IC 242 – Assignment (30 marks)

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Instructions:

- You may work in groups of two
- The work must be submitted in the form of a report (in PDF format), with the computer code(s) and plotting script(s) included in the appendix. Each group should submit only one report. The report should be named in the following format: *group_xx.pdf*, where *xx* is the two-digit group number that will be allotted to your group in due time.
- The report must be written in the IEEE format (`https://www.ieee.org/conferences/publishing/templates.html`). It should have the following sections: title (including author names and roll numbers), abstract, introduction, method, results and discussions, conclusion, and references. All work, including equations, must be typed; handwritten reports will not be accepted. The report should include answers to all the questions posed below, at appropriate sections.
- Each graph should be accompanied with appropriate discussion
- You will also be required to submit the computer code(s) and script(s) separately for evaluation, details of which are available on Moodle.

Continuum mechanics typically involves the prediction of the displacements (or velocities) for a continuum body under a specified external stress field. In the present case you are dealing with a fluid flow that is irrotational everywhere, i.e. the vorticity tensor is identically zero ($W_{ij} = 0$) for the flow (ideal fluids, which are inviscid, show the irrotational behaviour, e.g. the flow past an airfoil). Moreover, the fluid under consideration is incompressible, i.e. the rate of change of an infinitesimal volume in the fluid is identically zero.

1. Using the irrotational condition of the flow and the incompressible nature of the fluid *show* that the governing equation for the flow is the Laplace's equation

$$\nabla^2 \phi = 0, \tag{1}$$

   where ϕ is the scalar velocity potential, i.e. the velocity field $\mathbf{v} = \nabla \phi$.

2. The aim of this assignment is to solve for the velocity field for flow in a 2D rectangular channel, with specified boundary conditions (BCs). The rectangular domain is shown in Fig. 1, with the flow inlet and outlet shown using blue arrows.

The flow field at the inlet and outlet boundaries can be assumed to be aligned with the x_1 direction. All walls are impenetrable, i.e. the normal velocity is zero at the walls. Using this information *write* the boundary conditions for all six boundaries in terms of the velocity potential ϕ. (*Hint:* you will obtain Neumann BCs at all boundaries)

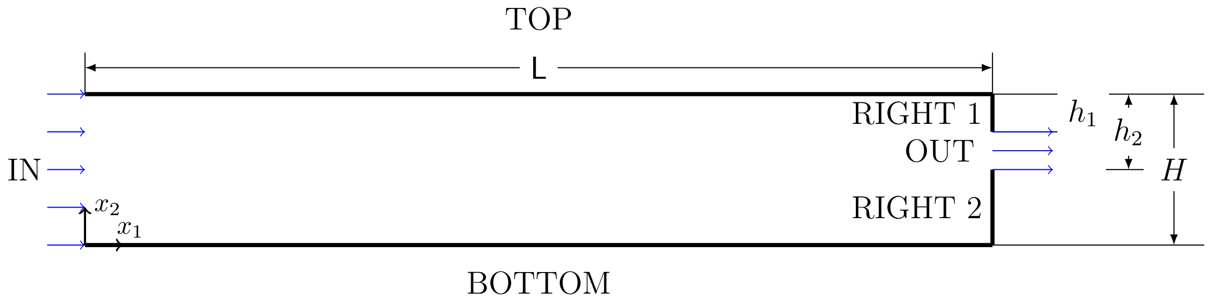

Figure 1: Two-dimensional rectangular channel.

3. It is clear that we are solving the Laplace's equation for ϕ in the interior of the rectangular domain, with the BCs for ϕ at the boundaries. The desired velocity field in the domain can therefore be obtained from the scalar velocity potential ϕ using $\mathbf{v} = \nabla\phi$.

   Since the Laplace's equation above (with Neumann boundary conditions) does not have an analytical solution, we will be using a numerical method to obtain the solution. The continuous field $\phi(x_1, x_2)$ is first discretised as $\phi^{i,j}$ on a structured grid of size $N_1 \times N_2$, as shown in Fig. 2. The easiest method to discretise the governing equation (and the boundary conditions) is the finite difference (FD) method.

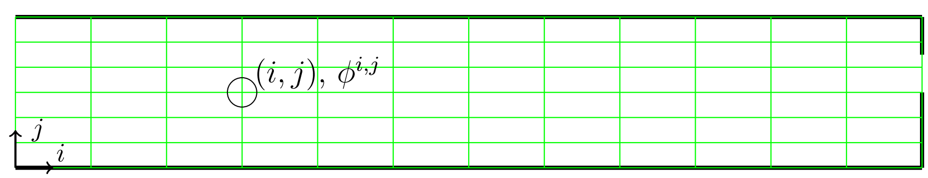

Figure 2: Structured grid for the two-dimensional rectangular channel.

   First- and second-order derivatives at point (i, j) can be approximated using the FD method as follows

$$\frac{\partial \phi}{\partial x_1} = \frac{\phi^{i,j} - \phi^{i-1,j}}{\Delta x_1},$$

   and

$$\frac{\partial^2 \phi}{\partial x_1^2} = \frac{\phi^{i-1,j} - 2\phi^{i,j} + \phi^{i+1,j}}{\Delta x_1^2}.$$

   Here, $\Delta x_1 = L/(N_1 - 1)$ and $\Delta x_2 = H/(N_2 - 1)$. It is evident that the FD approximations are obtained from the Taylor series expansion about the point (i, j) with the higher-order terms neglected. The discretisation in the first equation is backward in space and the second discretisation is centre in space.

(a) Using the centre-space discretisation in both x_1 and x_2 directions show that the governing equation at point (i, j) can be discretised as

$$\phi^{i-1,j} + \alpha^2\phi^{i,j-1} - 2(1 + \alpha^2)\phi^{i,j} + \alpha^2\phi^{i,j+1} + \phi^{i+1,j} = 0,$$

where $\alpha^2 = \Delta x_1^2/\Delta x_2^2$. There will be a total of $(N_1 - 2) \times (N_2 - 2)$ such equations for the interior points.

(b) Discretise the Neumann BCs, at the different walls, using the backward-space / forward-space discretisation method. Write these discretisations.

(c) The discrete equations can be written in the form of a set of linear equations $A\mathbf{x} = \mathbf{b}$, with the vector of unknowns $\mathbf{x}$ as the discrete ϕ values on the grid nodes. Write the linear system $A\mathbf{x} = \mathbf{b}$, showing the matrix A and vector $\mathbf{b}$. Remember to include all the boundary points.

(d) Write a computer code (in a high-level computer language such as Python, C, Fortran, Matlab, etc.) to solve the above linear system. You may use a direct matrix inversion method or (preferably) an indirect iterative method to solve for the vector $\mathbf{x}$. You may use standard linear algebra packages to solve the linear system. See `https://docs.scipy.org/doc/numpy-1.13.0/reference/generated/numpy.linalg.solve.html#numpy.linalg.solve` and `https://in.mathworks.com/help/matlab/math/systems-of-linear-equations.html#brzoiix`, as examples of direct and iterative methods, respectively. When using iterative methods (such as the Gauss-Seidel method) you may have to modify the inlet boundary condition for ϕ to a Dirichlet BC (such as $\phi = 1$) to get diagonal dominance.

(e) Use the FD method to compute the velocity field $\mathbf{v}$ from the velocity potential ϕ. Write the ϕ and $\mathbf{v}$ fields in to a text file.

4. Plot the contours of ϕ and the quiver plot for the velocity field $\mathbf{v}$. You may use any plotting software/library such as matplotlib, Matlab, Tecplot, gnuplot, etc.

Solve the above problem (and present results) for the following two cases:

(i) $L = 15$, $H = 10$, $h_1 = 2$, $h_2 = 4$, $N_1 = 301$, $N_2 = 201$ and $\mathbf{v}_{\text{in}} = (1, 0)$,

(ii) $L = 15$, $H = 10$, $h_1 = 2$, $h_2 = 4$, $N_1 = 301$, $N_2 = 201$ and $\mathbf{v}_{\text{in}} = (4, 0)$.

Note that $\mathbf{v}_{\text{out}}$ should be calculated using the conservation of mass equation. Please use Figure 3 for checking your results.

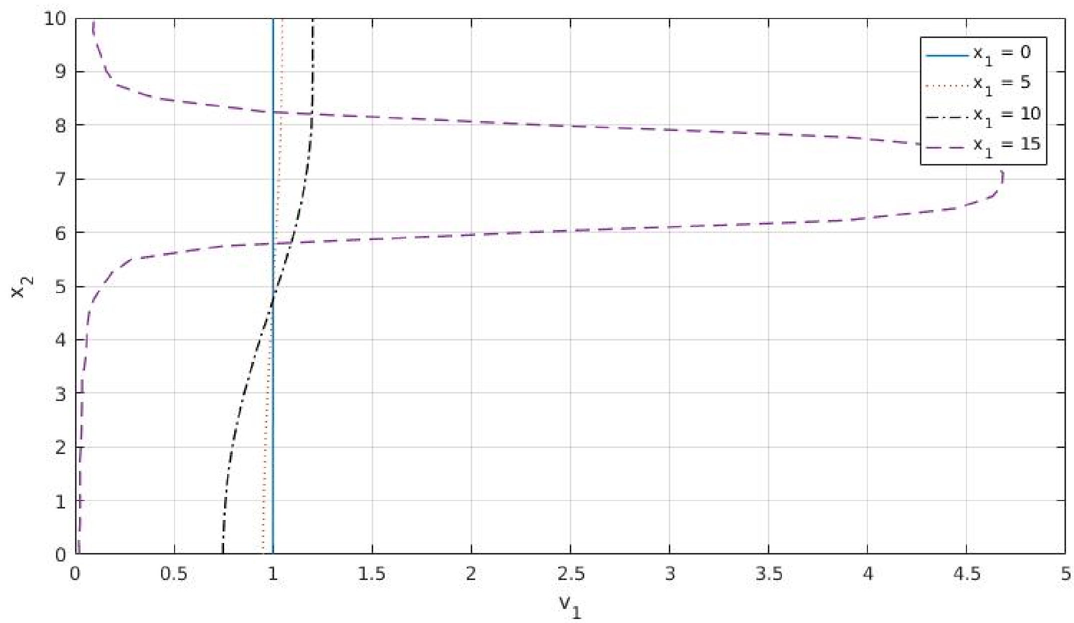

Figure 3: v_1 versus x_2 for four x_1 values, for the Case (i).