Unit II

Partially Ordered Sets and Lattice Theory

Contents:

Partial Ordering Relations, Partially Ordered Sets

Comparable and Noncomparable elements, Chain and Antichain

Hasse diagrams of Poset, Supremum and Infimum.

Lattice

Bounded, Distributive, Complete and Complemented Lattices.

- **Partial ordering(POSET) :** A relation R on a set S is called a ***partial ordering*** or ***partial order*** if it is reflexive, antisymmetric and transitive.
- NOTE : A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S,R). Members of S are called *elements* of the poset.

- Eg : Greater than or equal" relation ($\geq$) is a partial ordering on the set of integers. Prove it.

Sol : Because $a \geq a$ for every integer a, $\geq$ is reflexive.

If $a \geq b$ and $b \geq a$, then $a = b$. Hence, $\geq$ is antisymmetric.

Finally, $\geq$ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$.

It follows that $\geq$ is a partial ordering on the set of integers and $(\mathbf{Z}, \geq)$ is a poset.

- The divisibility relation | is a partial ordering on the set of positive integers. **Verify !!!**
- Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S. **Verify !!!**

Comparable and Noncomparable Elements :

- **Comparable and In comparable**:

The elements *a* and *b* of a poset *(S,≤)* are called ***comparable*** if either $a \leq b$ or $b \leq a$. When

- *a* and *b* are elements of *S* such that neither $a \leq b$ nor $b \leq a$, *a* and *b* are called ***Noncomparable***.
- Eg : In the poset *(**Z**+, |)*, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

- **Ans :** The integers 3 and 9 are comparable, because 3 | 9. The integers 5 and 7 are Noncomparable, because 5 doesn't divide 7 and 7 doesn't divide 5.

Examples Continues....

Let X = {1, 2, 3, 4, 5}.

- (a) The identity relation Id on X is reflexive, transitive and anti-symmetric and is therefore a partial order. However, no two elements of X are comparable.
- (b) The relation “Id **U** {(1, 2)}”is also a partial order on X. Here 1 and 2 are comparable.
- (c) The relation “Id **U** {(1, 2),(2,1)}” is both reflexive and transitive, but not anti-symmetric. (why)
- (d) The relation “Id **U** {(1, 2),(3,4)}” is a partial order on X. Here, 1 and 2 are comparable and so are 3 and 4.

Totally Ordered :

- The adjective "partial" is used to describe partial orderings because pairs of elements may be incomparable. When every two elements in the set are comparable, the relation is called a **total ordering**.

i.e., If $(S,\leq)$ is a poset and every two elements of S are comparable then S is called a ***totally ordered* or *linearly ordered set***.

Eg : The poset $(\mathbf{Z},\leq)$ is totally ordered (Verify)

- The poset $(\mathbf{Z}+, \mid)$ is not totally ordered because it contains elements that are incomparable.

- Note : A totally ordered set is also called a ***chain***.

Chain and Anti chain :

- **Chain** : Let (X, ≤) be a poset,

A subset, C of X, is called a **chain** if and only if induces a linear order on C. If C is a finite set, then the length of C is equal to the number of elements if C. If C is not a finite set, then the length of C is said to be infinite.

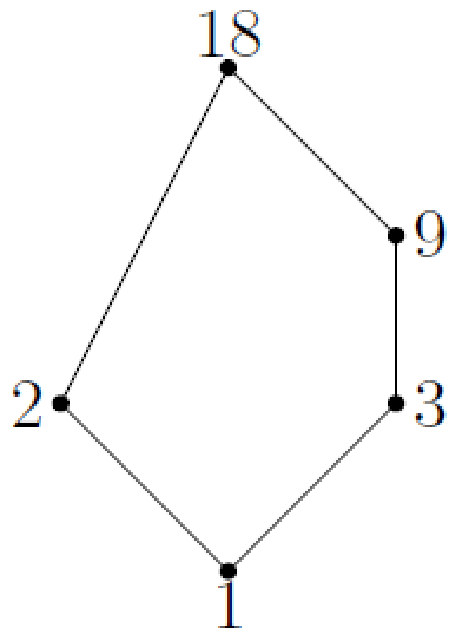
- **Anti chain :** A subset, A of X, is called an antichain if and only if no two elements of A are comparable. The length of an antichain is defined in precisely the same manner as that of the chain finite.
- Eg : The set N with the partial order f defined by "(a,b) ∈ f if a divides b" is not linearly ordered. However, the set {1,2,4,8,16} is a chain. This is just a linearly ordered subset of the poset.

Hasse Diagram :

The Hasse diagram of a finite poset $(X,\leq)$ is a picture drawn in the following way :

- 1. Each element of X is represented by a point and is labeled with the element.
- 2. If $a \leq b$ then the point labeled a must appear at a lower height than the point labeled b and further the two points are joined by a line.
- 3. If $a \leq b$ and $b \leq c$ then the line between a and c is removed.

Eg : Hasse diagram for the poset (A,≤) with A = {1, 2, 3, 9, 18} and "≤" as the `divides' relation is given here :



Work : 1. {1, 2, 3, 6, 9, 18} (all positive divisors of 18) with the relation as `divides'.

2. {2,3 4,5, 6, 7, 8} with the `divides' relation.

3. A = {1,2,3}, Draw Hasse diagram for the POSET (P(A), ⊆)

Upper bound and lower bounds:

- Let **(X,≤)** be a poset and let $A \subseteq X$.

Upper bound : We say that an element $x \in X$ is an **upper bound of A** if for each $z \in A$; $z \leq x$; or equivalently,when each element of A is less than or equal to x.

Lower bound : An element $y \in X$ is called a lower bound of A if for each $z \in A$; $y \leq z$; or equivalently, when y is less than or equal to each element of A.

It is very easy to identify the Upper and lower bounds for any subset of given POSET by using Hasse Diagram.

Maximum,Minimum,Maximal and Minimal Elements in the given subset of a POSET :

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- An element $x \in A$ is called the **maximum of A**, if x is an upper bound of A. Thus, maximum of A is an upper bound of A which is contained in A. Such an element is unique provided it exists. In this case, we denote $x = \max\{z : z \in A\}$.

Similarly,

Minimum of A is an element $y \in A$ which is a lower bound of A. If minimum of A exists, then it is unique and we write $y = \min\{z : z \in A\}$

- An element $x \in A$ is a **maximal element of A** if $x \leq z$ for some $z \in A$ implies $x = z$; or equivalently, when no element in A is larger than x.

An element $y \in A$ is called a **minimal element of A** if $z \leq y$ for some $z \in A$ implies $y = z$; or equivalently, when no element in A is lessthan y

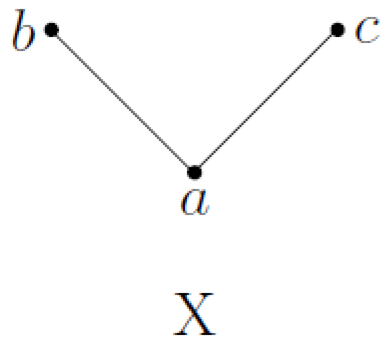
LUB and GLB :

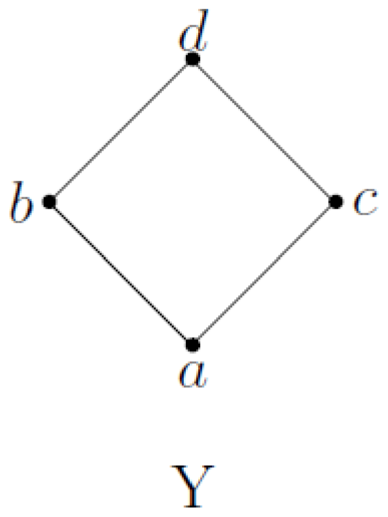
- **Least Upper Bound :** An element $x \in X$ is called the **least upper bound** (lub) of A in X if x is an upper bound of A and for each upper bound y of A, we have $x \leq y$; i.e., when x is the minimum (least) element of the set of all upper bounds of A.

- **Greatest lower bound :** Similarly, the **greatest lower bound** (glb) of A is a lower bound of A which is greater than or equal to all upper bounds of A; it is the maximum (largest) of the set of all lower bounds of A.

All the above bounds are very easy to verify by using Hasse Diagram.

- Consider the two posets X = {a, b, c} and Y = {a, b, c, d} described by the following Hasse diagrams:





Now We will see examples for the above all definitions below :

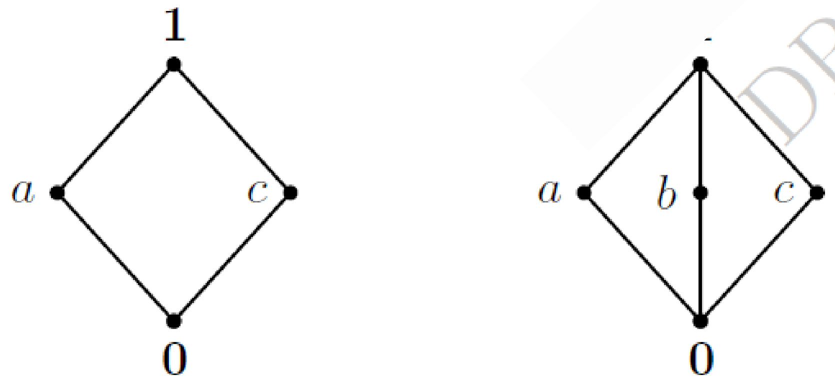
The following table illustrates the definitions by taking different subsets A of X, and also considering the same A as a subset of Y.

	$A = \{b, c\} \subseteq X$	$A = \{a, c\} \subseteq X$	$A = \{b, c\} \subseteq Y$
Maximal element(s) of A	b, c	c	b, c
Minimal element(s) of A	b, c	a	b, c
Lower bound(s) of A in X	a	a	a
Lower bound(s) of A in Y	a	a	a
Upper bound(s) of A in X	does not exist	c	d
Upper bound(s) of A in Y	does not exist	c	d
Maximum element of A	does not exist	c	does not exist
Minimum element of A	does not exist	a	does not exist
lub of A in X	does not exist	c	d
lub of A in Y	does not exist	c	d
glb of A in X	a	a	a
glb of A in Y	a	a	a

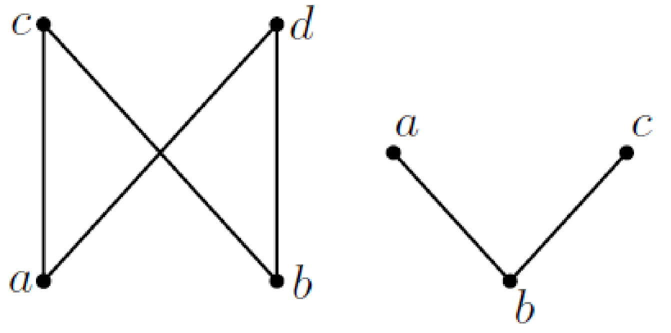
Lattice:

- **Def :** A poset (L,≤) is called a **lattice** if each pair x,y ∈ L has an LUB and also a glb

Eg :



(Lattices)



(not a lattices)

Which of the above POSets are Lattices… and why? (hint: just use the lattice definition) .

Ans : first two are lattices. Bcz, every pair of elements in it are having Lub and Glb

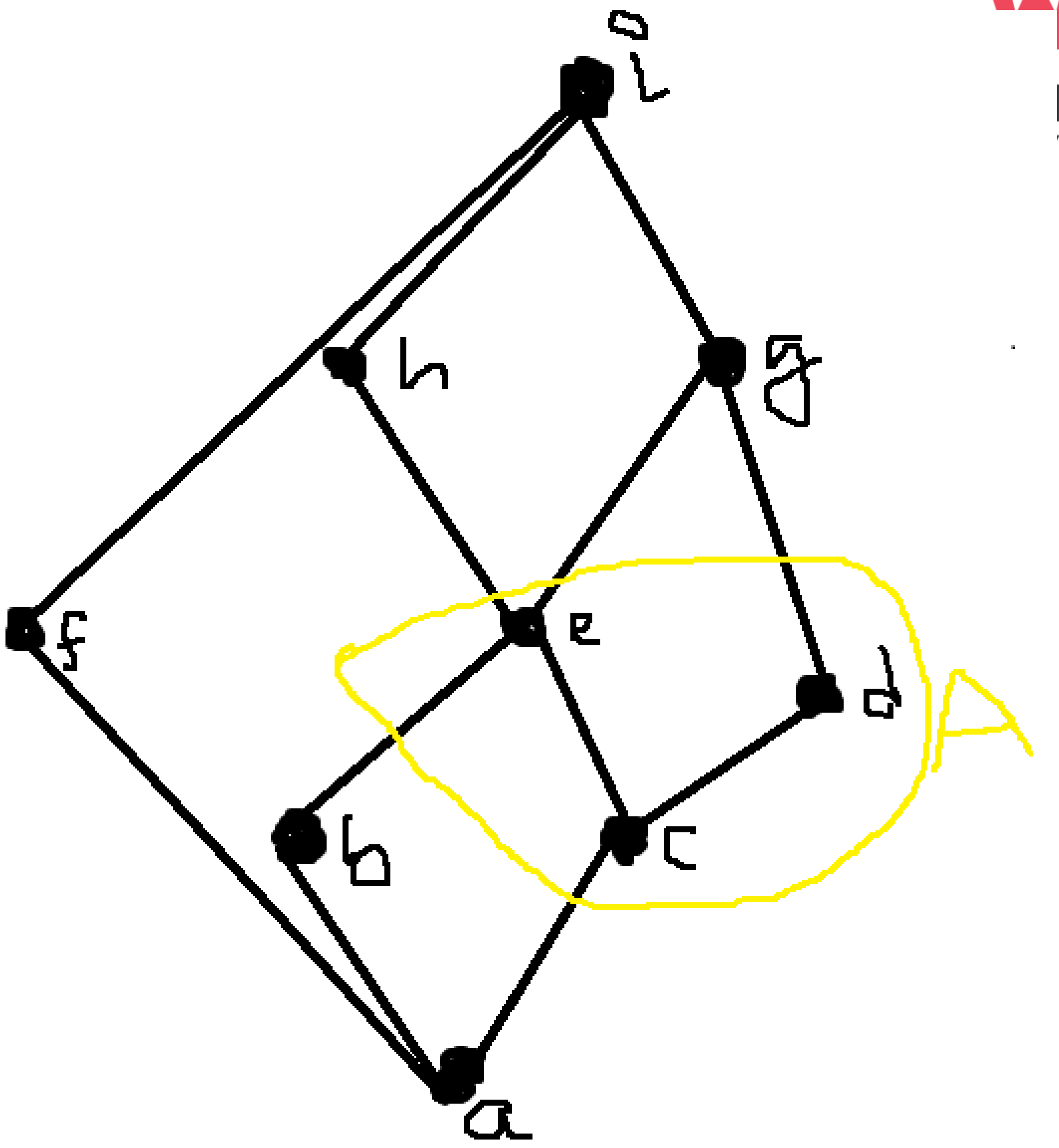
In the third fig, consider the subset {a,b}. Find its upper bounds, lower bounds, lub and glb, Max, min elements. Calculate the same for all figures.

Work out problems….

- Verify whether the adjacent POSET is lattice ?

If yes, consider the subset A and Calucate all the lower bounds, Upper bounds, lub and glb for A. Identify Max. and Min. elements, chain and anti chains..

Cal. The same by considering some other subset



Distributive Lattice :

- An lub of x,y is also written as $x \vee y$ (read as \`x or y' / \`join of x and y’) a.nd a glb of x,y as $x \wedge y$ (read as \`x and y' / \`meet of x and y’).
- Def : A lattice is called a **distributive lattice** if for all pairs of elements x,y,z in a lattice, the following conditions are holds, called distributive laws, are satised :

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Verify, given above all examples for distributive lattice.

Also verify for $A=\{1,2,3\}$, $(P(A), \subseteq)$ and $(\{1,2,3,4,6,12,30\},|)$ is distributive lattice or not.

(Hint : verify by taking any three elements(mostly consider from the anti chain) and verify any one above property.)

Bounded Lattice :

- Let $(L, \leq)$ be a lattice. It is called a **bounded lattice** if there exist elements $\alpha, \beta \in L$ such that for each $x \in L$; we have $x \leq \alpha$ and $\beta \leq x$. Such an element α is called the largest element of L, and is denoted by **I**. The element $\beta \in L$ satisfying $\beta \leq x$ for all $x \in L$ is called the smallest element of L and is denoted by **O**.

Notice that if a lattice is bounded, then I is the lub of the lattice and O is the glb of the lattice.

Now identity all Bounded lattices in the previous all examples.

Complete Lattice :

- **Def :** A lattice (L, ≤) is said to be complete if for each nonempty subset of L has lub and glb in L.

It follows that each complete lattice is a bounded lattice.

Note : Every finite lattice is **complete** and hence, **bounded.** (why?)

Eg : all above mentioned finite lattices are complete.

Now we will have look on infinite lattices.

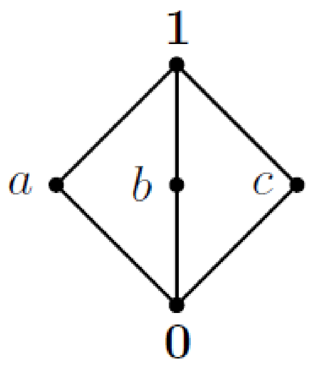
- The set [0 5] with the usual order is a lattice which is both bounded and complete. So, is the set [0 1), [2 3].
- The set (0 5] with the usual order is a lattice which is neither bounded nor complete.
- The set [0 1)U(2 3] with the usual order is a lattice which is bounded but not complete.

Complemented Lattice :

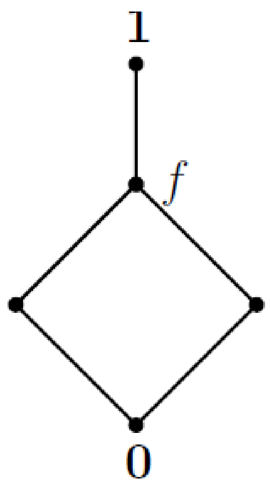
- **Def :** Let $(L, \leq)$ be a bounded lattice. We say that $(L, \leq)$ is a complemented lattice if for each $x \in L$; there exists $y \in L$ such that

$$x \vee y = I \text{ and } \qquad x \wedge y = O.$$

Such an element y corresponding to the element x is called a complement of x, and is denoted by $\neg x$.



Complemented but NOT distributive



Distributive but NOT complemented

Few Laws : Let (L, $\leq$) be a lattice. then

1. [Idempotence] : $a \vee a = a, \quad a \wedge a = a$
2. [Commutativity] : $a \vee b = b \vee a, \quad a \wedge b = b \wedge a$
3. [Associativity] : $a \vee (b \vee c) = (a \vee b) \vee c, \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c$
4. $a \leq b \Leftrightarrow a \vee b = b$. *Similarly,* $a \leq b \Leftrightarrow a \wedge b = a$
5. [Absorption] : $a \vee (a \wedge b) = a = a \wedge (a \vee b)$
6. [Isotonicity] : $b \leq c \Rightarrow a \vee b \leq a \vee c, \quad b \leq c \Rightarrow a \wedge b \leq a \wedge c$
7. $a \leq b,\ c \leq d \Rightarrow a \vee c \leq b \vee d, \quad a \leq b,\ c \leq d \Rightarrow a \wedge c \leq b \wedge d$
8. [Distributive Inequality] : $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c), \quad a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$
9. [Modularity] : $a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$

Boolean Algebra :

Definition A **Boolean algebra** is a nonempty set S which is closed under the binary operations $\vee$ (called **join**), $\wedge$ (called **meet**), and the unary operation $\neg$ (called **inverse** or **complement**) satisfying the following properties for all $x, y, z \in S$:

1. **[Commutativity]** : $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$.
2. **[Distributivity]** : $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
3. **[Identity elements]** : There exist elements $\mathbf{0},\ \mathbf{1} \in S$ such that $x \vee \mathbf{0} = x$ and $x \wedge \mathbf{1} = x$.
4. **[Inverse]** : $x \vee \neg x = \mathbf{1}$ and $x \wedge \neg x = \mathbf{0}$.

When required, we write the Boolean algebra S as $(S, \vee, \wedge, \neg)$ showing the operations explicitly.