CASELLA-BERGER
STATISTICAL INFERENCE SOLUTION:
CHAPTER 6

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Contents

1. Exercise 6.1

We have

$$\begin{aligned} f_X\left(x\middle|\sigma^2\right) &= \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}(x)^2} \\ &= \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}(|x|)^2} \\ &= f_X\left(|x|\middle|\sigma^2\right)\cdot 1. \end{aligned}$$

Therefore, by [BC01, Factorization Theorem 6.2.6 on page 276], the variable $|X|$ is a sufficient statistic for σ^2.

2. Exercise 6.2

The joint distribution is given by

$$\begin{aligned}
f_{X_1,\cdots,X_n}(x_1,\cdots,x_n|\theta) &= \prod_{i=1}^{n} f_{X_i}(x_i|\theta) \\
&= \prod_{i=1}^{n} \chi_{[i\theta,\infty)}(x_i)\, e^{i\theta - x_i} \\
&= \chi_{[\theta,\infty)}\left(\min_{1\le i\le n}(x_i)\right) \cdot e^{\displaystyle\sum_{i=1}^{n} i\theta - x_i} \\
&= \underbrace{\chi_{[\theta,\infty)}\left(\min_{1\le i\le n}(x_i)\right) e^{in\theta}}_{g\left(\min_{1\le i\le n}(x_i)\middle|\theta\right)} \cdot \underbrace{e^{\displaystyle -\sum_{i=1}^{n} x_i}}_{h(x)}.
\end{aligned}$$

The result then follows from [BC01, Factorization Theorem 6.2.6 on page 276].

3. Exercise 6.3

The joint distribution is given by

$$
\begin{aligned}
f_{X_1,\cdots,X_n}(x_1,\cdots,x_n|\mu,\sigma) &= \prod_{i=1}^{n} \chi_{(\mu,\infty)}(x_i)\,\frac{1}{\sigma}e^{\frac{-(x_i-\mu)}{\sigma}} \\
&= \underbrace{\chi_{(\mu,\infty)}\left(\min_{1\le i\le n} x_i\right)\cdot\frac{1}{\sigma^n}e^{\dfrac{-\sum_{i=1}^{n} x_i + n\mu}{\sigma}}}_{g\left(\min_{1\le i\le n} x_i,\,\sum_{i=1}^{n} x_i\,\middle|\,\mu,\sigma\right)}\cdot\underbrace{1}_{h(x)}.
\end{aligned}
$$

[BC01, Factorization Theorem 6.2.6 on page 276] says the pair $\left(\min_{1\le i\le n} X_i, \sum_{i=1}^{n} X_i\right)$ gives a sufficient statistic for (μ,σ).

4. Exercise 6.5

The joint distribution is given by

$$\begin{aligned} f(x_1, \cdots, x_n|\theta) &= \prod_{i=1}^{n} \chi_{(-i(\theta-1), i(\theta+1))}(x_i)\, \frac{1}{2i\theta} \\ &= \prod_{i=1}^{n} \chi_{(\theta-1,\theta+1)}\left(\frac{x_i}{i}\right) \frac{1}{2i\theta} \\ &= \chi_{(\theta-1,\theta+1)}\left(\min_{1\le i\le n} \frac{x_i}{i}\right) \cdot \chi_{(\theta-1,\theta+1)}\left(\max_{1\le i\le n} \frac{x_i}{i}\right) \cdot \frac{1}{(2\theta)^n n!}. \end{aligned}$$

[BC01, Factorization Theorem 6.2.6 on page 276] says the pair $\left(\min_{1\le i\le n} \frac{X_i}{i}, \sum_{i=1}^{n} \frac{X_i}{i}\right)$ gives a sufficient statistic for θ.

5. Exercise 6.6

The joint distribution is given by

$$\begin{aligned} f_{X_1,\cdots,X_n}(x_1,\cdots,x_n|\alpha,\beta) &= \prod_{i=1}^{n} \frac{x_i^{\alpha-1}e^{-\beta x_i}\beta^\alpha}{\Gamma(\alpha)} \\ &= \left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)^n \cdot \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \cdot e^{-\beta \sum_{i=1}^{n} x_i}. \end{aligned}$$

[BC01, Factorization Theorem 6.2.6 on page 276] says the pair $\left(\prod_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i\right)$ gives a sufficient statistic for (α, β).

6. Exercise 6.7

The joint distribution is given by

$$\begin{aligned} f_{X_1,Y_1,\cdots,X_n,Y_n}(x_1,y_1,\cdots,x_n,y_n|\theta_1,\theta_2,\theta_3,\theta_4) &= \prod_{i=1}^{n} \frac{1}{(\theta_4-\theta_2)(\theta_3-\theta_1)} \cdot \chi_{(\theta_1,\theta_3)}(x_i) \cdot \chi_{\theta_2,\theta_4}(y_i) \\ &= \frac{1}{(\theta_4-\theta_2)^n(\theta_3-\theta_1)^n} \cdot \chi_{(\theta_1,\theta_3)}\left(\min_{1\le i\le n} x_i\right) \cdot \chi_{(\theta_1,\theta_3)}\left(\max_{1\le i\le n} x_i\right) \cdot \\ &\qquad \chi_{(\theta_2,\theta_4)}\left(\min_{1\le i\le n} y_i\right) \cdot \chi_{(\theta_2,\theta_4)}\left(\max_{1\le i\le n} y_i\right) \end{aligned}$$

[BC01, Factorization Theorem 6.2.6 on page 276] says the tuple $\left(\min_{1\le i\le n} X_i, \max_{1\le i\le n} X_i, \min_{1\le i\le n} Y_i, \max_{1\le i\le n} Y_i\right)$ gives a sufficient statistic for $(\theta_1,\theta_2,\theta_3,\theta_4)$.

7. Exercise 6.8

We want to show the ordered statistic $(X_{(1)}, \cdots, X_{(n)})$ is a minimal sufficient statistic for θ. The joint distribution is given by

$$f_{X_1,\cdots,X_n}(x_1, \cdots, x_n|\theta) = \prod_{i=1}^{n} f(x_i - \theta)\,.$$

If the tuples $(x_i)_{i=1}^n$, $(y_i)_{i=1}^n$ are two sample points, such that $T((x_i)_{i=1}^n) = T((y_i)_{i=1}^n)$, then the two sets

$$\{x_i\}_{i=1}^n = \{y_i\}_{i=1}^n$$

are the same. Hence,

$$f_{X_1,\cdots,X_n}(x_1, \cdots, x_n|\theta) = f_{X_1,\cdots,X_n}(y_1, \cdots, y_n|\theta)\,.$$

Therefore, [BC01, Theorem 6.2.13 on page 281] says the ordered statistic is minimal sufficient for θ.

8. Exercise 6.9

Refer to [BC01, Theorem 6.2.13 on page 281].

(a) The joint distribution is given by

$$\begin{aligned} f_{X_1,\cdots,X_n}(x_1,\cdots x_n|\theta) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\theta)^2} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2\right]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{f_{X_1,\cdots,X_n}(x_1,\cdots x_n|\theta)}{f_{X_1,\cdots,X_n}(y_1,\cdots y_n|\theta)} &= \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2-(y_i-\theta)^2\right] \\ &= \exp\left[-\frac{1}{2}\sum_{i=1}^{n}x_i^2-y_i^2-2\theta(y_i-x_i)\right] \\ &= \exp\left[-\frac{1}{2}\left(\sum_{i=1}^{n}x_i^2-y_i^2\right)+n\theta(\overline{y}-\overline{x})\right], \end{aligned}$$

which is independent of θ if $\overline{x}=\overline{y}$. Therefore, $T(X)=\overline{X}$ is a minimal sufficient statistic for θ.

(b) The joint distribution is given by

$$\begin{aligned} f_{X_1,\cdots,X_n}(x_1,\cdots,x_n|\theta) &= \prod_{i=1}^{n} \chi_{(\theta,\infty)}(x_i)\cdot e^{-(x_i-\theta)} \\ &= \chi_{(\theta,\infty)}\left(\min_{1\le i\le n} x_i\right)\cdot e^{-n\overline{x}}\cdot e^{n\theta}. \end{aligned}$$

Therefore, we have

$$\frac{f_{X_1,\cdots,X_n}(x_1,\cdots x_n|\theta)}{f_{X_1,\cdots,X_n}(y_1,\cdots y_n|\theta)} = \frac{\chi_{(\theta,\infty)}\left(\min\limits_{1\le i\le n} x_i\right)\cdot e^{-n\overline{x}}}{\chi_{(\theta,\infty)}\left(\min\limits_{1\le i\le n} y_i\right)\cdot e^{-n\overline{y}}},$$

which is independent of θ if $\min\limits_{1\le i\le n} x_i = \min\limits_{1\le i\le n} y_i$. Therefore, $T(X)=\min\limits_{1\le i\le n} X_i$ is a minimal sufficient statistic for θ.

(c) The joint distribution is given by

$$f_{X_1,\cdots,X_n}(x_1,\cdots,x_n|\theta) = \prod_{i=1}^{n} \frac{e^{-(x_i-\theta)}}{(1+e^{-(x_i-\theta)})^2}$$
$$= \frac{e^{-n\overline{x}}}{\left[\prod_{i=1}^{n} 1+e^{-(x_i-\theta)}\right]^2}.$$

Therefore, we have

$$\frac{f_{X_1,\cdots,X_n}(x_1,\cdots x_n|\theta)}{f_{X_1,\cdots,X_n}(y_1,\cdots y_n|\theta)} = \frac{e^{n\overline{y}}}{e^{n\overline{x}}} \cdot \left[\prod_{i=1}^{n} \frac{1+e^{-(y_i-\theta)}}{1+e^{-(x_i-\theta)}}\right]^2$$
$$= \frac{e^{n\overline{y}}}{e^{n\overline{x}}} \cdot \left[\prod_{i=1}^{n} \frac{1+e^{-(y_{(i)}-\theta)}}{1+e^{-(x_{(i)}-\theta)}}\right]^2$$

which is independent of θ if the ordered statistics of x and y agree. Therefore, the ordered statistic is a minimal sufficient statistic for θ.

(d) A computation similar to part (c) shows the ordered statistic is a minimal sufficient statistic for θ.

(e) The joint distribution is given by

$$f_{X_1,\cdots,X_n}(x_1,\cdots,x_n|\theta) = \prod_{i=1}^{n} \frac{1}{2} e^{-|x_i-\theta|}$$
$$= \frac{1}{2^n} e^{-\sum_{i=1}^{n} |x_i-\theta|}$$

Therefore, the ordered statistic gives a minimal sufficient statistic for θ.

9. Exercise 6.11

C.f [BC01, Section 3.5]. We see that if $Z \sim f(x|0)$, then the two variables

$$Z + \theta \sim X$$

are equal in distribution. It follows that the ordered statistics

$$Z_{(i)} + \theta \sim X_{(i)}$$

of random samples from X and Z are equivalent for $1 \leq i \leq n$. Therefore,

$$\begin{aligned}(Y_1, \cdots Y_{n-1}) &= \left(X_{(n)} - X_{(1)}, \cdots, X_{(n)} - X_{(n-1)}\right) \\ &\sim \left(Z_{(n)} + \theta - X_{(1)} - \theta, \cdots, Z_{(n)} + \theta - Z_{(n-1)} - \theta\right) \\ &= \left(Z_{(n)} - Z_{(1)}, \cdots, Z_{(n)} - Z_{(n-1)}\right).\end{aligned}$$

The joint distribution of the last tuple is a function of the joint distribution of $(Z_1, \cdots, Z_n)$, which is a function of $f(x|0)$. In other words, the joint distribution of the last tuple does not depend on θ, proving the claim as desired.

10. Exercise 6.12

(a) Firstly, to prove minimal sufficiency, we compute the joint distribution

$$\begin{aligned} f_{X,N}(x,n|\theta) &= f_X(x|\theta,n)\cdot f_N(n) \\ &= \binom{n}{x}\theta^x(1-\theta)^{n-x}\cdot p_n. \end{aligned}$$

This gives the quotient

$$\begin{aligned} \frac{f_{X,N}(x,n|\theta)}{f_{X,N}(y,m|\theta)} &= \frac{\binom{n}{x}\theta^x(1-\theta)^{n-x}\cdot p_n}{\binom{m}{y}\theta^y(1-\theta)^{m-y}\cdot p_m} \\ &= \frac{\binom{n}{x}p_n\theta^{x-y}(1-\theta)^{n-m+y-x}}{\binom{m}{y}p_m}, \end{aligned}$$

which is independent of θ if $x=y$ and $n=m$. As a result, the tuple (X,N) gives a minimal sufficient statistic for θ.

Next, N is ancillary for θ because its distribution

$$f_N(n)=p_n$$

does not depend on θ.

(b) For the uniase part, we need to show the expected values for X/N and θ agree. We compute

$$\begin{aligned} E(X/N) &= E(E(X/N|N)) \\ &= E\big(N^{-1}E(X|N)\big) && \text{(c.f. solution to Exercise 4.30 (a))} \\ &= E\big(N^{-1}\cdot E(\text{Binomial}\,(N,\theta))\big) \\ &= E\big(N^{-1}\cdot N\theta\big) \\ &= E(\theta)\,. \end{aligned}$$

We now compute the variance with [BC01, Theorem 4.4.7 page 167]:

$$\begin{aligned} \text{Var}\,(X/N) &= E(\text{Var}\,(X/N|N)) + \text{Var}\,(E(X/N|N)) \\ &= E\big(N^{-2}\,\text{Var}\,(X|N)\big) + \text{Var}\,\big(N^{-1}E(X|N)\big) \\ &= E\big(N^{-2}\,\text{Var}\,(\text{Binomial}\,(N,\theta))\big) + \text{Var}\,\big(N^{-1}E(\text{Binomial}\,(N,\theta))\big) \\ &= E\left(\frac{N\theta(1-\theta)}{N^2}\right) + \text{Var}\,(\theta) \\ &= \theta(1-\theta)E\left(\frac{1}{N}\right). \end{aligned}$$

11. Exercise 6.13

From [BC01, Theorem 4.3.5 on page 161], the variables$Y_i = \log(X_i)$ are iid observations from the pdf

$$\begin{aligned} f_Y(y|\alpha) &= f_X(e^y|\alpha) \cdot |e^y| \\ &= \alpha e^{\alpha y - e^{\alpha y}}, \end{aligned}$$

In particular, if $\hat{Y} \sim f_Y(y|1)$, then $Y \sim \dfrac{1}{\alpha}\hat{Y}$. Therefore,

$$\begin{aligned} \frac{\log(X_1)}{\log(X_2)} &\sim \frac{Y_1}{Y_2} \\ &\sim \frac{\hat{Y}_1}{\hat{Y}_2}. \end{aligned}$$

The distribution of the last quotient is a function of $f_Y(y|1)$, which does not depend on α, proving the claim as desired.

12. Exercise 6.14

Fix $n \geq 1$, and write MY the sample median computed from n random sample from X. Next, define the variable

$$Z = \frac{X - EX}{\sqrt{\mathrm{Var}\,(X)}}.$$

Then Z is in the location family, and does not depend on the location parameter. Moreover, we have

$$X = Z + EX.$$

It follows that

$$MX = MZ + EX,$$
$$\overline{X} = \overline{Z} + EX.$$

Hence, we obtain

$$MX - \overline{X} = MZ - \overline{Z},$$

and the claim holds as desired.

References

[BC01] Roger Berger and George Casella. *Statistical Inference*. 2nd edition. Florence, AL: Duxbury Press, June 2001.