Unit III

Introduction to Propositional Logic

Proposition :

Our discussion begins with an introduction to the basic building blocks of logic :—propositions.

- **Def :** A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.
- **Eg :** 1. Washington, D.C., is the capital of the United States of America.

  2. Toronto is the capital of Canada.

  3. $1 + 1 = 2$.

  4. $2 + 2 = 3$
- **Non Eg :** 1. What time is it?

  2. Read this carefully.

  3. $x + 1 = 2$.

  4. $x + y = z$.

Note :

- We represent the Propositions with small letters *p, q, r, s, ...*
- *The truth values for any proposition are either T or F.*
- The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic**. It was first developed systematically by the Greek philosopher Aristotle more than 2300 years ago.
- **Compound Propositions** :Many mathematical statements are constructed by combining one or more propositions forming New propositions, called **compound propositions**, are formed from existing propositions using logical operators. We Discuss : Negation,

  Conjunction

  Disjunction and

  Implication.

Negation :

- **<u>Def :</u>** Let *p* be a proposition. The *negation of p*, denoted by $\neg p$ is the statement "It is not the case that *p*."
- The proposition $\neg p$ is read "not *p*." The truth value of the negation of *p*, $\neg p$, is the oppositeof the truth value of *p*.
- Eg : Find the negation of the proposition "Michael's PC runs Linux".

Ans :"It is not the case that Michael's PC runs Linux." This negation can be more simply expressed as "Michael's PC does not run Linux."

<u>Truth Table :</u>

p	$\neg p$
T	F
F	T

- The negation operator constructs a new proposition from a single existing proposition. We will now introduce the logical operators that are used to form new propositions from two or more existing propositions. These logical operators are also called **connectives**.

Conjunction and Disjunction :

- **Def :** Let *p* and *q* be propositions. The *conjunction* of *p* and *q*, denoted by $p \wedge q$, is the proposition "*p* and *q*." The conjunction $p \wedge q$ is true when both *p* and *q* are true and is false otherwise.
- Truth Table for Conjunction :

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjuction: Let *p* and *q* be propositions. The *disjunction* of *p* and *q*, denoted by $p \vee q$, is the proposition "*p* or *q*." The disjunction $p \vee q$ is false when both *p* and *q* are false and is true otherwise. Truth table is

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional Statements :

- **<u>Def :</u>** Let p and q be propositions. The *conditional statement* $p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).
- *$P \to q$ can also be expressed as follows :*

"if p, then q"
"if p, q"
"p is sufficient for q"
"q if p"
"q when p"
"a necessary condition for p is q"
"q unless $\neg p$"
"p implies q"
"p only if q"
"a sufficient condition for q is p"
"q whenever p"
"q is necessary for p"
"q follows from p"

Truth Table for Implication :

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

CONVERSE, CONTRAPOSITIVE, AND INVERSE

- We can form some new conditional statements starting with a conditional statement $p \to q$.
- The proposition $q \to p$ is called the **converse** of $p \to q$.
- The **contrapositive** of $p \to q$ is the proposition $\neg q \to \neg p$.
- The proposition $\neg p \to \neg q$ is called the **inverse** of $p \to q$.

Everyone Draw the truth tables for all the mentioned above and Notice the conditions that are having same truth values(i.e Identify the Logical Equivalence)

Def of Logical Equivalance : When two compound propositions always have the same truth value we call them equivalent(will discuss after few lectures)

- Eg : What are the contrapositive, the converse, and the inverse of the conditional statement "The home team wins whenever it is raining"
- Ans : Because "*q* whenever *p*" is one of the ways to express the conditional statement $p \rightarrow q$, the original statement can be rewritten as

  "If it is raining, then the home team wins."

  Consequently, the **contrapositive** of this conditional statement is

  "If the home team does not win, then it is not raining."

  The **converse** is "If the home team wins, then it is raining."

  The **inverse** is

  "If it is not raining, then the home team does not win."

Biconditional statement :

- **Def :** Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values and is false otherwise. Biconditional statements are also called *bi-implications*.
- There are some other common ways to express $p \leftrightarrow q$:

"p is necessary and sufficient for q"

"if p then q, and conversely"

"p iff q"

- The last way of expressing the biconditional statement $p \leftrightarrow q$ uses the abbreviation "iff" for "if and only if." Note that $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$

Truth Table For Bi Conditional :

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Work out Problems : (has to be done in the class)

- Let p and q be the propositions “The election is decided” and “The votes have been counted,” respectively. Express each of these compound propositions as an English sentence :

  a) $\neg p$
  b) $p \vee q$
  c) $\neg p \wedge q$
  d) $q \rightarrow p$
  e) $\neg q \rightarrow \neg p$
  f) $\neg p \rightarrow \neg q$
  g) $p \leftrightarrow q$
  h) $\neg q \vee (\neg p \wedge q)$

- Construct a truth table for each of these compound propositions:

  a) $p \wedge \neg p$
  b) $p \vee \neg p$
  c) $(p \vee \neg q) \rightarrow q$
  d) $(p \vee q) \rightarrow (p \wedge q)$
  e) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
  f) $(p \rightarrow q) \rightarrow (q \rightarrow p)$

Tautology, Contradiction and Contingency :

- A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*.

  Eg : $p \vee \neg p$ *and* $(p \wedge q) \rightarrow (p \vee q)$ *(verify with truth tables)*

- A compound proposition that is always false is called a *contradiction*.

  Eg : $p \wedge \neg p$ *(verify with truth tables)*

- A compound proposition that is neither a tautology nor a contradiction is called a *contingency.*

  Give an Example on your own………

Logical Equivalence Illustrations :

- The compound propositions *p* and *q* are called *logically equivalent* if $p \leftrightarrow q$ is a tautology.

The notation $p \equiv q$ denotes that *p* and *q* are logically equivalent.

Eg : 1)Identify Logical Equivalent compound Statements for P=>q and for its converse, inverse and contrapositive.

Eg 2 :

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

De Morgan Laws in Propositional Logic :

- $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Proof :

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Prove the other De Morgan Law………

Show that $p \to q$ and $\neg p \vee q$ are logically equivalent.

Equivalence Laws :

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Logical Equivalences Involving Conditional & Bi-Conditional Statements :

$$p \to q \equiv \neg p \vee q$$
$$p \to q \equiv \neg q \to \neg p$$
$$p \vee q \equiv \neg p \to q$$
$$p \wedge q \equiv \neg(p \to \neg q)$$
$$\neg(p \to q) \equiv p \wedge \neg q$$
$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$
$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$
$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$
$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Verify All Above-Mentioned Laws By using the Truth Tables…..

Argument and Valid ness:

- An ***argument*** in propositional logic is a *sequence of propositions*. All but the preceding propositions in the argument are called ***premises*** and the final proposition is called the ***conclusion***.

  i.e,. Premises ⊦ Conclusion ("⊦" is symbol for argument)

  Eg 1: $p \to q, q \to r \vdash p \to r$
- Eg 2: Consider the following argument involving propositions

  $p \to q$:"If you have a current password, then you can log onto the network."

  q :You have a current password."

  Therefore,

  r : "You can log onto the network."

  In the above example Premises : $p \to q$,r

  Conclusion : r

  Hence it is an argument.

Validness of an argument:

- An Argument is ***valid*** if the truth of all its premises implies that the conclusion is true.

i.e, the argument form with premises $p1$, $p2, \ldots, p_n$ and conclusion q is valid, when $(p1 \land p2 \land \cdots \land p_n) \to q$ is a tautology.

Means, To check the given argument is valid or not we need to follow:

a) Take conjunction to the Premises and write implication to conclusion.

b) If the last implication is tautology, then we can say that the given Argument is valid.

Note : An argument is not valid then we can call it as “Fallacy”

Checking Validness of the argument: $p \rightarrow q, q \rightarrow r \vdash p \rightarrow r$

p	q	r	p→q	q→r	p→r	(p→q) ∧(q→r)	**(p→q) ∧(q→r) → p→r**
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
F	T	T	T	T	T	T	T
T	F	F	F	T	F	F	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

In the Above ex, the last column we got Tatology. Hence this argument is valid.

In the similar manner, Check the validness of the following Arguments:

- ***HOME -WORK:***
- $(p \vee q), (\neg p \vee r) \vdash (q \vee r)$
- $p \vee q, \neg p \vdash q$
- $\neg q, (p \to q) \vdash \neg p$
- $p \vee q, \neg p \vee r \vdash q \vee r$
- $(p \vee q), (p \to r), (q \to r) \vdash r$

Proof Systems :

- **Methods of Proving Theorems :**

  1)Direct Proofs

  2)Proof by Contraposition

  3)Proofs by Contradiction

We are going to discuss these three techniques in proving the theorems in mathematics.

Some Terminology :

- **Theorem : Theorem** is a statement that can be shown to be true. In mathematical writing, the term theorem is usually reserved for a statement that is considered at least somewhat important. Theorem may be the universal quantification of a conditional statement with one or more premises and a conclusion. However, it may be some other type of logical statement.
- **Proof of a theorem :**A **proof** is a valid argument that establishes the truth of a mathematical statement. A proof can use the hypotheses of the theorem, if any, axioms assumed to be true, and previously proven theorems.
- Less important theorems sometimes are called **propositions.**
- A less important theorem that is helpful in the proof of other results is called a **lemma.**
- **Corollary** is a theorem that can be established directly from a theorem that has been proved.

Direct Proof Method :

- A **direct proof** of a conditional statement $p \to q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true.
- **Eg :If *n* is an odd integer, then n^2 is odd.**

Proof : Let us assume that n is odd.

By the definition of an odd integer, it follows that $n = 2k + 1$, where k is some integer.

We want to show that n^2 is also odd.

We can square both sides of the equation $n = 2k + 1$ to obtain a new equation that expresses n^2.

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

$$=2m+1 \text{ ; where } m=2k^2 + 2k \textit{ for all integer } k.$$

Theorem: If *m* and *n* are both perfect squares, then *nm* is also a perfect square.

- **Proof :** We assume that *m* and *n* are both perfect squares.

By the definition of a perfect square, it follows that there are integers *s* and *t* such that $m = s^2$ and $n = t^2$.

The goal of the proof is to show that *mn* must also be a perfect square.

Now consider $mn = s^2\, t^2 = (ss)(t\,t) = (st)(st) = (st)^2 = (ts)^2 = nm$ (using commutativity and associativity of multiplication)

By the definition of perfect square, it follows that *mn* is also a perfect square.

Home-Work:

Theorem: Prove that the sum of two rational numbers is rational.

Proof by Contraposition :

- An extremely useful type of indirect proof is known as **proof by contraposition**.
- Proofs by contraposition make use of the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$.
- This means that the conditional statement $p \rightarrow q$ can be proved by showing that its contrapositive, $\neg q \rightarrow \neg p$, is true.
- In a proof by contraposition of $p \rightarrow q$, we take $\neg q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference, we show that $\neg p$ must follow.

Theorem: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Proof :

- We first attempt a direct proof.
- To construct a direct proof, we first assume that $3n + 2$ is an odd integer. This means that $3n + 2 = 2k + 1$ for some integer k.
- We see that $3n + 1 = 2k$, but there does not seem to be any direct way to conclude that n is odd.
- Because our attempt at a direct proof failed, we next try a proof by contraposition.
- By contraposition is to assume that the conclusion of the conditional statement "If $3n + 2$ is odd, then n is odd" is false; namely, assume that n is even.
- Then, by the definition of an even integer, $n = 2k$ for some integer k. Substituting $2k$ for n, we find that

  $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. **This tells us that $3n + 2$ is even**
- Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded; we have proved the theorem "If $3n + 2$ is odd, then n is odd."

Theorem: Prove that if n is an integer and n^2 is odd, then n is odd.

Proof :

- Direct Method won't work here (Why!!)

- Now by contraposition method, consider n is not odd.

  implies.. N is even i.e $n = 2k$ for some integer k.

  keep $n= 2k$ in n^2 => $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$

  $= 2m$ where $m = 2k^2$ for k is an intiger.

Hence ,

We have proved that if n is an integer and $n2$ is odd, then n is odd. Our attempt to find a proof by contraposition succeeded.

Proofs by Contradiction:

- A proof by contradiction does not prove a result directly, it is another type of indirect proof in which **we assume** the negation of the given statement is true. So that we finally arrive at a contradiction (error) because of our wrong assumption.

- Hence, we conclude that our assumption is wrong and the given one is true….

Theorem: Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

- **Proof :** To prove by contradiction, Let us assume that the negation of the given statement is true.

i. e., if possible, suppose that $\sqrt{2}$ is Rational.

$\Rightarrow \sqrt{2} = a/b$; $b \neq 0$ and a,b have no common factors.

$\Rightarrow 2 = a^2/b^2 \Rightarrow a^2 = 2b^2$ -----→(1)

$\Rightarrow$ a is even (n^2 is even $\Rightarrow$ n is even)

$\Rightarrow a = 2k$ (by def of even)for some integer k

put in (1) $\Rightarrow (2k)^2 = 2b^2$

$\Rightarrow 4k^2 = 2b^2$

$\Rightarrow b^2 = 2k^2$

$\Rightarrow$ b is even ,which is a contradiction to our supposition as we already have a is even.

Hence our supposition is wrong $\Rightarrow$ **$\sqrt{2}$ is irrational**

Give a proof by contradiction of the theorem "If $3n + 2$ is odd, then n is odd."

- **Proof:** Since it is p=>q, let us assume that the negation of the statement is true. i.e $\neg(p \Rightarrow q) \equiv p \land \neg q$ is true.

$p \land \neg q$ means 3n+2 is odd and n is not odd----→(supposition)

n is not odd => n is even => n = 2k for every integer k

then $3n+2 = 3(2k)+2$

$=2(3k+1)$

$= 2m$ where $m = 3k+1$ for intiger k

i.e, 3n+2 is even which is contradiction to our supposition. Hence our supposition is wrong.

so, If $3n + 2$ is odd, then n is odd.

End of the Chapter