitly.

3. Kan Extension and Induction of Group Actions

Let $\varphi : H \rightarrow G$ be a homomorphism of groups, and let BH, BG denote the one-object categories with endomorphism monoids H, G respectively (the *classifying categories* of the groups). A functor $X : BH \rightarrow \mathbf{Set}$ is equivalently a (left) H -set. The restriction functor $\varphi^* : \text{Fun}(BG, \mathbf{Set}) \rightarrow \text{Fun}(BH, \mathbf{Set})$ sends a G -set to the same set with H -action via φ .

By general theory (using the Grothendieck construction), φ^* has a left adjoint given by the left Kan extension along $B\varphi : BH \rightarrow BG$, see [math.stackexchange.com](#).

In concrete terms:

- (a) Show that the left Kan extension $\text{Lan}_{B\varphi}(X)$ of an H -set X along $B\varphi : BH \rightarrow BG$ is a G -set whose underlying set can be identified with

$$\int^{h \in BH} G(\varphi(h), *) \times X(h) \cong G \times_H X,$$

the *coend* or quotient set $G \times X / \sim$ with $(g\varphi(h), x) \sim (g, hx)$. (This describes the usual “induced G -set” or *induction of X from H to G* .)

- (b) Verify explicitly (by unraveling the Kan extension coend or by a universal property argument) that $\text{Lan}_{B\varphi}(X)$ is isomorphic to the usual induced set $G \times_H X$. In particular, describe the G -action on $G \times_H X$ and show it satisfies the universal property of the Kan extension.

Hint: The general fact cited above (see [math.stackexchange.com](#)) implies the existence of this Kan extension as a left adjoint. The computation via coends gives a constructive formula:

$$(\text{Lan}_{B\varphi} X)(*) = \int^{h \in BH} G(*, \varphi(h)) \times X(h).$$

4. Kan Extensions in ∞ -Categories

Let $f: \mathcal{C}' \rightarrow \mathcal{C}$ be a functor of small ∞ -categories, and let \mathcal{D} be a presentable ∞ -category (so \mathcal{D} has all small colimits math.ias.edu). Consider the precomposition functor

$$f^*: \text{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}', \mathcal{D}), \quad F \mapsto F \circ f.$$

1. **Left Kan extension.** For a given functor $F: \mathcal{C}' \rightarrow \mathcal{D}$, define the *pointwise left Kan extension* $f_! F: \mathcal{C} \rightarrow \mathcal{D}$ by the formula

$$(f_! F)(c) = \text{colim}_{(c' \rightarrow c) \in \mathcal{C}'_c} F(c').$$

Show that this construction indeed produces a functor $f_! F$ and that $f_!: \text{Fun}(\mathcal{C}', \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})$ is left adjoint to f^* . In particular, verify the universal property

$$\text{Map}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(f_! F, G) \simeq \text{Map}_{\text{Fun}(\mathcal{C}', \mathcal{D})}(F, G \circ f)$$

by constructing the unit and counit maps and checking the triangle identities. (Here \mathcal{C}'_c denotes the ∞ -category of objects of \mathcal{C}' over c .) Use the fact that \mathcal{D} has all colimits math.ias.edu to ensure these Kan extensions exist.

2. **Right Kan extension.** Assume further that \mathcal{D} admits all small limits (for instance \mathcal{D} could be presentable and complete). Define the *pointwise right Kan extension* $f_* F: \mathcal{C} \rightarrow \mathcal{D}$ by

$$(f_* F)(c) = \lim_{(c \rightarrow c') \in (\mathcal{C}'_c)^\circ} F(c').$$

Show that this formula yields a functor $f_*: \text{Fun}(\mathcal{C}', \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})$ which is right adjoint to f^* . Again verify the adjunction homotopy equivalence

$$\text{Map}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(G, f_* F) \simeq \text{Map}_{\text{Fun}(\mathcal{C}', \mathcal{D})}(G \circ f, F).$$

Hints: In each case, you should identify the comma (over-)categories \mathcal{C}'_c and $(c \downarrow \mathcal{C}')$ and use the universal property of colimits/limits in \mathcal{D} . You may cite that in HA Lurie shows [colimit-preserving functors](http://math.ias.edu) between presentable ∞ -categories are left adjoints math.ias.edu.

5. The Cofiber Functor in a Stable ∞ -Category

Let \mathcal{C} be a stable ∞ -category (so \mathcal{C} is pointed and every morphism admits both a fiber and a cofiber, see [HA, Definition 1.1.1.9](http://math.ias.edu)).

Consider the ∞ -category of arrows $\text{Fun}(\Delta^1, \mathcal{C})$, whose objects are morphisms $f: X \rightarrow Y$ in \mathcal{C} . There is a functor

$$\text{cofib}: \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$$

which sends a morphism $f: X \rightarrow Y$ to its cofiber, i.e., the pushout of the span $X \leftarrow X \rightarrow Y$. There is also a functor

$$Z: \mathcal{C} \rightarrow \text{Fun}(\Delta^1, \mathcal{C}),$$

which sends an object X to the zero morphism $0 \rightarrow X$ (where 0 denotes the zero object in \mathcal{C} , see [HA, Remark 1.1.1.8](#)).

1. **Adjunction.** Show that cofib is left adjoint to Z . That is, construct natural transformations

$$\eta: \text{id}_{\mathcal{C}} \rightarrow Z \circ \text{cofib}, \quad \varepsilon: \text{cofib} \circ Z \rightarrow \text{id}_{\mathcal{C}}$$

exhibiting the unit and counit of the adjunction, and verify the triangular identities. Intuitively, this expresses that a morphism $f: X \rightarrow Y$ is equivalent to specifying an object $\text{cofib}(f)$ together with a map $0 \rightarrow \text{cofib}(f)$ (cf. [HA, Remark 1.1.1.8](#)).

2. **Kan Extension Perspective.** Deduce from [HA, Remark 1.1.1.8](#) that cofib arises as the left Kan extension of the inclusion of the fiber category. Consequently, cofib preserves all colimits which exist in $\text{Fun}(\Delta^1, \mathcal{C})$. In particular, explain why cofib preserves small colimits as a left adjoint.

Remark. According to Lurie, every morphism in a pointed ∞ -category admits a cofiber section up to contractible choice. Consequently, the forgetful functor

$$\theta: \{\text{cofiber squares in } \mathcal{C}\} \rightarrow \text{Fun}(\Delta^1, \mathcal{C})$$

is a trivial fibration and thus admits a homotopy section. This provides a functorial cofiber construction up to equivalence, identifying cofib as a left adjoint to the embedding $Z: \mathcal{C} \rightarrow \text{Fun}(\Delta^1, \mathcal{C})$ (cf. [HA, Remark 1.1.1.8](#)).

6. Extension of Scalars for Module ∞ -Categories

Let $(\mathcal{C}^{\otimes}, \otimes, \mathbf{1})$ be a presentable symmetric monoidal ∞ -category which is also stable — for example, Sp , the ∞ -category of spectra, or the derived ∞ -category of chain complexes. Suppose we are given a morphism $f: A \rightarrow B$ of E_1 -algebra objects in \mathcal{C} (that is, an associative algebra map). This induces a restriction (or forgetful) functor

$$f^*: \text{Mod}_B(\mathcal{C}) \longrightarrow \text{Mod}_A(\mathcal{C}),$$

which sends a B -module N to its underlying A -module via f . Here, $\text{Mod}_A(\mathcal{C})$ and $\text{Mod}_B(\mathcal{C})$ denote the ∞ -categories of left modules over A and B , respectively. These are known to be presentable and stable (cf. *Higher Algebra*, Chapter 4).

- **Left adjoint (Extension of Scalars).** Define the extension-of-scalars functor

$$f_!: \text{Mod}_A(\mathcal{C}) \longrightarrow \text{Mod}_B(\mathcal{C}), \quad M \mapsto B \otimes_A M,$$

where $B \otimes_A M$ denotes the relative tensor product in \mathcal{C} . Show that $f_!$ is left adjoint to f^* , i.e., for every $M \in \text{Mod}_A(\mathcal{C})$ and $N \in \text{Mod}_B(\mathcal{C})$, there is a natural equivalence

$$\text{Map}_{\text{Mod}_B(\mathcal{C})}(B \otimes_A M, N) \simeq \text{Map}_{\text{Mod}_A(\mathcal{C})}(M, f^*N).$$

Use the fact that \mathcal{C} is symmetric monoidal and that the relative tensor product satisfies the expected universal property. You may assume standard results from *Higher Algebra*, such as the presentability and stability of $\mathrm{Mod}_A(\mathcal{C})$, and the colimit-preserving nature of the tensor product.

- **Right adjoint (Coextension of Scalars).** Suppose that B is dualizable as an A -module object in \mathcal{C} . Then show that f^* also admits a right adjoint given by

$$f_* : \mathrm{Mod}_A(\mathcal{C}) \longrightarrow \mathrm{Mod}_B(\mathcal{C}), \quad M \mapsto \mathrm{Hom}_A(B, M),$$

where $\mathrm{Hom}_A(B, M)$ denotes the internal Hom in $\mathrm{Mod}_A(\mathcal{C})$, assuming that \mathcal{C} is closed symmetric monoidal. Verify formally that this is right adjoint to f^* , i.e., for every $M \in \mathrm{Mod}_A(\mathcal{C})$ and $N \in \mathrm{Mod}_B(\mathcal{C})$, there is a natural equivalence

$$\mathrm{Map}_{\mathrm{Mod}_A(\mathcal{C})}(f^*N, M) \simeq \mathrm{Map}_{\mathrm{Mod}_B(\mathcal{C})}(N, \mathrm{Hom}_A(B, M)).$$

Remark. The existence of both the left adjoint $f_!$ and the right adjoint f_* can be understood as instances of (left and right) Kan extensions in the ∞ -categorical setting. The foundational theory developed in *Higher Algebra* ensures the existence of such adjoints under the stated conditions, particularly that $\mathrm{Mod}_A(\mathcal{C})$ is presentable and that the tensor product preserves colimits in each variable.

7. ∞ -Operads and Monoidal ∞ -Categories

An ∞ -operad is defined by Lurie ([HA Definition 2.1.1.10](#)) as a coCartesian fibration

$$p : \mathcal{O}^\otimes \rightarrow \mathrm{N}(\mathrm{Fin}_*)$$

satisfying certain inert-lifting and Segal conditions. Equivalently, a symmetric monoidal ∞ -category \mathcal{C}^\otimes is an ∞ -operad with essentially surjective map to $\mathrm{N}(\mathrm{Fin}_*)$, and a *monoidal functor* corresponds to a map of ∞ -operads over $\mathrm{N}(\mathrm{Fin}_*)$.

Using this perspective, prove the following:

- Show that if $i : \mathcal{A}^\otimes \rightarrow \mathcal{B}^\otimes$ is a map of ∞ -operads over \mathcal{O}^\otimes , then the induced forgetful functor

$$\theta : \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C}) \rightarrow \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C})$$

(between algebras in a fixed coCartesian fibration $\mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$) **admits a left adjoint** (a “free \mathcal{B} -algebra” functor) whenever certain operadic colimits exist. Specifically, combine [HA Proposition 3.1.3.3](#) and [Corollary 3.1.3.4](#) to formulate the necessary criterion (involving the existence of operadic colimit diagrams) and prove that this criterion is sufficient.

- Apply this to the inclusion of ∞ -operads $\mathrm{Assoc} \hookrightarrow \mathrm{Comm}$ (the associative into the commutative operad). Show that if \mathcal{C}^\otimes admits sifted colimits (e.g. \mathcal{C} presentable), then the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \rightarrow \mathrm{Alg}(\mathcal{C})$ has a left adjoint — the **free commutative algebra** on a given associative algebra. Describe (at least informally) how this free commutative algebra can be constructed by forming the relevant colimits in \mathcal{C} .

8. Derived ∞ -Categories and Sheaf Theory

Let \mathcal{A} be a Grothendieck abelian category (for instance, sheaves of abelian groups on a space). Its derived ∞ -category $D(\mathcal{A})$ can be constructed by inverting quasi-isomorphisms in chain complexes. Prove that:

- $D(\mathcal{A})$ is a **presentable stable ∞ -category** and admits a natural t -structure whose heart is equivalent to \mathcal{A} (HA Proposition 1.3.5.21).
(Hint: Use the model of bounded-above complexes of injectives or projectives.)
- Deduce that the homotopy category $hD(\mathcal{A})$ is a **triangulated category** (with translation given by the suspension in $D(\mathcal{A})$) by applying HA Theorem 1.1.2.14.
Verify that the distinguished triangles in $hD(\mathcal{A})$ correspond exactly to the usual exact triangles of complexes.
- As an illustration, let X be a topological space. Show that $D(\mathrm{Shv}(X))$ (the derived ∞ -category of sheaves of abelian groups on X) **satisfies descent**: for any open cover $\{U_i\}$, the global sections functor can be computed as the limit of the Čech nerve of the cover.
(Equivalently, prove that sheaf cohomology can be computed via the ∞ -categorical Čech complex, using left Kan extensions and the Beck–Chevalley condition.)

9. Stable ∞ -Categories and Triangulated Structures

Recall Lurie’s definition: an ∞ -category \mathcal{C} is **stable** if it has a zero object and every map admits a fiber and a cofiber, with fibers and cofibers agreeing (Def. 1.1.1.9). Prove:

- \mathcal{C} has all finite limits and colimits, and a square in \mathcal{C} is a pushout if and only if it is a pullback (HA Prop. 1.1.3.4). Conclude that the suspension functor $\Sigma: \mathcal{C} \rightarrow \mathcal{C}$ (taking cofibers of identity maps) is an equivalence of \mathcal{C} .
- Using HA Theorem 1.1.2.14, show that $h\mathcal{C}$ carries a natural triangulated structure. Namely, verify that given any morphism $f: X \rightarrow Y$ in \mathcal{C} one can form a fiber-cofiber sequence

$$X \rightarrow Y \rightarrow Z \rightarrow X[1]$$

whose image in $h\mathcal{C}$ is a distinguished triangle. Check that Verdier’s axioms (TR1–TR4) hold.

- Finally, let $F: \mathcal{C} \rightarrow \mathcal{D}$ be an exact functor between stable ∞ -categories. Prove that the induced functor

$$hF: h\mathcal{C} \rightarrow h\mathcal{D}$$

is an exact functor of triangulated categories (it commutes with the shift and sends distinguished triangles to distinguished triangles). *Example:* verify these statements for $\mathcal{C} = \mathrm{Mod}_R$ where R is an E_∞ -ring and for $\mathcal{C} = \mathrm{Perf}(A)$ for a ring A .

10. Spectral Algebra and E_n -Ring Structures

Let E_n^\otimes denote the little n -cubes ∞ -operad (so E_1 is the associative operad and E_∞ is the commutative one). Using Lurie's Dunn Additivity Theorem (HA 5.1.2.2: mathias.edu), show that:

- For nonnegative k, ℓ , there is an equivalence of ∞ -operads $E_k \otimes E_\ell \simeq E_{k+\ell}$.
Conclude that equipping an ∞ -category with an $E_{k+\ell}$ -monoidal structure is equivalent to giving it compatible E_k - and E_ℓ -monoidal structures. In particular, deduce that an E_n -algebra in a symmetric monoidal ∞ -category \mathcal{C} can be viewed as an associative algebra object in the $(n-1)$ -fold monoidal ∞ -category of E_{n-1} -algebras.
- Let R be an E_n -ring spectrum (an E_n -algebra in $\mathcal{S}p$).
Prove that the ∞ -category Mod_R of left R -modules is naturally an E_{n-1} -monoidal stable ∞ -category. Describe this tensor product (often given by the relative tensor over R) and check that for $n = 2$ it is (braided) monoidal, while for $n = \infty$ it becomes symmetric monoidal.
Show that Mod_R is presentable and stable, and that restriction-of-scalars along a map of E_n -rings has both adjoints (extension and coextension of scalars).

Bonus: Analyze the homotopy groups $\pi_* R$ of an E_n -ring; for instance, explain why for $n \geq 2$ the graded ring $\pi_*(R)$ is graded-commutative.

11. ∞ -Topoi and Descent

Consider a flat morphism of commutative rings $f : A \rightarrow B$. Let $B^\bullet = B^{\otimes_A \bullet}$ denote the Čech nerve (the cosimplicial ring with $B^{\otimes_A n}$ in degree n). Using the Barr–Beck argument as in HA §4.7.5 (mathias.edu), prove the following descent statement:

- Define the ∞ -category of **descent data** $\mathrm{Desc}(f)$ as the limit of the cosimplicial ∞ -category

$$\cdots \rightrightarrows \mathrm{Mod}_{B^{\otimes_A 2}} \rightarrow \mathrm{Mod}_B \rightarrow \mathrm{Mod}_A.$$

Show that $\mathrm{Desc}(f) \simeq \varprojlim \mathrm{Mod}_{B^{\otimes_A \bullet}}$.

- Prove that if $f : A \rightarrow B$ is faithfully flat, then the canonical functor $\mathrm{Mod}_A \rightarrow \mathrm{Desc}(f)$ is an equivalence of ∞ -categories (generalizing Grothendieck's theorem). Conclude that quasi-coherent sheaves (or vector bundles) satisfy faithfully flat descent.
- **(Related)** More generally, let \mathcal{X} be an ∞ -topos. Prove that for any hypercover $U_\bullet \rightarrow *_{\mathcal{X}}$, the limit of the diagram of ∞ -categories $\mathrm{Shv}(X/U_n)$ recovers $\mathrm{Shv}(\mathcal{X})$, giving descent for sheaves of spaces or spectra on \mathcal{X} .

12. Presentability, Adjunctions, and Kan Extensions

Let \mathcal{C} be a small ∞ -category and \mathcal{D} a presentable ∞ -category.

- (a) Show that any functor $F : \mathcal{C} \rightarrow \mathcal{D}$ admits a **left Kan extension** $\tilde{F} : \mathcal{P}(\mathcal{C}) \rightarrow \mathcal{D}$ along the Yoneda embedding $j : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})$, which preserves all (small) colimits. Deduce the universal property $\text{Fun}^{\mathcal{L}}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \simeq \text{Fun}(\mathcal{C}, \mathcal{D})$, and hence that colimit-preserving functors out of a presheaf category are determined by their restriction to the dense subcategory \mathcal{C} .

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- (b) Using Lurie’s adjoint functor theorem for ∞ -categories, prove that any colimit-preserving functor between presentable ∞ -categories admits a right adjoint (this follows since in the presentable setting “having a right adjoint” is equivalent to preserving colimits).

Example: Consider the inclusion of the subcategory of compact/projective generators $i : \mathcal{C}_0 \rightarrow \mathcal{C}$ in a presentable ∞ -category \mathcal{C} ; then any functor $F : \mathcal{C}_0 \rightarrow \mathcal{D}$ to a cocomplete ∞ -category \mathcal{D} extends uniquely (as a left Kan extension) to a colimit-preserving functor $\mathcal{C} \rightarrow \mathcal{D}$.

- (c) Finally, analyze left Kan extensions along fully faithful functors: if $j : \mathcal{C}_0 \rightarrow \mathcal{C}$ has a left adjoint, show $\text{Lan}_j(F)$ exists for every $F : \mathcal{C}_0 \rightarrow \mathcal{E}$ and can be computed by composing F with that left adjoint. In general, formulate necessary and sufficient conditions for a left Kan extension along an inclusion j to exist in terms of (co)limits in the target.

13. Tensor Products of ∞ -Operads

The ∞ -category of ∞ -operads is known to be presentable mathoverflow.net.

Let $\mathcal{O}_1^{\otimes}, \mathcal{O}_2^{\otimes}$ be two small ∞ -operads. *Using the existence of (small) colimits in Op_{∞}* , show that there is a Boardman–Vogt tensor product operad

$$\mathcal{O}_1^{\otimes} \otimes \mathcal{O}_2^{\otimes},$$

characterized by the universal property that for any symmetric monoidal ∞ -category \mathcal{C}^{\otimes} , algebras over $\mathcal{O}_1^{\otimes} \otimes \mathcal{O}_2^{\otimes}$ in \mathcal{C} are the same as pairs of an \mathcal{O}_1^{\otimes} -algebra and an \mathcal{O}_2^{\otimes} -algebra in \mathcal{C} .

In particular, describe the set of colors (objects over $\langle 1 \rangle$) of $\mathcal{O}_1^{\otimes} \otimes \mathcal{O}_2^{\otimes}$ in terms of those of \mathcal{O}_1^{\otimes} and \mathcal{O}_2^{\otimes} , and verify explicitly that when one of the operads is the commutative operad Comm^{\otimes} , the tensor product yields the other operad (i.e., $\text{Comm}^{\otimes} \otimes = \text{the unit}$) mathoverflow.net.

14. Monoidal ∞ -Categories and CoCartesian Fibrations

Recall that a symmetric monoidal ∞ -category can be defined as a coCartesian fibration

$$p : \mathcal{C}^{\otimes} \rightarrow \Gamma^{\text{op}}$$

(over the category of pointed finite sets) satisfying two axioms (M1) and (M2) [idrissi.eu](#).

For instance, if \mathcal{C} is an ordinary category with finite products, one constructs \mathcal{C}^\otimes so that the fiber over $\langle n \rangle_+$ is \mathcal{C}^n , and p is coCartesian.

- (a) Prove in detail that in this construction p indeed is coCartesian and that the fiber over $\langle n \rangle_+$ is (equivalent to) \mathcal{C}^n (properties (M1) and (M2) in Lurie’s language) [idrissi.eu](#).
- (b) Deduce from this that an algebra over the associative operad (an E_1 -algebra) in \mathcal{C}^\otimes is the same as a monoid object in \mathcal{C} , and an algebra over Comm^\otimes is the same as a commutative monoid object in \mathcal{C} . (In other words, recover classical notions of algebra objects via the fibration language.)

15. Dunn’s Additivity Theorem for Little Cubes

The Little Cubes operads E_k satisfy a fundamental additivity property: the tensor product $E_m \otimes E_n$ is equivalent to E_{m+n} . State and prove Dunn’s Additivity Theorem (cf. [HA §5.1.2](#)), which asserts that the functor

$$E_m \otimes E_n \longrightarrow E_{m+n}$$

constructed by sending a pair of configurations of m -cubes and n -cubes into a configuration of $(m+n)$ -cubes is an equivalence of ∞ -operads ([math.ias.edu](#)).

Use the operadic tensor product and the little cubes model to construct this equivalence. As an application, show that giving an E_{m+n} -algebra in a symmetric monoidal ∞ -category is equivalent to giving an E_n -algebra in the category of E_m -algebras (and vice versa).

16. Iterated Loop Spaces as E_n -Algebras

Let X be a pointed topological space and let $\Omega^k X$ be its k -fold loop space. Using the little k -cubes operad model, show that $\Omega^k X$ carries a canonical E_k -algebra structure. More precisely, describe the action of the topological little k -cubes operad on $\Omega^k X$ by sending a collection of rectilinear embeddings of k -cubes with disjoint images to the operation that “concatenates” loops (see [HA 5.2.6](#), [math.ias.edu](#)).

Prove that this indeed defines an E_k -monoid structure on $\Omega^k X$. Conversely, show that any *grouplike* E_k -space is (up to homotopy) an iterated loop space of some connected space (May’s Recognition Principle in the ∞ -categorical context).

17. Faithfully Flat Descent via Barr–Beck

In *Higher Algebra* §4.7.5,¹ Lurie sets up an ∞ -categorical version of descent theory for ring spectra. Let

$$f^* : A \rightarrow B$$

¹[HA §4.7.5 reference link](#)

be a map of E_∞ -rings which is *faithfully flat* (i.e., B is a flat A -module and $\pi_0(f)$ is faithfully flat). Consider the extension-of-scalars functor

$$f^* : \text{Mod}_A \rightarrow \text{Mod}_B$$

and its left adjoint

$$f_* : \text{Mod}_B \rightarrow \text{Mod}_A$$

(i.e., the restriction-of-scalars functor).

Use the ∞ -categorical Barr–Beck theorem to show that f^* exhibits Mod_A as the limit (in Cat_∞) of the simplicial diagram:

$$\text{Mod}_B \rightrightarrows \text{Mod}_{B \otimes_A B} \rightrightarrows \text{Mod}_{B \otimes_A B \otimes_A B} \cdots$$

i.e.,

$$\text{Mod}_A \simeq \text{Tot}(\text{Mod}_{B \otimes_A (\bullet+1)}).$$

In particular, prove that the forgetful functor with descent data

$$\text{Desc}(f) \simeq \lim \left(\text{Mod}_B \rightrightarrows \text{Mod}_{B \otimes_A B} \rightrightarrows \cdots \right)$$

is equivalent to Mod_A .

(You may assume or prove the key hypotheses of the Barr–Beck theorem in this context.)

18. Adjoint Functor Theorem for ∞ -Categories

State and prove the Adjoint Functor Theorem in the setting of presentable ∞ -categories (as sketched in SAG, Remark 2.6; see rezk.web.illinois.edu).

In particular, let

$$F : \mathcal{A} \rightarrow \mathcal{B}$$

be a functor between presentable ∞ -categories.

- Prove that F admits a *right adjoint* if and only if it preserves all small colimits.
- Prove that F admits a *left adjoint* if and only if it preserves all small limits and is accessible.

As an application, use this to characterize when a symmetric monoidal functor between presentable symmetric monoidal ∞ -categories admits a (lax symmetric monoidal) right adjoint.

19. Sheafification and ∞ -Topoi

Let X be a topological space (or more generally a small ∞ -site). Show that the ∞ -category $\mathrm{Shv}(X)$ of space-valued sheaves on X is an ∞ -topos. Equivalently, prove that the inclusion

$$\mathrm{Shv}(X) \hookrightarrow \mathrm{PSh}(X)$$

of sheaves into all presheaves admits a left exact left adjoint (the sheafification functor). In other words, the localization $\mathrm{PSh}(X) \rightarrow \mathrm{Shv}(X)$ is left exact.

(For guidance, recall that for a cover $\{U_i \rightarrow U\}$ in X , sheafification enforces the usual Čech descent limit condition, and one shows this localization preserves finite limits — see rezk.web.illinois.edu.)

Conclude that $\mathrm{Shv}(X)$ satisfies Lurie’s axioms for an ∞ -topos (SAG Def. 2.4).

[Hint: In the case of an ordinary topological space this is classical; you may use the fact that the sheafification of presheaves of spaces preserves finite limits.]

20. Slices of ∞ -Topoi

Let \mathcal{X} be an ∞ -topos and let $U \in \mathcal{X}$ be any object. Prove that the slice ∞ -category $\mathcal{X}_{/U}$ is again an ∞ -topos.

(For example, one can use the fact that if $\mathcal{X} = \mathrm{Shv}(T)$ is presented by a site T , then $\mathcal{X}_{/U} \simeq \mathrm{Shv}(T_{/U})$, where $T_{/U}$ is the “slice site” of objects over U .)

As part of the solution, verify Example 2.9 of SAG: when $\mathcal{X} = \mathcal{S}$ (the ∞ -category of spaces) and U is a space, the slice $\mathcal{S}_{/U}$ is an ∞ -topos whose underlying 1-topos is $\mathrm{Fun}(\pi_1 U, \mathrm{Set})$ — see rezk.web.illinois.edu.

Deduce in general that the underlying 1-topos of $\mathcal{X}_{/U}$ depends only on the fundamental groupoid of U .

21. Stabilization and Spectrum Objects

Let \mathcal{C} be a pointed presentable ∞ -category with all finite limits and colimits. Define its stabilization $\mathrm{Sp}(\mathcal{C})$ as the ∞ -category of spectrum objects in \mathcal{C} (as in HA Ch. 6). Show that $\mathrm{Sp}(\mathcal{C})$ is a stable presentable ∞ -category and that the canonical functor

$$\Sigma^\infty : \mathcal{C} \rightarrow \mathrm{Sp}(\mathcal{C})$$

exhibits $\mathrm{Sp}(\mathcal{C})$ as the *stable envelope* of \mathcal{C} .

That is, prove that for any stable presentable ∞ -category \mathcal{D} , functors $\mathrm{Sp}(\mathcal{C}) \rightarrow \mathcal{D}$ are equivalent to functors $\mathcal{C} \rightarrow \mathcal{D}$ that take finite colimits in \mathcal{C} to colimits in \mathcal{D} . Equivalently, show that $\mathrm{Sp}(\mathcal{C})$ is initial among stable presentable ∞ -categories receiving a colimit-preserving functor from \mathcal{C} .

(In *Higher Algebra* 7.3.1.4, Lurie proves precisely that Σ^∞ exhibits $\mathrm{Sp}(\mathcal{C})$ as the stable envelope — see math.ias.edu.)

22. Localizations of E_∞ -Rings

Let R be a (connective) E_∞ -ring and let $x \in \pi_0(R)$ be an element. Define the localization $R[x^{-1}]$ by freely inverting x in the category of E_∞ -rings (see section 7.2.3 in *Higher Algebra*, [Lurie 2017](#)).

Prove that this localization exists and is characterized by the usual universal property: any map $R \rightarrow R'$ of E_∞ -rings in which x becomes invertible factors uniquely through $R[x^{-1}]$.

Show that on homotopy groups this agrees with the classical formula

$$\pi_*(R[x^{-1}]) \cong \pi_*(R)[x^{-1}],$$

by verifying that $\pi_0(R[x^{-1}]) \cong \pi_0(R)[x^{-1}]$ and that the localization behaves correctly on higher homotopy groups.

As an example, compute the localization of the p -local sphere spectrum at p , and compare it to the Eilenberg–MacLane spectrum of the localized ring.

23. Cotangent Complex of E_∞ -Rings

In *Higher Algebra* §7.3, Lurie defines the cotangent complex $L_{A/R}$ for a map $R \rightarrow A$ of E_∞ -rings. Give this definition—for example, via the stabilization of the slice category $\mathrm{Alg}_{R/}$ —and prove its basic properties.

Show in particular that if A is an ordinary discrete commutative ring (viewed as an E_∞ -ring), then

$$L_A \simeq \Omega_{A/\mathbb{Z}}^1$$

in degree 0, where $\Omega_{A/\mathbb{Z}}^1$ is the classical module of Kähler differentials, and that L_A vanishes in higher homotopy groups.

More generally, prove that

$$L_{B/R} \simeq 0$$

precisely when B is étale over R (assuming connective, finitely presented hypotheses).

Use these computations to derive obstruction-theoretic consequences for maps of rings (e.g. formal smoothness and unramifiedness in the spectral sense).

24. Algebras and Modules in Presentable Categories

Let \mathcal{C}^\otimes be a presentable symmetric monoidal ∞ -category whose tensor product preserves colimits separately in each variable. Show that for any small ∞ -operad \mathcal{O} , the ∞ -category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ of \mathcal{O} -algebra objects in \mathcal{C} is itself presentable.

Concretely, construct colimits in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ by taking suitable operadic (lax) colimit diagrams in \mathcal{C} , and show that the forgetful functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \rightarrow \mathcal{C}$$

creates these colimits.

Deduce in particular that $\mathrm{Alg}_{E_1}(\mathcal{C})$ and $\mathrm{Alg}_{E_\infty}(\mathcal{C})$ admit all small colimits, and that the free algebra functors preserve colimits.

Finally, analyze the case of module categories: if A is an algebra in \mathcal{C} , prove that the ∞ -category of A -modules in \mathcal{C} is also presentable, and that base-change along maps of algebras satisfies the expected adjointability properties (as discussed in *Higher Algebra*, Ch. 4).

Remarks are welcome at `virat[dot]algebraicgeometry[at]gmail[dot]com`