WOXSEN UNIVERSITY

UNIT-2

Introduction to Counting

Syllabus:UNIT-5

- Basic counting techniques – inclusion and exclusion
- Pigeon-hole principle and examples
- Permutations, Combinations
- Introduction to recurrence relation
- Generating functions

Basic Counting Principles:

We first present two basic counting principles, the **product rule** and the **sum rule**.

Then we will show how they can be used to solve many different counting problems.

The product rule applies when a procedure is made up of separate tasks.

Counting Functions:

- How many functions are there from a set with m elements to a set with n elements?

Clue: Afunction corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain.

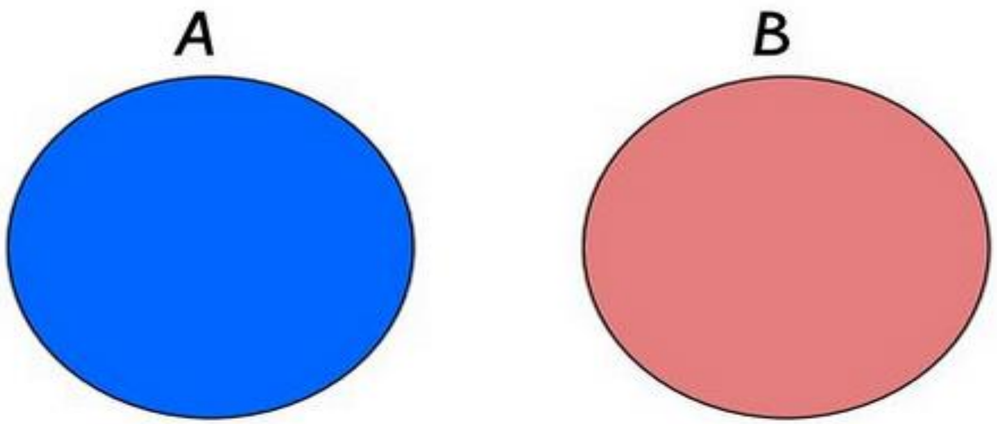
Hence, by the product rule there are $n \cdot n \cdot \cdots \cdot n = n^m$ functions from a set with m elements to one with n elements.

For example, there are $5^3 = 125$ different functions from a set with three elements to a set with five elements.

Sum Rule

If sets A and B are disjoint, then

$$|A \cup B| = |A| + |B|$$



What if A and B are not disjoint?

Principle of inclusion–exclusion:

- THE SUBTRACTION RULE
- If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

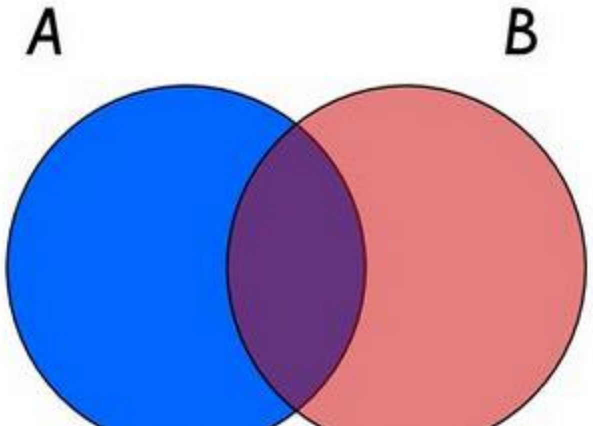
- The subtraction rule is also known as the **principle of inclusion–exclusion**, especially when it is used to count the number of elements in the union of two sets.

Principle of inclusion–exclusion(Union)

Inclusion-Exclusion (2 sets)

For two arbitrary sets A and B

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Example-1

- A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?
- Sol: Let A_1 be the set of students who majored in computer science and A_2 the set of students who majored in business.
- Then $A_1 \cup A_2$ is the set of students who majored in computer science or business (or both), and
- $A_1 \cap A_2$ is the set of students who majored both in computer science and in business. $A_1 = 220$, $A_2 = 147$ and $A_1 \cap A_2 = 51$

By the subtraction rule the number of students who majored either in computer science or in business (or both) equals

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$
$$= 220 + 147 - 51 = 316.$$

We conclude that 350 − 316 = 34 of the applicants majored neither in computer science nor in business.

Home Work: Example-1

1. Every student in a discrete mathematics class is either a computer science or a mathematics major or is a joint major in these two subjects. How many students are in the class if there are 38 computer science majors (including joint majors), 23 mathematics majors (including joint majors), and 7 joint majors?

Home Work: Example 2

- In a college, there are 20 students who enrolled for commerce only,90 students enrolled for mathematics only, 30 students enrolled for commerce and mathematics both and 60 students enrolled for others.
- Find

(a) The total number of students in college.

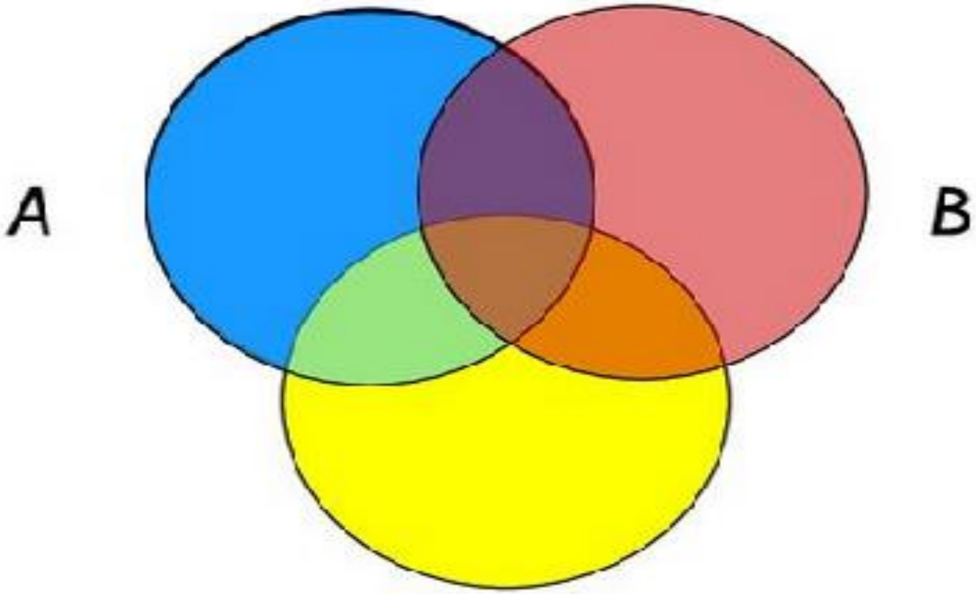
(b) Total number of students enrolled either for commerce or mathematics.

(c) Total number of students enrolled for commerce.

(d) Total number of students enrolled for mathematics.

Inclusion-Exclusion (3 sets)

$$\begin{aligned}|A \cup B \cup C| = |A| + |B| + |C| \\ - |A \cap B| - |A \cap C| - |B \cap C| \\ + |A \cap B \cap C|\end{aligned}$$



Example 1

- In a college the strength of the students are 50. Among 30 students known Java, 18 C++, 26 known C-language, and 9 students known Java and C++, 16 Java and C-language, 8 students practiced both C++ and C-language, 47 students known at least one language, and how many students know none.

Inclusion-Exclusion (3 sets)

From a total of 50 students:

$|A| \longrightarrow$ 30 know Java

$|B| \longrightarrow$ 18 know C++

$|C| \longrightarrow$ 26 know C#

$|A \cap B| \longrightarrow$ 9 know both Java and C++

$|A \cap C| \longrightarrow$ 16 know both Java and C#

$|B \cap C| \longrightarrow$ 8 know both C++ and C#

$|A \cup B \cup C| \longrightarrow$ 47 know at least one language.

How many know none?

How many know all? $\uparrow$ $|A \cap B \cap C|$

$$\boxed{|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|}$$

$$47 = 30 + 18 + 26 - 9 - 16 - 8 + |A \cap B \cap C|$$

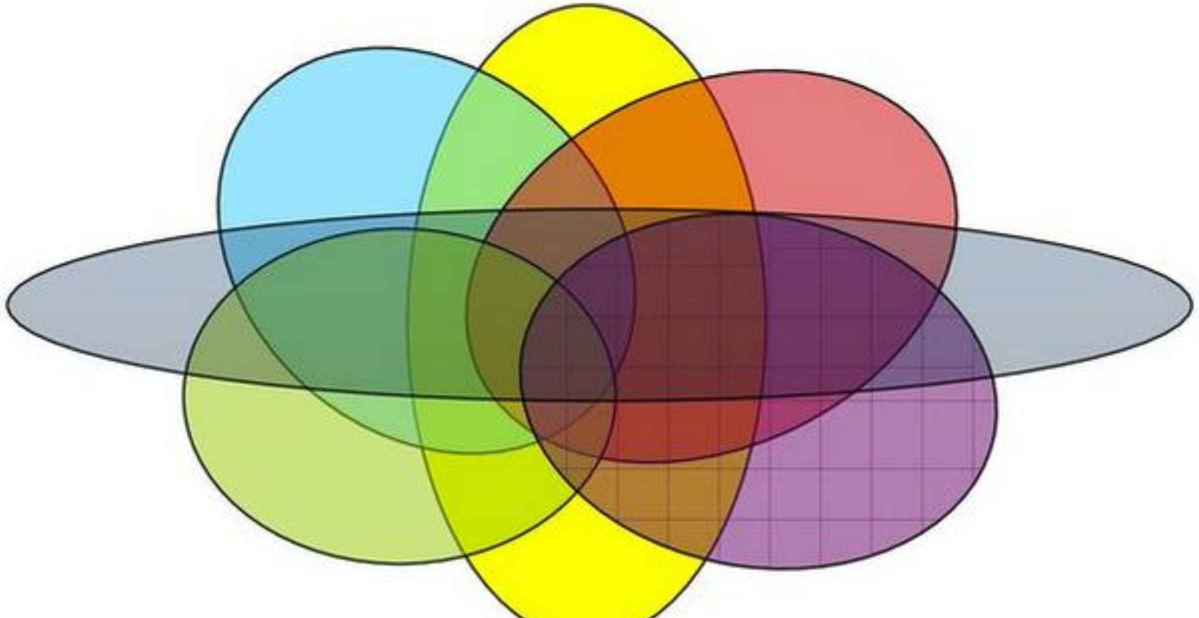
$$|A \cap B \cap C| = 6$$

Home Work: Example 1

- In a shop, 380 people buy socks, 150 people buy shoes and 200 people buy belt. If there are total 580 people who bought either socks or shoes or belt and only 30 people bought all the three things? So how many people bought exactly two things.

Inclusion-Exclusion (n sets)

What is the inclusion-exclusion formula for the union of n sets?



THE PIGEONHOLE PRINCIPLE

Statement: If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

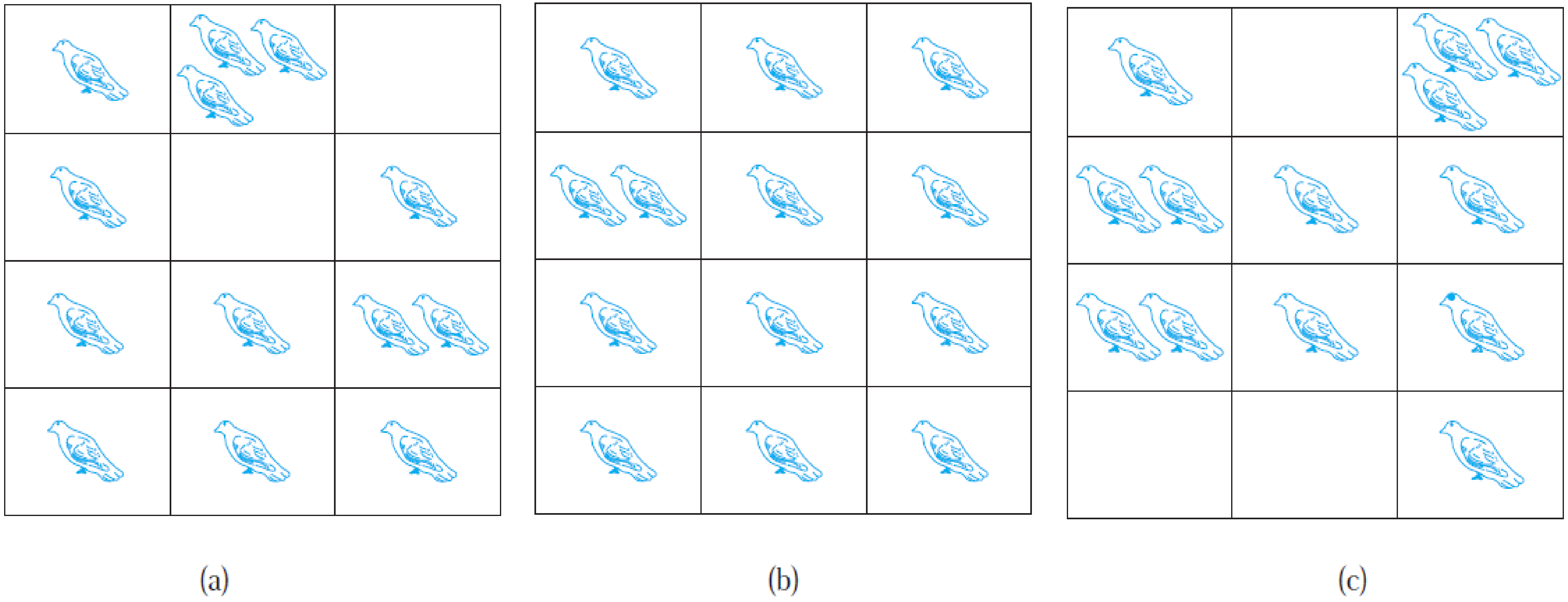


FIGURE 1 There Are More Pigeons Than Pigeonholes.

Example-2

- During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.
- Sol:

Solution: Let a_j be the number of games played on or before the jth day of the month. Then $a_1, a_2, \ldots, a_{30}$ is an increasing sequence of distinct positive integers, with $1 \leq a_j \leq 45$. Moreover, $a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$ is also an increasing sequence of distinct positive integers, with $15 \leq a_j + 14 \leq 59$.

The 60 positive integers $a_1, a_2, \ldots, a_{30}, a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$ are all less than or equal to 59. Hence, by the pigeonhole principle two of these integers are equal. Because the integers a_j, $j = 1, 2, \ldots, 30$ are all distinct and the integers $a_j + 14$, $j = 1, 2, \ldots, 30$ are all distinct, there must be indices i and j with $a_i = a_j + 14$. This means that exactly 14 games were played from day $j + 1$ to day i.

Permutations and Combinations

- **Introduction**
- Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters. Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected does not matter.

Definition

- A **permutation** of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of *r* elements of a set is called an ***r*-permutation**.

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

If n and r are integers with $0 \leq r \leq n$, then $P(n, r) = \dfrac{n!}{(n-r)!}$.

Example-1

- For example, in how many ways can we select three students from a group of five students to stand in line for a picture? How many different committees of three students can be formed from a group of four students?
- Sol: First, note that the order in which we select the students matters. There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line. By the product rule, there are $5 \cdot 4 \cdot 3 = 60$ ways to select three students from a group of five students to stand in line for a picture.
- To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way. Consequently, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to arrange all five students in a line for a picture.

Example-2

Q:How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Sol: Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements.

Consequently, the answer is

$P(100, 3) = 100 \cdot 99 \cdot 98 = 970{,}200.$

HOME WORK

How many permutations of the letters $ABCDEFGH$ contain the string ABC ?

Combination

- In mathematics, a **combination** is a way of selecting items from a collection where the order of selection does not matter.

Suppose we have a set of three numbers P, Q and R. Then in how many ways we can select two numbers from each set, is defined by combination.

- or

- The combination is defined as “An arrangement of objects where the order in which the objects are selected does not matter.” The combination means “Selection of things”, where the order of things has no importance.
- More formally, a *k*-combination of a set is a subset of *k* distinct elements of *S*. If the set has *n* elements, the number of *k*-combinations is equal to the binomial coefficient.

which can be written as;

- $\mathbf{{}^{n}C_{k} = n!/k!(n-k)!}$, **when n>k**
- Where:
- **n** – the total number of elements in a set
- **k** – the number of selected objects (the order of the objects is not important)
- **!** – factorial

Example-1:

- **Example 1: A group of 3 lawn tennis players S, T, U. A team consisting of 2 players is to be formed. In how many ways can we do so?**
- **Solution-** In a combination problem, we know that the order of arrangement or selection does not matter.
- Thus ST= TS, TU = UT, and SU=US.
- Thus we have 3 ways of team selection.
- By combination formula we have-
- ${}^{3}C_{2} = 3!/2!\,(3-2)!$
- $= (3.2.1)/(2.1.1) = 3$

Example-2

- **Example 2: Find the number of subsets of the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} having 3 elements.**
- Solution: The set given here have 10 elements. We need to form subsets of 3 elements in any order. If we select {1,2,3} as first subset then it is same as {3,2,1}. Hence, we will use the formula of combination here.
- Therefore, the number of subsets having 3 elements

= ${}^{10}C_3$

= 10!/(10-3)!3!

= 10.9.87!/7!.3!

= 10.9.8/3.2

= 120 ways.

Defintion of Recurrence Relations:

Concept of Recurrence relation :

A recurrence relation relates the nth term of a sequence to some of its preceding terms. Consider a function a(n), which is a sequence of terms a_0 , a_1, ... $a_{n-1.}$ The term a_n depends upon the previous terms a_0 , a_1, ... $a_{n-1.}$ This way of study is called recurrence relation

Example: The recurrence relation that defines the famous Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, ... is given by $f(n) = f(n-1) + f(n-2)$ with initial conditions $f(1) = 1$, $f(0) = 1$.