CS310 Lecture 14
Introduction to Turing Machines

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28 September 2022

1 Recap

- We have seen FSA and PDA, and their limitations. For example, $\{a^n b^n \mid n \geq 1\}$ and $\{a^n b^n c^n \mid n \geq 1\}$ cannot be expressed using FSA and PDA, respectively.
- Deterministic PDA (DPDA) accept all regular languages, but only a proper subset of CFLs. For instance, $\{ww^R \mid w \in \{0,1\}^*\}$ has no DPDA, but is context-free. In this sense, non-deterministic PDA are more powerful than DPDA.
- PDA with two stacks are as powerful as Turing Machines. For example, $\{a^n b^n c^n \mid n \geq 1\}$ can be expressed as follows: Push a's and b's in separate stacks (ensuring that b's follow a's). For every c, pop from each stack. Accept iff both stacks are empty and the entire input string is consumed. Other data structures, such as queues, can also be used.

2 Turing Machines

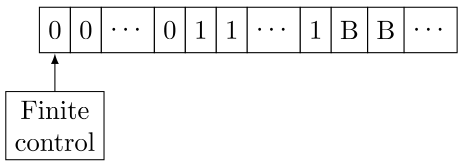
- Formal notation: 7-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$
  - Q: Finite set of states.
  - Σ: Finite set of input symbols.
  - Γ: Complete set of tape symbols. That is, Σ, B, and other tape symbols.
  - δ: Transition function ($Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$). This is a partial mapping. $\delta(q, X) = (p, Y, D)$ where q, X are current state and tape symbol, p, Y, D are next state, replacement symbol, and direction, respectively.
  - q_0: Start state.
  - B: Blank symbol. $B \in \Gamma, B \notin \Sigma$.
  - F: Set of final or accepting states.

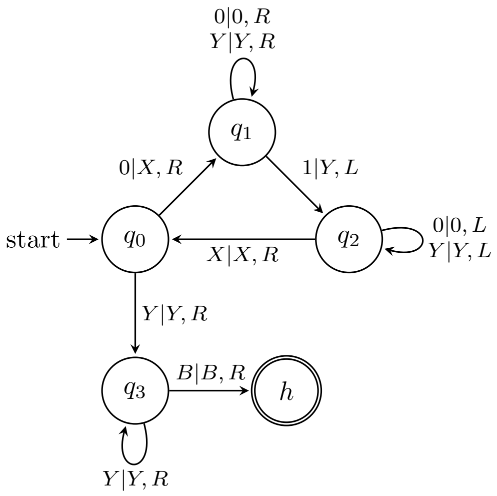
  The tape is divided into *cells*, each holding an element of Γ. The tape can be doubly-infinite, or bounded at one end (this does not change the computational power). The *tape head* is always positioned at one of the tape cells.

- Instantaneous Description (ID): At any point, the TM can be represented as $X_1 X_2 \cdots X_{i-1} q X_i X_{i+1} \cdots X_n$, where
  - q is the current state.
  - Tape head points to X_i.
  - $X_1 \cdots X_n$ is the portion of the tape between the leftmost and rightmost blank cells. If the head points to a blank cell, then some prefix or suffix of $X_1 \cdots X_n$ will be blank.

  ID's are used to describe moves of the TM. If $\delta(q, X_i) = (p, Y, L)$, then $X_1 \cdots X_{i-1} q X_{i+1} \cdots X_n \vdash X_1 \cdots X_{i-2} p X_{i-1} Y X_{i+1} \cdots X_n$. If $i = 1$, we get $pBYX_2 \cdots X_n$. If $i = n$ and $Y = B$, we get $X_1 \cdots X_{n-2} p X_{n-1}$. $\vdash^*$ is used for zero or more moves.

- Example: $L = \{0^n1^n \mid n \geq 1\}$





  - Formally, the TM is $M = (\{q_0, q_1, q_2, q_3, h\}, \{0, 1\}, \{0, 1, X, Y, B\}, \delta, q_0, B, \{h\})$. δ is given by the state transitions, and h is the *halt state*. A left-bound is assumed on the tape. As mentioned earlier, this has no effect on the computational power.
  - Starting at the left end, the loop $q_0 - q_1 - q_2 - q_0$ changes a 0 to X, and moves right over 0's and Y's. Then, it changes a corresponding 1 to Y, and moves left over Y's and 0's. In this process, if some other symbol is encountered, the machine 'crashes'. After finding an X, it checks if a 0 is present at its immediate right. If present, the entire loop repeats. Else, M moves to q_3 (if the symbol is a Y). Then, after moving right over several Y's, it accepts on a B.
  - Note: We cannot merge q_0 and q_3. The resulting TM would accept strings such as $0101 \notin L$ too.

3 Computing using a Turing Machine

Suppose we wish to compute an n-ary function on the Turing machine. Then, the arguments (which are assumed to be integers) are provided on the tape in unary format. The unary alphabet is $\{0\}$ and we seperate the arguments using 1. The function's return value is written at the beginning of the tape when it finishes execution.
Example: Consider the *monus function*, defined as $f(x_1, x_2) = \max(0, x_1 - x_2)$. The following machine will compute the monus function.

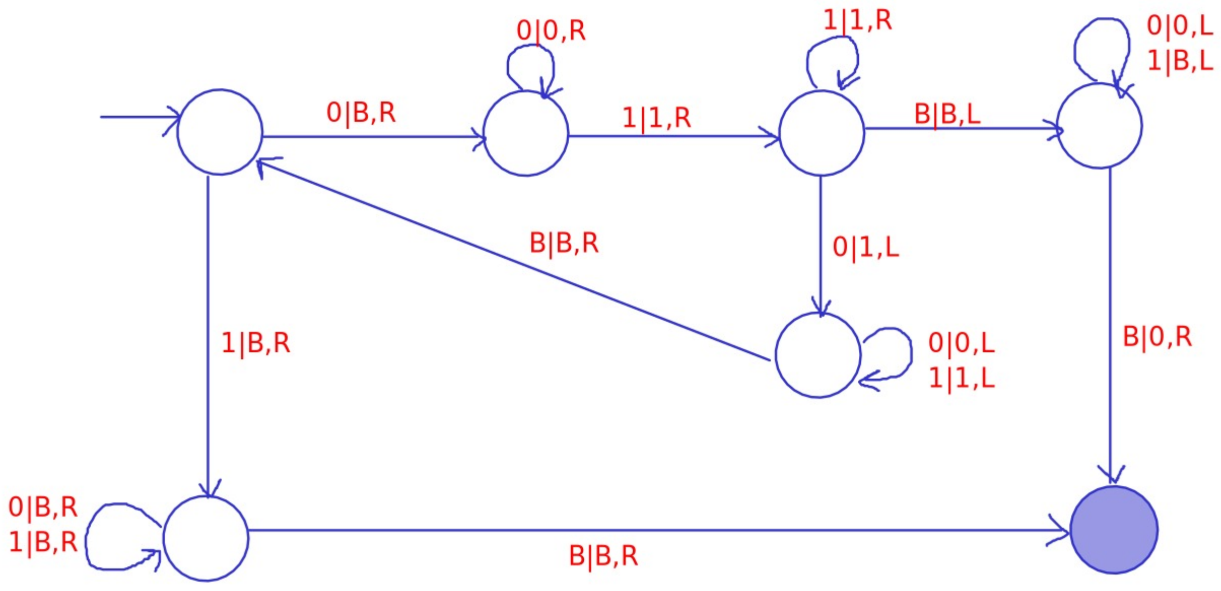


Figure 1: monus function

The Turing machine cancels out the 0's from the arguments one by one, till one of them runs out. Then it does some tidying up to get the output in the desired format.
Example: Consider a Turing machine that performs addition on positive numbers. To do this, we must simply get rid of the 1 that seperates the two arguments so that the 0s become contiguous.

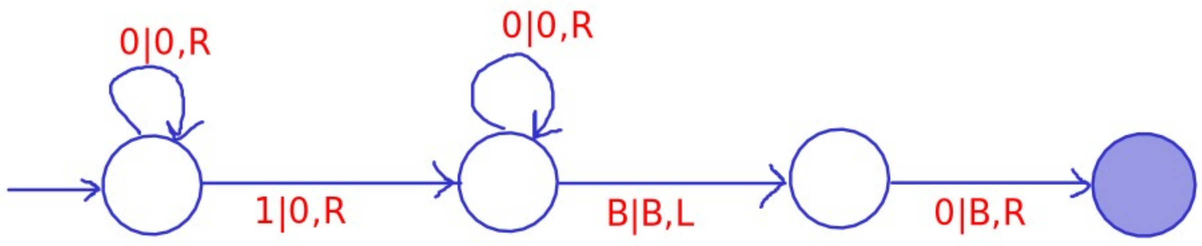


Figure 2: unary adder

4 References

- Section 8.2 of Hopcroft, Motwani, Ullman.