CASELLA-BERGER STATISTICAL INFERENCE SOLUTION: CHAPTER 4

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1. Exercise 4.1

(a) We want to know the probability for (X, Y) to land inside the circle

$$X^2 + Y^2 = 1.$$

This circle has area π, so the probability is $\frac{\pi}{4}$.

(b) We want to know the probability for (X, Y) to land below the line

$$2X - Y = 0.$$

This line divides the square into two uniform trapeziums. One of them has vertices $\left(\pm\frac{1}{2}, \pm 1\right)$, and has area 2. Therefore, the probability is $\frac{1}{2}$.

(c) The region $|X + Y| < 2$ contains the square, so the probability is 1.

2. Exercise 4.4

(a)

$$\begin{aligned} 1 &= \int_0^1 \int_0^2 C(x+2y)\ dx\ dy \\ &= 4C \end{aligned}$$

So $C = \dfrac{1}{4}$.

(b)

$$\begin{aligned} f_X(x) &= \int_0^1 \frac{x+2y}{4}\ dy \\ &= \frac{x+1}{4} \end{aligned}$$

on $\mathcal{X} = (0,2)$.

(c)

$$\begin{aligned} F_{XY}(x,y) &= P(X \le x, Y \le y) \\ &= \begin{cases} 0 & \text{if } x \le 0 \text{ or } y \le 0 \\ 1 & \text{if } x \ge 2,\ y \ge 1 \\ \int_0^y \int_0^x f(u,v)\ du\ dv & \text{if else} \end{cases} \end{aligned}$$

The if else case requires some work.

$$\int_0^y \int_0^x f(u,v)\ du\ dv = \begin{cases} \int_0^y \int_0^x f(u,v)\ du\ dv & \text{if } 0 < x < 2,\ 0 < y < 1 \\ \int_0^1 \int_0^x f(u,v)\ du\ dv & \text{if } 0 < x < 2,\ y \ge 1 \\ \int_0^y \int_0^2 f(u,v)\ du\ dv & \text{if } 0 < x < 2,\ y \ge 1 \end{cases}$$

$$= \begin{cases} \dfrac{xy(x+2y)}{8} & \text{if } 0 < x < 2,\ 0 < y < 1 \\ \dfrac{x(x+2)}{8} & \text{if } 0 < x < 2,\ y \ge 1 \\ \dfrac{y(1+y)}{2} & \text{if } x \ge 2,\ 0 < y < 1 \end{cases}$$

(d) Let $z = g(x) = \dfrac{9}{(x+1)^2}$, then $g^{-1}(z) = \dfrac{3}{\sqrt{z}} - 1$, and $\dfrac{d}{dz} g^{-1}(z) = \dfrac{3}{-2z^{\frac{3}{2}}}$. Therefore,

$$\begin{aligned} f_Z(z) &= f_X(g^{-1}(z)) \cdot \left| \frac{d}{dz} g^{-1}(z) \right| \\ &= \frac{9}{8z^2} \end{aligned}$$

3. Exercise 4.5

(a)

$$\begin{aligned} P\left(X > \sqrt{Y}\right) &= \int_0^1 \int_{\sqrt{y}}^1 x + y \ dx \ dy \\ &= \frac{7}{20} \end{aligned}$$

(b)

$$\begin{aligned} P\left(X^2 < Y < X\right) &= \int_0^1 \int_{x^2}^x 2x \ dy \ dx \\ &= \frac{1}{6} \end{aligned}$$

4. Exercise 4.6

Let X (resp. Y) be the arrival time of A (resp. B), so that $X \sim \text{Uniform}([0,1]) \sim Y$. Let T be the waiting time. Then

$$T = \max\{Y - X, 0\}.$$

Therefore,

$$P(T < t) = P(Y - X < t, Y \geq X) + P(Y < X).$$

The first summand represents the area inside the square $[0,1] \times [0,1]$, bounded between the lines $y = x + t$ and $y = x$. The second summand represents the area of half of the square. Thus,

$$\begin{aligned} P(T < t) &= P(Y - X < t, Y \geq X) + P(Y < X) \\ &= \int_0^{1-t} t\ dx + \int_{1-t}^{1} 1 - x\ dx + \frac{1}{2} \\ &= -\frac{t^2}{2} + t + \frac{1}{2} \end{aligned}$$

5. Exercise 4.7

We represent the period from 8 AM to 9 AM by the closed interval $[0,1]$. Then $X \sim \text{Uniform}\left(\left[0,\frac{1}{2}\right]\right)$, and $Y \sim \text{Uniform}\left(\left[\frac{2}{3},\frac{5}{6}\right]\right)$. The arrival time is given by $X+Y$, with

$$f_{X+Y}(x,y) = 12.$$

Therefore

$$\begin{aligned}P(X+Y<1) &= \int_R f_{X+Y}(x,y)\ dA \\ &= 12\,(\text{area of } R)\,.\end{aligned}$$

The region R is bounded by the functions:

$$\begin{cases} x+y &= 1 \\ y &= \frac{1}{2} \\ y &= \frac{5}{6} \\ x &= 0 \end{cases}$$

It is a trapezium with vertices $\left(0,\frac{5}{6}\right),\left(\frac{1}{6},\frac{5}{6}\right),\left(0,\frac{2}{3}\right),\left(\frac{1}{3},\frac{2}{3}\right)$, and has area $\frac{1}{24}$.

Therefore,

$$P(X+Y<1) = 12\cdot\frac{1}{24} = \frac{1}{2}.$$

6. Exercise 4.9

$$\begin{aligned}
P(a \le X \le b, c \le Y \le d) &= P(X \le b, c \le Y \le d) - P(X \le a, c \le Y \le d) \\
&= [P(X \le b, Y \le d) - P(X \le b, Y \le c)] - \\
&\quad [P(X \le a, Y \le d) - P(X \le a, Y \le c)] \\
&= F_{X,Y}(b,d) - F_{X,Y}(b,c) - F_{X,Y}(a,d) + F_{X,Y}(a,c) \\
&= [F_X(b) - F_X(a)]\, F_Y(d) - [F_X(b) - F_X(a)]\, F_Y(c) \\
&= [F_X(b) - F_X(a)]\, [F_Y(d) - F_Y(c)] \\
&= P(a \le X \le b)\, P(c \le Y \le d)
\end{aligned}$$

7. Exercise 4.10

(a) The marginal pdfs are given by

$$f_X(1) = \frac{1}{4} \qquad f_Y(2) = \frac{1}{3}$$
$$f_X(2) = \frac{1}{2} \qquad f_Y(3) = \frac{1}{3}$$
$$f_X(3) = \frac{1}{4} \qquad f_Y(4) = \frac{1}{3}$$

We see that

$$\begin{aligned} P(X = 2, Y = 3) &= 0 \\ &\neq \frac{1}{2} \cdot \frac{1}{3} \\ &= f_X(2)\, f_Y(3) \end{aligned}$$

Therefore, they are dependent.

(b) Let $U = X$, $V = Y$, and the pair (U, V) has distribution

$$f_{U,V}(u, v) = f_U(u)\, f_V(v)\,.$$

8. Exercise 4.11

Both V and V follow negative binomial distribution:

$$U \sim \text{NegBinomial}\,(1, p)\,,$$
$$V \sim \text{NegBinomial}\,(2, p)\,.$$

In particular,

$$P(V = k) = p \cdot P(U = k - 1)\,.$$

This shows they are dependent.

9. Exercise 4.12

Without loss of generality, say the stick is given by the interval $(0,1)$. Let X, Y be points chosen from $(0,1)$. Then $X \sim \text{Uniform}((0,1)) \sim Y$, and

$$f_{X,Y}(x,y) = 1$$

on $(0,1) \times (0,1)$.

By symmetry, we may assume $y > x$ first. The points x, y divide $(0,1)$ into three pieces of length x, $y-x$, $1-y$ respectively. A triangle can be formed if and only if they satisfy the triangle inequality:

$$\begin{cases} x + (y-x) & \geq 1 - y \\ x + (1-y) & \geq y - x \\ (y-x) + (1-y) & \geq x \end{cases}$$

or equivalently,

$$\begin{cases} y & \geq \frac{1}{2} \\ y - x & \leq \frac{1}{2} \\ x & \leq \frac{1}{2} \end{cases}$$

This is the triangle given by

$$\begin{cases} 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2} \leq y \leq x + \frac{1}{2} \end{cases}$$

Combining with the case $x > y$, the required probability is

$$\begin{aligned} 2\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{x+\frac{1}{2}} f_{X,Y}(x,y)\; dy\; dx &= 2 \cdot (\text{area of the triangle}) \\ &= \frac{1}{4}. \end{aligned}$$

10. Exercise 4.14

Since X and Y are independent, the joint distribution is given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}$$

(a)

$$\begin{aligned} P\big(X^2 + Y^2 < 1\big) &= \int_{x^2+y^2<1} f_{X,Y}(x,y)\ dA \\ &= \frac{1}{2\pi} \cdot \int_0^{2\pi} \int_0^1 r e^{-\frac{r^2}{2}}\ dr\ d\theta \\ &= 1 - e^{-\frac{1}{2}} \end{aligned}$$

(b) Let $Y = X^2$, then

$$\begin{aligned} f_Y(y) &= f_X(\sqrt{y}) + f_X(-\sqrt{y}) \\ &= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} \end{aligned}$$

which is the pdf for χ_1^2. Therefore,

$$\begin{aligned} P\big(X^2 < 1\big) &= \int_0^1 f_Y(y)\ dy \\ &\approx 0.682689 \end{aligned}$$

11. Exercise 4.15

Let $U = X + Y$, and $V = X$. Then $U \sim \text{Poisson}\,(\theta + \lambda)$; and U, V are independent by [BC01, Theorem 4.3.2 on page 158].

We repeat the same computation as in [BC01, Example 4.3.1 on page 157] to find the joint pdf

$$f(v, u) = \frac{\lambda^{u-v}\theta^v e^{-(\theta+\lambda)}}{(u-v)!\,v!}$$

for (U, V). Therefore, the conditional distribution is given by

$$\begin{aligned} f(v|u) &= \frac{f(v, u)}{f(u)} \\ &= \frac{\lambda^{u-v}\theta^v e^{-(\theta+\lambda)}}{(u-v)!\,v!} \cdot \frac{u!}{(\theta+\lambda)^u e^{-(\theta+\lambda)}} \\ &= \binom{u}{v} \left(\frac{\theta}{\theta+\lambda}\right)^v \left(\frac{\lambda}{\theta+\lambda}\right)^{u-v} \\ &\sim \text{Binomial}\left(u, \frac{\theta}{\theta+\lambda}\right). \end{aligned}$$

Likewise, $Y|X+Y \sim \text{Binomial}\left(u, \dfrac{\lambda}{\theta+\lambda}\right)$.

12. Exercise 4.16

Write $X \sim \text{Geometric}\,(p) \sim Y$.

(a) The joint distribution is given by

$$\begin{aligned} f(u,v) &= P(U = u,\ V = v) \\ &= P(\min(X,Y) = u,\ X - Y = v) \\ &= \begin{cases} P(Y = u,\ X = v + u) & \text{if } v \geq 0 \\ P(X = u,\ Y = u - v) & \text{if } v < 0 \end{cases} \\ &= \begin{cases} (1-p)^{2u+v-2}p^2 & \text{if } v \geq 0 \\ (1-p)^{2u-v-2}p^2 & \text{if } v < 0 \end{cases} \\ &= (1-p)^{2u+|v|-2}p^2 \\ &= \underbrace{\left[(1-p)^{2u-1}p\right]}_{g(u)} \underbrace{\left[(1-p)^{|v|-1}p\right]}_{h(v)}. \end{aligned}$$

[BC01, Lemma 4.2.7 on page 153] then says U, V are independent.

(b) We begin by noting Z takes values in $\mathbb{Q}$. Therefore, we represent all possible values of Z by fractions $\dfrac{r}{s}$ with $\gcd(r,s) = 1$. We then compute

$$\begin{aligned} P\Big(Z = \frac{r}{s}\Big) &= P\bigg(\frac{X}{X+Y} = \frac{r}{s}\bigg) \\ &= \sum_{n=1}^{\infty} P(X = nr,\ X + Y = ns) \\ &= \sum_{n=1}^{\infty} P(X = nr,\ Y = n(s-r)) \\ &= \sum_{n=1}^{\infty} (1-p)^{ns-2}p^2 \\ &= \frac{(1-p)^{s-2}p^2}{1-(1-p)^{s-2}} \end{aligned}$$

(c)

$$\begin{aligned} P(X = u,\ X + Y = v) &= P(X = u,\ Y = v - u) \\ &= (1-p)^{v-2}p^2 \end{aligned}$$

13. Exercise 4.17

(a)

$$\begin{aligned} P(Y = y) &= P(y \le X < y+1) \\ &= \int_y^{y+1} e^{-x}\ dx \\ &= (1 - e^{-1}) \left(e^{-1}\right)^y \\ &\sim \text{Geometric}\left(e^{-1}\right) \end{aligned}$$

(b) Let $Z = X - 4$. We compute the cdf first.

$$\begin{aligned} P(\,Z \le z|Y \ge 5\,) &= \frac{P(Z \le z,\ Y \ge 5)}{P(Y \ge 5)} \\ &= \frac{P(Z \le z,\ X \ge 4)}{P(X \ge 4)} \\ &= \frac{P(4 \le X \le z+4)}{P(X \ge 4)} \\ &= \frac{e^{-4} - e^{-4-z}}{e^{-4}} \\ &= 1 - e^{-z} \end{aligned}$$

Therefore, the pdf is given by

$$\begin{aligned} P(\,Z = z|Y \ge 5\,) &= \frac{d}{dz} 1 - e^{-z} \\ &= e^{-z} \end{aligned}$$

on $\mathcal{Z} = [0, \infty)$.

14. Exercise 4.18

Polar coordinates.

15. Exercise 4.19

(a) By [BC01, Theorem 4.2.14 on page 156], if $X \sim \text{Normal}\left(\mu_X, \sigma_X^2\right)$ and $Y \sim \text{Normal}\left(\mu_Y, \sigma_Y^2\right)$, then $X - Y \sim \text{Normal}\left(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2\right)$. In particular, when X and Y are both standard normal, the difference

$$\frac{X - Y}{\sqrt{2}} \sim \text{Normal}\,(0,1)$$

is standard normal as well. It follows from Exercise 4.14 that

$$\begin{aligned}\frac{(X-Y)^2}{2} &= \left(\frac{X-Y}{\sqrt{2}}\right)^2 \\ &\sim \left(\text{Normal}\,(0,1)\right)^2 \\ &\sim \chi_1^2\end{aligned}$$

(b) Refer to [BC01, page 158].
Define

$$\begin{cases} y_1 &= \dfrac{x_1}{x_1 + x_2}, \\ y_2 &= x_1 + x_2, \end{cases}$$

so that

$$\begin{cases} x_1 &= y_1 y_2, \\ x_2 &= y_2(1 - y_1). \end{cases}$$

Next, the Jacobi determinant is given by

$$\begin{aligned}|J| &= \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix} \\ &= \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1 - y_1 \end{vmatrix} \\ &= |y_2|\,. \end{aligned}$$

Therefore, the joint distribution for $Y_1 = \dfrac{X_1}{X_1 + X_2}$ and $Y_2 = X_1 + X_2$ is given by

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1,X_2}(y_1y_2, y_2(1-y_1)) \cdot |y_2| \\
&= f_{X_1}(y_1y_2) \cdot f_{X_2}(y_2(1-y_1)) \cdot |y_2| && \text{(since X_1 and X_2 are independent.)} \\
&= \frac{(y_1y_2)^{\alpha_1-1}e^{-y_1y_2}}{\Gamma(\alpha_1)} \cdot \frac{(y_2(1-y_1))^{\alpha_2-1}e^{-y_2(1-y_1)}}{\Gamma(\alpha_2)} \cdot |y_2| \\
&= \left[\frac{y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}\right] \cdot \left[y_2^{\alpha_1+\alpha_2-1}e^{-y_2}\right] \\
&= \underbrace{\left[\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)} y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1}\right]}_{f_{Y_1}(y_1)} \cdot \underbrace{\left[\frac{y_2^{\alpha_1+\alpha_2-1}e^{-y_2}}{\Gamma(\alpha_1+\alpha_2)}\right]}_{f_{Y_2}(y_2)}.
\end{aligned}
$$

In particular, this shows $Y_1 \sim \operatorname{Beta}(\alpha_1, \alpha_2)$. Finding the pdf of $\dfrac{X_2}{X_1+X_2} = 1 - Y_1$ is similar.

16. Exercise 4.20

We can think of the variables as Cartesian coordinates versus polar coordinates on $\mathbb{R}^2$. The variables are related as:

$$x_1 = \sqrt{y_1}y_2,$$
$$x_2 = \pm\sqrt{y_1 - y_1 y_2^2},$$

and we have two Jacobi matrices:

$$J_\pm = \begin{bmatrix} \dfrac{y_2}{2\sqrt{y_1}} & \sqrt{y_1} \\ \dfrac{\pm\sqrt{y_1 - y_1 y_2^2}}{2y_1} & \mp\dfrac{y_1 y_2}{\sqrt{y_1 - y_1 y_2^2}} \end{bmatrix},$$

with $|J_\pm| = \dfrac{1}{2\sqrt{1-y_2^2}}$. As a result, the joint distribution is given by

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= \left[f_{X_1,X_2}\left(\sqrt{y_1}y_2, \sqrt{y_1 - y_1 y_2^2}\right) + f_{X_1,X_2}\left(\sqrt{y_1}y_2, -\sqrt{y_1 - y_1 y_2^2}\right)\right] \cdot \frac{1}{2\sqrt{1-y_2^2}} \\ &= \left[\frac{1}{2\pi\sigma^2} e^{-\frac{y_1}{2\sigma^2}}\right] \cdot \left[\frac{1}{\sqrt{1-y_2^2}}\right], \end{aligned}$$

proving Y_1, Y_2 are independent as well.

17. Exercise 4.21

Write $\mathcal{R} = R^2$. Then

$$
\begin{aligned}
f_{X,Y}(x,y) &= f_{\mathcal{R},\theta}\Big(\mathcal{R} = x^2+y^2, \theta = \arctan\Big(\frac{y}{x}\Big)\Big) \cdot \left| \begin{matrix} \dfrac{\partial \mathcal{R}}{\partial x} & \dfrac{\partial \mathcal{R}}{\partial dy} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial dy} \end{matrix} \right| \\
&= \left[\frac{1}{2} e^{-\frac{x^2+y^2}{2}}\right] \cdot \frac{1}{2\pi} \cdot \left| \begin{matrix} 2x & 2y \\ -\dfrac{y}{x^2+y^2} & -\dfrac{x}{x^2+y^2} \end{matrix} \right| \\
&= \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right] \cdot \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\right]
\end{aligned}
$$

18. Exercise 4.22

We have

$$\begin{cases} x &= \dfrac{u-b}{a}, \\ y &= \dfrac{v-d}{c}. \end{cases}$$

The Jacobi determinant is then given by

$$\begin{aligned} |J| &= \begin{vmatrix} \dfrac{1}{a} & 0 \\ 0 & \dfrac{1}{c} \end{vmatrix} \\ &= \frac{1}{ac}. \end{aligned}$$

Therefore, the result follows immediately from [BC01, page 158].

19. Exercise 4.27

Let

$$\begin{cases} u &= x + y, \\ v &= x - y \end{cases}$$

Then

$$\begin{aligned}
f_{U,V}(u,v) &= f_{X,Y}\left(x(u,v)\,, y(u,v)\right) \cdot |J| \\
&= f_X(x(u,v)) \cdot f_Y(y(u,v)) \cdot |J| \\
&\quad \text{(since } X \text{ and } Y \text{ are independent)} \\
&= f_X(x(u,v)) \cdot f_Y(y(u,v)) \cdot \left| \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{array} \right| \\
&= \frac{1}{2} \cdot f_X\left(\frac{u+v}{2}\right) \cdot f_Y\left(\frac{u-v}{2}\right) \\
&= \frac{1}{4\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2}\left[\left(\frac{u+v}{2} - \mu\right)^2 + \left(\frac{u-v}{2} - \gamma\right)^2\right]\right) \\
&= \frac{1}{4\pi\sigma^2} \exp\left(-\frac{1}{8\sigma^2}\left[\left[(u+v) - 2\mu\right]^2 + \left[(u-v) - 2\gamma\right]^2\right]\right) \\
&= \frac{1}{4\pi\sigma^2} \exp\left(-\frac{1}{8\sigma^2}\left[2\left[u - (\gamma+\mu)\right]^2 - 2(\gamma+\mu)^2 + 2v^2 + 4(\gamma-\mu)v + 4\mu^2 + 4\gamma^2\right]\right) \\
&= \frac{1}{4\pi\sigma^2} \exp\left(-\frac{1}{8\sigma^2}\left[2\left[u - (\gamma+\mu)\right]^2 + 2\left[v - (\mu-\gamma)\right]^2\right]\right) \\
&= \underbrace{\frac{1}{\sqrt{2\pi}\cdot\sqrt{2}\sigma} \exp\left(-\frac{1}{2} \cdot \frac{\left[u - (\gamma+\mu)\right]^2}{2\sigma^2}\right)}_{f_U(u)} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\cdot\sqrt{2}\sigma} \exp\left(-\frac{1}{2} \cdot \frac{\left[v - (\gamma-\mu)\right]^2}{2\sigma^2}\right)}_{f_V(v)} \\
&\sim \text{Normal}\left(\gamma+\mu, 2\sigma^2\right) \cdot \text{Normal}\left(\gamma-\mu, 2\sigma^2\right)
\end{aligned}$$

20. Exercise 4.30

(a) Firstly,

$$\begin{aligned} EY &= E(E(Y|X)) \\ &\quad \text{([BC01, Theorem 4.4.3 on page 164])} \\ &= E\big(E\big(\text{Normal}\left(x, x^2\right)\big)\big) \\ &= E(X) \\ &= \frac{1}{2}. \end{aligned}$$

Secondly,

$$\begin{aligned} \text{Var}\,(Y) &= E(\text{Var}\,(Y|X)) + \text{Var}\,(E(Y|X)) \\ &= E\big(\text{Var}\,\big(\text{Normal}\left(x, x^2\right)\big)\big) + \text{Var}\,\big(E\big(\text{Normal}\left(x, x^2\right)\big)\big) \\ &= E\big(X^2\big) + \text{Var}\,(X) \\ &= 2\,\text{Var}\,(X) + E(X)^2 \\ &= \frac{5}{12} \end{aligned}$$

Finally, to compute $\text{Cov}\,(X, Y)$, we notice we have to deal with the random variable XY. Let $U = XY$, $V = X$. [BC01, page 158] gives the joint distribution

$$f_{U,V}(u, v) = \frac{1}{v} f_{X,Y}\left(v, \frac{u}{v}\right), \tag{20.0.1}$$

and (hence) the conditional distribution is given by

$$\begin{aligned} f_{U|V}\,(u|v) &= \frac{f_{U,V}(u, v)}{f_V(v)} \\ &= \frac{\frac{1}{v} f_{X,Y}\left(v, \frac{u}{v}\right)}{f_X(v)} \\ &= \frac{1}{v} f_{Y|X}\left(\frac{u}{v}\middle| v\right). \end{aligned} \tag{20.0.2}$$

This allows us to prove the following formula for expectation:

$$
\begin{aligned}
E(E(XY|X)) &= E(E(U|V)) \\
&\quad (U = XY,\, V = X) \\
&= E\left(\int u f_{U|V}(u|v)\, du\right) \\
&= E\left(\int \frac{u}{v} f_{Y|X}\left(\frac{u}{v}\middle| v\right)\, du\right) \\
&= E\left(\int xy f_{Y|X}(y|x)\, dy\right) \\
&\quad \left(y = \frac{u}{v},\, x = v,\, dy = \frac{1}{v} du\right) \\
&= E(XE(Y|X)) \,.
\end{aligned} \tag{20.0.3}
$$

As a result,

$$
\begin{aligned}
\operatorname{Cov}(X,Y) &= E(XY) - (EX)(EY) \\
&= E(XY) - \frac{1}{4} \\
&= E(XY|X) - \frac{1}{4} \\
&= E(XE(Y|X)) - \frac{1}{4} \\
&= E\big(X \cdot E\big(\text{Normal}\left(x, x^2\right)\big)\big) - \frac{1}{4} \\
&= E\big(X^2\big) - \frac{1}{4} \\
&= \operatorname{Var}(X) + (EX)^2 - \frac{1}{4} \\
&= \frac{1}{12}.
\end{aligned}
$$

(b) Let $U = \dfrac{Y}{X}$, $V = X$. The Jacobi matrix is given by

$$
\begin{bmatrix} 0 & 1 \\ v & u \end{bmatrix},
$$

and the joint distribution is given by

$$
\begin{aligned}
f_{U,V}(u,v) &= f_{X,Y}(v,uv)\cdot|J| \\
&\quad ([\text{BC01}, \text{page } 158]) \\
&= v\cdot f_X(v)\cdot f_{Y|X}(uv|v) \\
&= v\cdot 1\cdot\frac{1}{\sqrt{2\pi v}}e^{-\frac{1}{2}\left(\frac{uv-v}{v}\right)^2} \\
&= \underbrace{\frac{1}{\sqrt{2\pi 1}}e^{-\frac{1}{2}\left(\frac{u-1}{1}\right)^2}}_{f_U(u)}\cdot\underbrace{1}_{f_V(v)} \\
&\sim \text{Normal}\,(1,1)\cdot\text{Uniform}\,(0,1)\,.
\end{aligned}
$$

References

[BC01] Roger Berger and George Casella. *Statistical Inference*. 2nd edition. Florence, AL: Duxbury Press, June 2001.