CASELLA-BERGER
STATISTICAL INFERENCE SOLUTION:
CHAPTER 3

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August 5, 2022

Contents

1. Exercise 3.1

X has pmf

$$f_X(n) = \frac{1}{N_1 - N_0 + 1}$$

on $\mathcal{X} = [N_0, N_1] \cap \mathbb{Z}$. Therefore,

$$EX = \sum_{n=N_0}^{N_1} n f_X(n) = \frac{N_0 + N_1}{2},$$

$$\begin{aligned}
EX^2 &= \sum_{n=N_0}^{N_1} n^2 f_X(n) \\
&= \frac{1}{N_1 - N_0 + 1}\left[\frac{(2N_1+1)N_1(N_1+1)}{6} - \frac{(2N_0-1)(N_0-1)N_0}{6}\right], \\
&\quad \left(\text{where } \sum_{k=1}^{n} k^2 = \frac{(2n+1)n(n+1)}{6}.\right)
\end{aligned}$$

$$\begin{aligned}
\operatorname{Var}(X) &= EX^2 - (EX)^2 \\
&= \frac{(N_1 - N_0)(N_1 - N_0 + 2)}{12}
\end{aligned}$$

2. Exercise 3.2

Let X be the number of defective parts found in K samples; and M be the number of defective parts in the lot.

(a) We want

$$P(X = 0 \mid M > 5) < 0.1,$$

where

$$P(X = 0 \mid M > 5) = \frac{\binom{100-M}{K}}{\binom{100}{K}}.$$

The probability of a defective part being included in the K samples is proportional to M. Therefore, we may choose $M = 6$. This gives

$$P(X = 0 \mid M = 6) = \frac{\binom{94}{K}}{\binom{100}{K}}.$$

Using numerical method, we get $K \geq 32$.

(b) We want

$$P(X \leq 1 \mid M = 6) < 0.1.$$

We compute

$$P(X = 1 \mid M = 6) = \frac{6\binom{94}{K-1}}{\binom{100}{K}}$$

and get

$$P(X \leq 1 \mid M = 6) = \frac{\binom{94}{K}}{\binom{100}{K}} + \frac{6\binom{94}{K-1}}{\binom{100}{K}}.$$

Using numerical method, we find $K \geq 51$.

3. Exercise 3.3

The entire crossing takes requires 3 seconds to execute. If the pedestrian has to wait for exactly 4 seconds before starting to cross, then the entire process takes 7 seconds to complete. Therefore, we can represent the sample space by

$$S = \{(x_1, \cdots, x_7) \mid x_i = 0, 1\}$$

where 1 represents a car is passing.

Secondly, the crossing takes place at x_7. Therefore, we must have $x_5 = x_6 = x_7 = 0$. It follows that $x_4 = 1$, otherwise the crossing will occur before x_7.

Now, the pedestrian cannot cross at x_3. This means the sequence cannot start with $(0, 0, 0, 1)$.

As a result, the required probability is

$$\left[1 - (1 - p)^3 p\right] (1 - p)^3.$$

4. Exercise 3.4

Let X be the number of trials it takes to open the door.

(a) We have

$$P(X = k) = \left(1 - \frac{1}{n}\right)^{k-1} \frac{1}{n}$$

for $k = 1, \cdots, n$. This is the geometric distribution, hence

$$\begin{aligned} EX &= \frac{1}{\frac{1}{n}} \\ &= n. \end{aligned}$$

(b) At the k-th trial, we are selecting from $n - k + 1$ keys. The probability of choosing the right key is then $\dfrac{1}{n-k+1}$. Hence,

$$\begin{aligned} P(X = k) &= \left[\prod_{i=1}^{k-1} \left(1 - \frac{1}{n-i+1}\right)\right] \frac{1}{n-k+1} \\ &= \prod_{i=1}^{k-1} \frac{n-i}{n-i+1} \cdot \frac{1}{n-k+1} \\ &= \frac{n-k+1}{n} \cdot \frac{1}{n-k+1} \\ &= \frac{1}{n}. \end{aligned}$$

As a result,

$$\begin{aligned} EX &= \sum_{k=1}^{n} \frac{k}{n} \\ &= \frac{n+1}{2}. \end{aligned}$$

5. Exercise 3.5

If the new and old drugs are equally effective, then the old drug can also have 85 or more successes observed in a 100 patients trial. Let us calculate the probability for this event to happen.

Let X be the number of observing successes in a 100 patients trial. Then

$$\begin{aligned} P(X \geq 85) &= \sum_{k=85}^{100} \binom{100}{k} (0.8)^k (0.2)^{100-k} \\ &\approx 0.1285 \end{aligned}$$

This means there is (approximately) a chance of 13% for the old drug to produce the same result. Therefore, we cannot conclude the new drug is superior.

6. Exercise 3.6

(a) Binomial distribution Binomial (2000) 0.01, with

$$P(X = k) = \binom{2000}{k}(0.01)^k(0.99)^{2000-k}$$

(b)

$$P(X < 100) = \sum_{k=0}^{99}\binom{2000}{k}(0.01)^k(0.99)^{200-k}$$

(c) We find $\min(np, n(1-p)) = 20$, which is at least 5. Hence, the normal approximation to binomial is good. We can therefore approximate part (b) with Normal $(np, np(1-p))$. As a result,

$$\begin{aligned} P(X < 100) &= P\left(\frac{X - 20}{\sqrt{20 \cdot 0.99}} < \frac{100 - 20}{\sqrt{20 \cdot 0.99}}\right) \\ &= P(Z < 17.98) \\ &\approx 1. \end{aligned}$$

7. Exercise 3.7

Let X be the number of chocolate chips in a cookie. Then

$$X \sim \text{Poisson}\,(\lambda)\,.$$

Next,

$$\begin{aligned} P(X \geq 2) &= 1 - P(X < 1) \\ &= 1 - e^{-\lambda} - \lambda e^{-\lambda} \\ &> 0.99 \end{aligned}$$

So $\lambda \approx 6.64$.

8. Exercise 3.8

(a) Let X be the number of people in a theatre. Then $X \sim \text{Binomial}\,(1000, 0.5)$, and

$$\begin{aligned} P(X > N) &= \sum_{k=N+1}^{1000} \binom{1000}{k} (0.5)^k (0.5)^{1000-k} \\ &= (0.5)^{1000} \sum_{k=N+1}^{1000} \binom{1000}{k} \end{aligned}$$

Therefore,

$$(0.5)^{1000} \sum_{k=N+1}^{1000} \binom{1000}{k} < 0.01.$$

(b) We check $\min(np, n(1-p)) = 500$, which is at least 5. So the normal approximation by $\text{Normal}\,(500, 250)$ is good.
Next,

$$\begin{aligned} P(X > N) &= P\left(\frac{X - 500}{\sqrt{250}} > \frac{N - 500}{\sqrt{250}}\right) \\ &= P\left(Z > \frac{N - 500}{\sqrt{250}}\right) \end{aligned}$$

Now, $Z \sim \text{Normal}\,(0, 1)$. By looking up the values, we see

$$\begin{aligned} 0.01 &> 0.099 \\ &\approx P(Z > 2.33) \end{aligned}$$

So $N \approx 537$.

9. Exerise 3.9

(a) Let $X \sim \text{Binomial}\,(60)\,\frac{1}{90}$. Then

$$\begin{aligned} P(X \geq 5) &= 1 - \sum_{k=0}^{4} \binom{60}{k} \left(\frac{1}{90}\right)^{k} \left(\frac{89}{90}\right)^{60-k} \\ &\approx 0.000556628 \end{aligned}$$

(b) Let X be the number of elementary schools in the state that has at least 5 pairs of twins. Then $X \sim \text{Binomial}\,(310)\,0.006$, where we rounded the probability computed in part (a).

As a result,

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - (0.9994)^{310} \\ &= 0.169773 \end{aligned}$$

(c) The probability of a state to have 5 pairs of twins in the same school is 0.17. Since there are 50 states, the probability of having at least one state to have 5 pairs of twins in the same school is

$$1 - (1 - 0.17)^{50} = 0.99991$$

10. Exercise 3.10

(a) Trivial.
(b) Let p be the probability in part (a). Then

$$\max_{M,N} p = \max_{M,N} \log(p)$$

since log is injective and monotone. In particular, because $M + N = 496$, the only term depending on M, N is the numerator. Thus

$$\begin{aligned}
\max_{M,N} \log(p) &= \max_{M,N} \log\left(\binom{N}{4}\binom{M}{2}\right) \\
&= \max_{4\leq N\leq 496} \log\left(\binom{N}{4}\binom{496-N}{2}\right) \\
&= \max_{4\leq N\leq 496} \log\left[N(N-1)(N-2)(N-3)(496-N)(495-N)\right]
\end{aligned}$$

Using calculus, we get $N \approx 330.834$.

11. Exercise 3.12

$$
\begin{aligned}
F_X(r-1) &= P(X \leq r-1) \\
&= P(\text{at most } r-1 \text{ successes in } n \text{ trials}) \\
&= P(\text{at least } n-r+1 \text{ failures before the } r\text{-th success}) \\
&= P(Y \geq n-r+1) \\
&= 1 - F_Y(n-r)
\end{aligned}
$$

12. Exercise 3.13

For a general random discrete variable X, we compute

$$\begin{aligned} EX_T^n &= \sum_{x=1}^{\infty} x^n P(X_T = x) \\ &= \sum_{x=1}^{\infty} x^n \frac{P(X = x)}{P(X > 0)} \\ &= \frac{EX^n}{P(X > 0)}, \end{aligned} \tag{12.0.1}$$

$$\begin{aligned} \mathrm{Var}\,(X_T) &= EX_T^2 - (EX_T)^2 \\ &= \frac{EX^2}{P(X > 0)} - \frac{(EX)^2}{P(X > 0)^2} \\ &= \frac{\mathrm{Var}\,(X) + (EX)^2}{P(X > 0)} - \frac{(EX)^2}{P(X > 0)^2} \end{aligned} \tag{12.0.2}$$

(a) If $X \sim \text{Poisson}\,(\lambda)$, then $EX = \lambda$ and $P(X > 0) = 1 - e^{-\lambda}$. So

$$\begin{aligned} EX_T &= \frac{\lambda}{1 - e^{-\lambda}}, \\ \mathrm{Var}\,(X_T) &= \frac{\lambda^2 + \lambda}{1 - e^{-\lambda}} - \frac{\lambda^2}{(1 - e^{-\lambda})^2} \end{aligned}$$

(b) If $X \sim \text{NegBinomial}\,(r, p)$, then

$$\begin{aligned} P(X > 0) &= 1 - P(X = 0) \\ &= 1 - \binom{r-1}{0} p^r (1-p)^0 \\ &= 1 - p^r, \end{aligned}$$

$$EX = \frac{rp}{1 - p},$$

$$\mathrm{Var}\,(X) = \frac{rp}{(p - 1)^2}$$

Therefore,

$$\begin{aligned} EX_T &= \frac{rp}{(1 - p)(1 - p^r)}, \\ \mathrm{Var}\,(X_T) &= -\frac{pr(p^r(1 + pr) - 1)}{(p^r - 1)^2 (p - 1)^2} \end{aligned}$$

13. Exercise 3.14

(a)

$$\begin{aligned}
\sum_{x=1}^{\infty} P(X=x) &= \sum_{x=1}^{\infty} \frac{-(1-p)^x}{x\log(p)} \\
&= \frac{1}{\log(p)} \sum_{x=1}^{\infty} \frac{-(1-p)^x}{x} \\
&= \frac{1}{\log(p)} \cdot \log(1-(1-p)) \\
&= 1.
\end{aligned}$$

(b)

$$\begin{aligned}
EX &= \sum_{x=1}^{\infty} \frac{-(1-p)^x}{\log(p)} \\
&= \frac{-(1-p)}{p\log(p)},
\end{aligned}$$

$$\begin{aligned}
EX^2 &= \sum_{x=1}^{\infty} \frac{-x(1-p)^x}{\log(p)} \\
&= \frac{-(1-p)}{\log(p)} \sum_{x=1}^{\infty} x(1-p)^{x-1} \\
&= \frac{-(1-p)}{\log(p)} \cdot \left.\frac{d}{dx}\right|_{x=1-p} \frac{1}{1-x} \\
&= \frac{-(1-p)}{p^2\log(p)},
\end{aligned}$$

$$\begin{aligned}
\operatorname{Var}(X) &= EX^2 - (EX)^2 \\
&= \frac{-(1-p)}{p^2\log(p)} - \frac{(1-p)^2}{[p\log(p)]^2} \\
&= \frac{-(1-p)\log(p) - (1-p)^2}{[p\log(p)]^2}
\end{aligned}$$

14. Exercise 3.17

$$
\begin{aligned}
EX^{\nu} &= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} x^{\nu} \cdot x^{\alpha-1} e^{-x/\beta}\, dx \\
&= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} x^{\alpha+\nu-1} e^{-x/\beta}\, dx \\
&= \frac{\Gamma(\alpha+\nu)\beta^{\alpha+\nu}}{\Gamma(\alpha)\beta^{\alpha}} \\
&= \frac{\Gamma(\alpha+\nu)\beta^{\nu}}{\Gamma(\alpha)}
\end{aligned}
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References

[BC01] Roger Berger and George Casella. *Statistical Inference*. 2nd edition. Florence, AL: Duxbury Press, June 2001.