Chapter 3. Growth of Functions

3.1-1.
$(1/2)(f(n) + g(n)) \le \max(f(n), g(n)) \le f(n) + g(n) \Rightarrow \max(f(n), g(n)) = \Theta(f(n) + g(n))$

3.1-2.
$n^b \le (n+a)^b \le (\binom{n}{0} + \cdots + \binom{n}{n})n^b \Rightarrow (n+a)^b = \Theta(n^b)$.

3.1-3.
Because it doesn't constrain the upper bound of the running time of algorithm A.

3.1-4.
Yes, $2^{n+1} = 2 \cdot 2^n$.
No, for every $c > 0$, eventually $2^{2n} > c2^n$.

3.1-5.
$\Rightarrow$: Trivial.
$\Leftarrow$: Let n_1, n_2 be two constants for definitions of $O(g(n))$ and $\Omega(g(n))$. Defining $n = \max(n_1, n_2)$ fulfills the definition of $\Theta(g(n))$.

3.1-6.
$\Rightarrow$: Trivial.
$\Leftarrow$: There are positive constants c_1, c_2, n_1, n_2 such that $0 \le c_1 g(n) \le f_1(n)$ for all $n \ge n_1$ and $0 \le f_2(n) \le c_2 g(n)$ for all $n \ge n_2$, where $f_1(n), f_2(n)$ is the best-case/worst-case running time. From the inequalities, we can deduce that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge \max(n_1, n_2)$, where $f(n)$ is an arbitrary case running time. Therefore the running time is $\Theta(g(n))$.

3.1-7.

Fix $k > 0$ and suppose there is a constant $n_1 > 0, n_2 > 0$ such that $0 \leq f(n) < kg(n)$ for all $n \geq n_1$, $0 \leq kg(n) < f(n)$ for all $n \geq n_2$. Then for $n \geq \max(n_1, n_2)$, $f(n) < kg(n) < f(n)$, which is impossible.

3.1-8.
$\Omega(g(n,m)) = \{f(n,m) : \exists c, n_0, m_0 > 0 s.t. 0 \leq cg(m,n) \leq f(n,m) \forall n, m s.t. n \geq n_0 \text{and} m \geq m_0\}$.
$\Theta(g(n,m)) = \{f(n,m) : \exists c_1, c_2, n_0, m_0 > 0 s.t. 0 \leq c_1 g(m,n) \leq f(n,m) \leq c_2 g(m,n) \forall n, m s.t. n \geq n_0 \text{and} m \geq m_0\}$.

3.2-1.
$m \leq n \Rightarrow f(m) + g(m) \leq f(n) + g(n)$.
$m \leq n \Rightarrow g(m) \leq g(n) \Rightarrow f(g(m)) \leq f(g(n))$.
$m \leq n$ and $f(n), g(n) \geq 0 \Rightarrow f(m)g(m) \leq f(n)g(n)$.

3.2-2.
$\lg_b(1/a) = \lg_b(a^{-1}) = -\lg_b(a)$.
$a = b^{\lg_b a} \Rightarrow 1 = \lg_a(b^{\lg_b a}) \Rightarrow \lg_b a = \frac{1}{\lg_a b}$.
$a^{\lg_b c} = c^{\lg_b c \lg_c a} = c^{\frac{\lg_c a}{\lg_c b}} = c^{\lg_b a}$.

3.2-3.
$2 \cdot 2 \cdots 2 < n! < n \cdot n \cdots n \Rightarrow n! = o(n^n), n! = \omega(2^n)$.
$\lg n! = \lg 1 + \cdots + \lg n \leq n \lg n$, and $\lg n! = \lg 1 + \cdots + \lg n \geq \lg(n/2) + \cdots + \lg n \geq \lg(n/2) + \cdots + \lg(n/2) = (n/2) \cdot \lg(n/2) \geq (1/4) n \lg n \Rightarrow \lg n! = \Theta(n \lg n)$.

3.2-4.
Let $n = e^m$, then $(\lceil \lg n \rceil)! = m!$. For any $k > 0$, we can pick large m such that $\lg(m/2) > 2k$, then $m! > m \cdot (m-1) \cdot (m/2) = e^{(m/2)\lg(m/2)} > e^{mk} = n^k$, so $(\lceil \lg n \rceil)!$ is not polynomially bounded.
Let $n = e^{e^m}$, then $(\lceil \lg \lg n \rceil)! = m!$. Since $\lg m! < \lg m^m = m \lg m < m^2 < e^m = \lg e^{e^m} = \lg n$, we have $(\lceil \lg \lg n \rceil)! < n$, so $(\lceil \lg \lg n \rceil)!$ is polynomially bounded.

3.2-5.
Let $\lg^* n = k$. Then $\lg(\lg^* n) = \lg k$, $\lg^*(\lg n) = k - 1$, therefore $\lg^*(\lg n)$ is asymptotically larger.

3.2-6.
For ϕ and $\hat{\phi}$,

$$x^2 = \frac{1+\sqrt{5}^2 \pm 2\sqrt{5}}{4} = \frac{3 \pm \sqrt{5}}{2} = 1 + x.$$

3.2-7.

$$F_1 = 1 = \frac{\phi - \hat{\phi}}{\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}}.$$

$\phi^{i+1} - \hat{\phi}^{i+1} = \phi(\phi^i - \hat{\phi}^i) + \hat{\phi}(\phi^i - \hat{\phi}^i) + \phi\hat{\phi}^i - \phi^i\hat{\phi} = (\phi + \hat{\phi})(\phi^i - \hat{\phi}^i) - \phi\hat{\phi}(\phi^{i-1} - \hat{\phi}^{i-1})$
Since $\phi + \hat{\phi} = 1$ and $\phi\hat{\phi} = -1$, $F_{i+1} = F_i + F_{i-1}$.

3.2-8.

$$c_1 n \le k \lg k \le c_2 n$$

If k is large enough, $k \lg k < k^2$,

$$\Rightarrow c_1 n \le k^2$$

$$\Rightarrow \lg n < 2 \lg k$$

$$\Rightarrow \frac{k}{2} < k \frac{\lg k}{\lg n} <\le c_2 \frac{n}{\lg n}$$

$$\Rightarrow k < \frac{2c_2 n}{\lg n}$$

Since $k \lg k > k$,

$$\Rightarrow k < c_2 n$$

$$\Rightarrow \lg k < \lg c_2 + \lg n < 2 \lg n$$

$$\Rightarrow \frac{\lg k}{\lg n} < 2$$

$$\Rightarrow \frac{c_1 n}{\lg n} < \frac{k \lg k}{\lg n} < 2k$$

$$\Rightarrow \frac{c_1 n}{2 \lg n} < k$$

$$\Rightarrow k = \Theta(\frac{n}{\lg n})$$

3-1.
a. $p(n) \le d \max a_i n^d \le d \max a_i n^k \Rightarrow p(n) = O(n^k)$.
b. $p(n) \ge a_d n^d \ge a_d n^k \Rightarrow p(n) = \Omega(n^k)$.
c. $p(n) \le d \max a_i n^d$, $p(n) \ge a_d n^d \Rightarrow p(n) = \Theta(n^k)$.
d. $p(n) \le d \max a_i n^d < d \max a_i n^k \Rightarrow p(n) = o(n^k)$.
e. $p(n) \ge a_d n^d > a_d n^k \Rightarrow p(n) = \omega(n^k)$.

3-2.
a. $(\lg n)^k = \Omega(n^\epsilon), \omega(n^\epsilon)$.
b. $n^k = \Omega(c^n), \omega(c^n)$.
c. No one of the five relationships holds.
d. $2^n = O(n^\epsilon), o(n^\epsilon)$.
e. Since $n^{\lg c} = c^{\lg n}$, $n^{\lg c} = O(c^{\lg n}), o(c^{\lg n}), \Omega(c^{\lg n}), \omega(c^{\lg n}), \Theta(c^{\lg n})$.
f. As proved in 3.2-3 $\lg n! = O(\lg(n^n)), o(\lg n!\,), \Omega(\lg(n^n)), \omega(\lg n!\,), \Theta(\lg(n^n))$.

3-3.
a. $1 < \lg(\lg^* n) < \lg^*(\lg n) = \lg^* n < \ln \ln n < 2^{\lg^* n} < \sqrt{\lg n} < \lg^2 n < 2^{\sqrt{2 \lg n}} < n^{1/\lg n} < \sqrt{2}^{\lg n} < n = 2^{\lg n} < n^2 = 4^{\lg n} = n^3 < n^{\lg \lg n} < (\lg n)^{\lg n} < (3/2)^n < 2^n < n2^n < e^n < (\lg n)! < n! < (n+1)! < 2^{2^n} < 2^{2^{n+1}}$.
b. $f(n) = 2^{2^{n+1}}(1 + \sin n)$ is neither $O(g_i(n))$ nor $\Omega(g_i(n))$ for all $g_i(n)$.

3-4.
a. False. $f(n) = n$, $g(n) = n^2$
b. False. $f(n) = n$, $g(n) = n^2$
c. False. $f(n) = 2$, $g(n) = 1$.
d. False. $f(n) = 2 \lg n$, $g(n) = \lg n$.
e. False. $f(n) = 1/n$.
f. True. We have $c, n_0 > 0$ such that $f(n) < cg(n)$ for all $n > n_0$. Since $(1/c)f(n) < g(n)$, $g(n) = \Omega(f(n))$.
g. False. $f(n) = n^n$.
h. True. Let $g(n) = o(f(n))$. For $n_0 > 0$, we have $c > 0$ such that $g(n) < cf(n)$ for $n > n_0$. This gives $(1/c)(f(n) + g(n)) < (c+1)/(c)f(n) < (c+1)/(c)(f(n) + g(n)) < (c+1)^2/(c)f(n) \Rightarrow f(n) + g(n) = \Theta(f(n))$.

3-5.

a. If $f(n) \notin \overset{\infty}{\Omega}(g(n))$, then for all $c > 0$, $f(n) \leq cg(n)$ for large enough n, therefore $f(n) \in O(g(n))$.
b. The definition of $\overset{\infty}{\Omega}$ enables you to handle with "fluctuating" functions. However, it does not ensure asymptotic behavior over the whole set after some n_0.
c. The both directions are uneffected. $f(n) = \Theta(g(n))$ ensures $f(n) = O'(g(n))$ and $f(n) = \Omega(g(n))$. The converse direction also holds: $f(n) = \Omega(g(n))$ ensures $f(n)$ is asymptotically nonnegative. Combining this with $f(n) = O'(g(n))$ gives $f(n) = O(g(n))$.
d. $\widetilde{\Omega}(g(n)) = \{f(n) : \exists c, k, n_0 > 0 s.t. 0 \leq cg(n) \lg^{-k}(n) \leq f(n) \forall n \geq n_0\}$.
$\widetilde{\Theta}(g(n)) = \{f(n) : \exists c_1, k_1, c_2, k_2, n_0 > 0 s.t. 0 \leq c_1 g(n) \lg^{-k_1}(n) \leq f(n) \leq c_2 g(n) \lg^{k_2}(n) \forall n \geq n_0\}$.
The analog to Theorem 3.1 would be $f(n) = \widetilde{\Theta}(g(n)) \Leftrightarrow f(n) = \widetilde{\Omega}(g(n))$ and $f(n) = \widetilde{O}(g(n))$. The proof is trivial.

3-6.
a. $f_c^*(n) = \Theta(n)$.
b. $f_c^*(n) = \Theta(\lg^* n)$.
c. $f_c^*(n) = \Theta(\lg n)$.
d. $f_c^*(n) = \Theta(\lg n)$.
e. $n^{2^{-i}} \geq 2 \Rightarrow 2^i = \Theta(\lg n) \Rightarrow f_c^*(n) = \Theta(\lg \lg n)$.
f. The iterated functions do not converge.
g. $n^{3^{-i}} \geq 2 \Rightarrow 3^i = \Theta(\lg n) \Rightarrow f_c^*(n) = \Theta(\lg \lg n)$.
h. Since each iteration, $f(n)$ decreases value by at least $\lfloor \lg n \rfloor$ times, for fixed n, $\lfloor \lg n \rfloor$ iteration of f gives $f^{(k)}(n) \leq 2 \Rightarrow f_c^*(n) = o(\lg n)$.
Since each iteration, for fixed n, applying f decreases value by at most $\lceil \lg n \rceil$ times, hence at least $\lceil \lg \lg n \rceil$ iterations of f is needed to make $f^{(k)}(n) \leq 2$.