POLITECNICO
MILANO 1863

DEPARTMENT OF
MECHANICAL ENGINEERING

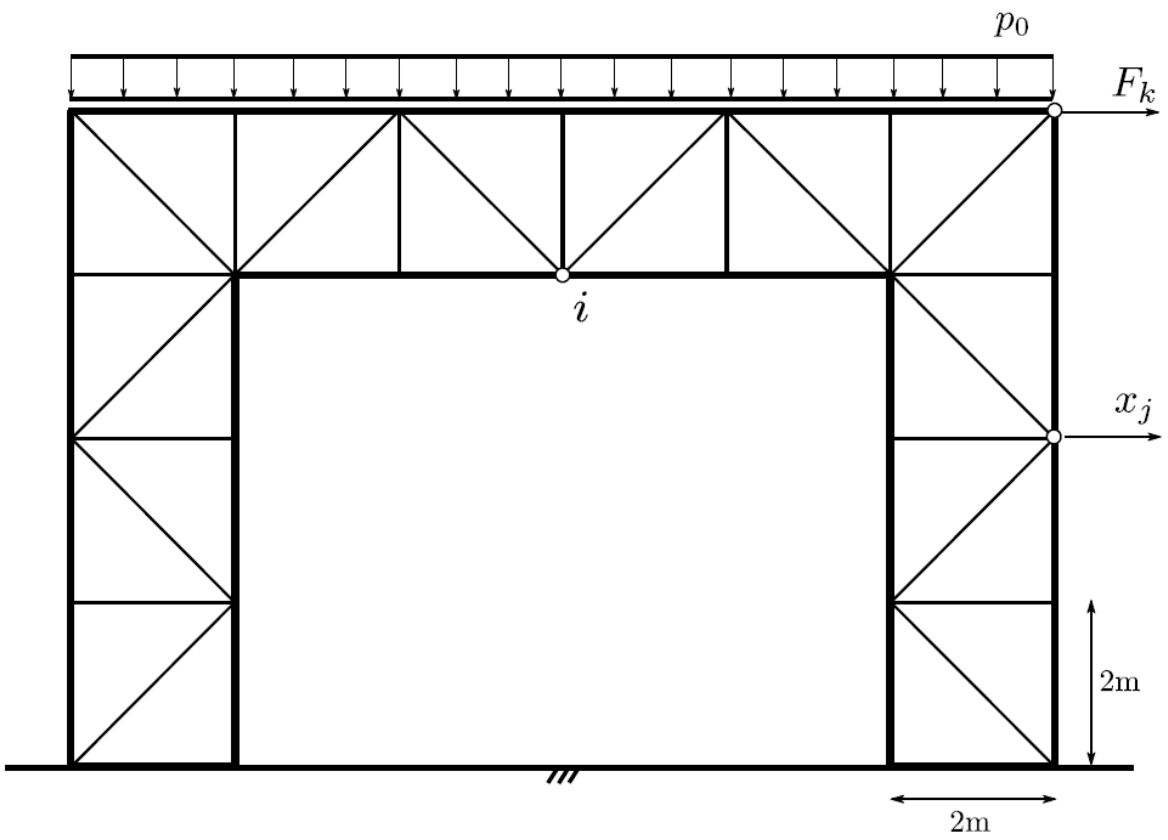
ADVANCED DYNAMICS OF MECHANICAL SYSTEMS

Finite Elements Method (FEM) in structural dynamics: software implementation in Matlab environment – part 5

I. LA PAGLIA

EXAM SOLUTION EXAMPLE

Mechanical System Dynamics - Proff. Bruni, Corradi
12 February 2016



	$m\ [kg/m]$	$EA\ [N]$	$EJ\ [Nm^2]$
External beams	4	4e7	9e3
Internal beams	0.7	1e5	5e3

EXAM SOLUTION EXAMPLE

Damping is defined according to the proportional damping assumption: $[C] = \alpha\,[M] + \beta\,[K]$.

1. Define a FE model of the structure and save the image of the undeformed structure in a .fig file.
2. Calculate the structure vibration modes. Save the images of the first 4 mode shapes in 4 distinct .fig files with the indication of the associated natural frequencies.
3. Calculate the structure frequency response function which relates the input force at the node *k* to the output horizontal displacement *j* (assume the input force to vary in the 0 - 10 Hz frequency range and set the frequency resolution to 0.01 Hz). Plot the magnitude and phase diagrams and save the Matlab figures in two files. Provide a short comment to the diagrams (in the table at the back of this paper).
4. Calculate the same frequency response function specified in item 3, by developing a model in modal coordinates, limited to the first two modes. Plot the magnitude and phase diagrams superimposed to those of item 3 and save the Matlab figure. Provide a short comment to the diagrams (in the table at the back of this paper).
5. Compute the modal mass and stiffness of the first two modes and reassign the values of α and β so that they result in the following damping ratios for the first two vibration modes: $\xi_1 = 2\ \%$, $\xi_2 = 3\ \%$.
6. Calculate the structure static response to the distributed load *p0 = 1e2 N/m* and provide the vertical displacement of the node *i*.

EXAM SOLUTION EXAMPLE

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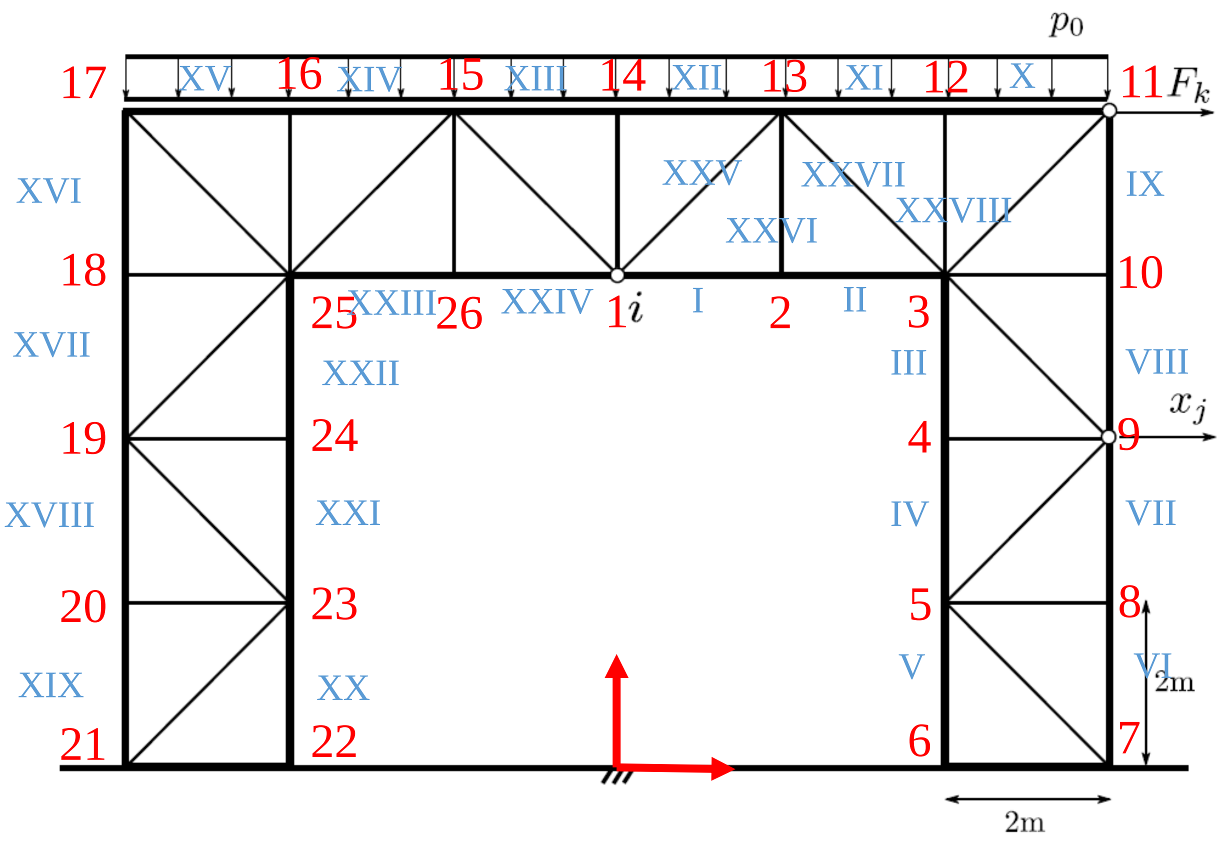
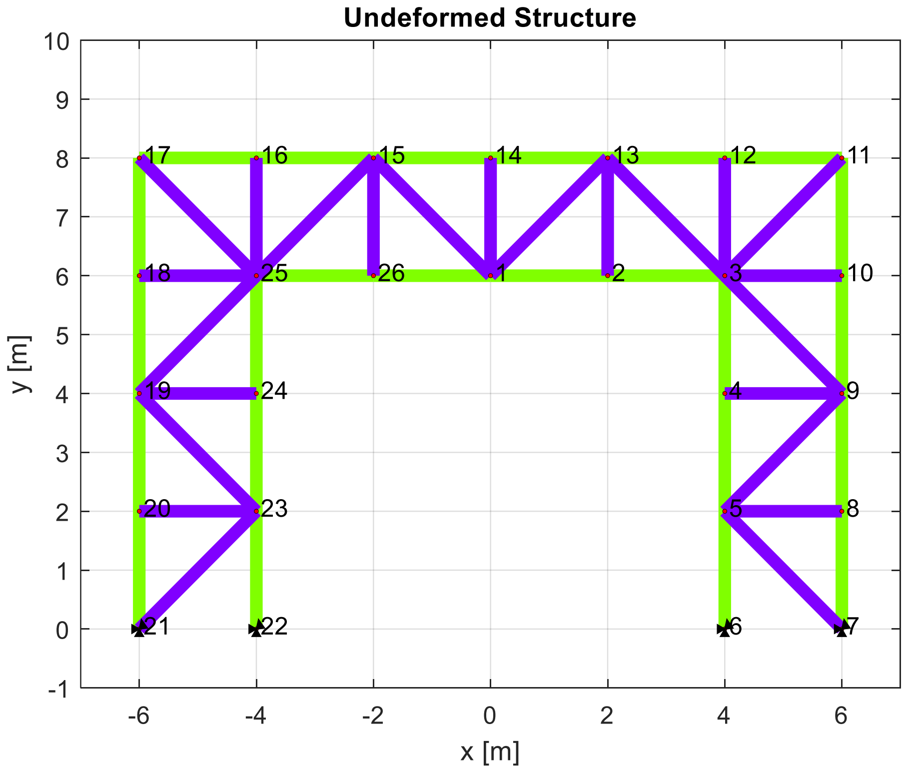


	$m\ [kg/m]$	$EA\ [N]$	$EJ\ [Nm^2]$
External beams	4	4e7	9e3
Internal beams	0.7	1e5	5e3

QUESTION 1

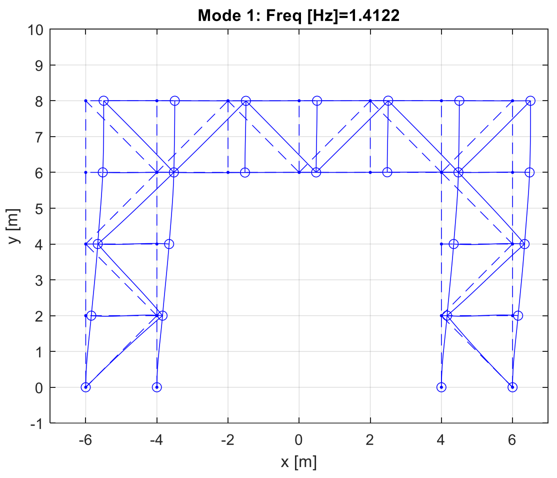
Define a FE model of the structure and save the image of the undeformed structure in a .fig file.

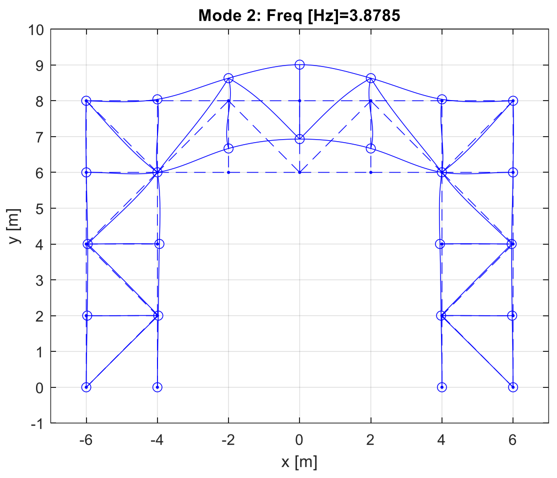
```
[file_i,xy,nnod,sizee,idb,ndof,incid,l,gamma,m,EA,EJ,posiz,nbeam,pr]=loadstructure;

dis_stru(posiz,l,gamma,xy,pr,idb,ndof)
```

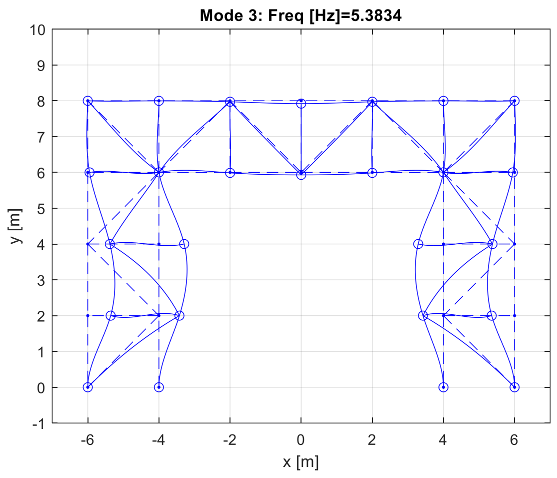


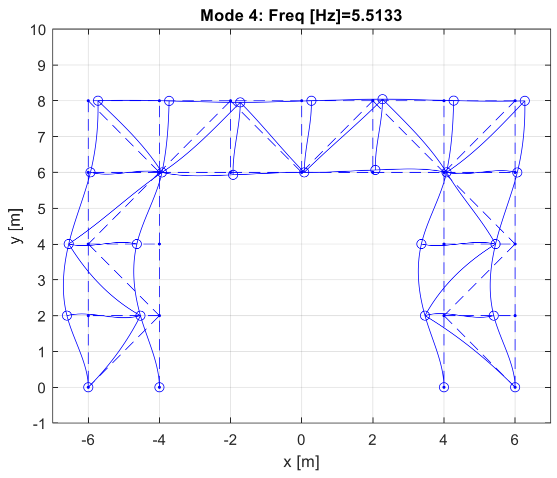
QUESTION 2

Calculate the structure vibration modes. Save the images of the first 4 mode shapes in 4 distinct .fig files with the indication of the associated natural frequencies.



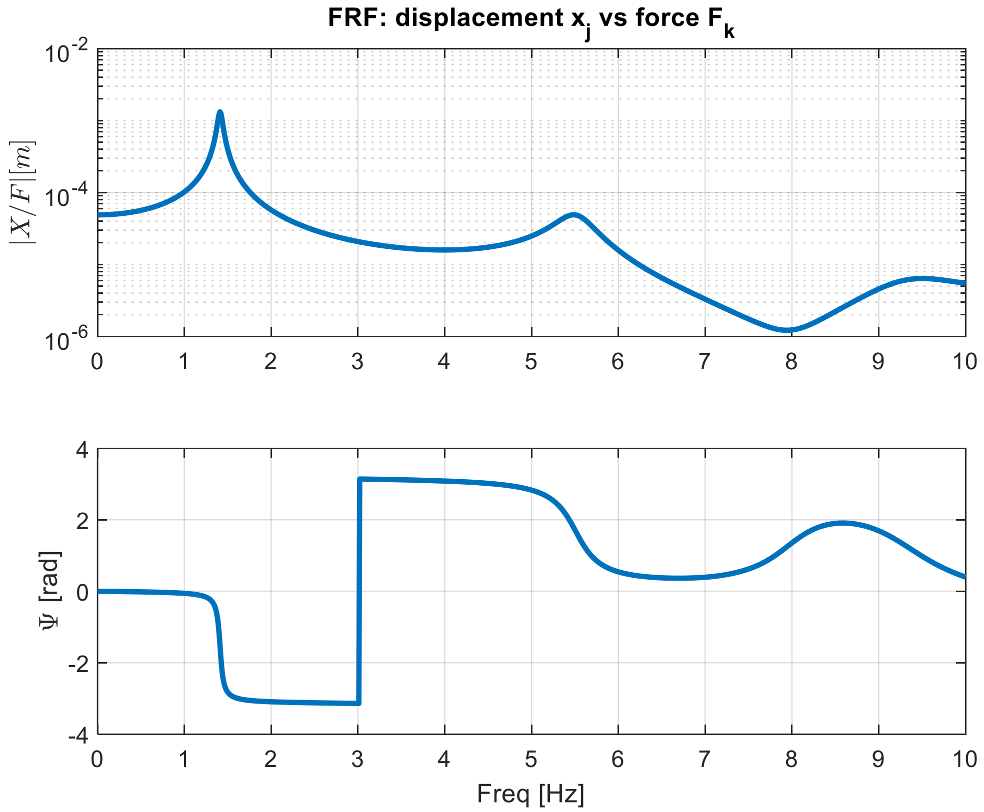






QUESTION 3

Calculate the structure frequency response function which relates the input force at the node *k* to the output horizontal displacement *j* (assume the input force to vary in the 0 - 10 Hz frequency range and set the frequency resolution to 0.01 Hz). Plot the magnitude and phase diagrams and save the Matlab figures in two files. Provide a short comment to the diagrams (in the table at the back of this paper).



Damping contribution
$$\begin{cases} \xi_1 = \dfrac{\alpha}{2\omega_1} + \dfrac{\beta\omega_1}{2} \\ \xi_2 = \dfrac{\alpha}{2\omega_2} + \dfrac{\beta\omega_2}{2} \end{cases}$$

$$[A] = \begin{bmatrix} \dfrac{1}{2\omega_1} & \dfrac{\omega_1}{2} \\ \dfrac{1}{2\omega_2} & \dfrac{\omega_2}{2} \end{bmatrix} \quad \underline{B} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = [A]^{-1}\underline{B}$$

QUESTION 4

Calculate the same frequency response function specified in item 3, by developing a model in modal coordinates, limited to the first two modes. Plot the magnitude and phase diagrams superimposed to those of item 3 and save the Matlab figure. Provide a short comment to the diagrams (in the table at the back of this paper).

Modal coordinate transformation

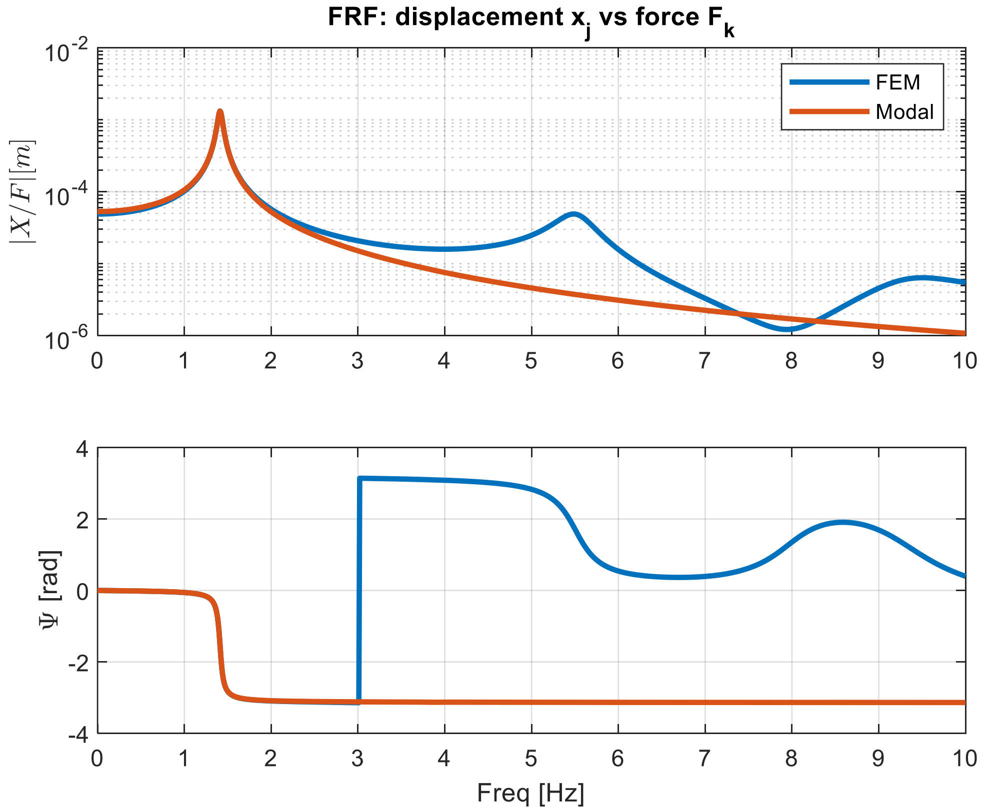
$$[\Phi] = \left[\underline{X}^{(1)} \quad \underline{X}^{(2)}\right]$$

$$\left[M_q\right] = [\Phi]^T\,[M]\,[\Phi]$$

$$\left[K_q\right] = [\Phi]^T\,[K]\,[\Phi]$$

$$\left[C_q\right] = [\Phi]^T\,[C]\,[\Phi]$$

$$\left[Q_q\right] = [\Phi]^T\underline{F}$$



QUESTION 5

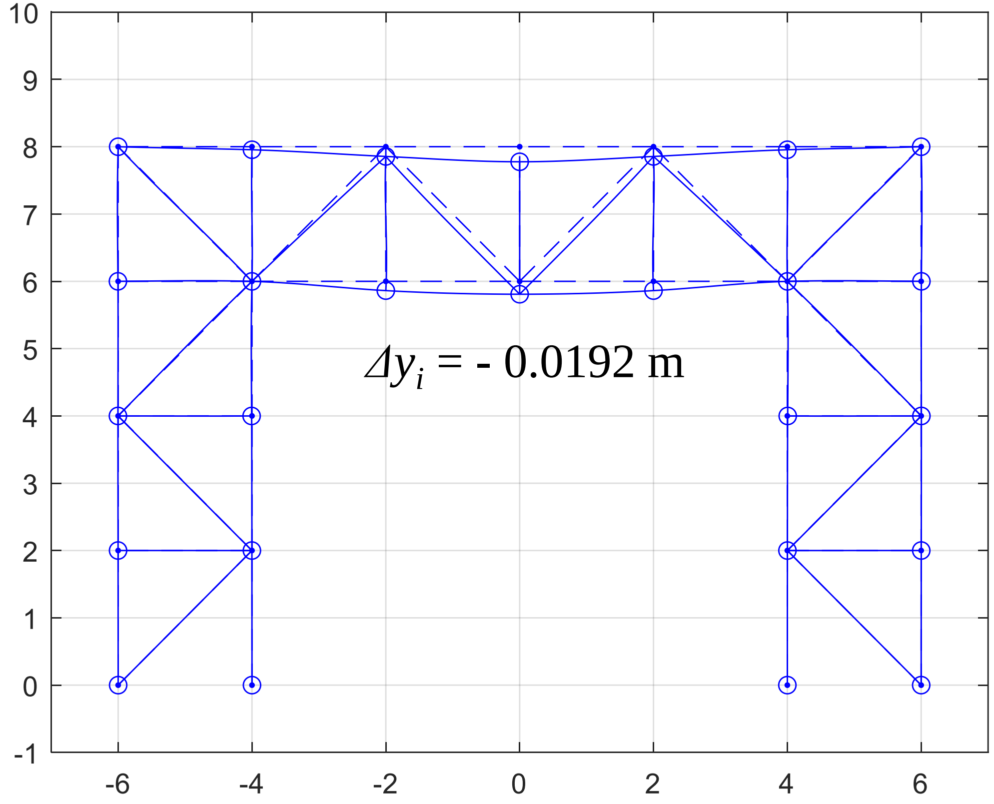
Compute the modal mass and stiffness of the first two modes and reassign the values of α and β so that they result in the following damping ratios for the first two vibration modes: $\xi_1 = 2\ \%$, $\xi_2 = 3\ \%$.

```
% Damping Matrix

om1   = 2*pi*frqord(1);
om2   = 2*pi*frqord(2);
ab    = [1/2/om1 om1/2;
         1/2/om2 om2/2]^-1*[0.02; 0.03];
alpha = ab(1);
beta  = ab(2);

C     = alpha*M + beta*K;
CFF   = C(1:ndof,1:ndof);

% Modal matrices

for ii = 1:2 % for the first 2 mode shapes
    mode = modes(:,ordmode(ii));
    Phi(:,ii) = mode;
end

Mmodal = Phi'*MFF*Phi;
Kmodal = Phi'*KFF*Phi;
Cmodal = Phi'*CFF*Phi;
```

QUESTION 6

Calculate the structure static response to the distributed load $p0 = 1e2\ N/m$ and provide the vertical displacement of the node i.

Static deformation of the structure
(scale factor = 10)



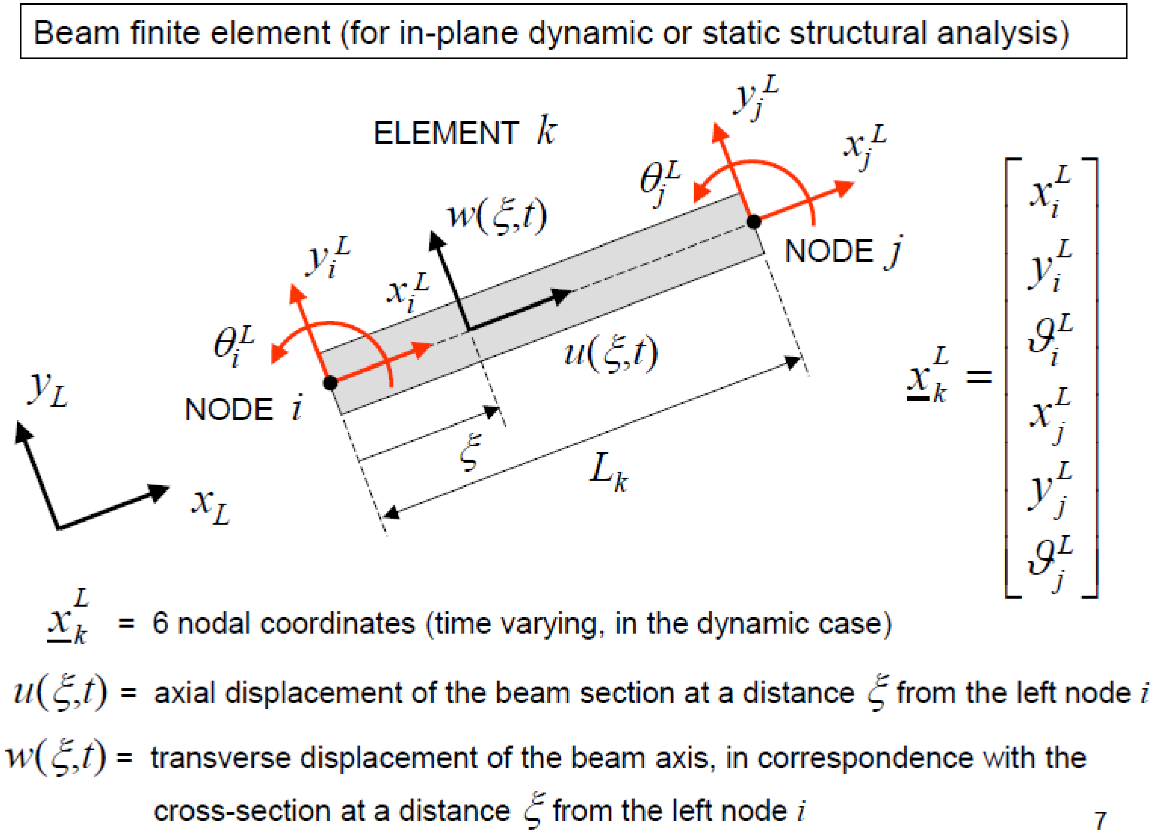
$\Delta y_i = -\ 0.0192\ \mathrm{m}$

INTERNAL FORCES

According to the FE formulation, the axial and the transverse displacement of a beam section are defined as function of the nodal displacements through the <u>shape functions</u> (which are defined in the local reference frame of each finite element).

$$u(\xi,t) = a + b\xi$$

$$w(\xi,t) = a + b\xi + c\xi^2 + d\xi^3$$



Considering the Euler-Bernoulli beam, the internal actions can be computed as:

$$N = EA\frac{\partial u}{\partial \xi} \qquad M = EJ\frac{\partial^2 w}{\partial \xi^2} \qquad T = EJ\frac{\partial^3 w}{\partial \xi^3}$$

INTERNAL FORCES

- The shape functions for axial deformation are assumed to be **linear functions** of ξ

$$u(\xi,t) = a + b\xi \quad \Rightarrow \quad \begin{cases} a = x_i^L \\ b = \dfrac{x_j^L - x_i^L}{L_k} \end{cases} \quad \Rightarrow \quad \epsilon_x = \frac{\partial u}{\partial \xi} = b = \frac{x_j^L - x_i^L}{L_k} = const$$

  - ➢ The axial force is constant along the beam element

$$N = EA\frac{\partial u}{\partial \xi}$$

- The shape functions for bending deformation are assumed to be **cubic functions** of ξ

$$w(\xi,t) = a + b\xi + c\xi^2 + d\xi^3 \quad \Rightarrow \quad \begin{cases} a = y_i^L \\ b = \theta_i^L \\ c = -\frac{3}{L_k^2}y_i^L + \frac{3}{L_k^2}y_j^L - \frac{2}{L_k}\theta_i^L - \frac{1}{L_k}\theta_j^L \\ d = \frac{2}{L_k^3}y_i^L - \frac{2}{L_k^3}y_j^L + \frac{1}{L_k^2}\theta_i^L + \frac{1}{L_k^2}\theta_j^L \end{cases}$$

  - ➢ The bending moment and the shear force are proportional to the second and third partial derivatives of w with respect to ξ

$$M = EJ\frac{\partial^2 w}{\partial \xi^2}$$

$$T = EJ\frac{\partial^3 w}{\partial \xi^3}$$

INTERNAL FORCES

$N = EA\dfrac{\partial u}{\partial \xi}$ $\dfrac{\partial u}{\partial \xi} = b$	$M = EJ\dfrac{\partial^2 w}{\partial \xi^2}$ $\dfrac{\partial^2 w}{\partial \xi^2} = 2c + 6d\xi$	$T = EJ\dfrac{\partial^3 w}{\partial \xi^3}$ $\dfrac{\partial^3 w}{\partial \xi^3} = 6d$

By properly rearranging the partial derivatives according to matrix formulation, the internal forces can be written as the product of the vector collecting the shape functions derivatives and the vector of the nodal coordinates.

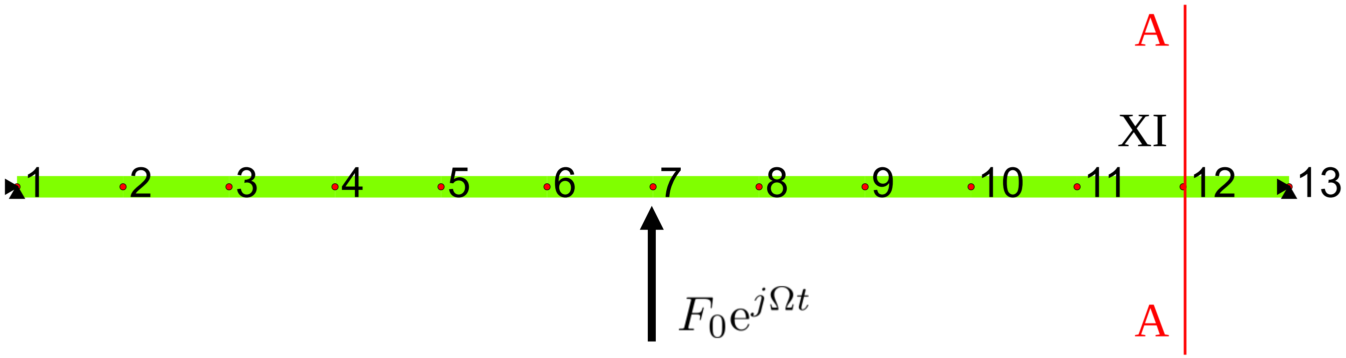
As an example, consider the bending moment M at the specific $\hat{\xi}$ axial coordinate along the local reference frame of the finite element.

$$w(\xi, t) = \underline{f}_w^T(\xi)\, \underline{x}^L(t)$$

$$\frac{\partial^2 w}{\partial \xi^2}(\xi, t) = \underline{f}_w''^{\,T}(\xi)\, \underline{x}^L(t) \qquad \Rightarrow \qquad M(\hat{\xi}) = EJ\frac{\partial^2 w}{\partial \xi^2}\bigg|_{\hat{\xi}} = EJ\, \underline{f}_w''^{\,T}(\hat{\xi})\, \underline{x}^L(t) \qquad \text{where} \qquad \underline{f}_w'' = \begin{bmatrix} 0 \\ \dfrac{12\,\xi}{L_k^3} - \dfrac{6}{L_k^2} \\ \dfrac{6\,\xi}{L_k^2} - \dfrac{4}{L_k} \\ 0 \\ -\dfrac{12\,\xi}{L_k^3} + \dfrac{6}{L_k^2} \\ \dfrac{6\,\xi}{L_k^2} - \dfrac{2}{L_k} \end{bmatrix}$$

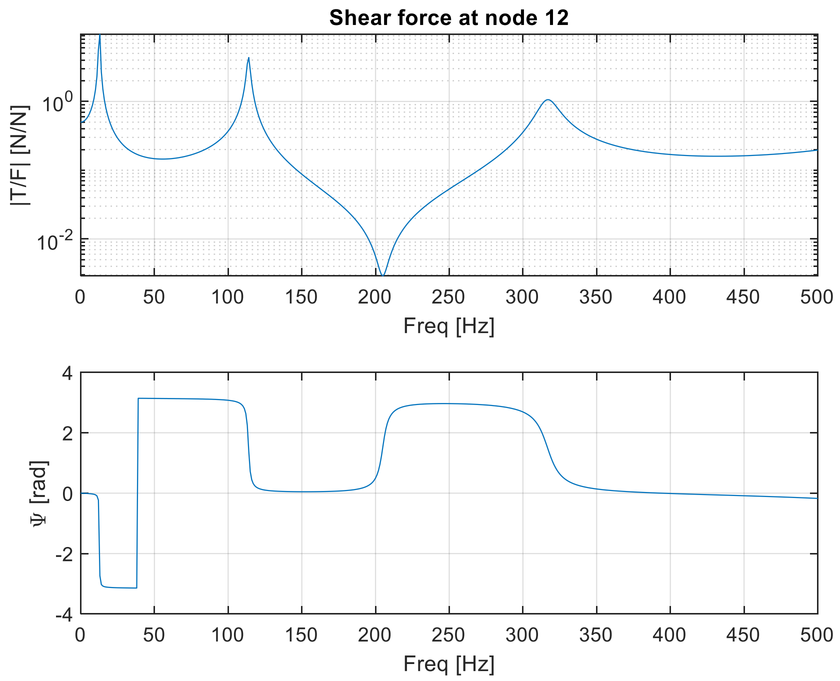
- Similar results can be obtained considering the axial and shear force

INTERNAL FORCES



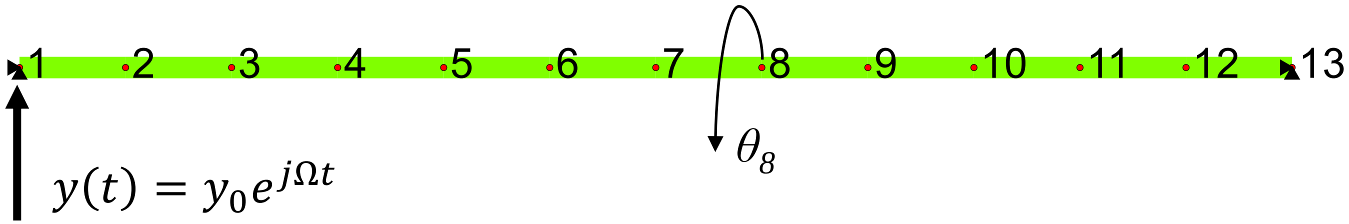
Shear force in section A-A (element XI close to node 12)

```
n_el = 11;
L_el = l(n_el);

idof_i = idb(11,:);
idof_j = idb(12,:);
lambda = [cos(gamma(n_el)) sin(gamma(n_el)) 0;
         -sin(gamma(n_el)) cos(gamma(n_el)) 0;
          0                0                1];
Xi = lambda*X(idof_i,:);
Xj = lambda*X(idof_j,:);

d =  2/L_el^3*Xi(2,:) -2/L_el^3*Xj(2,:) ...
    +1/L_el^2*Xi(3,:) +1/L_el^2*Xj(3,:);
T = EJ(n_el)*(6*d);
```



MOTION IMPOSED AT THE CONSTRAINTS

Input: node 1 y imposed displacement
Output: node 8 rotation



Start from the EOMs

$$[M]\underline{\ddot{x}} + [C]\underline{\dot{x}} + [K]\underline{x} = \underline{F}$$

and perform the partitions

$$\begin{bmatrix} [M_{FF}] & [M_{FC}] \\ [M_{CF}] & [M_{CC}] \end{bmatrix} \begin{bmatrix} \underline{\ddot{x}_F} \\ \underline{\ddot{x}_C} \end{bmatrix} + \begin{bmatrix} [C_{FF}] & [C_{FC}] \\ [C_{CF}] & [C_{CC}] \end{bmatrix} \begin{bmatrix} \underline{\dot{x}_F} \\ \underline{\dot{x}_C} \end{bmatrix} + \begin{bmatrix} [K_{FF}] & [K_{FC}] \\ [K_{CF}] & [K_{CC}] \end{bmatrix} \begin{bmatrix} \underline{x_F} \\ \underline{x_C} \end{bmatrix} = \begin{bmatrix} \underline{F_F} \\ \underline{F_C} + \underline{R} \end{bmatrix}$$

$$\underline{F} = \begin{bmatrix} \underline{F_F} \\ \underline{F_C} + \underline{R} \end{bmatrix}$$

Equivalent nodal forces which account for the assigned time-dependent concentrated/distributed loads

Unknown constraint forces

In the case under study:

$\underline{F_F} = 0$

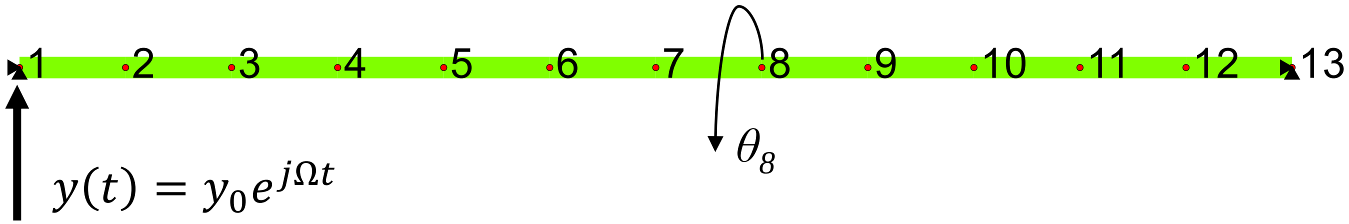
$\underline{F_C} = 0$

$\underline{x_C} = y(t)$

$$\begin{cases} [M_{FF}]\underline{\ddot{x}_F} + [M_{FC}]\underline{\ddot{x}_C} + [C_{FF}]\underline{\dot{x}_F} + [C_{FC}]\underline{\dot{x}_C} + [K_{FF}]\underline{x_F} + [K_{FC}]\underline{x_C} = \cancel{\underline{F_F}} \\ [M_{CF}]\underline{\ddot{x}_F} + [M_{CC}]\underline{\ddot{x}_C} + [C_{CF}]\underline{\dot{x}_F} + [C_{CC}]\underline{\dot{x}_C} + [K_{CF}]\underline{x_F} + [K_{CC}]\underline{x_C} = \cancel{\underline{F_C}} + \underline{R} \end{cases}$$

MOTION IMPOSED AT THE CONSTRAINTS

Input: node 1 y imposed displacement
Output: node 8 rotation



$$[M_{FF}]\ddot{\underline{x}}_F + [C_{FF}]\dot{\underline{x}}_F + [K_{FF}]\underline{x}_F = -([M_{FC}]\ddot{\underline{x}}_C + [C_{FC}]\dot{\underline{x}}_C + [K_{FC}]\underline{x}_C)$$

being $\underline{x}_C$ an harmonic function of time t

$$[M_{FF}]\ddot{\underline{x}}_F + [C_{FF}]\dot{\underline{x}}_F + [K_{FF}]\underline{x}_F = -(-\Omega^2[M_{FC}] + j\Omega[C_{FC}] + [K_{FC}])\,\underline{X}_{C0}\, e^{j\Omega t}$$

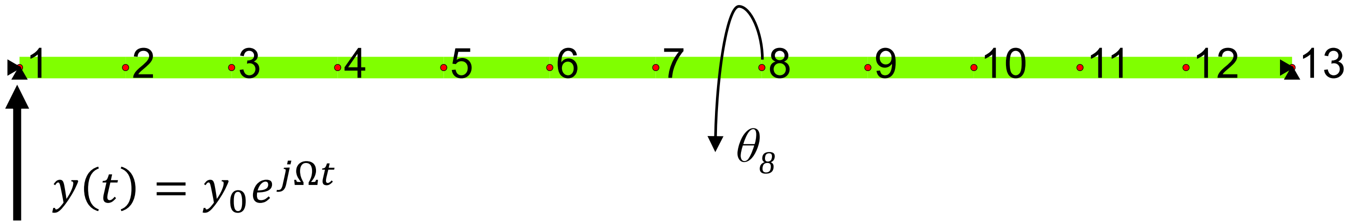
$\underline{x}_F$ will be harmonic too with the same frequency

$$(-\Omega^2[M_{FF}] + j\Omega[C_{FF}] + [K_{FF}])\,\underline{X}_{F0}\, e^{j\Omega t} = -(-\Omega^2[M_{FC}] + j\Omega[C_{FC}] + [K_{FC}])\,\underline{X}_{C0}\, e^{j\Omega t}$$

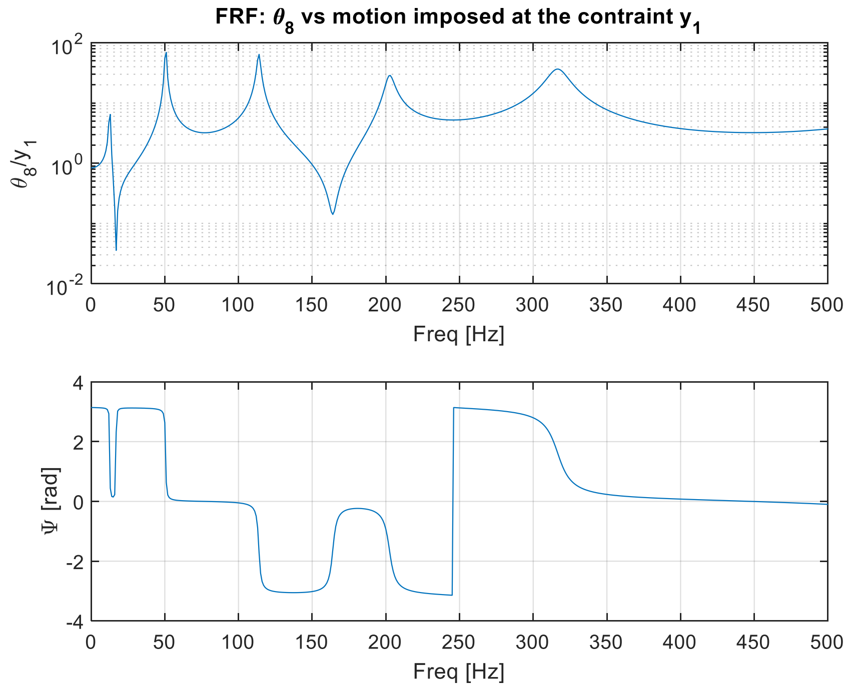
$$\frac{\underline{X}_{F0}}{\underline{X}_{C0}} = -(-\Omega^2[M_{FF}] + j\Omega[C_{FF}] + [K_{FF}])^{-1}\,(-\Omega^2[M_{FC}] + j\Omega[C_{FC}] + [K_{FC}]) \longrightarrow \frac{\theta_8}{y}$$

MOTION IMPOSED AT THE CONSTRAINTS

Input: node 1 y imposed displacement
Output: node 8 rotation

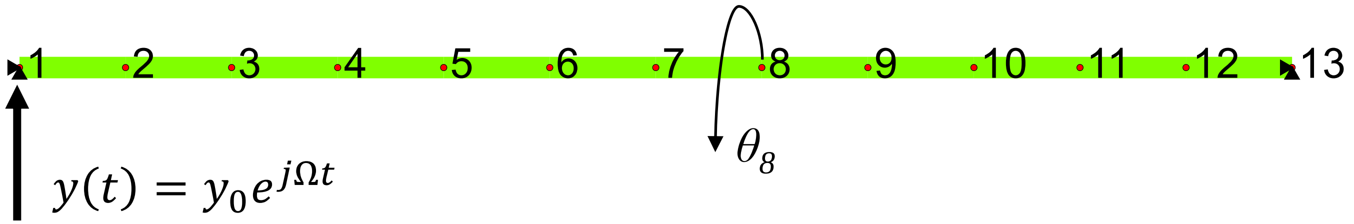



```
ncons = nnod*3-ndof;

Y = zeros(ncons,1);
Y(idb(1,2)-ndof) = 1;

for ii = 1:length(om)
  X(:,ii) = -(-MFF*om(ii)^2 + 1i*CFF*om(ii) + KFF)\ ...
           (-MFC*om(ii)^2 + 1i*CFC*om(ii) + KFC)*Y;
end

FRF = X(idb(8,3),:);
```



TIME RESPONSE IN STEADY-STATE CONDITIONS

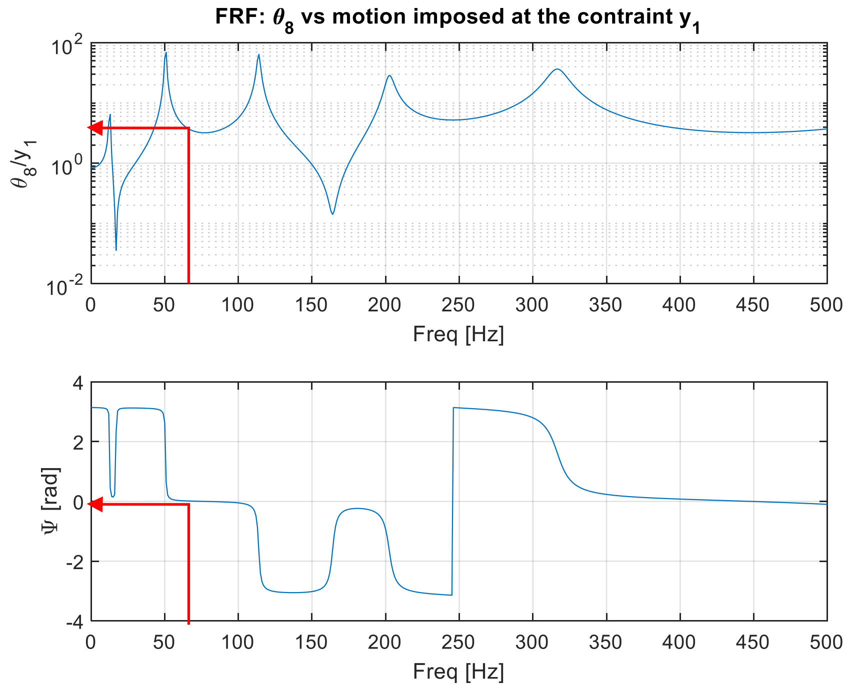
Input: node 1 y imposed displacement
Output: node 8 rotation



Time response in steady-state condition if $y_0 = 0.01\ m$ and $f = 60\ Hz$

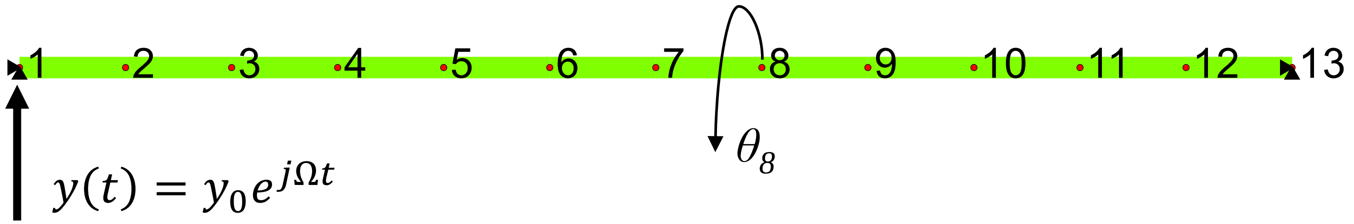
```
ncons = nnod*3-ndof;

Y = zeros(ncons,1);
Y(idb(1,2)-ndof) = 0.01;       %[m]

om = 2*pi*60;                   %[rad/s]
xx = -(-MFF*om^2 + 1i*CFF*om + KFF)\ ...
       (-MFC*om^2 + 1i*CFC*om + KFC)*Y;

x  = xx(idb(8,3));
```

1x1 complex vector



TIME RESPONSE IN STEADY-STATE CONDITIONS

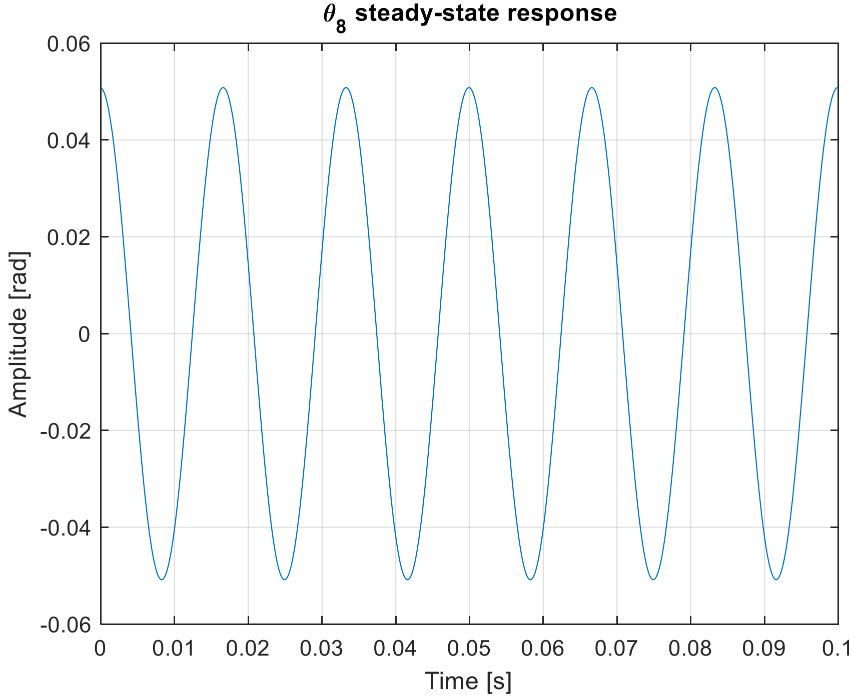
Input: node 1 y imposed displacement
Output: node 8 rotation



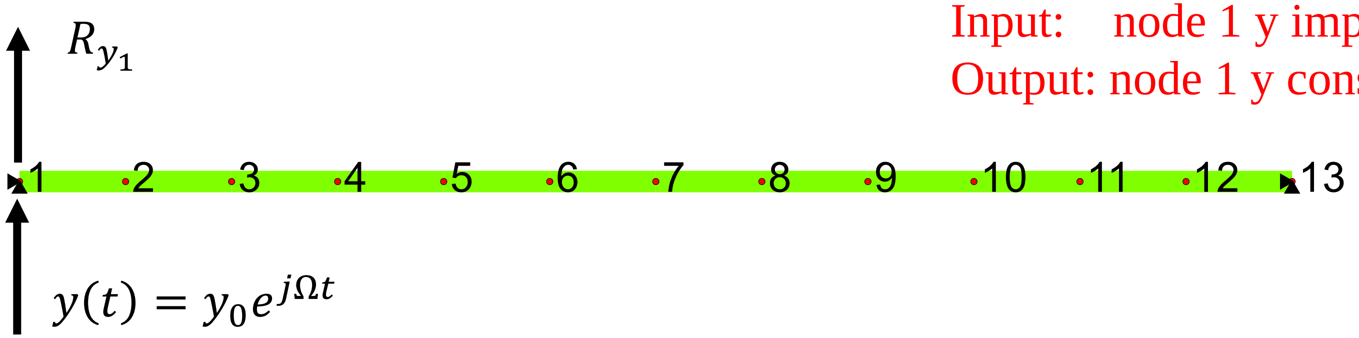
Time response in steady-state condition if $y_0 = 0.01\ m$ and $f = 60\ Hz$

```
dt  = 0.0001;
t   = 0:dt:1;

x_t = zeros(length(t),1);

for ii = 1:length(t)
   x_t(ii) = abs(x)*cos(om*t(ii)+angle(x));
end
```

Rely on the IDFT to evaluate the time response



CONSTRAINT FORCES



Input: node 1 y imposed displacement
Output: node 1 y constraint force

$$[M]\ddot{\underline{x}} + [C]\dot{\underline{x}} + [K]\underline{x} = \underline{F}$$

Start from the EOMs

$$\begin{bmatrix} [M_{FF}] & [M_{FC}] \\ [M_{CF}] & [M_{CC}] \end{bmatrix} \begin{bmatrix} \ddot{\underline{x}}_F \\ \ddot{\underline{x}}_C \end{bmatrix} + \begin{bmatrix} [C_{FF}] & [C_{FC}] \\ [C_{CF}] & [C_{CC}] \end{bmatrix} \begin{bmatrix} \dot{\underline{x}}_F \\ \dot{\underline{x}}_C \end{bmatrix} + \begin{bmatrix} [K_{FF}] & [K_{FC}] \\ [K_{CF}] & [K_{CC}] \end{bmatrix} \begin{bmatrix} \underline{x}_F \\ \underline{x}_C \end{bmatrix} = \begin{bmatrix} \underline{F}_F \\ \underline{F}_C + \underline{R} \end{bmatrix}$$

and perform the partitions

$$\underline{F} = \begin{bmatrix} \underline{F}_F \\ \underline{F}_C + \underline{R} \end{bmatrix}$$

Equivalent nodal forces which account for the assigned time-dependent concentrated/distributed loads

Unknown constraint forces

In the case under study:

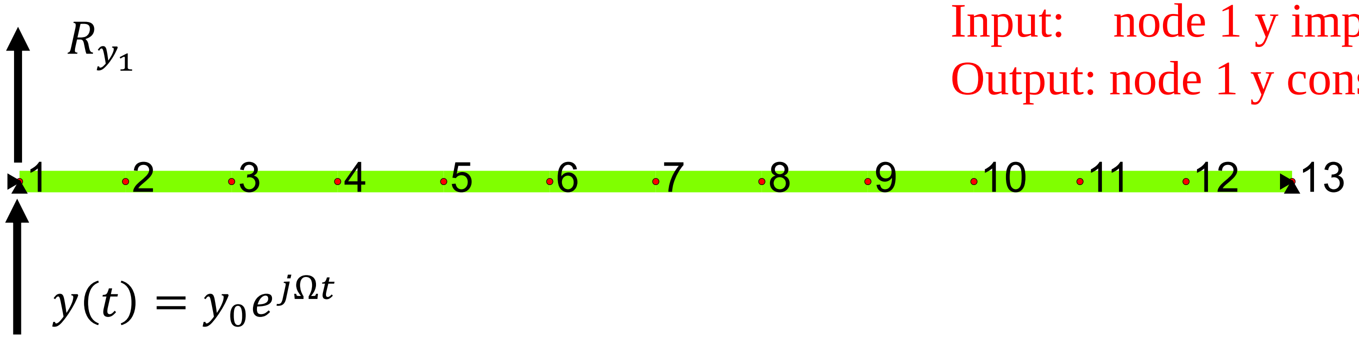
$\underline{F}_F = 0$

$\underline{F}_C = 0$

$\underline{x}_C = y(t)$

$$\begin{cases} [M_{FF}]\ddot{\underline{x}}_F + [M_{FC}]\ddot{\underline{x}}_C + [C_{FF}]\dot{\underline{x}}_F + [C_{FC}]\dot{\underline{x}}_C + [K_{FF}]\underline{x}_F + [K_{FC}]\underline{x}_C = \cancel{\underline{F}_F} \\ [M_{CF}]\ddot{\underline{x}}_F + [M_{CC}]\ddot{\underline{x}}_C + [C_{CF}]\dot{\underline{x}}_F + [C_{CC}]\dot{\underline{x}}_C + [K_{CF}]\underline{x}_F + [K_{CC}]\underline{x}_C = \cancel{\underline{F}_C} + \underline{R} \end{cases}$$

CONSTRAINT FORCES



Input: node 1 y imposed displacement
Output: node 1 y constraint force

Compute the force necessary to apply the motion imposed at the constraint with $y_0 = 0.01\ m$

```
Y = zeros(ncons,1);
Y(idb(1,2)-ndof) = 0.01;

for ii = 1:length(om)
  X(:,ii)= -(-MFF*om(ii)^2 + 1i*CFF*om(ii) + KFF)\...
            (-MFC*om(ii)^2 + 1i*CFC*om(ii) + KFC)*Y;

  R(:,ii)=  (-MCF*om(ii)^2 + 1i*CCF*om(ii) + KCF)*X(:,ii)+...
            (-MCC*om(ii)^2 + 1i*CCC*om(ii) + KCC)*Y;
end

Ry1 = R(idb(1,2)-ndof,:);
```