CASELLA-BERGER STATISTICAL INFERENCE SOLUTION: CHAPTER 5

VIRGIL CHAN

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Contents

1. Exercise 5.1

Let X_n be the number of blind people in a sample of size n. Then $X \sim \text{Binomial}\,(n, 0.01)$. As a result,

$$\begin{aligned} 0.95 &\leq P(\text{the sample contains a blind people}) \\ &= P(X > 0) \\ &= 1 - P(X = 0) \\ &= 1 - 0.99^n, \end{aligned}$$

or equivalently,

$$\begin{aligned} n &\geq \frac{\log(0.05)}{\log(0.99)} \\ &\approx 298.073. \end{aligned}$$

We need a sample size of at least 299 people.

2. Exercise 5.2

Let K be the number of years until X_1 is exceeded for the first time.

(a)

$$\begin{aligned}
P(K=k) &= P(X_k > X_1,\, X_i \leq X_1 \text{ for all } 1 < i < k) \\
&= \int_{\mathbb{R}} P(X_1 = x,\, X_k > x,\, X_i \leq x \text{ for all } 1 < i < k)\; dx \\
&= \int_{\mathbb{R}} P(\, X_k > x,\, X_i \leq x \text{ for all } 1 < i < k | X_1 = x\,) \cdot P(X_1 = x)\; dx \\
&= \int_{\mathbb{R}} P(X_k > x) \cdot \prod_{i=2}^{k-1} P(X_i \leq x) \cdot P(X_1 = x)\; dx \\
&= \int_{\mathbb{R}} [1 - F(x)] \cdot \prod_{i=2}^{k-1} F(x) \cdot f(x)\; dx \\
&= \int_{\mathbb{R}} F(x)^{k-2} f(x)\; dx - \int_{\mathbb{R}} F(x)^{k-1} f(x)\; dx \\
&= \left. \frac{F(x)^{k-1}}{k-1} \right|_{x=-\infty}^{x=\infty} - \left. \frac{F(x)^{k}}{k} \right|_{x=-\infty}^{x=\infty} \\
&= \frac{1}{k-1} - \frac{1}{k} \\
&= \frac{1}{k(k-1)}
\end{aligned}$$

(b)

$$\begin{aligned}
ET &= \sum_k k \cdot \frac{1}{k(k-1)} \\
&= \sum_k \frac{1}{k-1} \\
&= \infty
\end{aligned}$$

3. Exercise 5.3

In other words,

$$Y_i \sim \text{Bernoulli}\,(p)\,,$$

where $p = P(X_i > \mu)$. The probabilities are the same for all i because of the iid assumption on X_i's. This iid assumption, together with the definition of Y_i, imply that Y_i's are independent as well. Therefore,

$$\begin{aligned}\sum_{i=1}^{n} Y_i &\sim \sum_{i=1}^{n} \text{Bernoulli}\,(p) \\ &\sim \text{Binomial}\,(n, p)\end{aligned}$$

4. Exercise 5.4

(a) Let σ be a permutation on k objects for $k \le n$. Then

$$\begin{aligned}
P\big(X_1 = x_{\sigma(1)}, \cdots, X_k = x_{\sigma(k)}\big) &= \int_0^1 P\big(X_1 = x_{\sigma(1)}, \cdots, X_k = x_{\sigma(k)} \big| P = p\big) \cdot f_P(p)\ dp \\
&= \int_0^1 \prod_{i=1}^{k} P\big(X_i = x_{\sigma(i)} \big| P = p\big) \cdot 1\ dp \\
&\quad (X_i|P\text{'s are iid}) \\
&= \int_0^1 \prod_{i=1}^{k} p^{x_{\sigma(i)}} (1-p)^{1-x_{\sigma(i)}}\ dp \\
&= \int_0^1 p^{\sum_{i=1}^{k} x_{\sigma(i)}} (1-p)^{k - \sum_{i=1}^{k} x_{\sigma(i)}}\ dp \\
&= \int_0^1 p^{\sum_{i=1}^{k} x_i} (1-p)^{k - \sum_{i=1}^{k} x_i}\ dp \\
&= P(X_1 = x_1, \cdots, X_k = x_k) \\
&= \int_0^1 p^t (1-p)^{k-t}\ dp \\
&\quad \left(\text{where } t = \sum_{i=1}^{k} x_i\right) \\
&= \frac{\Gamma(t+1) \cdot \Gamma(k-t+1)}{\Gamma(k+2)} \\
&= \frac{t!\,(k-t)!}{(k+1)!}
\end{aligned}$$

(b) We compute

$$\begin{aligned}
\prod_{i=1}^{n} P(X_i = x_i) &= \prod_{i=1}^{n} \int_0^1 P(X_i = x_i | P = p) \cdot f_P(p)\ dp \\
&= \prod_{i=1}^{n} \int_0^1 p^{x_i} (1-p)^{1-x_i}\ dp,
\end{aligned}$$

which is not the same as we computed in part (a) for $k = n$.

5. Exercise 5.5

Let $Y = \sum_{i=1}^{n} X_i$, then $Y = n\hat{X}$. Thus

$$\begin{aligned} f_{\hat{X}}(x) &= f_Y\big(x^{-1}(y)\big) \cdot \left|\frac{dx}{dy}\right| \\ &= n \cdot f_Y(nx) \end{aligned}$$

6. Exercise 5.6

The problem statement has a typo: it should be (5.2.9) instead.

(a) Let $Z = X - Y$ and $W = X$. Then

$$\begin{aligned} f_{Z,W}(z,w) &= f_{X,Y}(x(z,w), y(z,w)) \cdot \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} \\ &= f_{X,Y}(w, w-z) \cdot \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} \\ &= f_X(w)\, f_Y(w-z) \end{aligned}$$

As a result,

$$\begin{aligned} f_Z(z) &= \int_{\mathcal{W}} f_{Z,W}(z,w)\; dw \\ &= \int_{\mathcal{W}} f_X(w) \cdot f_Y(w-z)\; dw. \end{aligned}$$

(b) Let $Z = XY$ and $W = X$. Then

$$\begin{aligned} f_{Z,W}(z,w) &= f_{X,Y}(x(z,w), y(z,w)) \cdot \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} \\ &= f_{X,Y}\left(w, \frac{z}{w}\right) \cdot \left|\frac{1}{w}\right| \\ &= \left|\frac{1}{w}\right| f_X(w)\, f_Y\left(\frac{z}{w}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} f_Z(z) &= \int_{\mathcal{W}} f_{Z,W}(z,w)\; dw \\ &= \int_{\mathcal{W}} f_{Z,W}(z,w)\; dw \\ &= \int_{\mathcal{W}} \left|\frac{1}{w}\right| f_X(w)\, f_Y\left(\frac{z}{w}\right)\; dw \end{aligned}$$

(c) Let $Z = X/Y$ and $W = X$. Then

$$
\begin{aligned}
f_Z(z) &= \int_{\mathcal{W}} f_{Z,W}(z,w)\ dw \\
&= \int_{\mathcal{W}} f_{X,Y}(x(z,w)\,, y(z,w)) \cdot \left| \begin{array}{cc} \dfrac{\partial x}{\partial z} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial z} & \dfrac{\partial y}{\partial w} \end{array} \right| dw \\
&= \int_{\mathcal{W}} \left| \frac{w}{z^2} \right| f_{X,Y}\left(w, \frac{w}{z}\right)\ dw \\
&= \int_{\mathcal{W}} \left| \frac{w}{z^2} \right| f_X(w)\, f_Y\left(\frac{w}{z}\right)\ dw
\end{aligned}
$$

7. Exercise 5.7

(a) We have

$$\begin{aligned}\frac{1}{1+(w/\sigma)^2}\cdot\frac{1}{1+((z-w)/\tau)^2} \equiv{}& \frac{Aw}{1+(w/\sigma)^2}+\frac{B}{1+(w/\sigma)^2}\\ &-\frac{Cw}{1+((z-w)/\tau)^2}-\frac{D}{1+((z-w)/\tau)^2}\\ \iff 1 \equiv{}& \left[B\left(1+\frac{z^2}{\tau^2}\right)-D\right]+\left[A\left(1+\frac{z^2}{\tau^2}\right)-B\left(\frac{2z}{\tau^2}\right)-C\right]w\\ &\left[A\left(\frac{-2z}{\tau^2}\right)+B\left(\frac{1}{\tau^2}\right)+D\left(\frac{-1}{\sigma^2}\right)\right]w^2+\left[A\left(\frac{1}{\tau^2}\right)+C\left(\frac{-1}{\sigma^2}\right)\right]w^3.\end{aligned}$$

Comparing coefficients of w gives the following system

$$\begin{cases} A\left(\frac{1}{\tau^2}\right)+C\left(\frac{-1}{\sigma^2}\right) &= 0\\ A\left(\frac{-2z}{\tau^2}\right)+B\left(\frac{1}{\tau^2}\right)+D\left(\frac{-1}{\sigma^2}\right) &= 0\\ A\left(1+\frac{z^2}{\tau^2}\right)-B\left(\frac{2z}{\tau^2}\right)-C &= 0\\ B\left(1+\frac{z^2}{\tau^2}\right)-D &= 1\end{cases}$$

of equations with unknowns A, B, C, D. Solving this gives

$$\begin{cases} A &= \frac{2z\tau^2}{\mathcal{D}}\\ B &= \frac{\tau^2(z^2-\sigma^2+\tau^2)}{\mathcal{D}}\\ C &= \frac{2z\sigma^2}{\mathcal{D}}\\ D &= \frac{\sigma^2(-3z^2-\sigma^2+\tau^2)}{\mathcal{D}}\end{cases}$$

for which

$$\mathcal{D}=(z^2+\sigma^2)^2+2(z-\sigma)(z+\sigma)\tau^2+\tau^4.$$

8. Exercise 5.8

(a) The desired result follows from

$$
\begin{aligned}
2n\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2 &= 2n\sum_{i=1}^{n}\left(X_i-\frac{1}{n}\sum_{j=1}^{n}X_j\right)^2\\
&= 2n\sum_{i=1}^{n}\left[X_i^2-\frac{2}{n}X_i\sum_{j=1}^{n}X_j+\frac{1}{n^2}\left(\sum_{j=1}^{n}X_j\right)^2\right]\\
&= 2n\left[\sum_{i=1}^{n}X_i^2-\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}X_iX_J+\frac{1}{n}\left(\sum_{j=1}^{n}X_j\right)^2\right]\\
&= 2n\left[\sum_{i=1}^{n}X_i^2-\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}X_iX_J\right]\\
&= 2n\sum_{i=1}^{n}X_i^2-2\sum_{i=1}^{n}\sum_{j=1}^{n}X_iX_j\\
&= \sum_{i=1}\sum_{j=1}\left(X_i-X_j\right)^2.
\end{aligned}
$$

(b) We first compute

$$
\begin{aligned}
E\left(S^2\right) &= E\left(\frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(X_i-X_j\right)^2\right)\\
&= \frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left(X_i-X_j\right)^2\\
&= \frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left(X_i-\theta_1-X_j+\theta_1\right)^2\\
&= \frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left(X_i-\theta_1\right)^2-2E\left(\left(X_i-\theta_1\right)\left(X_j-\theta_1\right)\right)+E\left(X_j-\theta_1\right)^2\\
&= \frac{1}{2n(n-1)}\sum_{i\neq j}E\left(X_i-\theta_1\right)^2+E\left(X_j-\theta_1\right)^2\\
&= \frac{1}{2n(n-1)}\cdot n(n-1)\cdot 2\theta_2\\
&= \theta_2.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
E\big(S^4\big) &= \left[\frac{1}{2n(n-1)}\right]^2 \sum_{i,j,k,\ell=1}^{n} E\big((X_i - X_j)^2 (X_k - X_\ell)^2\big) \\
&= \left[\frac{1}{2n(n-1)}\right]^2 \sum_{i,j,k,\ell=1}^{n} E\big((X_i - \theta_1 - X_j + \theta_1)^2 (X_k - \theta_1 - X_\ell + \theta_1)^2\big) \\
&= \left[\frac{1}{2n(n-1)}\right]^2 \cdot \Bigg[\sum_{i,j,k,\ell=1}^{n} E\big((X_i-\theta_1)^2(X_j-\theta_1)^2\big) + E\big((X_i-\theta_1)^2(X_\ell-\theta_1)^2\big) \\
&\qquad + E\big((X_j-\theta_1)^2(X_k-\theta_1)^2\big) + E\big((X_j-\theta_1)^2(X_\ell-\theta_1)^2\big) \\
&\qquad - 2\sum_{i,j,k,\ell=1}^{n} E\big((X_i-\theta_1)^2(X_k-\theta_1)(X_\ell-\theta_1)\big) + E\big((X_j-\theta_1)^2(X_k-\theta_1)(X_\ell-\theta_1)\big) \\
&\qquad + E\big((X_k-\theta_1)^2(X_i-\theta_1)(X_j-\theta_1)\big) + E\big((X_\ell-\theta_1)^2(X_i-\theta_1)(X_j-\theta_1)\big) \\
&\qquad + 4\sum_{i,j,k,\ell=1}^{n} E((X_i-\theta_1)(X_j-\theta_1)(X_k-\theta_1)(X_\ell-\theta_1))\Bigg] \\
&= \left[\frac{1}{2n(n-1)}\right]^2 \cdot \Bigg[4\sum_{\substack{i\neq j\\ k\neq \ell\\ i=k}} E(X_i-\theta_1)^4 + 4\sum_{\substack{i\neq j\\ k\neq \ell\\ i\neq k}} E(X_i-\theta_1)^2(X_j-\theta_1)^2 \\
&\qquad - 0 \\
&\qquad + 8\sum_{\substack{i\neq j,\\ k\neq \ell\\ (i,j)=(k,\ell)}} E(X_i-\theta_1)^2(X_j-\theta_1)^2\Bigg] \\
&= \left[\frac{1}{2n(n-1)}\right]^2 \cdot \big[4n(n-1)^2\theta_4 + 4n(n-1)^3\theta_2^2 + 8n(n-1)\theta_2^2\big] \\
&= \frac{1}{n}\theta_4 + \frac{n^2-2n+3}{n(n-1)}\theta_2^2.
\end{aligned}
$$

Therefore,

$$\begin{aligned}\operatorname{Var}\left(S^2\right) &= ES^4 - (ES^2)^2 \\ &= \frac{1}{n}\theta_4 + \frac{n^2-2n+3}{n(n-1)}\theta_2^2 - \theta_2^2 \\ &= \frac{1}{n}\left(\theta_4 + \frac{n-3}{n-1}\theta_2^2\right)\end{aligned}$$

(c) First assume $\theta_1 = 0$. Then

$$\begin{aligned}\operatorname{Cov}\left(\overline{X}, S^2\right) &= E\big(\overline{X}S^2\big) - E\big(\overline{X}\big)\,E\big(S^2\big) \\ &= E\big(\overline{X}S^2\big) - \theta_1 E\big(S^2\big) \\ &= E\big(\overline{X}S^2\big) \\ &= E\left(\left(\frac{1}{n}\sum_{i=1}^n X_i\right)\cdot\left(\frac{1}{2n(n-1)}\sum_{i=1}^n\sum_{j=1}^n (X_i - X_j)^2\right)\right) \\ &= \frac{1}{2n^2(n-1)}\sum_{i,j,k=1}^n E\big(X_i(X_j - X_k)^2\big) \\ &= \frac{1}{2n^2(n-1)}\sum_{i,j,k=1}^n E\big(X_iX_j^2\big) - 2E(X_iX_jX_k) + E\big(X_iX_k^2\big) \\ &= \frac{1}{2n^2(n-1)}\Big[n^2E\big(X_i^3\big) + n^2(n-1)E\big(X_iX_j^2\big) \\ &\qquad - 2n(n-1)(n-2)E(X_iX_jX_k) + 3n(n-1)E\big(X_iX_j^2\big) + nE\big(X_i^3\big) \\ &\qquad n^2E\big(X_i^3\big) + n^2(n-1)E\big(X_iX_k^2\big)\Big] \\ &= \frac{1}{2n^2(n-1)}\Big[2n(n-1)E\big(X_i^3\big) \\ &\qquad +2n(n-1)(n-3)E\big(X_iX_j^2\big) - 2n(n-1)(n-2)E(X_iX_jX_k)\Big] \\ &= \frac{1}{2n^2(n-1)}\cdot 2n(n-1)E\big(X_i^3\big) \\ &= \frac{\theta_3}{n}.\end{aligned}$$

For general θ_1, we observe that

$$\begin{aligned}\operatorname{Cov}(X,Y) &= E((X-EX)(Y-EY)) \\ &= E((X-EX-0)(Y-EY)) \\ &= E((X-EX-E(X-EX))(Y-EY)) \\ &= \operatorname{Cov}(X-EX, Y).\end{aligned}$$

In other words, by replacing X_i by $X_i - \theta_1$, we return to the case $\theta_1 = 0$. Moreover, this replacement preserves covariance. Therefore, $\mathrm{Cov}\left(\overline{X}, S^2\right) = \dfrac{\theta_3}{n}$, and it vanishes iff θ_1 vanishes.

9. Exercise 5.10

See [BC01, Theorem 5.3.1 on page 218].

$$\begin{aligned}\theta_1 &= EX_i \\ &= \mu,\end{aligned}$$

$$\begin{aligned}\theta_2 &= ES^2 && \text{(Exercise 5.8 (b))} \\ &= E\left(\frac{\sigma^2}{n-1}\chi^2_{n-1}\right) \\ &= \sigma^2,\end{aligned}$$

$$\begin{aligned}\theta_3 &= n\operatorname{Cov}\left(\overline{X}, S^2\right) && \text{(Exercise 5.8 (c))} \\ &= 0,\end{aligned}$$

$$\begin{aligned}\theta_4 &= n\operatorname{Var}\left(S^2\right) + \frac{n-3}{n-1}\theta_2^2 && \text{(Exercise 5.8 (b))} \\ &= n\operatorname{Var}\left(\frac{\sigma^2}{n-1}\chi^2_{n-1}\right) + \frac{n-3}{n-1}\theta_2^2 \\ &= \frac{2n\sigma^4}{n-1} + \frac{(n-3)\sigma^4}{n-1} \\ &= 3\sigma^4.\end{aligned}$$

10. Exercise 5.12

Firstly, we have

$$\begin{aligned}
E\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right|\right) &= E\left(\left|\overline{X}\right|\right) \\
&= E\left(\left|\text{Normal}\left(0, \frac{1}{n}\right)\right|\right) \\
&\quad \text{([BC01, Theorem 5.3.1 on page 218])} \\
&= \int_{-\infty}^{\infty} |x| \cdot \frac{n}{\sqrt{2\pi}} e^{-\frac{1}{2}(nx)^2}\ dx \\
&= \int_{0}^{\infty} x \cdot \frac{n}{\sqrt{2\pi}} e^{-\frac{1}{2}(nx)^2}\ dx - \int_{-\infty}^{0} x \cdot \frac{n}{\sqrt{2\pi}} e^{-\frac{1}{2}(nx)^2}\ dx \\
&= \frac{1}{\sqrt{2\pi}n} + \frac{1}{\sqrt{2\pi}n} \\
&= \frac{\sqrt{\frac{2}{\pi}}}{n}.
\end{aligned}$$

Secondly, we have

$$\begin{aligned}
E\left(\frac{1}{n}\sum_{i=1}^{n} |X_i|\right) &= \frac{1}{n}\sum_{i=1}^{n} E(|X_i|) \\
&= E(|X_i|) \\
&= E(|\text{Normal}\,(0,1)|) \\
&= \sqrt{\frac{2}{\pi}}
\end{aligned}$$

11. Exercise 5.24

In other words, $X \sim \text{Uniform}\,(0, \theta)$. Hence, it has cdf

$$F_X(x) = \frac{x}{\theta}.$$

[BC01, Theorem 5.4.6 page 230] then gives us the joint distribution

$$\begin{aligned} f_{X_{(1)},X_{(n)}}(u,v) &= \frac{n!}{(n-2)!} f_X(u)\, f_X(v)\, [F_X(v) - F_X(u)]^{n-2} \\ &= \frac{n!}{(n-2)!} \frac{(v-u)^{n-2}}{\theta^n}. \end{aligned} \tag{11.0.1}$$

Now, let $Z = \dfrac{X_{(1)}}{X_{(n)}}$ and $W = X_{(n)}$. Then

$$\begin{aligned} f_{Z,W}(z,w) &= f_{X_{(1)},X_{(n)}}(x(z,w), y(z,w)) \cdot |J| \\ &= f_{X_{(1)},X_{(n)}}(wz, w) \cdot \begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix} \\ &= \frac{n!}{(n-2)!} \frac{w^{n-1}(1-z)^{n-2}}{\theta^n}. \end{aligned}$$

Since the variables can be separated in the joint distribution, Z, W are independent.

12. Exercise 5.25

From [BC01, page 230], the joint distribution of all the order statistics is given by

$$f_{X_{(1)},\cdots,X_{(1)}}(x_1 < \cdots < x_n) = n! \prod_{i=1}^{n} f_X(x_i) = \frac{a^n n!}{\theta^{an}} \left(\prod_{i=1}^{n} x_i \right)^{a-1}. \tag{12.0.1}$$

Define the new variables

$$Y_i = \begin{cases} \dfrac{X_{(i)}}{X_{(i+1)}} & \text{if } i < n, \\ X_{(n)} & \text{if } i = n. \end{cases}$$

We then compute the joint distribution

$$\begin{aligned} f_{Y_1,\cdots,Y_n}(y_1,\cdots,y_n) &= f_{X_{(1)},\cdots,X_{(n)}}(x_1(y_1,\cdots,y_n),\cdots,x_n(y_1,\cdots,y_n)) \cdot |J| \\ &= f_{X_{(1)},\cdots,X_{(n)}}\left(\prod_{k=1}^{n} y_k, \cdots, \prod_{k=i}^{n} y_k, \cdots, y_n \right) \cdot \left(\prod_{i=1}^{n-1} \prod_{j=i+1}^{n} y_j \right) \\ &= \left(\frac{a^n n!}{\theta^{an}} \prod_{i=1}^{n} y_i^{i(a-1)} \right) \left(\prod_{i=2}^{n} y_i^{i-1} \right) \\ &= \frac{a^n n!}{\theta^{an}} \prod_{i=1}^{n} y_i^{ia-1}. \end{aligned}$$

This proves the independency.

13. Exercise 5.27

(a) By [BC01, Theorem 5.4.4 on page 229] and Theorem 5.4.6 on page 230, loc. cit., we have

$$\begin{aligned}
f_{i|j}(u|v) &= \frac{f_{X_{(i)},X_{(j)}}(u,v)}{f_{X_{(j)}}(v)} \\
&= \frac{\frac{n! f_X(u) f_X(v) [F_X(u)]^{i-1} [F_X(v)-F_X(u)]^{j-1-i} [1-F_X(v)]^{n-j}}{(i-1)!(j-1-i)!(n-j)!}}{\frac{n! f_X(v) [F_X(v)]^{j-1} [1-F_X(v)]^{n-j}}{(j-1)!(n-j)!}} \\
&= \frac{(j-1)!\, f_X(u)\, [F_X(u)]^{i-1}\, [F_X(v)-F_X(u)]^{j-1-i}}{(i-1)!\,(j-1-i)!\,[F_X(v)]^{j-1}}
\end{aligned}$$

(b) $f_{V|R}(v|r) = \dfrac{1}{a-r}$.

References

[BC01] Roger Berger and George Casella. *Statistical Inference*. 2nd edition. Florence, AL: Duxbury Press, June 2001.