

**CASELLA-BERGER  
STATISTICAL INFERENCE SOLUTION:  
CHAPTER 4**

VIRGIL CHAN

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## 1. EXERCISE 4.1

- (a) We want to know the probability for  $(X, Y)$  to land inside the circle

$$X^2 + Y^2 = 1.$$

This circle has area  $\pi$ , so the probability is  $\frac{\pi}{4}$ .

- (b) We want to know the probability for  $(X, Y)$  to land below the line

$$2X - Y = 0.$$

This line divides the square into two uniform trapeziums. One of them has vertices

$\left(\pm\frac{1}{2}, \pm 1\right)$ , and has area 2. Therefore, the probability is  $\frac{1}{2}$ .

- (c) The region  $|X + Y| < 2$  contains the square, so the probability is 1.

## 2. EXERCISE 4.4

(a)

$$\begin{aligned}
 1 &= \int_0^1 \int_0^2 C(x+2y) \, dx \, dy \\
 &= 4C
 \end{aligned}$$

So  $C = \frac{1}{4}$ .

(b)

$$\begin{aligned}
 f_X(x) &= \int_0^1 \frac{x+2y}{4} \, dy \\
 &= \frac{x+1}{4}
 \end{aligned}$$

on  $\mathcal{X} = (0, 2)$ .

(c)

$$\begin{aligned}
 F_{XY}(x, y) &= P(X \leq x, Y \leq y) \\
 &= \begin{cases} 0 & \text{if } x \leq 0 \text{ or } y \leq 0 \\ 1 & \text{if } x \geq 2, y \geq 1 \\ \int_0^y \int_0^x f(u, v) \, du \, dv & \text{if else} \end{cases}
 \end{aligned}$$

The if else case requires some work.

$$\begin{aligned}
 \int_0^y \int_0^x f(u, v) \, du \, dv &= \begin{cases} \int_0^y \int_0^x f(u, v) \, du \, dv & \text{if } 0 < x < 2, 0 < y < 1 \\ \int_0^1 \int_0^x f(u, v) \, du \, dv & \text{if } 0 < x < 2, y \geq 1 \\ \int_0^y \int_0^2 f(u, v) \, du \, dv & \text{if } 0 < x < 2, y \geq 1 \end{cases} \\
 &= \begin{cases} \frac{xy(x+2y)}{8} & \text{if } 0 < x < 2, 0 < y < 1 \\ \frac{x(x+2)}{8} & \text{if } 0 < x < 2, y \geq 1 \\ \frac{y(1+y)}{2} & \text{if } x \geq 2, 0 < y < 1 \end{cases}
 \end{aligned}$$

(d) Let  $z = g(x) = \frac{9}{(x+1)^2}$ , then  $g^{-1}(z) = \frac{3}{\sqrt{z}} - 1$ , and  $\frac{d}{dz}g^{-1}(z) = \frac{3}{-2z^{\frac{3}{2}}}$ . Therefore,

$$\begin{aligned}
 f_Z(z) &= f_X(g^{-1}(z)) \cdot \left| \frac{d}{dz} g^{-1}(z) \right| \\
 &= \frac{9}{8z^2}
 \end{aligned}$$

## 3. EXERCISE 4.5

(a)

$$\begin{aligned} P(X > \sqrt{Y}) &= \int_0^1 \int_{\sqrt{y}}^1 x + y \, dx \, dy \\ &= \frac{7}{20} \end{aligned}$$

(b)

$$\begin{aligned} P(X^2 < Y < X) &= \int_0^1 \int_{x^2}^x 2x \, dy \, dx \\ &= \frac{1}{6} \end{aligned}$$

## 4. EXERCISE 4.6

Let  $X$  (resp.  $Y$ ) be the arrival time of  $A$  (resp.  $B$ ), so that  $X \sim \text{Uniform}([0, 1]) \sim Y$ .  
Let  $T$  be the waiting time. Then

$$T = \max \{Y - X, 0\}.$$

Therefore,

$$P(T < t) = P(Y - X < t, Y \geq X) + P(Y < X).$$

The first summand represents the area inside the square  $[0, 1] \times [0, 1]$ , bounded between the lines  $y = x + t$  and  $y = x$ . The second summand represents the area of half of the square. Thus,

$$\begin{aligned} P(T < t) &= P(Y - X < t, Y \geq X) + P(Y < X) \\ &= \int_0^{1-t} t \, dx + \int_{1-t}^1 1 - x \, dx + \frac{1}{2} \\ &= -\frac{t^2}{2} + t + \frac{1}{2} \end{aligned}$$

## 5. EXERCISE 4.7

We represent the period from 8 AM to 9 AM by the closed interval  $[0, 1]$ . Then  $X \sim \text{Uniform}\left(\left[0, \frac{1}{2}\right]\right)$ , and  $Y \sim \text{Uniform}\left(\left[\frac{2}{3}, \frac{5}{6}\right]\right)$ . The arrival time is given by  $X + Y$ , with

$$f_{X+Y}(x, y) = 12.$$

Therefore

$$\begin{aligned} P(X + Y < 1) &= \int_R f_{X+Y}(x, y) \, dA \\ &= 12 (\text{area of } R). \end{aligned}$$

The region  $R$  is bounded by the functions:

$$\begin{cases} x + y &= 1 \\ y &= \frac{1}{2} \\ y &= \frac{5}{6} \\ x &= 0 \end{cases}$$

It is a trapezium with vertices  $\left(0, \frac{5}{6}\right)$ ,  $\left(\frac{1}{6}, \frac{5}{6}\right)$ ,  $\left(0, \frac{2}{3}\right)$ ,  $\left(\frac{1}{3}, \frac{2}{3}\right)$ , and has area  $\frac{1}{24}$ .

Therefore,

$$P(X + Y < 1) = 12 \cdot \frac{1}{24} = \frac{1}{2}.$$

## 6. EXERCISE 4.9

$$\begin{aligned}
P(a \leq X \leq b, c \leq Y \leq d) &= P(X \leq b, c \leq Y \leq d) - P(X \leq a, c \leq Y \leq d) \\
&= [P(X \leq b, Y \leq d) - P(X \leq b, Y \leq c)] - \\
&\quad [P(X \leq a, Y \leq d) - P(X \leq a, Y \leq c)] \\
&= F_{X,Y}(b, d) - F_{X,Y}(b, c) - F_{X,Y}(a, d) + F_{X,Y}(a, c) \\
&= [F_X(b) - F_X(a)] F_Y(d) - [F_X(b) - F_X(a)] F_Y(c) \\
&= [F_X(b) - F_X(a)] [F_Y(d) - F_Y(c)] \\
&= P(a \leq X \leq b) P(c \leq Y \leq d)
\end{aligned}$$



## 7. EXERCISE 4.10

(a) The marginal pdfs are given by

$$\begin{array}{ll} f_X(1) = \frac{1}{4} & f_Y(2) = \frac{1}{3} \\ f_X(2) = \frac{1}{2} & f_Y(3) = \frac{1}{3} \\ f_X(3) = \frac{1}{4} & f_Y(4) = \frac{1}{3} \end{array}$$

We see that

$$\begin{aligned} P(X = 2, Y = 3) &= 0 \\ &\neq \frac{1}{2} \cdot \frac{1}{3} \\ &= f_X(2) f_Y(3) \end{aligned}$$

Therefore, they are dependent.

(b) Let  $U = X$ ,  $V = Y$ , and the pair  $(U, V)$  has distribution

$$f_{U,V}(u, v) = f_U(u) f_V(v).$$

## 8. EXERCISE 4.11

Both  $U$  and  $V$  follow negative binomial distribution:

$$U \sim \text{NegBinomial}(1, p),$$

$$V \sim \text{NegBinomial}(2, p).$$

In particular,

$$P(V = k) = p \cdot P(U = k - 1).$$

This shows they are dependent.

## 9. EXERCISE 4.12

Without loss of generality, say the stick is given by the interval  $(0, 1)$ . Let  $X, Y$  be points chosen from  $(0, 1)$ . Then  $X \sim \text{Uniform}((0, 1)) \sim Y$ , and

$$f_{X,Y}(x, y) = 1$$

on  $(0, 1) \times (0, 1)$ .

By symmetry, we may assume  $y > x$  first. The points  $x, y$  divide  $(0, 1)$  into three pieces of length  $x, y - x, 1 - y$  respectively. A triangle can be formed if and only if they satisfy the triangle inequality:

$$\begin{cases} x + (y - x) & \geq 1 - y \\ x + (1 - y) & \geq y - x \\ (y - x) + (1 - y) & \geq x \end{cases}$$

or equivalently,

$$\begin{cases} y & \geq \frac{1}{2} \\ y - x & \leq \frac{1}{2} \\ x & \leq \frac{1}{2} \end{cases}$$

This is the triangle given by

$$\begin{cases} 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2} \leq y \leq x + \frac{1}{2} \end{cases}$$

Combining with the case  $x > y$ , the required probability is

$$\begin{aligned} 2 \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{x+\frac{1}{2}} f_{X,Y}(x, y) \, dy \, dx &= 2 \cdot (\text{area of the triangle}) \\ &= \frac{1}{4}. \end{aligned}$$

## 10. EXERCISE 4.14

Since  $X$  and  $Y$  are independent, the joint distribution is given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}$$

(a)

$$\begin{aligned} P(X^2 + Y^2 < 1) &= \int_{x^2+y^2 < 1} f_{X,Y}(x,y) \, dA \\ &= \frac{1}{2\pi} \cdot \int_0^{2\pi} \int_0^1 r e^{-\frac{r^2}{2}} \, dr \, d\theta \\ &= 1 - e^{-\frac{1}{2}} \end{aligned}$$

(b) Let  $Y = X^2$ , then

$$\begin{aligned} f_Y(y) &= f_X(\sqrt{y}) + f_X(-\sqrt{y}) \\ &= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} \end{aligned}$$

which is the pdf for  $\chi_1^2$ . Therefore,

$$\begin{aligned} P(X^2 < 1) &= \int_0^1 f_Y(y) \, dy \\ &\approx 0.682689 \end{aligned}$$

## 11. EXERCISE 4.15

Let  $U = X + Y$ , and  $V = X$ . Then  $U \sim \text{Poisson}(\theta + \lambda)$ ; and  $U, V$  are independent by [BC01, Theorem 4.3.2 on page 158].

We repeat the same computation as in [BC01, Example 4.3.1 on page 157] to find the joint pdf

$$f(v, u) = \frac{\lambda^{u-v} \theta^v e^{-(\theta+\lambda)}}{(u-v)! v!}$$

for  $(U, V)$ . Therefore, the conditional distribution is given by

$$\begin{aligned} f(v|u) &= \frac{f(v, u)}{f(u)} \\ &= \frac{\lambda^{u-v} \theta^v e^{-(\theta+\lambda)}}{(u-v)! v!} \cdot \frac{u!}{(\theta + \lambda)^u e^{-(\theta+\lambda)}} \\ &= \binom{u}{v} \left( \frac{\theta}{\theta + \lambda} \right)^v \left( \frac{\lambda}{\theta + \lambda} \right)^{u-v} \\ &\sim \text{Binomial} \left( u, \frac{\theta}{\theta + \lambda} \right). \end{aligned}$$

Likewise,  $Y|X + Y \sim \text{Binomial} \left( u, \frac{\lambda}{\theta + \lambda} \right)$ .

## 12. EXERCISE 4.16

Write  $X \sim \text{Geometric}(p) \sim Y$ .

(a) The joint distribution is given by

$$\begin{aligned}
 f(u, v) &= P(U = u, V = v) \\
 &= P(\min(X, Y) = u, X - Y = v) \\
 &= \begin{cases} P(Y = u, X = v + u) & \text{if } v \geq 0 \\ P(X = u, Y = u - v) & \text{if } v < 0 \end{cases} \\
 &= \begin{cases} (1 - p)^{2u+v-2} p^2 & \text{if } v \geq 0 \\ (1 - p)^{2u-v-2} p^2 & \text{if } v < 0 \end{cases} \\
 &= (1 - p)^{2u+|v|-2} p^2 \\
 &= \underbrace{[(1 - p)^{2u-1} p]}_{g(u)} \underbrace{[(1 - p)^{|v|-1} p]}_{h(v)}.
 \end{aligned}$$

[BC01, Lemma 4.2.7 on page 153] then says  $U, V$  are independent.

(b) We begin by noting  $Z$  takes values in  $\mathbb{Q}$ . Therefore, we represent all possible values of  $Z$  by fractions  $\frac{r}{s}$  with  $\gcd(r, s) = 1$ . We then compute

$$\begin{aligned}
 P\left(Z = \frac{r}{s}\right) &= P\left(\frac{X}{X+Y} = \frac{r}{s}\right) \\
 &= \sum_{n=1}^{\infty} P(X = nr, X+Y = ns) \\
 &= \sum_{n=1}^{\infty} P(X = nr, Y = n(s-r)) \\
 &= \sum_{n=1}^{\infty} (1 - p)^{ns-2} p^2 \\
 &= \frac{(1 - p)^{s-2} p^2}{1 - (1 - p)^{s-2}}
 \end{aligned}$$

(c)

$$\begin{aligned}
 P(X = u, X + Y = v) &= P(X = u, Y = v - u) \\
 &= (1 - p)^{v-2} p^2
 \end{aligned}$$

## 13. EXERCISE 4.17

(a)

$$\begin{aligned}
 P(Y = y) &= P(y \leq X < y + 1) \\
 &= \int_y^{y+1} e^{-x} dx \\
 &= (1 - e^{-1}) (e^{-1})^y \\
 &\sim \text{Geometric}(e^{-1})
 \end{aligned}$$

(b) Let  $Z = X - 4$ . We compute the cdf first.

$$\begin{aligned}
 P(Z \leq z | Y \geq 5) &= \frac{P(Z \leq z, Y \geq 5)}{P(Y \geq 5)} \\
 &= \frac{P(Z \leq z, X \geq 4)}{P(X \geq 4)} \\
 &= \frac{P(4 \leq X \leq z + 4)}{P(X \geq 4)} \\
 &= \frac{e^{-4} - e^{-4-z}}{e^{-4}} \\
 &= 1 - e^{-z}
 \end{aligned}$$

Therefore, the pdf is given by

$$\begin{aligned}
 P(Z = z | Y \geq 5) &= \frac{d}{dz} 1 - e^{-z} \\
 &= e^{-z}
 \end{aligned}$$

on  $\mathcal{Z} = [0, \infty)$ .

## 14. EXERCISE 4.18

Polar coordinates.



## 15. EXERCISE 4.19

- (a) By [BC01, Theorem 4.2.14 on page 156], if  $X \sim \text{Normal}(\mu_X, \sigma_X^2)$  and  $Y \sim \text{Normal}(\mu_Y, \sigma_Y^2)$ , then  $X - Y \sim \text{Normal}(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2)$ . In particular, when  $X$  and  $Y$  are both standard normal, the difference

$$\frac{X - Y}{\sqrt{2}} \sim \text{Normal}(0, 1)$$

is standard normal as well. It follows from Exercise 4.14 that

$$\begin{aligned} \frac{(X - Y)^2}{2} &= \left( \frac{X - Y}{\sqrt{2}} \right)^2 \\ &\sim (\text{Normal}(0, 1))^2 \\ &\sim \chi_1^2 \end{aligned}$$

- (b) Refer to [BC01, page 158].  
Define

$$\begin{cases} y_1 &= \frac{x_1}{x_1 + x_2}, \\ y_2 &= x_1 + x_2, \end{cases}$$

so that

$$\begin{cases} x_1 &= y_1 y_2, \\ x_2 &= y_2(1 - y_1). \end{cases}$$

Next, the Jacobi determinant is given by

$$\begin{aligned} |J| &= \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} \\ &= \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1 - y_1 \end{vmatrix} \\ &= |y_2|. \end{aligned}$$

Therefore, the joint distribution for  $Y_1 = \frac{X_1}{X_1 + X_2}$  and  $Y_2 = X_1 + X_2$  is given by

$$\begin{aligned}
f_{Y_1, Y_2}(y_1, y_2) &= f_{X_1, X_2}(y_1 y_2, y_2(1 - y_1)) \cdot |y_2| \\
&= f_{X_1}(y_1 y_2) \cdot f_{X_2}(y_2(1 - y_1)) \cdot |y_2| && \text{(since } X_1 \text{ and } X_2 \text{ are independent.)} \\
&= \frac{(y_1 y_2)^{\alpha_1 - 1} e^{-y_1 y_2}}{\Gamma(\alpha_1)} \cdot \frac{(y_2(1 - y_1))^{\alpha_2 - 1} e^{-y_2(1 - y_1)}}{\Gamma(\alpha_2)} \cdot |y_2| \\
&= \left[ \frac{y_1^{\alpha_1 - 1} (1 - y_1)^{\alpha_2 - 1}}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \right] \cdot [y_2^{\alpha_1 + \alpha_2 - 1} e^{-y_2}] \\
&= \underbrace{\left[ \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1) \Gamma(\alpha_2)} y_1^{\alpha_1 - 1} (1 - y_1)^{\alpha_2 - 1} \right]}_{f_{Y_1}(y_1)} \cdot \underbrace{\left[ \frac{y_2^{\alpha_1 + \alpha_2 - 1} e^{-y_2}}{\Gamma(\alpha_1 + \alpha_2)} \right]}_{f_{Y_2}(y_2)}.
\end{aligned}$$

In particular, this shows  $Y_1 \sim \text{Beta}(\alpha_1, \alpha_2)$ . Finding the pdf of  $\frac{X_2}{X_1 + X_2} = 1 - Y_1$  is similar.

## 16. EXERCISE 4.20

We can think of the variables as Cartesian coordinates versus polar coordinates on  $\mathbb{R}^2$ . The variables are related as:

$$\begin{aligned} x_1 &= \sqrt{y_1}y_2, \\ x_2 &= \pm\sqrt{y_1 - y_1y_2^2}, \end{aligned}$$

and we have two Jacobi matrices:

$$J_{\pm} = \begin{bmatrix} \frac{y_2}{2\sqrt{y_1}} & \sqrt{y_1} \\ \frac{\pm\sqrt{y_1 - y_1y_2^2}}{2y_1} & \mp\frac{y_1y_2}{\sqrt{y_1 - y_1y_2^2}} \end{bmatrix},$$

with  $|J_{\pm}| = \frac{1}{2\sqrt{1 - y_2^2}}$ . As a result, the joint distribution is given by

$$\begin{aligned} f_{Y_1, Y_2}(y_1, y_2) &= \left[ f_{X_1, X_2}\left(\sqrt{y_1}y_2, \sqrt{y_1 - y_1y_2^2}\right) + f_{X_1, X_2}\left(\sqrt{y_1}y_2, -\sqrt{y_1 - y_1y_2^2}\right) \right] \cdot \frac{1}{2\sqrt{1 - y_2^2}} \\ &= \left[ \frac{1}{2\pi\sigma^2} e^{-\frac{y_1}{2\sigma^2}} \right] \cdot \left[ \frac{1}{\sqrt{1 - y_2^2}} \right], \end{aligned}$$

proving  $Y_1, Y_2$  are independent as well.

## 17. EXERCISE 4.21

Write  $\mathcal{R} = R^2$ . Then

$$\begin{aligned}
 f_{X,Y}(x,y) &= f_{\mathcal{R},\theta}\left(\mathcal{R} = x^2 + y^2, \theta = \arctan\left(\frac{y}{x}\right)\right) \cdot \left| \begin{array}{cc} \frac{\partial \mathcal{R}}{\partial x} & \frac{\partial \mathcal{R}}{\partial dy} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial dy} \end{array} \right| \\
 &= \left[ \frac{1}{2} e^{-\frac{x^2+y^2}{2}} \right] \cdot \frac{1}{2\pi} \cdot \left| \begin{array}{cc} 2x & 2y \\ \frac{y}{x^2+y^2} & -\frac{x}{x^2+y^2} \end{array} \right| \\
 &= \left[ \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right] \cdot \left[ \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \right]
 \end{aligned}$$

## 18. EXERCISE 4.22

We have

$$\begin{cases} x &= \frac{u-b}{a}, \\ y &= \frac{v-d}{c}. \end{cases}$$

The Jacobi determinant is then given by

$$\begin{aligned} |J| &= \begin{vmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{c} \end{vmatrix} \\ &= \frac{1}{ac}. \end{aligned}$$

Therefore, the result follows immediately from [BC01, page 158].

## 19. EXERCISE 4.27

Let

$$\begin{cases} u &= x + y, \\ v &= x - y \end{cases}$$

Then

$$\begin{aligned} f_{U,V}(u, v) &= f_{X,Y}(x(u, v), y(u, v)) \cdot |J| \\ &= f_X(x(u, v)) \cdot f_Y(y(u, v)) \cdot |J| \\ &\quad (\text{since } X \text{ and } Y \text{ are independent}) \\ &= f_X(x(u, v)) \cdot f_Y(y(u, v)) \cdot \left| \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{array} \right| \\ &= \frac{1}{2} \cdot f_X\left(\frac{u+v}{2}\right) \cdot f_Y\left(\frac{u-v}{2}\right) \\ &= \frac{1}{4\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2} \left[ \left(\frac{u+v}{2} - \mu\right)^2 + \left(\frac{u-v}{2} - \gamma\right)^2 \right]\right) \\ &= \frac{1}{4\pi\sigma^2} \exp\left(-\frac{1}{8\sigma^2} [(u+v) - 2\mu]^2 + [(u-v) - 2\gamma]^2\right) \\ &= \frac{1}{4\pi\sigma^2} \exp\left(-\frac{1}{8\sigma^2} [2[u - (\gamma + \mu)]^2 - 2(\gamma + \mu)^2 + 2v^2 + 4(\gamma - \mu)v + 4\mu^2 + 4\gamma^2]\right) \\ &= \frac{1}{4\pi\sigma^2} \exp\left(-\frac{1}{8\sigma^2} [2[u - (\gamma + \mu)]^2 + 2[v - (\mu - \gamma)]^2]\right) \\ &= \underbrace{\frac{1}{\sqrt{2\pi} \cdot \sqrt{2}\sigma} \exp\left(-\frac{1}{2} \cdot \frac{[u - (\gamma + \mu)]^2}{2\sigma^2}\right)}_{f_U(u)} \cdot \underbrace{\frac{1}{\sqrt{2\pi} \cdot \sqrt{2}\sigma} \exp\left(-\frac{1}{2} \cdot \frac{[v - (\gamma - \mu)]^2}{2\sigma^2}\right)}_{f_V(v)} \\ &\sim \text{Normal}(\gamma + \mu, 2\sigma^2) \cdot \text{Normal}(\gamma - \mu, 2\sigma^2) \end{aligned}$$

## 20. EXERCISE 4.30

(a) Firstly,

$$\begin{aligned}
 EY &= E(E(Y|X)) \\
 &\quad ([\text{BC01, Theorem 4.4.3 on page 164}]) \\
 &= E(E(\text{Normal}(x, x^2))) \\
 &= E(X) \\
 &= \frac{1}{2}.
 \end{aligned}$$

Secondly,

$$\begin{aligned}
 \text{Var}(Y) &= E(\text{Var}(Y|X)) + \text{Var}(E(Y|X)) \\
 &= E(\text{Var}(\text{Normal}(x, x^2))) + \text{Var}(E(\text{Normal}(x, x^2))) \\
 &= E(X^2) + \text{Var}(X) \\
 &= 2\text{Var}(X) + E(X)^2 \\
 &= \frac{5}{12}
 \end{aligned}$$

Finally, to compute  $\text{Cov}(X, Y)$ , we notice we have to deal with the random variable  $XY$ . Let  $U = XY$ ,  $V = X$ . [BC01, page 158] gives the joint distribution

$$f_{U,V}(u, v) = \frac{1}{v} f_{X,Y}\left(v, \frac{u}{v}\right), \quad (20.0.1)$$

and (hence) the conditional distribution is given by

$$\begin{aligned}
 f_{U|V}(u|v) &= \frac{f_{U,V}(u, v)}{f_V(v)} \\
 &= \frac{\frac{1}{v} f_{X,Y}\left(v, \frac{u}{v}\right)}{f_X(v)} \\
 &= \frac{1}{v} f_{Y|X}\left(\frac{u}{v} \middle| v\right). \quad (20.0.2)
 \end{aligned}$$

This allows us to prove the following formula for expectation:

$$\begin{aligned}
E(E(XY|X)) &= E(E(U|V)) \\
&\quad (U = XY, V = X) \\
&= E\left(\int u f_{U|V}(u|v) du\right) \\
&= E\left(\int \frac{u}{v} f_{Y|X}\left(\frac{u}{v}|v\right) du\right) \\
&= E\left(\int xy f_{Y|X}(y|x) dy\right) \\
&\quad \left(y = \frac{u}{v}, x = v, dy = \frac{1}{v} du\right) \\
&= E(XE(Y|X)). \tag{20.0.3}
\end{aligned}$$

As a result,

$$\begin{aligned}
\text{Cov}(X, Y) &= E(XY) - (EX)(EY) \\
&= E(XY) - \frac{1}{4} \\
&= E(XY|X) - \frac{1}{4} \\
&= E(XE(Y|X)) - \frac{1}{4} \\
&= E(X \cdot E(\text{Normal}(x, x^2))) - \frac{1}{4} \\
&= E(X^2) - \frac{1}{4} \\
&= \text{Var}(X) + (EX)^2 - \frac{1}{4} \\
&= \frac{1}{12}.
\end{aligned}$$

(b) Let  $U = \frac{Y}{X}$ ,  $V = X$ . The Jacobi matrix is given by

$$\begin{bmatrix} 0 & 1 \\ v & u \end{bmatrix},$$

and the joint distribution is given by



$$\begin{aligned}
f_{U,V}(u, v) &= f_{X,Y}(v, uv) \cdot |J| \\
&\quad ([\text{BC01, page 158}]) \\
&= v \cdot f_X(v) \cdot f_{Y|X}(uv|v) \\
&= v \cdot 1 \cdot \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2}\left(\frac{uv-v}{v}\right)^2} \\
&= \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u-1)^2}}_{f_U(u)} \cdot \underbrace{1}_{f_V(v)} \\
&\sim \text{Normal}(1, 1) \cdot \text{Uniform}(0, 1).
\end{aligned}$$

## REFERENCES

- [BC01] Roger Berger and George Casella. *Statistical Inference*. 2nd edition. Florence, AL: Duxbury Press, June 2001.