CASELLA-BERGER
STATISTICAL INFERENCE SOLUTION:
CHAPTER 7

VIRGIL CHAN

September 26, 2022

Contents

1. Exercise 7.1

For each x, we find θ so that $f(x|\theta)$ is maximised.

x	0	1	2	3	4
θ	1	1	Either 2 or 3 works	3	3
$f(x\|\theta)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Therefore,

$$\hat{\theta}_1(x) = \begin{cases} 1 & \text{if } x = 0, 1, \\ 2 & \text{if } x = 2, \\ 3 & \text{if } x = 3, 4 \end{cases} \qquad \hat{\theta}_2(x) = \begin{cases} 1 & \text{if } x = 0, 1, \\ 3 & \text{if } x = 2, 3, 4, \end{cases}$$

are the possible MLEs' for θ.

2. Exercise 7.2

The likelihood function for β is given by

$$\begin{aligned} L(\beta|x_1,\cdots,x_n) &= \prod_{i=1}^{n} f(x_i;\,\alpha,\beta) \\ &= \prod_{i=1}^{n} \frac{x_i^{\alpha-1}e^{-\frac{x_i}{\beta}}}{\Gamma(\alpha)\,\beta^{\alpha}} \\ &= \frac{\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}}{[\Gamma(\alpha)]^n}\cdot\exp\left[-\frac{\sum_{i=1}^{n} x_i}{\beta}\right]\cdot\beta^{\alpha n}, \end{aligned}$$

and has derivative

$$\frac{\partial L}{\partial\beta} = \frac{\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}}{[\Gamma(\alpha)]^n}\cdot\exp\left[-\frac{\sum_{i=1}^{n} x_i}{\beta}\right]\cdot\beta^{-\alpha n-2}\cdot\left(\sum_{i=1}^{n} x_i-\alpha n\beta\right).$$

In particular,

$$\begin{aligned} \frac{\partial L}{\partial\beta}=0 &\iff \sum_{i=1}^{n} x_i-\alpha n\beta=0 \\ &\iff \beta=\frac{\sum_{i=1}^{n} x_i}{\alpha n}. \end{aligned}$$

It remains to show this β maximises L.

We note that the sign of $\dfrac{\partial L}{\partial\beta}$ is completely determined by the sign of the term $\sum_{i=1}^{n} x_i-\alpha n\beta$. Furthermore,

$$\begin{aligned} \left.\frac{\partial L}{\partial\beta}\right|_{\beta=\hat\beta+\varepsilon} &= -\alpha n\varepsilon \\ &= \begin{cases} >0 & \text{if } \varepsilon<0, \\ <0 & \text{if } \varepsilon>0. \end{cases} \end{aligned}$$

Thus, the First Derivative Test says $\hat\beta=\dfrac{\sum_{i=1}^{n} x_i}{\alpha n}$ is the MLE for β.

3. Exercise 7.3

Pre-calculus.

4. Exercise 7.6

(a) The joint distribution is given by

$$\begin{aligned} f_{X_1,\cdots,X_n}(x_1,\cdots,x_n|\theta) &= \prod_{i=1}^{n} \chi_{[\theta,\infty)}(x_i)\cdot\frac{\theta}{x_i^2} \\ &= \frac{\chi_{[\theta,\infty)}\left(\min\limits_{1\le i\le n} x_i\right)}{\left(\prod\limits_{i=1}^{n} x_i\right)^2}\cdot\theta^n. \end{aligned}$$

[BC01, Factorization Theorem 6.2.6 on page 276] says the tuple $\left(\min\limits_{1\le i\le n} X_i, \prod\limits_{i=1}^{n} X_i\right)$ is a sufficient statistic for θ.

(b) Up to a positive multiplier, the likelihood function $L(\theta|x_1,\cdots,x_n)$ is just the monomial θ^n. But θ^n has no maximum on the interval $(0,\infty)$. However, there is also the characteristic function $\chi_{[\theta,\infty)}\left(\min\limits_{1\le i\le n} x_i\right)$, which restricts θ inside the interval $\left(0, \min\limits_{1\le i\le n} x_i\right]$. We summarise this discussion as the following optimisation problem

$$\max_{\theta\in\left(0,\ \min\limits_{1\le i\le n} x_i\right]}\left[\frac{\chi_{[\theta,\infty)}\left(\min\limits_{1\le i\le n} x_i\right)}{\left(\prod\limits_{i=1}^{n} x_i\right)^2}\cdot\theta^n\right],$$

and its solution is $\hat{\theta} = \min\limits_{1\le i\le n} x_i$.

(c) We compute the moments

$$\begin{aligned} E_\theta[X^n] &= \int_0^\infty x^n\cdot f(x|\theta)\,dx \\ &= \int_0^\infty \theta x^{n-2}\,dx \\ &= \infty && (\text{since } n\ge 1)\,. \end{aligned}$$

Therefore, the desired estimator does not exist.

5. Exercise 7.7

The likelihood function is given by

$$\begin{aligned}
L(\theta|x_1,\cdots,x_n) &= \prod_{i=1}^n f(x_i|\theta) \\
&= \begin{cases} \displaystyle\prod_{i=1}^n 1 & \text{if } \theta = 0 \text{ and } 0 < x < 1, \\ \displaystyle\prod_{i=1}^n \frac{1}{2\sqrt{x_i}} & \text{if } \theta = 1 \text{ and } 0 < x < 1, \\ 0 & \text{if else.} \end{cases} \\
&= \begin{cases} 1 & \text{if } \theta = 0 \text{ and } 0 < x < 1, \\ \displaystyle\prod_{i=1}^n \frac{1}{2\sqrt{x_i}} & \text{if } \theta = 1 \text{ and } 0 < x < 1, \\ 0 & \text{if else.} \end{cases}
\end{aligned}$$

Therefore, the MLE is

$$\hat{\theta}(x_1,\cdots,x_n) = \begin{cases} 0 & \text{if } \displaystyle\prod_{i=1}^n \frac{1}{2\sqrt{x_i}} \leq 1, \\ 1 & \text{if else.} \end{cases}$$

References

[BC01] Roger Berger and George Casella. *Statistical Inference*. 2nd edition. Florence, AL: Duxbury Press, June 2001.