CASELLA-BERGER
STATISTICAL INFERENCE SOLUTION:
CHAPTER 7

VIRGIL CHAN

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Contents

1. Exercise 7.1

For each x, we find θ so that $f(x|\theta)$ is maximised.

x	0	1	2	3	4
θ	1	1	Either 2 or 3 works	3	3
$f(x\|\theta)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Therefore,

$$\hat{\theta}_1(x) = \begin{cases} 1 & \text{if } x = 0, 1, \\ 2 & \text{if } x = 2, \\ 3 & \text{if } x = 3, 4 \end{cases} \qquad \hat{\theta}_2(x) = \begin{cases} 1 & \text{if } x = 0, 1, \\ 3 & \text{if } x = 2, 3, 4, \end{cases}$$

are the possible MLEs' for θ.

2. Exercise 7.2

The likelihood function for β is given by

$$\begin{aligned}
L(\beta|x_1,\cdots,x_n) &= \prod_{i=1}^{n} f(x_i;\,\alpha,\beta) \\
&= \prod_{i=1}^{n} \frac{x_i^{\alpha-1} e^{-\frac{x_i}{\beta}}}{\Gamma(\alpha)\,\beta^{\alpha}} \\
&= \frac{\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}}{[\Gamma(\alpha)]^n} \cdot \exp\left[-\frac{\sum_{i=1}^{n} x_i}{\beta}\right] \cdot \beta^{\alpha n},
\end{aligned}$$

and has derivative

$$\frac{\partial L}{\partial \beta} = \frac{\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}}{[\Gamma(\alpha)]^n} \cdot \exp\left[-\frac{\sum_{i=1}^{n} x_i}{\beta}\right] \cdot \beta^{-\alpha n-2} \cdot \left(\sum_{i=1}^{n} x_i - \alpha n\beta\right).$$

In particular,

$$\begin{aligned}
\frac{\partial L}{\partial \beta} = 0 \iff & \sum_{i=1}^{n} x_i - \alpha n \beta = 0 \\
\iff & \beta = \frac{\sum_{i=1}^{n} x_i}{\alpha n}.
\end{aligned}$$

It remains to show this β maximises L.

We note that the sign of $\dfrac{\partial L}{\partial \beta}$ is completely determined by the sign of the term $\sum_{i=1}^{n} x_i - \alpha n\beta$.

Thus, algebra says $\hat{\beta} = \dfrac{\sum_{i=1}^{n} x_i}{\alpha n}$ is the MLE for β.

References

[BC01] Roger Berger and George Casella. *Statistical Inference*. 2nd edition. Florence, AL: Duxbury Press, June 2001.