CASELLA-BERGER
STATISTICAL INFERENCE SOLUTION: CHAPTER 2

VIRGIL CHAN

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Contents

1. Problem 2.1

In each of the following find the pdf of Y. Show that the pdf integrates to 1.

(a) $Y = X^3$ and $f_X(x) = 42x^5(1-x)$, $0 < x < 1$
(b) $Y = 4X + 3$ and $f_X(x) = 7e^{-7x}$, $0 < x < \infty$
(c) $Y = X^2$ and $f_X(x) = 30x^2(1-x)^2$, $0 < x < 1$

Solution. We begin by noting all conditions of [BC01, Theorem 2.1.5 on page 51] are satisfied in each case. We leave it to the reader to verify the pdf integrates to 1.

(a) Let $g(x) = x^3$ for $x \in (0,1)$, then $g^{-1}(y) = y^{\frac{1}{3}}$ for $y \in (0,1)$, and

$$\left|\frac{d}{dy} g^{-1}(y)\right| = \frac{1}{3y^{\frac{2}{3}}}.$$

Hence,

$$\begin{aligned} f_Y(y) &= f_X\left(g^{-1}(y)\right) \cdot \left|\frac{d}{dy} g^{-1}(y)\right| \\ &= \left[42y^{\frac{5}{3}}\left(1 - y^{\frac{1}{3}}\right)\right] \cdot \frac{1}{3y^{\frac{2}{3}}} \\ &= 14\left(y - y^{\frac{4}{3}}\right) \end{aligned}$$

on $\mathcal{Y} = (0,1)$.

(b) Let $g(x) = 4x + 3$ for $x \in (0,\infty)$, then $g^{-1}(y) = \dfrac{y-3}{4}$ for $y \in (3,\infty)$, and

$$\left|\frac{d}{dy} g^{-1}(y)\right| = \frac{1}{4}.$$

Hence,

$$\begin{aligned} f_Y(y) &= f_X\left(g^{-1}(y)\right) \cdot \left|\frac{d}{dy} g^{-1}(y)\right| \\ &= \frac{7}{4} e^{-\frac{-7(y-3)}{4}} \end{aligned}$$

on $\mathcal{Y} = (3,\infty)$.

(c) Let $g(x) = x^2$ for $x \in (0,1)$, then $g^{-1}(y) = y^{\frac{1}{2}}$ for $y \in (0,1)$, and

$$\left|\frac{d}{dy} g^{-1}(y)\right| = \frac{1}{2} y^{-\frac{1}{2}}.$$

Hence,

$$\begin{aligned}
f_Y(y) &= f_X\left(g^{-1}\left(y\right)\right) \cdot \left|\frac{d}{dy} g^{-1}(y)\right| \\
&= 15y^{\frac{1}{2}}\left(1 - y^{\frac{1}{2}}\right)^2
\end{aligned}$$

$\square$

2. Problem 2.2

In each of the following find the pdf of Y.

(a) $Y = X^2$ and $f_X(x) = 1$, $0 < x < 1$

(b) $Y = -\log(X)$ and X has pdf

$$f_X(x) = \frac{(n+m+1)!}{n!\,m!} x^n (1-x)^m,\ 0 < x < 1,\ m,\ n \text{ positive integers}$$

(c) $Y = e^X$ and X has pdf

$$f_X(x) = \frac{1}{\sigma^2} x e^{-\frac{(x/\sigma)^2}{2}},\ 0 < x < \infty,\ \sigma^2 \text{ a positive constant}$$

Solution. We begin by noting all conditions of [BC01, Theorem 2.1.5 on page 51] are satisfied in each case.

(a) Let $g(x) = x^2$ for $x \in (0,1)$, then $g^{-1}(y) = y^{\frac{1}{2}}$ for $y \in (0,1)$, and

$$\left|\frac{d}{dy} g^{-1}(y)\right| = \frac{1}{2} y^{-\frac{1}{2}}.$$

Hence,

$$\begin{aligned} f_Y(y) &= f_X\left(g^{-1}(y)\right) \cdot \left|\frac{d}{dy} g^{-1}(y)\right| \\ &= \frac{1}{2} y^{-\frac{1}{2}} \end{aligned}$$

on $\mathcal{Y} = (0,1)$.

(b) Let $g(x) = -\log(x)$ for $x \in (0,1)$, then $g^{-1}(y) = e^{-y}$ for $y \in (0,\infty)$, and

$$\left|\frac{d}{dy} g^{-1}(y)\right| = e^{-y}.$$

Hence,

$$\begin{aligned} f_Y(y) &= f_X\left(g^{-1}(y)\right) \cdot \left|\frac{d}{dy} g^{-1}(y)\right| \\ &= \frac{(n+m+1)!}{n!\,m!} e^{-ny} (1-e^{-y})^m \end{aligned}$$

on $\mathcal{Y} = (0,\infty)$.

(c) Let $g(x) = e^x$ for $x \in (0,\infty)$, then $g^{-1}(y) = \log(y)$ for $y \in (1,\infty)$, and

$$\left|\frac{d}{dy} g^{-1}(y)\right| = \frac{1}{y}.$$

Hence,

$$
\begin{aligned}
f_Y(y) &= f_X\left(g^{-1}(y)\right) \cdot \left|\frac{d}{dy} g^{-1}(y)\right| \\
&= \frac{\log(y)}{y\sigma^2} e^{-\frac{(\log(y)/\sigma)^2}{2}}
\end{aligned}
$$

on $\mathcal{Y} = (0, \infty)$.

$\square$

3. Problem 2.3

Suppose X has the geometric pmf $f_X(x) = \frac{1}{3}\left(\frac{2}{3}\right)^x$, $x = 0, 1, 2, \cdots$. Determine the probability distribution of $Y = X/(X+1)$. Note that here both X and Y are discrete random variables. To specify the probability distribution of Y, specify its pmf.

Solution.

$$\begin{aligned}
f_Y(y) &= P(Y = y) \\
&= P\left(\frac{X}{X+1} = y\right) \\
&= P\left(X = \frac{y}{1-y}\right) \\
&= f_X\left(\frac{y}{1-y}\right) \\
&= \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}
\end{aligned}$$

on $\mathcal{Y} = \left\{ \frac{x}{x+1} \;\middle|\; x = \frac{1}{3}\left(\frac{2}{3}\right)^k \text{ for some } k \in \mathbb{N} \cup \{0\} \right\}$ $\square$

4. Problem 2.4

Let λ be a fixed positive constant, and define the function $f(x)$ by $f(x) = \frac{1}{2}\lambda e^{-\lambda x}$ if $x \geq 0$ and $f(x) = \frac{1}{2}\lambda e^{\lambda x}$ if $x < 0$.

(a) Verify that $f(x)$ is a pdf.
(b) If X is a random variable with pdf given by $f(x)$, find $P(X < t)$ for all t. Evaluate all integrals.
(c) Find $P(|X| < t)$ for all t. Evaluate all integrals.

Solution.

(a) Check the conditions listed on [BC01, Theorem 1.6.5 on page 36] for $f(x)$.
(b)

$$\begin{aligned} P(X < t) &= \int_{-\infty}^{t} f(x)\ dx \\ &= \begin{cases} \int_{-\infty}^{t} \frac{1}{2}\lambda e^{\lambda x}\ dx & \text{if } t < 0, \\ \int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x}\ dx + \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x}\ dx & \text{if else,} \end{cases} \\ &= \begin{cases} \frac{e^{\lambda t}}{2} & \text{if } t < 0, \\ 1 - \frac{1}{2}e^{-\lambda t} & \text{if else.} \end{cases} \end{aligned}$$

(c)

$$\begin{aligned} P(|X| < t) &= P(-t < X < t) \\ &= P(X < t) - P(X < -t) \\ &= \left(1 - \frac{1}{2}e^{-\lambda t}\right) - \frac{e^{-\lambda t}}{2} && \text{(part (b))} \\ &= 1 - e^{-\lambda t} \end{aligned}$$

□

5. Problem 2.5

Use [BC01, Theorem 2.1.8 on page 53] to find the pdf of Y in [BC01, Example 2.1.2 on page 49]. Show that the same answer is obtained by differentiating the cdf given in [BC01, Equation 2.1.6 on page 49].

Solution. Partition the interval $(0, 2\pi)$ into $\{A_i\}_{i=0}^4$, with

$$A_i = \begin{cases} \{0\} & \text{if } i = 0, \\ \left(\frac{(i-1)\pi}{2}, \frac{i\pi}{2}\right) & \text{if } i > 0. \end{cases}$$

For each i, write $g_i(x) = \sin^2(x)$ on A_i. Then

$$\begin{aligned} g_1^{-1}(y) &= \arcsin(\sqrt{y}) \\ g_2^{-1}(y) &= \pi - \arcsin(\sqrt{y}) \\ g_3^{-1}(y) &= \pi + \arcsin(\sqrt{y}) \\ g_4^{-1}(y) &= 2\pi - \arcsin(\sqrt{y}) \end{aligned}$$

Therefore,

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^4 f_X\left(g_i^{-1}(y)\right) \cdot \left| \frac{d}{dy} g_i^{-1}(y) \right| \\ &= 4\left(\frac{1}{2\pi}\right)\left[\frac{1}{2\sqrt{y - y^2}}\right] \\ &= \frac{1}{\pi\sqrt{y - y^2}} \end{aligned}$$

on $\mathcal{Y} = (0, 1)$. $\square$

6. Problem 2.6

In each of the following find the pdf of Y and show that the pdf integrates to 1.

(a) $f_X(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$; $Y = |X|^3$
(b) $f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$; $Y = 1 - X^2$
(c) $f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$; $Y = 1 - X^2$ if $X \leq 0$ and $Y = 1 - X$ if $X > 0$

Solution. We note that [BC01, Theorem 2.1.8 on page 53] applies to all cases, and let readers to verify the pdf integrates to 1.

(a) Parition $(-\infty, \infty)$ into

$$\begin{aligned} A_0 &= \{0\} \\ A_1 &= (-\infty, 0) \\ A_2 &= (0, \infty) \end{aligned}$$

and define

$$g_i(x) = \begin{cases} x^3 & \text{if } i \text{ even,} \\ -x^3 & \text{if } i \text{ odd} \end{cases}$$

on A_i. Then

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^{2} f_X\left(g_i^{-1}(y)\right) \cdot \left| \frac{d}{dy} g_i^{-1}(y) \right| \\ &= \frac{1}{3} y^{-\frac{2}{3}} e^{-y^{1/3}} \end{aligned}$$

on $\mathcal{Y} = (0, \infty)$.

(b) Partition $(-1, 1)$ into

$$\begin{aligned} A_0 &= \{0\} \\ A_1 &= (-1, 0) \\ A_2 &= (0, 1) \end{aligned}$$

and define

$$g_i(x) = 1 - x^2$$

on A_i. Then

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^{2} f_X\left(g_i^{-1}(y)\right) \cdot \left| \frac{d}{dy} g_i^{-1}(y) \right| \\ &= \frac{3}{8}\left(\frac{1}{\sqrt{1-y}} + \sqrt{1-y} \right) \end{aligned}$$

on $\mathcal{Y} = (0, 1)$.

(c) Partition $(-1,1)$ just as in part (b), and define

$$g_i(x) = \begin{cases} 1-x^2 & \text{on } A_1, \\ 1-x & \text{on } A_2. \end{cases}$$

Then

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^{2} f_X\left(g_i^{-1}(y)\right) \cdot \left|\frac{d}{dy} g_i^{-1}(y)\right| \\ &= \frac{3}{16} \frac{1}{\sqrt{1-y}} \left(1-\sqrt{1-y}\right)^2 + \frac{3}{8}(2-y)^2 \end{aligned}$$

on $\mathcal{Y} = (0,1)$.

$\square$

7. Problem 2.7

Let X have pdf $f_X(x) = \frac{2}{9}(x+1)$, $-1 \le x \le 2$.

(a) Find the pdf of $Y = X^2$. Note that [BC01, Theorem 2.1.8 on page 53] is not directly applicable in this problem.
(b) Show that [BC01, Theorem 2.1.8 on page 53] remains valid if the sets $A_0, A_1, \cdots, A_k$ contain $\mathcal{X}$, and apply the extension to solve part (a) using $A_0 = \varnothing$, $A_1 = (-2, 0)$, and $A_2 = (0, 2)$.

Solution.

(a)

$$\begin{aligned}
P(Y \le y) &= P\big(X^2 \le y\big) \\
&= \begin{cases} P(-\sqrt{y} \le X \le \sqrt{y}) & \text{if } y < 1, \\ P(-1 \le X \le \sqrt{y}) & \text{if } 1 \le y \le 4. \end{cases} \\
&= \begin{cases} \displaystyle\int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\, dx & \text{if } y < 1, \\ \displaystyle\int_{-1}^{\sqrt{y}} f_X(x)\, dx & \text{if } 1 \le y \le 4. \end{cases} \\
&= \begin{cases} \dfrac{4\sqrt{y}}{9} & \text{if } y < 1, \\ \dfrac{1}{9}\left(1+\sqrt{y}\right)^2 & \text{if } 1 \le y \le 4. \end{cases}
\end{aligned}$$

on $\mathcal{Y} = (0, 4)$.

(b) C.f. Problem 2.6.

$\square$

8. Problem 2.8

In each of the following show that the given function is a cdf and find $F_X^{-1}(y)$.

(a)

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-x} & \text{if } x \geq 0 \end{cases}$$

(b)

$$F_X(x) = \begin{cases} e^x/2 & \text{if } x < 0 \\ 1/2 & \text{if } 0 \leq x < 1 \\ 1 - (e^{1-x}/2) & \text{if } 1 \leq x \end{cases}$$

(c)

$$F_X(x) = \begin{cases} e^x/4 & \text{if } x < 0 \\ 1 - (e^{-x}/4) & \text{if } x \geq 0 \end{cases}$$

Solution. To show a function is cdf, we verify the conditions in [BC01, Theorem 1.5.3 on page 31], which are routine computations.

(a)

$$F_X^{-1}(y) = -\log(1-y)$$

(b)

$$F_X^{-1}(y) = \begin{cases} \log(2y) & \text{if } 0 \leq y \leq \frac{1}{2} \\ 1 - \log(2(1-y)) & \text{if } \frac{1}{2} \leq y \leq 1 \end{cases}$$

(c)

$$F_X^{-1}(y) = \begin{cases} \log(4y) & \text{if } 0 \leq y \leq \frac{1}{4} \\ -\log(4(1-y)) & \text{if } \frac{1}{4} \leq y \leq 1 \end{cases}$$

□

9. Problem 2.9

If the random variable X has pdf

$$f(x) = \begin{cases} \dfrac{x-1}{2} & 1 < x < 3, \\ 0 & \text{otherwise,} \end{cases}$$

find a monotone function $u(x)$ such that the random variable $Y = u(X)$ has uniform(0,1) distribution.

Solution. This is a direct application of [BC01, Theorem 2.1.10 on page 54]. The cdf is given by

$$\begin{aligned} F_X(x) &= \begin{cases} 0 & \text{if } x \leq 1 \\ \displaystyle\int_1^x f(t)\ dt & \text{if } 1 < x < 3 \\ 1 & \text{if else} \end{cases} \\ &= \begin{cases} 0 & \text{if } x \leq 1 \\ \dfrac{(x-1)^2}{4} & \text{if } 1 < x < 3 \\ 1 & \text{if else} \end{cases} \end{aligned}$$

which is clearly monotone. So $u(x) = F_X(x)$. □

10. Problem 2.11

Let X have the standard normal pdf, $f_X(x) = (1/\sqrt{2\pi})e^{-x^2/2}$.

(a) Find EX^2 directly, and then by using the pdf of $Y = X^2$ from [BC01, Example 2.1.7 on page 52] and calculating EY.
(b) Find the pdf of $Y = |X|$, and find its mean and variance.

Solution.

(a) First we have

$$\begin{aligned}
EX^2 &= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx && \text{([BC01, Definition 2.2.1 on page 55])}\\
&= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} e^{-x^2/2}\, dx\\
&= \frac{1}{\sqrt{2\pi}}\left[xe^{-x^2/2}\Big|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \right]\\
&= 1.
\end{aligned}$$

Secondly, by [BC01, Example 2.1.7 on page 52], the pdf of Y is given by

$$\begin{aligned}
f_Y(y) &= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]\\
&= \frac{1}{\sqrt{2\pi y}} e^{-y/2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
EY &= \int_0^{\infty} y f_Y(y)\, dy\\
&= \int_0^{\infty} \sqrt{\frac{y}{2\pi}} e^{-y/2}\, dy\\
&= 1
\end{aligned}$$

(b) Using [BC01, Theorem 2.1.8 on page 53], Y has pdf

$$\begin{aligned}
f_Y(y) &= f_X(y) + f_X(-y)\\
&= \sqrt{\frac{2}{\pi}} e^{-y^2/2}
\end{aligned}$$

Therefore,

$$EY = \int_0^\infty y f_Y(y)\, dy = \sqrt{\frac{2}{\pi}}$$

$$\mathrm{Var}\,(Y) = EY^2 - (EY)^2 = 1 - \frac{2}{\pi}$$

$\square$

11. Problem 2.12

See [BC01, page 77] for the problem statement.

Solution. We know

$$y = \underbrace{d\tan(x)}_{g(x)}$$

for $x \in (0, \pi/2)$, and

$$\begin{aligned}\frac{dg^{-1}}{dy} &= \frac{d}{dy}\arctan\left(\frac{y}{d}\right)\\ &= \frac{d}{d^2+y^2}.\end{aligned}$$

Therefore, [BC01, Theorem 2.1.5 on page 51] gives

$$\begin{aligned}f_Y(y) &= f_X(g^{-1}(y))\left|\frac{dg^{-1}}{dy}\right|\\ &= \frac{1}{\frac{\pi}{2}-0}\cdot\frac{d}{d^2+y^2}\\ &= \frac{2d}{\pi(d^2+y^2)}\end{aligned}$$

on $\mathcal{Y} = (0, \infty)$, which is the Cauchy distribution. In particular, $EY = \infty$. □

12. Problem 2.13

Consider a sequence of independent coin flips, each of which has probability p of being heads. Define a random variable X as the length of the run (of either heads or tails) started by the first trail. (For example, $X = 3$ if either TTTH or HHHT is observed.) Find the distribution of X, and find EX.

Solution. X has pmf

$$P(X = k) = (1-p)^k p + p^k(1-p).$$

Therefore,

$$\begin{aligned}
EX &= \sum_{k=1}^{\infty} k\left[(1-p)^k p + p^k(1-p)\right] \\
&= (1-p)p\left[\sum_{k=1}^{\infty} k(1-p)^{k-1} + \sum_{k=1}^{\infty} kp^{k-1}\right] \\
&= (1-p)p\left(\frac{1}{p^2} + \frac{1}{(1-p)^2}\right)
\end{aligned}$$

□

13. Probblem 2.14

(a) Let X be a continuous, nonnegative random variable [$f(x) = 0$ for $x < 0$]. Show that

$$EX = \int_0^\infty [1 - F_X(x)]\ dx,$$

where $F_X(x)$ is the cdf of X.

(b) Let X be a discrete random variable whose range is the nonnegative integers. Show that

$$EX = \sum_{k=0}^{\infty} (1 - F_X(k)),$$

where $F_X(k) = P(X \le k)$. Compare this with part (a).

Solution.

(a)

$$\begin{aligned}
\int_0^\infty [1 - F_X(x)]\ dx &= \int_0^\infty P(X > x)\ dx \\
&= \int_0^\infty \int_x^\infty f_X(y)\ dy\ dx \\
&= \int_0^\infty \int_0^y f_X(y)\ dx\ dy \\
&= \int_0^\infty y f_X(y)\ dy \\
&= EX
\end{aligned}$$

(b)

$$\begin{aligned}
EX &= \sum_{k=0}^{\infty} kP(X = k) \\
&= \sum_{k=1}^{\infty} P(X = k) + \sum_{k=2}^{\infty} P(X = k) + \sum_{k=3}^{\infty} P(X = k) + \cdots \\
&= P(X > 0) + P(X > 1) + P(X > 2) + \cdots \\
&= \sum_{k=0}^{\infty} 1 - F_X(k)
\end{aligned}$$

□

14. Problem 2.18

Show that if X is a continuous random variable, then

$$\min_a E|X-a| = E|X-m|,$$

where m is the median of X.

Solution. The expected value of $|X-a|$ is given by

$$\begin{aligned} E|X-a| &= \int_{-\infty}^{\infty} |x-a|\, f_X(x)\ dx \\ &= \int_a^{\infty} (x-a) f_X(x)\ dx - \int_{-\infty}^{a} (x-a) f_X(x)\ dx \end{aligned}$$

Differentiate with respect to a we have

$$\begin{aligned} \frac{d}{da} E|X-a| &= \frac{d}{da}\left[\int_a^{\infty} (x-a) f_X(x)\ dx\right] - \frac{d}{da}\left[\int_{-\infty}^{a} (x-a) f_X(x)\ dx\right] \\ &= \int_a^{\infty} \frac{\partial}{\partial a}\left[(x-a) f_X(x)\right]\ dx - \int_{-\infty}^{a} \frac{\partial}{\partial a}\left[(x-a) f_X(x)\right]\ dx \\ &= \int_{-\infty}^{a} f_X(x)\ dx - \int_a^{\infty} f_X(x)\ dx \\ &= P(X \le a) - P(X > a). \end{aligned}$$

In particular,

$$1 - 2P(X > a) = \frac{d}{da} E|X-a| = 1 - 2P(X \le a).$$

Therefore, the solution to

$$\frac{d}{da} E|X-a| = 0$$

is the median m. Moreover, m is a minima because

$$\left.\frac{d^2}{da^2}\right|_{a=m} E|X-a| = 2f_X(m) > 0.$$

□

15. Problem 2.19

Prove that

$$\frac{d}{da}E(X-a)^2 = 0 \iff EX = a$$

by differentiating the integral. Verify, using calculus, that $a = EX$ is indeed a minimum. List the assumptions about F_X and f_X are needed.

Solution. We have

$$\begin{aligned}
\frac{d}{da}E(X-a)^2 &= \frac{d}{da}\int_{-\infty}^{\infty} (x-a)^2 f_X(x)\ dx \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial a}\left[(x-a)^2 f_X(x)\right]\ dx \\
&= -2\int_{-\infty}^{\infty} (x-a) f_X(x)\ dx \\
&= -2E(X-a)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d}{da}E(X-a)^2 = 0 &\iff -2E(X-a) = 0 \\
&\iff E(X-a) = 0 \\
&\iff EX = a.
\end{aligned}$$

To verify $a = EX$ is minimum, we compute the second derivative

$$\frac{d^2}{da^2}E(X-a)^2 = 2 > 0.$$

□

16. Proboelm 2.21

Prove the "two-way" rule for expectations, [BC01, Equation (2.2.5) on page 58], which says $Eg(X) = EY$ where $Y = g(X)$. Assume that $g(x)$ is a monotone function.

Solution.

$$\begin{aligned} Eg(X) &= \int_{\mathbb{R}} g(x) f_X(x)\ dx \\ &= \int_{\mathbb{R}} y f_X(g^{-1}(y)) \cdot \frac{dg^{-1}}{dy}\ dy \\ &= \int_{\mathbb{R}} y f_Y(y)\ dy \\ &= EY \end{aligned}$$

$\square$

17. Problem 2.22

Let X have the pdf

$$f(x) = \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-x^2/\beta^2},\ 0 < x < \infty,\ \beta > 0.$$

(a) Verify that $f(x)$ is a pdf.
(b) Find EX and $\mathrm{Var}\,(X)$.

Solution.

(a) [BC01, Theorem 1.6.5 on page 36].
(b)

$$\begin{aligned} EX &= \int_0^\infty x f(x)\ dx \\ &= \int_0^\infty \frac{4}{\beta^3\sqrt{\pi}} x^3 e^{-x^2/\beta^2}\ dx \\ &= \frac{4}{\beta^3\sqrt{\pi}} \int_0^\infty x^3 e^{-x^2/\beta^2}\ dx \\ &= \left(\frac{4}{\beta^3\sqrt{\pi}}\right)\left(-\frac{\beta^2}{2}\right)\left(-\int_0^\infty 2x e^{-x^2/\beta^2}\ dx\right) \\ &= \left(\frac{4}{\beta^3\sqrt{\pi}}\right)\left(\frac{\beta^4}{2}\right) \\ &= \frac{2\beta}{\sqrt{\pi}} \end{aligned}$$

and similarly,

$$\begin{aligned} EX^2 &= \frac{3\beta^2}{2}, \\ \mathrm{Var}\,(X) &= EX^2 - (EX)^2 \\ &= \beta^2\left[\frac{3}{2} - \frac{4}{\pi}\right] \end{aligned}$$

$\square$

18. Problem 2.23

Let X have the pdf

$$f(x) = \frac{1}{2}(1+x),\ -1 < x < 1.$$

(a) Find the pdf of $Y = X^2$.
(b) Find EY and $\mathrm{Var}\,(Y)$.

Solution.

(a) Define $g_i(x) = x^2$ on $A_1 = (-1, 0)$ and $A_2 = (0, 1)$. Then

$$\begin{aligned} f_Y(y) &= [f(-\sqrt{y}) + f(\sqrt{y})] \cdot \frac{1}{2\sqrt{y}} \\ &= \frac{1}{2\sqrt{y}} \end{aligned}$$

on $\mathcal{Y} = (0, 1)$.
(b) We have

$$\begin{aligned} \int_0^1 y^n f_Y(y)\ dy &= \frac{1}{2}\int_0^1 y^{n-1/2}\ dy \\ &= \frac{1}{2n+1}. \end{aligned}$$

This gives

$$\begin{aligned} EY &= \frac{1}{3} \\ EY^2 &= \frac{1}{5} \\ \mathrm{Var}\,(Y) &= \frac{4}{45} \end{aligned}$$

□

19. Problem 2.26

Let $f(x)$ be a pdf and let a be a number such that, for all $\varepsilon > 0$, $f(a+\varepsilon) = f(a-\varepsilon)$. Such a pdf is said to be summetric about the point a.

(a) Give three examples of symmetric pdfs.
(b) Show that if $X \sim f(x)$, symmetric, then the median of X (see Exercise 2.17) is the number a.
(c) Show that if $X \sim f(x)$, symmetric and EX exists, then $EX = a$.
(d) Show that $f(x) = e^{-x}$, $x \geq 0$, is not a symmetric pdf.
(e) Show that for the pdf in part (d), the median is less than the mean.

Solution.

(a) Cauchy, Normal, Uniform.
(b) By change of variable, we may assume $a = 0$. The statement thus becomes: the median of an even pdf is 0, which is obvious because

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)\ dx \\ &= \int_{-\infty}^{0} f(x)\ dx + \int_{0}^{\infty} f(x)\ dx \\ &= 2\int_{-\infty}^{0} f(x)\ dx \\ &= 2P(X \leq 0) \end{aligned}$$

(c) Following the same logic in part (b), the statement becomes: the expected value of an even pdf $f(x)$ is 0.
This is true because the funcion $xf(x)$ is odd, hence

$$\begin{aligned} EX &= \int_{-\infty}^{\infty} xf(x)\ dx \\ &= 0 \end{aligned}$$

(d) If $f(x) = e^{-x}$ were symmetric, then it would be symmetric at $x = EX$ for $X \sim f(x)$ by part (c). In particular,

$$\begin{aligned} EX &= \int_{0}^{\infty} xe^{-x}\ dx \\ &= 1. \end{aligned}$$

However, a direct computation shows 1 is not the median of X, contradicting part (b). Therefore it is not symmetric.
(e) As computed in part (d), the mean is 1. The claim follows from the computation

$$\begin{aligned}
\int_0^{\text{mean}} f(x)\ dx &= \int_0^1 e^{-x}\ dx \\
&= \frac{e-1}{e} \\
&> \frac{1}{2}
\end{aligned}$$

$\square$

20. Exercise 2.32

We compute

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} S(t) &= \left.\frac{d}{dt}\right|_{t=0} \log(M_X(t))\\ &= \frac{\dot{M}_X(0)}{M_X(0)}\\ &= EX,\end{aligned}$$

and

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} S(t) &= \frac{\ddot{M}_X(0)\, M_X(0) - \dot{M}_X^2(0)}{M_X^2(0)}\\ &= EX^2 - (EX)^2\\ &= \operatorname{Var}(X)\,.\end{aligned}$$

21. Exercise 2.33

(a) The mgf is

$$\begin{aligned} M_X(t) &= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda}\lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\ &= e^{-\lambda} \cdot e^{\lambda e^t} \\ &= e^{\lambda\left(e^t - 1\right)}. \end{aligned}$$

The moments are

$$\begin{aligned} EX &= \frac{d}{dt}\bigg|_{t=0} M_X(t) \\ &= \lambda e^t \cdot e^{\lambda\left(e^t-1\right)}\bigg|_{t=0} \\ &= \lambda, \end{aligned}$$

$$\begin{aligned} EX^2 &= \frac{d^2}{dt^2}\bigg|_{t=0} M_X(t) \\ &= \lambda\left(1 + e^t\lambda\right) \cdot e^{\lambda\left(e^t-1\right)}\bigg|_{t=0} \\ &= \lambda^2 + \lambda. \end{aligned}$$

Therefore,

$$\operatorname{Var}(X) = \lambda.$$

(b) The mgf is

$$\begin{aligned} M_X(t) &= \sum_{x=0}^{\infty} e^{tx} p(1-p)^x \\ &= \sum_{x=0}^{\infty} p\left[e^t(1-p)\right]^x \\ &= \frac{p}{1 - e^t(1-p)} \end{aligned}$$

The moments are

$$EX = \frac{1-p}{p},$$

$$EX^2 = \frac{(2-p)(1-p)}{p^2}$$

Therefore,

$$\mathrm{Var}\,(X) = \frac{1-p}{p^2}.$$

(c) The mgf is

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sqrt{2\pi}\sigma}\ dx \\ &= \int_{-\infty}^{\infty} \frac{e^{-\left(x^2-2x\mu+\mu^2-2\sigma^2 tx\right)/(2\sigma^2)}}{\sqrt{2\pi}\sigma}\ dx \\ &= e^{\frac{2\mu\sigma^2 t+\sigma^4 t^2}{2\sigma^2}} \cdot \int_{-\infty}^{\infty} \frac{e^{-\frac{\left[x-\left(\mu+\sigma^2 t\right)\right]^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}\ dx \\ &= e^{\mu t+\frac{\sigma^2 t^2}{2}}. \end{aligned}$$

The moments are

$$\begin{aligned} EX &= \left.\frac{d}{dt}\right|_{t=0} M_X(t) \\ &= \mu, \end{aligned}$$

$$\begin{aligned} EX^2 &= \left.\frac{d^2}{dt^2}\right|_{t=0} M_X(t) \\ &= \sigma^2 + \mu^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{Var}\,(X) &= EX^2 - (EX)^2 \\ &= \sigma^2 \end{aligned}$$

References

[BC01] Roger Berger and George Casella. *Statistical Inference*. 2nd edition. Florence, AL: Duxbury Press, June 2001.