

**CASELLA-BERGER
STATISTICAL INFERENCE SOLUTION:
CHAPTER 6**

VIRGIL CHAN

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CONTENTS

1. Exercise 6.1	2
2. Exercise 6.2	3
3. Exercise 6.3	4
4. Exercise 6.5	5
5. Exercise 6.6	6
6. Exercise 6.7	7
7. Exercise 6.8	8
8. Exercise 6.9	9
References	11

1. EXERCISE 6.1

We have

$$\begin{aligned} f_X(x|\sigma^2) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(x)^2} \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(|x|)^2} \\ &= f_X(|x||\sigma^2) \cdot 1. \end{aligned}$$

Therefore, by [BC01, Factorization Theorem 6.2.6 on page 276], the variable $|X|$ is a sufficient statistic for σ^2 .

2. EXERCISE 6.2

The joint distribution is given by

$$\begin{aligned}
 f_{X_1, \dots, X_n}(x_1, \dots, x_n | \theta) &= \prod_{i=1}^n f_{X_i}(x_i | \theta) \\
 &= \prod_{i=1}^n \chi_{[i\theta, \infty)}(x_i) e^{i\theta - x_i} \\
 &= \chi_{[\theta, \infty)}\left(\min_{1 \leq i \leq n} (x_i)\right) \cdot e^{\sum_{i=1}^n i\theta - x_i} \\
 &= \underbrace{\chi_{[\theta, \infty)}\left(\min_{1 \leq i \leq n} (x_i)\right) e^{in\theta}}_{g\left(\min_{1 \leq i \leq n} (x_i) \middle| \theta\right)} \cdot \underbrace{e^{-\sum_{i=1}^n x_i}}_{h(x)}.
 \end{aligned}$$

The result then follows from [BC01, Factorization Theorem 6.2.6 on page 276].

3. EXERCISE 6.3

The joint distribution is given by

$$\begin{aligned}
 f_{X_1, \dots, X_n}(x_1, \dots, x_n | \mu, \sigma) &= \prod_{i=1}^n \chi_{(\mu, \infty)}(x_i) \frac{1}{\sigma} e^{\frac{-(x_i - \mu)}{\sigma}} \\
 &= \underbrace{\chi_{(\mu, \infty)}\left(\min_{1 \leq i \leq n} x_i\right) \cdot \frac{1}{\sigma^n} e^{\frac{-\sum_{i=1}^n x_i + n\mu}{\sigma}}}_{g\left(\min_{1 \leq i \leq n} x_i, \sum_{i=1}^n x_i \mid \mu, \sigma\right)} \cdot \underbrace{1}_{h(x)}.
 \end{aligned}$$

[BC01, Factorization Theorem 6.2.6 on page 276] says the pair $\left(\min_{1 \leq i \leq n} X_i, \sum_{i=1}^n X_i\right)$ gives a sufficient statistic for (μ, σ) .

4. EXERCISE 6.5

The joint distribution is given by

$$\begin{aligned}
 f(x_1, \dots, x_n | \theta) &= \prod_{i=1}^n \chi_{(-i(\theta-1), i(\theta+1))}(x_i) \frac{1}{2i\theta} \\
 &= \prod_{i=1}^n \chi_{(\theta-1, \theta+1)}\left(\frac{x_i}{i}\right) \frac{1}{2i\theta} \\
 &= \chi_{(\theta-1, \theta+1)}\left(\min_{1 \leq i \leq n} \frac{x_i}{i}\right) \cdot \chi_{(\theta-1, \theta+1)}\left(\max_{1 \leq i \leq n} \frac{x_i}{i}\right) \cdot \frac{1}{(2\theta)^n n!}.
 \end{aligned}$$

[BC01, Factorization Theorem 6.2.6 on page 276] says the pair $\left(\min_{1 \leq i \leq n} \frac{X_i}{i}, \sum_{i=1}^n \frac{X_i}{i}\right)$ gives a sufficient statistic for θ .

5. EXERCISE 6.6

The joint distribution is given by

$$\begin{aligned} f_{X_1, \dots, X_n}(x_1, \dots, x_n | \alpha, \beta) &= \prod_{i=1}^n \frac{x_i^{\alpha-1} e^{-\beta x_i} \beta^\alpha}{\Gamma(\alpha)} \\ &= \left(\frac{\beta^\alpha}{\Gamma(\alpha)} \right)^n \cdot \left(\prod_{i=1}^n x_i \right)^{\alpha-1} \cdot e^{-\beta \sum_{i=1}^n x_i}. \end{aligned}$$

[BC01, Factorization Theorem 6.2.6 on page 276] says the pair $\left(\prod_{i=1}^n X_i, \sum_{i=1}^n X_i \right)$ gives a sufficient statistic for (α, β) .

6. EXERCISE 6.7

The joint distribution is given by

$$\begin{aligned}
 f_{X_1, Y_1, \dots, X_n, Y_n}(x_1, y_1, \dots, x_n, y_n | \theta_1, \theta_2, \theta_3, \theta_4) &= \prod_{i=1}^n \frac{1}{(\theta_4 - \theta_2)(\theta_3 - \theta_1)} \cdot \chi_{(\theta_1, \theta_3)}(x_i) \cdot \chi_{(\theta_2, \theta_4)}(y_i) \\
 &= \frac{1}{(\theta_4 - \theta_2)^n (\theta_3 - \theta_1)^n} \cdot \chi_{(\theta_1, \theta_3)}\left(\min_{1 \leq i \leq n} x_i\right) \cdot \chi_{(\theta_1, \theta_3)}\left(\max_{1 \leq i \leq n} x_i\right) \cdot \\
 &\quad \chi_{(\theta_2, \theta_4)}\left(\min_{1 \leq i \leq n} y_i\right) \cdot \chi_{(\theta_2, \theta_4)}\left(\max_{1 \leq i \leq n} y_i\right)
 \end{aligned}$$

[BC01, Factorization Theorem 6.2.6 on page 276] says the tuple $\left(\min_{1 \leq i \leq n} X_i, \max_{1 \leq i \leq n} X_i, \min_{1 \leq i \leq n} Y_i, \max_{1 \leq i \leq n} Y_i\right)$ gives a sufficient statistic for $(\theta_1, \theta_2, \theta_3, \theta_4)$.

7. EXERCISE 6.8

We want to show the ordered statistic $(X_{(1)}, \dots, X_{(n)})$ is a minimal sufficient statistic for θ . The joint distribution is given by

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n | \theta) = \prod_{i=1}^n f(x_i - \theta).$$

If the tuples $(x_i)_{i=1}^n, (y_i)_{i=1}^n$ are two sample points, such that $T((x_i)_{i=1}^n) = T((y_i)_{i=1}^n)$, then the two sets

$$\{x_i\}_{i=1}^n = \{y_i\}_{i=1}^n$$

are the same. Hence,

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n | \theta) = f_{X_1, \dots, X_n}(y_1, \dots, y_n | \theta).$$

Therefore, [BC01, Theorem 6.2.13 on page 281] says the ordered statistic is minimal sufficient for θ .

8. EXERCISE 6.9

Refer to [BC01, Theorem 6.2.13 on page 281].

(a) The joint distribution is given by

$$\begin{aligned} f_{X_1, \dots, X_n}(x_1, \dots, x_n | \theta) &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i - \theta)^2} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \exp \left[-\frac{1}{2} \sum_{i=1}^n (x_i - \theta)^2 \right]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{f_{X_1, \dots, X_n}(x_1, \dots, x_n | \theta)}{f_{X_1, \dots, X_n}(y_1, \dots, y_n | \theta)} &= \exp \left[-\frac{1}{2} \sum_{i=1}^n (x_i - \theta)^2 - (y_i - \theta)^2 \right] \\ &= \exp \left[-\frac{1}{2} \sum_{i=1}^n x_i^2 - y_i^2 - 2\theta(y_i - x_i) \right] \\ &= \exp \left[-\frac{1}{2} \left(\sum_{i=1}^n x_i^2 - y_i^2 \right) + n\theta(\bar{y} - \bar{x}) \right], \end{aligned}$$

which is independent of θ if $\bar{x} = \bar{y}$. Therefore, $T(X) = \bar{X}$ is a minimal sufficient statistic for θ .

(b) The joint distribution is given by

$$\begin{aligned} f_{X_1, \dots, X_n}(x_1, \dots, x_n | \theta) &= \prod_{i=1}^n \chi_{(\theta, \infty)}(x_i) \cdot e^{-(x_i - \theta)} \\ &= \chi_{(\theta, \infty)} \left(\min_{1 \leq i \leq n} x_i \right) \cdot e^{-n\bar{x}} \cdot e^{n\theta}. \end{aligned}$$

Therefore, we have

$$\frac{f_{X_1, \dots, X_n}(x_1, \dots, x_n | \theta)}{f_{X_1, \dots, X_n}(y_1, \dots, y_n | \theta)} = \frac{\chi_{(\theta, \infty)} \left(\min_{1 \leq i \leq n} x_i \right) \cdot e^{-n\bar{x}}}{\chi_{(\theta, \infty)} \left(\min_{1 \leq i \leq n} y_i \right) \cdot e^{-n\bar{y}}},$$

which is independent of θ if $\min_{1 \leq i \leq n} x_i = \min_{1 \leq i \leq n} y_i$. Therefore, $T(X) = \min_{1 \leq i \leq n} X_i$ is a minimal sufficient statistic for θ .

(c) The joint distribution is given by

$$\begin{aligned}
f_{X_1, \dots, X_n}(x_1, \dots, x_n | \theta) &= \prod_{i=1}^n \frac{e^{-(x_i - \theta)}}{(1 + e^{-(x_i - \theta)})^2} \\
&= \frac{e^{-n\bar{x}}}{\left[\prod_{i=1}^n (1 + e^{-(x_i - \theta)}) \right]^2}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\frac{f_{X_1, \dots, X_n}(x_1, \dots, x_n | \theta)}{f_{X_1, \dots, X_n}(y_1, \dots, y_n | \theta)} &= \frac{e^{n\bar{y}}}{e^{n\bar{x}}} \cdot \left[\prod_{i=1}^n \frac{1 + e^{-(y_i - \theta)}}{1 + e^{-(x_i - \theta)}} \right]^2 \\
&= \frac{e^{n\bar{y}}}{e^{n\bar{x}}} \cdot \left[\prod_{i=1}^n \frac{1 + e^{-(y_{(i)} - \theta)}}{1 + e^{-(x_{(i)} - \theta)}} \right]^2
\end{aligned}$$

which is independent of θ if the ordered statistics of x and y agree. Therefore, the ordered statistic is a minimal sufficient statistic for θ .

- (d) A computation similar to part (c) shows the ordered statistic is a minimal sufficient statistic for θ .
- (e) The joint distribution is given by

$$\begin{aligned}
f_{X_1, \dots, X_n}(x_1, \dots, x_n | \theta) &= \prod_{i=1}^n \frac{1}{2} e^{-|x_i - \theta|} \\
&= \frac{1}{2^n} e^{-\sum_{i=1}^n |x_i - \theta|}
\end{aligned}$$

Therefore, the ordered statistic gives a minimal sufficient statistic for θ .

REFERENCES

- [BC01] Roger Berger and George Casella. *Statistical Inference*. 2nd edition. Florence, AL: Duxbury Press, June 2001.