HW 1

Due date: Sep 29, 2021

As always, “map” is used for “morphism”.

Étale spaces. For a presheaf F on a topological space, we will use the notations we used in class. Thus $\mathscr{E}(F)$ is the topological space associated with F, $\pi\colon \mathscr{E}(F) \to X$ the natural map, F^+ the sheafification of F etc.

In what follows, X is a topological space, and F a presheaf on X.

1) Show that $\pi\colon \mathscr{E}(F) \to X$ is a local homeomorphism.

2) Show that if U is open in X and $s \in F(U)$, and for $x \in U$, s_x the germ of s at x,[1] then the map

$$\sigma_s\colon U \to \mathscr{E}(F) = \coprod_{x\in X} F_x$$

given by $x \mapsto s_x$, $x \in U$, is a continuous map.

Recall that the natural map $\theta(=\theta_F)\colon F \to F^+$ is the map defined on every open set U of X by $s \mapsto \sigma_s$ with the notation as above.

3) If F is a sheaf, show that θ_F is an isomorphism.

4) Let E be a topological space, $p\colon E \to X$ a local homeomorphism such that for every $x \in X$, $p^{-1}(x)$ is an abelian group. Define $E \times_X E$ to be the subspace of $E \times E$ consisting of pairs (e, e') with $p(e) = p(e')$. Suppose the two maps $E \times_X E \to E$, $(e, e') \mapsto e + e'$ and $E \to E$, $e \mapsto -e$ are continuous. Let $\mathscr{F} = \mathscr{F}_E$ be the sheaf of sections of $p\colon E \to X$, i.e., for an open subset U of X, $\mathscr{F}(U)$ is the abelian group of continuous maps from $\sigma\colon U \to E$ such that $p \circ \sigma = 1_U$. Show the following.
   (a) For $x \in X$, there is a natural isomorphism of abelian groups $\psi_x\colon \mathscr{F}_x \xrightarrow{\sim} p^{-1}(x)$.
   (b) There is an isomorphism $\psi\colon \mathscr{E}(\mathscr{F}) \xrightarrow{\sim} E$ such that $p \circ \psi = \pi$.

5) Let X be a topological space, $\mathscr{F}$ a sheaf on X, U an open subset of X, and $\mathfrak{U} = \{U_\alpha\}$ an open cover of U. For every α and β set $U_{\alpha\beta} := U_\alpha \cap U_\beta$. Show that the sequence of abelian groups

$$0 \to \mathscr{F}(U) \xrightarrow{\epsilon} \prod_\alpha \mathscr{F}(U_\alpha) \xrightarrow{d^0} \prod_{\alpha,\beta} \mathscr{F}(U_{\alpha\beta})$$

[1] In other words, s_x is the image of $s \in F(U)$ in the stalk F_x under the natural map $F(U) \to F_x$ arising from the definition of a direct limit.

is exact, where ϵ is the "diagonal" map $s \mapsto (s|_{U_\alpha})_\alpha$ and the map d^0 is defined by $d^0((s_\alpha)_\alpha) = (\sigma_{\alpha\beta})_{\alpha,\beta}$ where $\sigma_{\alpha\beta} = s_\beta|_{U_{\alpha\beta}} - s_\alpha|_{U_{\alpha\beta}}$.

$\mathscr{B}$-sheaves. For the remaining problems consider the following. Let X be a topological space, $\mathscr{B}$ a basis for the topology on X with the extra condition that if B_1 and B_2 are in $\mathscr{B}$ then so is $B_1 \cap B_2$ (e.g. the standard basis for the topology on $\mathrm{Spec}(A)$, where A is a commutative ring). Let F be a $\mathscr{B}$-sheaf (defined in class). For U an open set of X set

$$\mathscr{F}(U) := \ker\left[\prod_\alpha F(U_\alpha) \xrightarrow{d^0} \prod_{\alpha,\beta} F(U_{\alpha\beta})\right] \qquad (*)$$

where (U_α) is an open cover of U with $U_\alpha \in \mathscr{B}$ for every α and d^0 is as in **5**).

6) Show that $\mathscr{F}(U)$ does not depend on the open cover (U_α) of U, i.e. any two covers by members of $\mathscr{B}$ give rise to isomorphic kernels as in $(*)$.

7) Show that the assignment $U \mapsto \mathscr{F}(U)$ gives us a sheaf, which we will denote $\mathscr{F}$.

8) Show that we have an isomorphism of $\mathscr{B}$-sheaves $\mathscr{F}|_{\mathscr{B}} \xrightarrow{\sim} F$.

9) If G is a $\mathscr{B}$-sheaf and $\varphi\colon F \to G$ a map of $\mathscr{B}$-sheaves and if $\mathscr{G}$ is the sheaf on X arising from G via the process outlined in **7**) then show that there is a map $\tilde{\varphi}\colon \mathscr{F} \to \mathscr{G}$ such that the diagram

$$\begin{tikzcd} \mathscr{F}|_{\mathscr{B}} \arrow[r, "\sim"] \arrow[d, "\tilde{\varphi}"'] & F \arrow[d, "\varphi"] \\ \mathscr{G}|_{\mathscr{B}} \arrow[r, "\sim"] & G \end{tikzcd}$$

commutes, where the horizontal isomorphisms are as in **8**).

10) Show that

$$\mathscr{F}(U) \xrightarrow{\sim} \varprojlim_{B} F(B)$$

where the inverse limit is taken over B such that $B \in \mathscr{B}$ and $B \subset U$.

HW 2

Due date: Oct 11, 2021

As always, "map" is used for "morphism". In particular a "map of complexes" is either chain map pr a co-chain map, depending on whether the complexes in question are chain complexes or co-chain complexes. For problems involving an abelian category $\mathscr{A}$, you may, if you feel like, assume $\mathscr{A} = \mathrm{Mod}_A$, the category of modules of a ring A. For a complex $C^\bullet$, $Z^p(C^\bullet)$ is the kernel of d_C^p and $B^p(C^\bullet)$ is the image of d_C^{p-1}, i.e. if we are dealing with $\mathscr{A} = \mathrm{Mod}_A$, $Z^p(C^\bullet)$ is the module of p-cocycles of $C^\bullet$ and $B^p(C^\bullet)$ is the module of p-coboundaries of $C^\bullet$. As always, the p^{th} cohomology $\mathrm{H}^p(C^\bullet)$ of $C^\bullet$ is the "quotient":

$$\mathrm{H}^p(C^\bullet) := Z^p(C^\bullet)/B^p(C^\bullet).$$

Please look at https://www.cmi.ac.in/~pramath/AGI/notes/CechNotes.pdf for various definitions involving Čech complexes and the $\mathrm{Hom}^\bullet$ complexes.

Homotopies. Let $\alpha\colon C^\bullet$ and $D^\bullet$ be two complexes in an abelian category $\mathscr{A}$. A map of complexes $\alpha\colon C^\bullet \to D^\bullet$ is said to be *homotopic to* 0 if there exist maps $k^p\colon C^p \to D^{p-1}$, $p \in \mathbf{Z}$, such that $\alpha^p = d_D^{p-1} \circ k^p + k^{p+1} \circ d_C^p$ for every $p \in \mathbf{Z}$. In this case we write $\alpha \sim 0$. Note that if $\alpha \sim 0$ then $-\alpha \sim 0$. Two maps $\alpha, \beta\colon C^\bullet \to D^\bullet$, are said to be homotopic to each other if $\alpha - \beta \sim 0$. Homotopy is clearly an equivalence relation between maps of complexes.

1. Show that if $\alpha \sim \beta$ then $\mathrm{H}^p(\alpha) = \mathrm{H}^p(\beta)$ for all $p \in \mathbf{Z}$.

2. Let $T^\bullet = \mathrm{Hom}^\bullet_{\mathscr{A}}(C^\bullet, D^\bullet)$.
- **(a)** Show that $Z^0(T^\bullet)$ is the group of maps of complexes from $C^\bullet$ to $D^\bullet$.
- **(b)** Show that $B^0(T^\bullet)$ is the group of maps of complexes from $C^\bullet$ to $D^\bullet$ which are homotopic to zero.

The sheaf Čech complex. Let X be a topological space, and $\mathfrak{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ an open cover of X with Λ totally ordered. For any open set V of X, set $\mathfrak{U} \cap V := \{U_\alpha \cap V\}$. Fix $p \in \{0, 1, 2, \dots, n, \dots\}$. If $C^\bullet(\mathfrak{U}, \mathscr{F})$ denotes the Cech complex of a sheaf of $\mathscr{F}$, let $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ be the presheaf given by $V \mapsto C^p(\mathfrak{U} \cap V, \mathscr{F}|_V)$, V open in X. It is easy to check that $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ is a sheaf and that the coboundaries in the Cech complex restrict well to open subsets, and hence we have a complex, the so called *sheaf Cech complex*, $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ as well as a map $\mathscr{F} \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$.

3. Show that the natural map $\mathscr{F} \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ is such that the induced map $\mathscr{F}_x \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})_x$ on stalks is a quasi-isomorphism for every $x \in X$. [**Hint:** Find a homotopy between the zero map and the identity map on the augumented complex

$$0 \to \mathscr{F}_x \to \mathscr{C}^0(\mathfrak{U}, \mathscr{F})_x \to \mathscr{C}^1(\mathfrak{U}, \mathscr{F})_x \to \cdots \to \mathscr{C}^n(\mathfrak{U}, \mathscr{F})_x \to \dots.$$

In greater detail, for simplicity assume that $x \in U_{\alpha^*}$, where α^* is the smallest element in the well-ordered set Λ. Let $\boldsymbol{\xi} \in \mathscr{C}^p(\mathfrak{U}, \mathscr{F})_x$. Then $\boldsymbol{\xi}$ is represented by a section $\sigma \in C^p(\mathfrak{U} \cap V, \mathscr{F})$ for some neighbourhood V of x. Without loss of generality, we may assume $V \subset U_{\alpha^*}$. Let $\kappa^p(\sigma) \in C^{p-1}(\mathfrak{U} \cap V, \mathscr{F})$ be the element whose $(\alpha_0, \ldots, \alpha_{p-1})^{\text{th}}$ component, with $\alpha_0 < \cdots < \alpha_{p-1}$, is $\sigma_{\alpha^* \alpha_0 \ldots \alpha_{p-1}} \in \mathscr{F}(U_{\alpha^* \alpha_0 \ldots \alpha_{p-1}} \cap V) = \mathscr{F}(U_{\alpha_0 \ldots \alpha_{p-1}} \cap V)$. Let $k^p(\boldsymbol{\xi})$ be the germ of $\kappa^p(\sigma)$ at x. Check everything is well-defined and $\{k^p\}$ is the required homotopy for such an x. What would you do if $x \notin U_{\alpha^*}$?]

The punctured affine plane. Let k be a field, $A = k[S, T]$, the polynomial ring over k in two variables, $\mathfrak{m}_\circ$ the maximal ideal $\langle S, T\rangle$ of A, $\mathbb{A}^2_k = \operatorname{Spec} A$, $X = \mathbb{A}^2_k \smallsetminus \{\mathfrak{m}_\circ\}$, $U_0 = \operatorname{Spec} A_S$, $U_1 = \operatorname{Spec} A_T$. In geometric terms, $\mathbb{A}^2_k$ is the affine plane over k, $\mathfrak{m}_\circ$ the origin of this affine plane, X the plane punctured at the origin, U_0 the affine plane minus the T-axis, U_1 the affine plane minus the S-axis. Note that $\mathfrak{U} = \{U_0, U_1\}$ is an open cover of the punctured plane X.The Čech complex $C^\bullet(\mathfrak{U}, \mathscr{O}_X)$ is clearly the complex

$$0 \longrightarrow A_S \oplus A_T \xrightarrow{\ d\ } A_{ST} \longrightarrow 0$$

where $d(a, b) = \frac{b}{1} - \frac{a}{1}$. The grading is such that $A_S \oplus A_T$ is in the 0^{th}-place.

4. Show that $\check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{O}_X)$ can be identified with the module of *inverse polynomials in S and T*, i.e., the A-module which as a k-vector space is generated by the linearly independent elements $S^\mu T^\nu$ where $\mu, \nu < 0$, and whose A-module structure is given by $S^m T^n (S^\mu T^\nu) = S^{m+\mu} T^{n+\nu}$ if $m + \mu$ and $n + \nu$ are both negative, and is zero otherwise.

Varieties and Schemes. The following problems deal with some material which will be covered soon.

5. Let t be the functor $t : \mathbb{V}\mathrm{ar}_{/k} \to \mathbb{S}\mathrm{ch}_{/k}$ from the category of varieties to the category of schemes over an algebraically closed field k. Show that for any two varieties V, W over k, the natural map

$$\operatorname{Hom}_{\mathbb{V}\mathrm{ar}_{/k}}(V, W) \longrightarrow \operatorname{Hom}_{\mathbb{S}\mathrm{ch}_{/k}}(t(V), t(W))$$

is bijective.

6. Let S be a graded ring and f be a homogenous element of S_+. For any homogenous ideal $\mathfrak{a} \subseteq S$, let $\phi(\mathfrak{a}) = \mathfrak{a} S_f \cap S_{(f)}$. Show that ϕ gives a bijective map from $D_+(f)$ to $\operatorname{Spec} S_{(f)}$.

HW 3

Due date: Oct 25, 2021

As always, "map" is used for "morphism". In particular a "map of complexes" is either chain map pr a co-chain map, depending on whether the complexes in question are chain complexes or co-chain complexes. For problems involving an abelian category $\mathscr{A}$, you may, if you feel like, assume $\mathscr{A} = \mathrm{Mod}_A$, the category of modules of a ring A.

The symbols $\mathscr{Psh}_X$ and $\mathscr{Sh}_X$ are as in the lectures.

The functor Γ. Let X be a topological space. The association $F \mapsto F(X)$, as F varies in $\mathscr{Psh}_X$, gives us a functor, the so called *global sections functor*, from $\mathscr{Psh}_X$ to $\mathscr{Ab}$. This functor is denoted $\Gamma(X, -)$. Thus $\Gamma(X, F) = F(X)$ for $F \in \mathscr{Psh}_X$. For an open subset U of X, the convention is to use the short hand $\Gamma(U, F)$ instead of $\Gamma(U, F|_U)$ for a presheaf F on X. We will follow that convention.

1. Let $(f, f^\#)\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y))$ be a morphism in the category of *locally ringed spaces*, and suppose Y is an affine scheme,[1] say $Y = \operatorname{Spec} A$. Let $x \in X$, and consider the map of rings $A \to \mathscr{O}_{X,x}$ given by the composite

$$A = \Gamma(Y, \mathscr{O}_Y) \xrightarrow{\Gamma(Y, f^\#)} \Gamma(X, \mathscr{O}_X) \longrightarrow \mathscr{O}_{X,x}$$

Let $\mathfrak{m}_x$ be the maximal ideal of $\mathscr{O}_{X,x}$ and $\mathfrak{p}$ the prime ideal in A which is the inverse image of $\mathfrak{m}_x$. Let $y \in Y$ be the point corresponding to $\mathfrak{p}$. Show that $f(x) = y$. [**Note:** We are not assuming X is a scheme. It is merely a locally ringed space.]

2. Let X be a locally ringed space and let $f \in \Gamma(X, \mathscr{O}_X)$. For $x \in X$, let $\mathfrak{m}_x$ be the maximal ideal of $\mathscr{O}_{X,x}$. Fix $x \in X$. Show that if the image of f in $\mathscr{O}_{X,x}$ does not lie in $\mathfrak{m}_x$ then there is an open neighbourhood U of x such that $f|_U$ is invertible in $\Gamma(U, \mathscr{O}_X)$.

Direct Limits. For a quick recall of the definitions and the existence of direct limits, look up these notes.

3. Let A be a ring, $t \in A$ an element, and Q be an A-module. Let $(Q_n)_{n\geq 0}$ be the directed system with $Q_n = Q$ for all n and with transition maps $\mu_{m,n}$ given by $x \mapsto t^{n-m}x$ for $m \leq n$. Show that

$$\varinjlim_n Q_n = Q_t$$

where Q_t is the localisation of Q at the multiplicative system $\{1, t, t^2, \dots, t^n, \dots\}$. [**Hint:** Let $\nu_n\colon Q_n \to Q_t$ be the map $x \mapsto x/t^n$. Check that $\nu_n \circ \mu_{m,n} = \nu_m$ for $m \leq n$. Show that the resulting map $\nu\colon \varinjlim_n Q_n \to Q_t$ is an isomorphism.]

[1] i.e. $(Y, \mathscr{O}_Y)$ is an affine scheme.

Koszul and Čech. In this subsection A is a ring, $\boldsymbol{t} = (t_1, \dots, t_d)$ a d-tuple of elements in A, I the ideal $\langle t_1, \dots, t_d\rangle$ generated by the t_i, X the affine scheme $X = \operatorname{Spec} A$, Z the closed subset $V(I)$ of X, $U = X \smallsetminus Z$.

For $f \in A$, we use the more standard notation $D(f)$ for the open set X_f we defined in §2.2 of Lecture 2.

Let U_i be the open subscheme $U_i = \operatorname{Spec} A_{t_i} = D(t_i)$ of X, $i = 1, \dots, d$ and $\mathfrak{U}$ the family of open sets $\{U_i\}_{i=1,\dots,d}$. Note that $\mathfrak{U}$ is an open cover of U.

Fix an A-module M. Set

$$K_\infty^\bullet = K_\infty^\bullet(\boldsymbol{t},\, M) := \varinjlim_{\boldsymbol{\nu}\in\mathbf{N}^d} K^\bullet(\boldsymbol{t}^{\boldsymbol{\nu}},\, M),$$

where $(K^\bullet(\boldsymbol{t}^{\boldsymbol{\nu}},\, M))_{\boldsymbol{\nu}}$ is the direct system defined in (3.1.2) of these supplementary notes on complexes. The p^{th} differential of this complex will be denoted d_∞^p.

4. Let $C^\bullet = C^\bullet(\mathfrak{U}, \widetilde{M}|_U)$ where $\widetilde{M}$ is the sheaf on X defined by the $\mathscr{B}$-sheaf $\Gamma(D(f), \widetilde{M}) = M_f$ for $f \in A$, and $\mathscr{B} = \{D(g)\}_{g\in A}$ the standard base for the topology on X. Show that
(a) Show that $C^p = K_\infty^{p+1}$, $p \geq 0$.
(b) Show that $d_C^p = d_\infty^{p+1}$, $p \geq 0$.
(c) Show that if $\boldsymbol{t}$ is an M-sequence then $\mathrm{H}^p(K_\infty^\bullet) = 0$ for $p \neq d$. [**Hint:** Use the fact that direct limits preserve exactness and hence commute with cohomology.]

Hint: Use Problem 3.

5. Let $R = A[T_0, \dots, T_n]$ be the polynomial ring in $n+1$ variables over A. Let $\mathbb{A}_A^{n+1} = \operatorname{Spec} R$, J the ideal generated by $T_0, \dots, T_n$, V the complement of $V(J)$ in $\mathbb{A}_A^{n+1}$, and $V_i = D(T_i)$, $i = 0, \dots, n$. Let $\mathfrak{V} = \{V_i\}_{i=0}^n$ be the cover of V given by the V_i. Show that
(a) $\check{\mathrm{H}}^i(\mathfrak{V},\, \mathscr{O}_V) = 0$ for $i \neq 0, n$.
(b) $\check{\mathrm{H}}^0(\mathfrak{V},\, \mathscr{O}_V) = R$.
(c) $\check{\mathrm{H}}^n(\mathfrak{V},\, \mathscr{O}_V)$ is the R-module P of *inverse polynomials* in $T_0, \dots, T_n$, i.e. as an A-module P is the direct sum of the free rank one A-modules $A_{\boldsymbol{\nu}} = A \cdot T_0^{\nu_0} \dots T_n^{\nu_n}$ with each $\nu_i < 0$ and the R-module structure is given as follows: For a monomial $\mathbf{T}^{\boldsymbol{\mu}} := T_0^{\mu_0} \dots T_n^{\mu_n}$ in $A[T_0, \dots, T_n]$ and an A-basis element $\mathbf{T}^{\boldsymbol{\nu}} := T_0^{\nu_0} \dots T_n^{\nu_n}$ of P (in the just mentioned direct sum decomposition of P) we have

$$\mathbf{T}^{\boldsymbol{\mu}} \cdot \mathbf{T}^{\boldsymbol{\nu}} = \begin{cases} T_0^{\mu_0+\nu_0} \dots T_n^{\mu_n+\nu_n} & \text{if } \mu_i + \nu_i < 0 \text{ for } i = 0, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

When you solve Problem 5 (and it is not difficult), you would have essentially proved the main result of this course, namely the so-called *Cohomology of Projective Space* which is in Chapter III, p. 225 of Hartshorne. This is a short cut to that result.

HW 4

Due date: Nov 3, 2021

As always, "map" is used for "morphism". In particular a "map of complexes" is either chain map pr a co-chain map, depending on whether the complexes in question are chain complexes or co-chain complexes. For problems involving an abelian category $\mathscr{A}$, you may, if you feel like, assume $\mathscr{A} = \mathrm{Mod}_A$, the category of modules of a ring A.

The symbols $\mathscr{Psh}_X$ and $\mathscr{Sh}_X$ are as in the lectures.

For a ringed space $(X, \mathscr{O}_X)$, the symbol $\mathrm{Mod}_{\mathscr{O}_X}$ will denote the category of $\mathscr{O}_X$-modules.

If $\mathfrak{U} = (U_\alpha)$ is an open cover of a topological space X, and V is an open subset of X, then $\mathfrak{U} \cap V$ denotes the open cover $(U_\alpha \cap V)$ of V.

Quasi-coherent sheaves on affine schemes. Recall that if A is a ring and $M \in \mathrm{Mod}_A$, then $\widetilde{M}$ is the sheaf of $\mathscr{O}_{\mathrm{Spec}\,A}$-modules defined by $D(f) \mapsto M_f$, $f \in A$, with restrictions given by further localisation. Sometimes it is useful to specify the ring A (e.g. if $A' \to A$ is a ring homomorphism, so that M is an A'-module and an A-module). In that case we use the symbol $\widetilde{M}_A$. Recall that an $\mathscr{O}_X$-module $\mathscr{F}$ on $X = \mathrm{Spec}\,A$ is said to be *quasi-coherent* if $\mathscr{F}$ is isomorphic to $\widetilde{M}$ as an $\mathscr{O}_X$-module for some $M \in \mathrm{Mod}_A$.

For problems in this section, fix a ring A and let $X = \mathrm{Spec}\,A$. For $f \in A$, $D(f)$ is identified with the scheme $\mathrm{Spec}\,A_f$. Note that $\mathscr{O}_{D(f)} = \mathscr{O}_X|_{D(f)}$

1. (a) Let $f \in A$ and $M \in \mathrm{Mod}_A$. Show that $(\widetilde{M}_A)|_{D(f)} = (\widetilde{M_f})_{A_f}$. Conclude that if $\mathscr{F}$ is quasi-coherent on X, then $\mathscr{F}|_{D(f)}$ is quasi-coherent for each $f \in A$.

(b) For a map of rings $A \to B$, with $Y = \mathrm{Spec}\,B$ and $\alpha\colon Y \to X$ the map of schemes induced by $A \to B$, show that $\alpha_*(\widetilde{M}_B) = \widetilde{M}_A$, for $M \in \mathrm{Mod}_B$. Here α_* is the direct image functor defined in (1.2.1) of Lecture 3. In particular, deduce that if $f \in A$ and $i\colon D(f) \to X$ is the natural open inclusion, then $i_*((\widetilde{M_f})_{A_f}) = (\widetilde{M_f})_A$.

2. Suppose we have elements $f_0, \dots, f_d \in A$ such that $X = \cup_{i=0}^d D(f_i)$ (equivalently $\langle f_0, \dots, f_d\rangle = A$). Let $\mathfrak{U}$ be the ordered open cover $(D(f_i))_{i=0}^d$ of X. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module such that $\mathscr{F}|_{D(f_i)}$ is quasi-coherent on the affine scheme $D(f_i)$ for $i = 0, \dots, d$.

(a) Show that the Čech complex $C^\bullet(\mathfrak{U}, \mathscr{F})$ is a complex of A-modules.

(b) Let $g \in A$. Show that the localisation $C^\bullet(\mathfrak{U}, \mathscr{F})_g$ of the Čech complex of A-modules $C^\bullet(\mathfrak{U}, \mathscr{F})$ at g is the Čech complex of $\mathscr{F}|_{D(g)}$ with respect to the cover $\mathfrak{U} \cap D(g)$.

(c) Show that $\mathscr{F}$ is quasi-coherent. [**Hint:** Let $M = \Gamma(X\,\mathscr{F})$. Localise the exact sequence $0 \to M \to C^0(\mathfrak{U},\, \mathscr{F}) \to C^1(\mathfrak{U},\, \mathscr{F})$ at various $g \in A$ and compute $\mathscr{F}(D(g))$. Use the fact that A_g is a flat A-algebra, whence localisation at g is an exact functor.]

3. Let $\mathfrak{U} = (U_\alpha)$ be an affine open cover[1] of X such that $\mathscr{F}|_{U_\alpha}$ is quasi-coherent for every α. Show that $\mathscr{F}$ is quasi-coherent. [**Hint:** You might need to use the fact that X is quasi-compact since it is an affine scheme.]

4. Suppose Z is a scheme, $\mathscr{F}$ an $\mathscr{O}_Z$-module, and $\mathfrak{U} = (U_\alpha)$ an affine open cover of Z such that $\mathscr{F}|_{U_\alpha}$ is quasi-coherent for every α. Let $X = \operatorname{Spec} A$ be an affine open subscheme of Z.[2] Show that $\mathscr{F}|_X$ is quasi-coherent. [**Hint:** Cover X by affine open subschemes on which $\mathscr{F}$ is quasi-coherent and use the previous problem.]

$\mathscr{O}_X$-modules on a scheme. In this section we fix a scheme X, not necessarily affine. Let $\mathscr{F} \in \operatorname{Mod}_{\mathscr{O}_X}$. We say $\mathscr{F}$ is a *quasi-coherent $\mathscr{O}_X$-module* if there exists an affine open cover $\mathfrak{U}$ of X such that $\mathscr{F}|_U$ is quasi-coherent for each $U \in \mathfrak{U}$. Equivalently (by Problem **4**.) $\mathscr{F}$ is quasi-coherent if $\mathscr{F}|_U$ is quasi-coherent for every affine open subscheme of X.

Fix a scheme X.

5. Let X be affine, say $X = \operatorname{Spec} A$. Let $\mathscr{B} = \{D(f)\}$ be the standard base for the topology on X. Let $\mathscr{F}$ be a $\operatorname{Mod}_{\mathscr{O}_X}$-module and $M = \Gamma(X,\, \mathscr{F})$.

(a) Show that the natural map $M_f \to \mathscr{F}(D(f))$, for $f \in A$, arising from the universal property of localisation, gives a map of $\mathscr{B}$-sheaves $\widetilde{M}|_{\mathscr{B}} \to \mathscr{F}|_{\mathscr{B}}$.

(b) Let $\varphi_{\mathscr{F}} \colon \widetilde{M} \to \mathscr{F}$ be the resulting map of sheaves. It is clearly a map of $\mathscr{O}_X$-modules (you don't have to prove this). Show that $\varphi_{\mathscr{F}}$ is functorial in $\mathscr{F}$. In greater detail, writing $M_{\mathscr{F}}$ for $\Gamma(X,\, \mathscr{F})$, show that given a map $\mathscr{F} \to \mathscr{G}$ in $\operatorname{Mod}_{\mathscr{O}_X}$, the following diagram commutes

$$\begin{tikzcd} \widetilde{M_{\mathscr{F}}} \arrow[r, "\text{via } \varphi"] \arrow[d, "\varphi_{\mathscr{F}}"'] & \widetilde{M_{\mathscr{G}}} \arrow[d, "\varphi_{\mathscr{G}}"] \\ \mathscr{F} \arrow[r, "\varphi"'] & \mathscr{G} \end{tikzcd}$$

6. Let

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

be a short exact sequence of $\mathscr{O}_X$-modules.

(a) Show that if $\mathscr{F}_2$ and $\mathscr{F}_3$ are quasi-coherent, then so is $\mathscr{F}_1$.

(b) Show that if $\mathscr{F}_1$ and $\mathscr{F}_2$ are quasi-coherent, then so is $\mathscr{F}_3$.

(c) Show that if $\mathscr{F}_1$ and $\mathscr{F}_3$ are quasi-coherent, then so is $\mathscr{F}_2$.

Hint: Since quasi-coherence is a local property, without loss of generality assume thar X is affine. Let $M_i = \Gamma(X\,\mathscr{F}_i)$. Apply Problem **5**. For part (a) use the fact that $\Gamma(X,\, -)$ is left exact and that $\widetilde{(-)}$ is exact. For parts (b) and (c) use the fact that on an affine scheme Z any short exact sequence of sheaves

[1]i.e. an open cover such that every $(U_\alpha,\, \mathscr{O}_X|_{U_\alpha})$ is an affine scheme.

[2]This means X is open in Z and $\mathscr{O}_X = \mathscr{O}_Z|_X$.

$0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$, with $\mathscr{A}$ quasi-coherent, gives us a short exact sequence of abelian groups $0 \to \mathscr{A}(Z) \to \mathscr{B}(Z) \to \mathscr{C}(Z) \to 0$. We will prove this result later, and it rests on the fact that that the cohomology modules of quasi-coherent sheaves on affine schemes vanish. We don't need $\mathscr{B}$ or $\mathscr{C}$ to be in $\mathrm{Mod}_{\mathscr{O}_X}$ for the validity of this statement on exact sequences of global sections on affine schemes. The following commutative diagram with exact rows may be useful.

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{M}_1 \arrow[r] \arrow[d, "\varphi_{\mathscr{F}_1}"'] & \widetilde{M}_2 \arrow[r] \arrow[d, "\varphi_{\mathscr{F}_2}"'] & \widetilde{M}_3 \arrow[d, "\varphi_{\mathscr{F}_3}"] & \\
0 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_3 \arrow[r] & 0
\end{tikzcd}$$

You will have to argue that the top row is exact. And that sometimes one can put an arrow to 0 on the right of that row, and retain exactness.

The d-uple embedding. Please look at your tutorial notes for the intuitive definition of the d-uple embedding of a projective space into another projective space. This set of exercises is meant to make that rigorous. For a graded ring $S = \bigoplus_n S_n$, let S_+ be the ideal $S_+ = \bigoplus_{n>0} S_n$. For $d \geq 1$ we denote by $S^{(d)}$ be the graded ring given by $(S^{(d)})_n = S_{nd}$. It is well known that if $\phi\colon R \to S$ is a map of graded rings, $X = \mathrm{Proj}(S)$, $Y = \mathrm{Proj}(R)$, and $U(\phi) = \bigcup_{f \in R_+} D_+(\phi(f)) \subset X$, then we have a canonical map

$$r_\phi\colon U(\phi) \to Y.$$

In greater detail, the ring homomorphisms $R_{(f)} \to S_{(\phi(f))}$ give us maps $D_+(\phi(f)) \to D_+(f)$ which patch as f varies over R_+, to give r_ϕ. You may assume this easily provable fact. It is in this sense that we sometimes say that $\mathrm{Proj}(-)$ is functorial.

Note that the inclusion $S^{(d)} \subset S$ is *not* a graded ring map.

By a *closed embedding* of schemes we mean a map of schemes $j\colon Y \to Z$ such that the topological map $Y \to j(Y)$ induced by j is a homeomorphism, $j(Y)$ is a closed subset of Z, and the map $j^\#\colon \mathscr{O}_Y \to j_*\mathscr{O}_Z$ is a surjective map of sheaves (i.e. a map of sheaves whose cokernel sheaf is zero), and the kernel of $j^\#$ is quasi-coherent. For affine schemes this amounts to a surjective map of rings $A \to B$, with the scheme structure on $V(I)$ being provided by $\mathrm{Spec}\, B$, where $I = \ker(A \to B)$.

For us the projective space $\mathbb{P}^n_k$ over a field k is $\mathbb{P}^n_k = \mathrm{Proj}(k[T_0, \ldots, T_n])$.

7. Show that we have an isomorphism $\Psi\colon X \xrightarrow{\sim} Y$ where $X = \mathrm{Proj}(S)$ and $Y = \mathrm{Proj}(S^{(d)})$. Heed the warning given above. [**Hint:** Let $R = S^{(d)}$. The inclusion $R \subset S$ induces a scheme map $j\colon \mathrm{Spec}\, S \to \mathrm{Spec}\, R$. Show that if $\mathfrak{p}$ is a graded prime ideal of S then $j(\mathfrak{p}) = \mathfrak{p} \cap R$ is a graded prime ideal of R. Show that if $f \in S_+$ is homogeneous and $f \notin \mathfrak{p}$, then $f^d \in R_+$ and $f^d \notin j(\mathfrak{p})$. Show further that if $\mathfrak{q} \subset R$ is a graded prime ideal and $I := \mathfrak{q}S$ the ideal of S generated by elements of $\mathfrak{q}$, then I is homogeneous, $\mathfrak{q} = I \cap R$, $\mathfrak{p} := \sqrt{I}$ is a graded prime ideal of S, $j(\mathfrak{p}) = \mathfrak{q}$, and if $\mathfrak{q}$ does not contain R_+ then $\mathfrak{p}$ does not contain S_+. Conclude that j induces a homeomorphism $i\colon X \to Y$. Next show that $S_{(f)} \cong R_{(f^d)}$ for f homogeneous. Conclude that one has an isomorphism $\mathscr{O}_Y \to i_*\mathscr{O}_X$.]

8. As above, let d be a positive integer. Let k be a field. Consider the subring $S = k[T_0^{\mu_0} \dots T_n^{\mu_d} \mid \sum_{i=0}^{n} \mu_i = d]$ of the ring $k[T_0, \dots, T_n]$ and give R the grading which gives each monomial $\mathbf{T}^{\boldsymbol{\mu}}$ degree 1 (in other words, $S = k[\mathbf{T}]^{(d)}$, with $k[\mathbf{T}]$ being the the standard graded ring of polynomials in $(n+1)$-variables). Let R be the polynomial ring in $\binom{n+d}{n}$ variables and write R as $R = k[Z_{\boldsymbol{\mu}} \mid \sum_{i=0}^{n} \mu_i = d]$, where the μ_i are non-negative integers and the $Z_{\boldsymbol{\mu}}$ are indeterminates (we are using the well known combinatorial fact that the number of $\boldsymbol{\mu}$ of the kind we specified is $\binom{n+d}{d}$). Show that the graded ring map $\phi\colon R \to S$ given by $Z_{\boldsymbol{\mu}} \to \mathbf{T}^{\boldsymbol{\mu}}$ gives a closed embedding of $\mathbb{P}_k^n \hookrightarrow \mathbb{P}_k^N$ where $N = \binom{n+d}{d} - 1$. [**Hint:** First show that $U(\phi)$ defined in the beginning of this section is $\mathrm{Proj}(S)$. Then show that $R_{(Z_{\boldsymbol{\mu}})} \to S_{(\mathbf{T}^{\boldsymbol{\mu}})}$ is surjective.]

It is worth pointing out that the field k above is not algebraically closed. In our course we will de-emphasise algebraic closure (we will sometimes use it, but generally it won't be necessary). In fact note that the problem really did not require k to be a field. It could have been a ring, and one would then be getting results for W-schemes, where $W = \operatorname{Spec} k$. One can go one step further and actually see that everything goes through for arbitrary W, not necessarily affine W. All this is food for thought, not to be submitted.

HW 5

Due date: Nov 10, 2021

1. Let $\mathscr{A}$ be an abelian category and $E^\bullet$ a bounded below injective complex of objects in $\mathscr{A}$ such that $\mathrm{H}^n(E^\bullet) = 0$ for $n \neq 0$ and $\mathrm{H}^0(E^\bullet) = A$. Show that there is a quasi-isomorphism $A \to E^\bullet$, where A is regarded as a complex in the usual way. [**Hint:** You may use the fact that if E is an injective object which is a subobject of an object X, then X must be of the form $X = E \oplus E'$. You do not have to prove this easy fact in the quiz. But see if you can prove it for yourself later.]

2. Let A_0 be a ring, and let A and B be A_0-algebras. Let $S = \operatorname{Spec} A_0$, $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. Let $\pi\colon X \to S$ and $\varpi\colon Y \to S$ be the natural scheme maps. Show that there exists a scheme W together with scheme maps $p\colon W \to X$ and $q\colon W \to Y$ such that $\pi\circ p = \varphi\circ q$ and such that if we have a scheme T together with scheme maps $f\colon T \to X$, $g\colon T \to Y$ satisfying $\pi\circ f = \varpi\circ g$, then there exists a unique map $(f,g)\colon T \to W$ satisfying $p\circ(f,g) = f$ and $q\circ(f,g) = g$.

In other words, given a commutative diagram with solid arrows as below, the broken arrow can be filled in exactly one way to make the resulting diagram commute:

$$\begin{tikzcd}
T \arrow[dr, dashed, "\exists!", "(f,g)"'] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& W \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "\varpi"] \\
& X \arrow[r, "\pi"'] & S
\end{tikzcd}$$

Comment: The datum (W, p, q) is clearly unique up to unique isomorphism because of the universal property the datum has. It is usually denoted $X \times_S Y$. As is common in mathematics, the symbol is used for the datun (W, p, q) as well as for the scheme W. The scheme or the datum $X \times_S Y$ is called the *fibre product* of X and Y over S.

HW 6

Due date: Nov 22, 2021

The inverse image of a sheaf. Let $f\colon X \to Y$ be a continuous map of topological spaces, and $\mathscr{G} \in \mathcal{S}\!h_Y$. Then the inverse image $f^{-1}\mathscr{G}$ of $\mathscr{G}$ is the sheaf on X obtained from the fibre-product $X \times_Y \mathscr{E}(\mathscr{G})$ consisting of points $(x, e) \in X \times \mathscr{E}(\mathscr{G})$ such that $f(x) = \pi_{\mathscr{G}}(e)$, where $\pi = \pi_{\mathscr{G}}\colon \mathscr{E}(\mathscr{G}) \to Y$ is the standard projection map. The space $X \times \mathscr{E}(\mathscr{G})$ is given the product topology, and $X \times_Y \mathscr{E}(\mathscr{G})$ the subspace topology. It is easy to see that the natural map $X \times_Y \mathscr{E}(\mathscr{G}) \to X$ males $X \times_Y \mathscr{E}(\mathscr{G})$ into an étale space over X and hence gives a sheaf, which we denote $f^{-1}\mathscr{G}$. Equivalently, consider the presheaf $f^{\#}\mathscr{G}$ given by $U \mapsto \varinjlim_{V} \mathscr{G}(V)$ where the direct limit is taken over open subsets V of Y containing $f(U)$. Then $f^{-1}\mathscr{G}$ is the sheafification of $f^{\#}\mathscr{G}$. It is not hard to show that f^{-1} is a left adjoint to f_*, i.e. one has a bifunctorial isomorphism $\mathrm{Hom}_{\mathcal{S}\!h_X}(f^{-1}\mathscr{G}, \mathscr{H}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{S}\!h_Y}(\mathscr{G}, f_*\mathscr{H})$. The (easy) details may be found in https://www.cmi.ac.in/~pramath/random_notes/upper-lower-star.

1. Let $f\colon X \to Y$ be a continuous map of topological spaces.

(a) Show that f^{-1} is exact. Using this, show that $f_*\mathscr{E}$ is an injective sheaf on Y if $\mathscr{E}$ is injective on X.

(b) Given $x \in X$ and $y = f(x)$, show that for any sheaf $\mathscr{G}$ on Y, there is a functorial map $\mathscr{G}_y \to (f^{-1}\mathscr{G})_x$.

(c) Show that the a map of ringed spaces $(f, f^{\#})\colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ is equivalent to the data $(f, f^{\flat})$ with $f^{\flat}\colon f^{-1}\mathscr{B} \to \mathscr{A}$ a map of sheaves of rings. Show further that if $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are locally ringed spaces then $(f, f^{\#})$ is a map of locally ringed spaces if and only if for every $x \in X$, with $y = f(x)$, the composite $\mathscr{B}_y \to (f^{-1}\mathscr{B})_x \xrightarrow{f^{\flat}_x} \mathscr{A}_x$ is a local homomorphism.

Tensor product of sheaves of modules and the upper-star functor. Let $(X, \mathscr{O}_X)$ be a ringed space, and let $\mathscr{F}, \mathscr{G} \in \mathrm{Mod}_{\mathscr{O}_X}$. We have a presheaf $\mathscr{F} \overset{P}{\otimes}_{\mathscr{O}_X} \mathscr{G}$ given by $U \rightsquigarrow \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$. We define the tensor product $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ to be the sheafification of $\mathscr{F} \overset{P}{\otimes}_{\mathscr{O}_X} \mathscr{G}$. Note that $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \in \mathrm{Mod}_{\mathscr{O}_X}$. Combining the universal property of sheafifications and the universal property of tensor products we see that there is a universal $\mathscr{O}_X$-bilinear map (i.e. over each open U is bilinear as a map of $\mathscr{O}_X(U)$-modules) $\mathscr{F} \times \mathscr{G} \to \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ such that if if $\mathscr{F} \times \mathscr{G} \to \mathscr{H}$ is a bilinear map of $\mathscr{O}_X$-modules, there is a unique map $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \to \mathscr{H}$ in $\mathrm{Mod}_{\mathscr{O}_X}$ such that our bilinear map factors as $\mathscr{F} \times \mathscr{G} \xrightarrow{\text{universal}} \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \to \mathscr{H}$.

If $f\colon X \to Y$ is a map of ringed spaces, and $\mathscr{G} \in \mathrm{Mod}_{\mathscr{O}_Y}$, we define $f^*\mathscr{G}$ as $f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$, where $\mathscr{O}_X$ is an $f^{-1}\mathscr{O}_Y$ algebra via $f^{\flat}$. Note that f^* is right exact, but need not be exact. It is easy to see that $\mathrm{Hom}_X(f^*\mathscr{G}, \mathscr{H}) \xrightarrow{\sim} \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{H})$

where $\mathrm{Hom}_X(-, ?)$ is the Hom in $\mathrm{Mod}_{\mathscr{O}_X}$. Once again see https://www.cmi.ac.in/~pramath/random_notes/upper-lower-star (at least for the case of schemes and quasi-coherent sheaves, though the proof generalises).

2. Let $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, $f\colon X \to Y$ a map of schemes, and M an A-module. Show that $f^*\widetilde{M}$ is the sheafificationof $M \otimes_A B$.

3. Suppose X is a scheme and $\mathscr{A}$ a sheaf of $\mathscr{O}_X$-algebras on X such that, as an $\mathscr{O}_X$-module, $\mathscr{A}$ is quasi-coherent. Show that there is an X-scheme $\mathbf{Spec}(\mathscr{A})$ such that the structure morphism $\Phi\colon \mathbf{Spec}(\mathscr{A}) \to X$ is an affine map, and such that $\Phi_*\mathscr{O}_{\mathbf{Spec}(\mathscr{A})}$ is canonically isomorphic to $\mathscr{A}$. Show further that the canonical map $\Phi^{\#}\colon \mathscr{O}_X \to v_*\mathscr{O}_{\mathbf{Spec}(\mathscr{A})}$ is, under this identification, the algebra map $\mathscr{O}_X \to \mathscr{A}$. Conversely, if $\Psi\colon S \to X$ is an affine map of schemes, show that $\mathbf{Spec}(\Psi_*\mathscr{O}_S)$ is canonically isomorphic to S and that under this identification, the structure map $\mathbf{Spec}(\Psi_*\mathscr{O}_S) \to X$ agrees with Ψ. I would like to remind you that according to item **4** of 3.1.1 of Lecture 13, $\Psi_*\mathscr{O}_S$ is quasi-coherent on X since Ψ is an affine morphism.