POLITECNICO
MILANO 1863

DEPARTMENT OF
MECHANICAL ENGINEERING

ADVANCED DYNAMICS OF MECHANICAL SYSTEMS

Assignment 1 – Part A
Cantilever beam

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TARGET

Compute the **natural frequencies** and the **mode shapes** of a cantilever beam, by making reference to the standing wave solution of a slender beam in bending vibration.

Study the forced response of the system, by computing its **Frequency Response Function** (for assigned input and output positions).

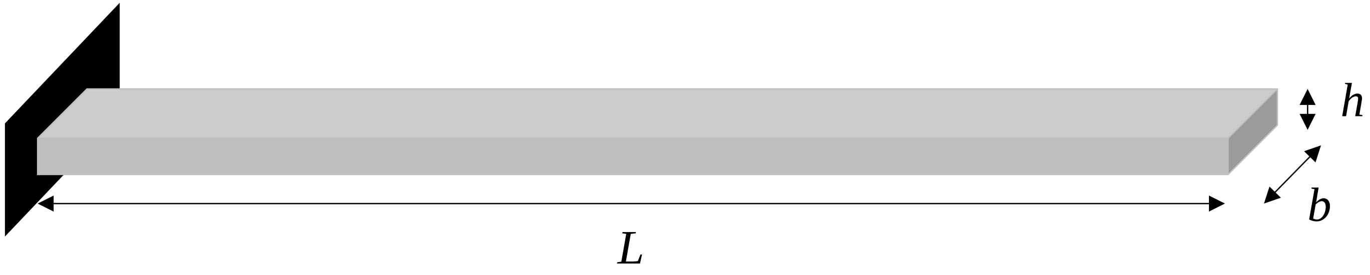
Assuming the FRFs to be representative of an experimental test, apply **modal parameters identification** algorithm according to the FRF-based multi-mode curve fitting method.

Contents:

- Data of the reference structure (geometry and material properties)
- Vibration modes computation
- Frequency Response Functions computation
- Modal parameters identification: FRF-based multi-mode curve fitting method (n = 1)

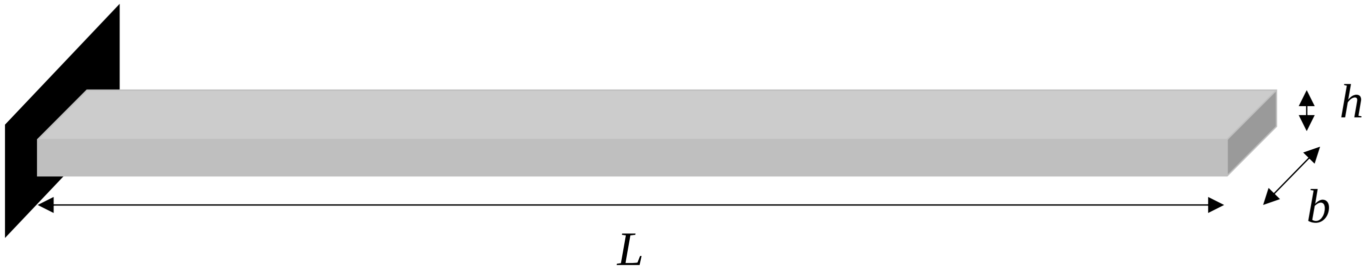
DATA OF THE REFERENCE STRUCTURE

Aluminum beam with rectangular cross-section



Parameter	symbol	unit	value
Lenght	L	mm	1200
Thickness	h	mm	8
Width	b	mm	40
Density	ρ	$\mathrm{kg/m^3}$	2700
Young's Modulus	E	GPa	68

VIBRATION MODES OF THE CANTILEVER BEAM

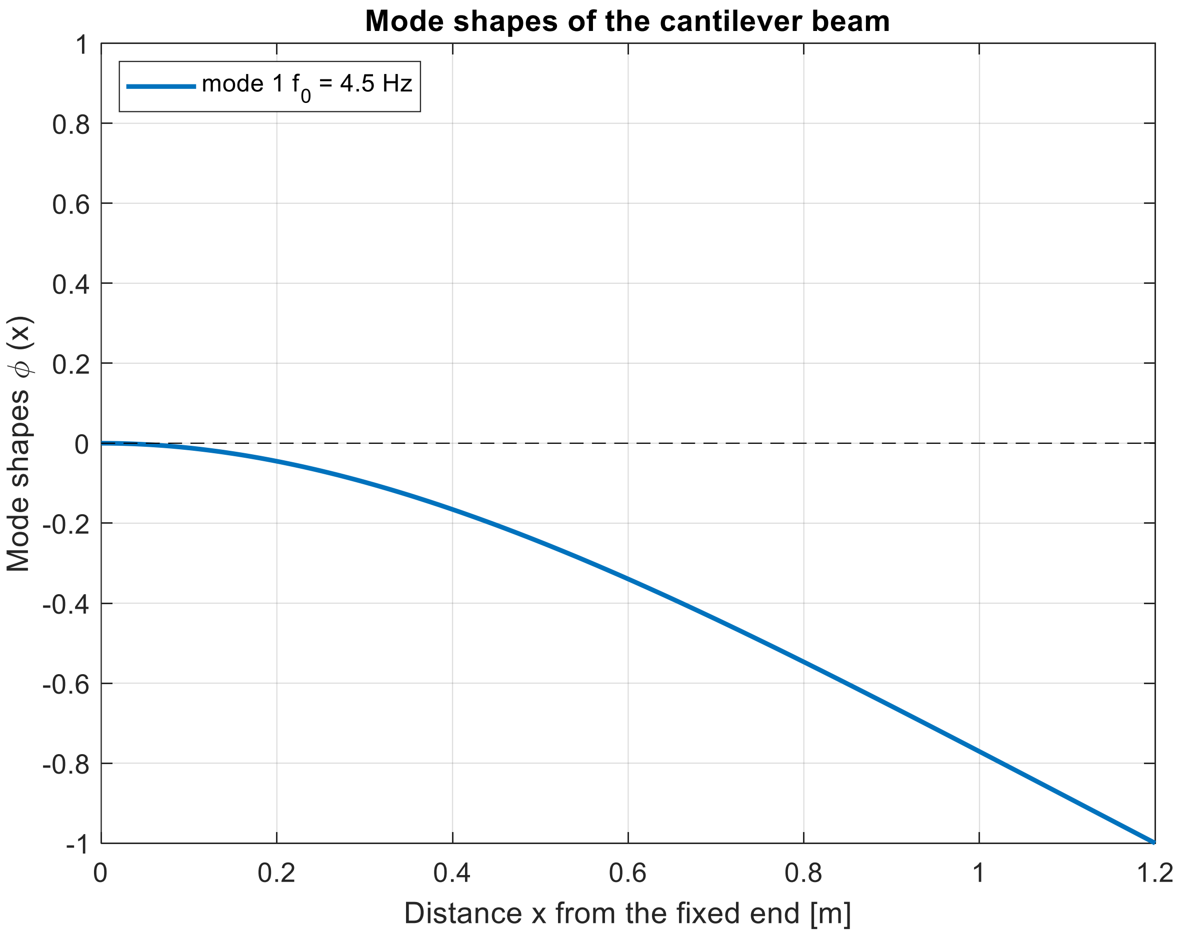


1. Standing wave solution

$$w(x,t) = [A\cos(\gamma x) + B\sin(\gamma x) + C\,\mathrm{Cosh}(\gamma x) + D\,\mathrm{Sinh}(\gamma x)]\cos(\omega t + \varphi)$$

2. Boundary conditions
3. Matrix formulation $\quad [H(\omega)]\,\underline{z} = \underline{0}$
   ($\underline{z}$ vector of unknown coefficients)
4. Solution of the characteristic equation $\quad \det[H(\omega)] = 0 \;\rightarrow\; \omega_i$
   (numerical solution in Matlab)
5. Mode shapes computation $\quad \omega_i \;\rightarrow\; [H(\omega_i)]\,\underline{z}^{(i)} = \underline{0} \;\rightarrow\; \Phi_i(x)$
6. Plot the mode shapes with the associated natural frequencies

VIBRATION MODES OF THE CANTILEVER BEAM
MODE SHAPES



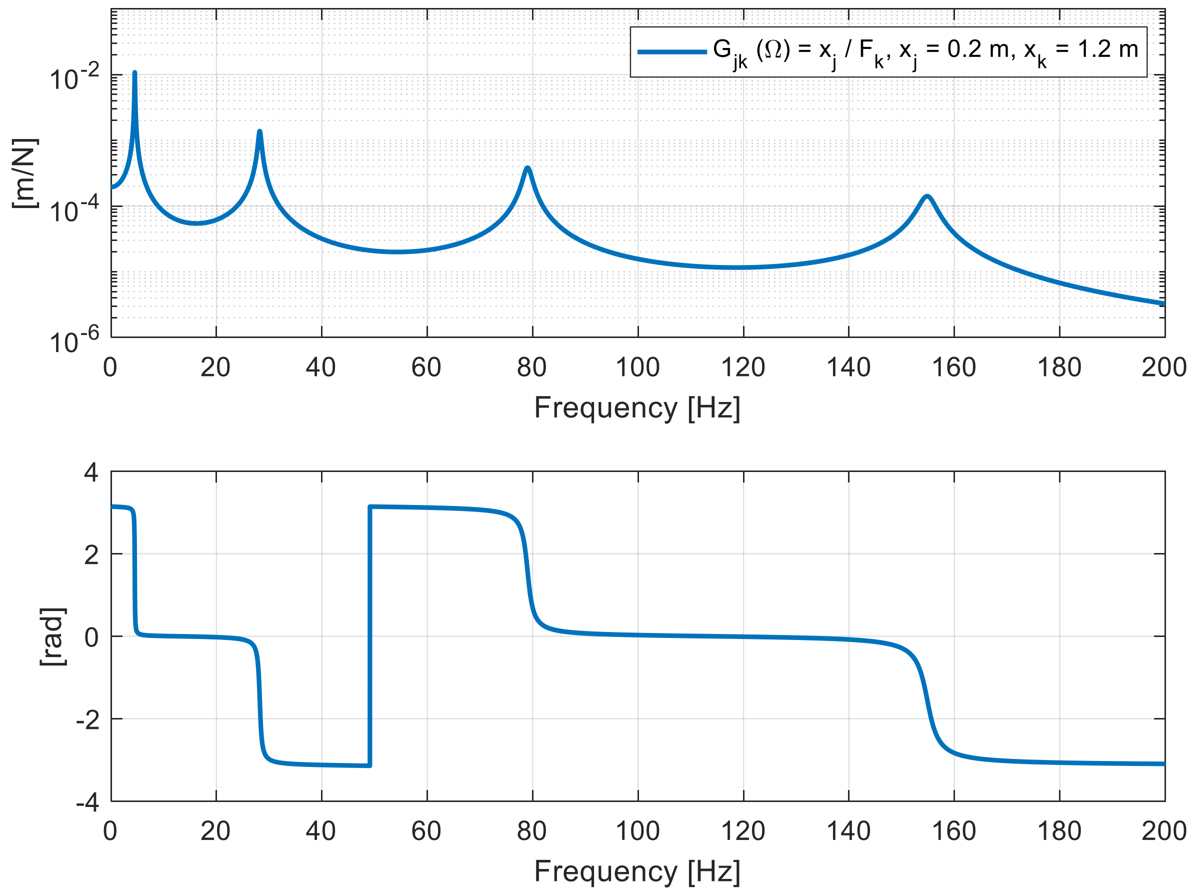
FREQUENCY RESPONSE FUNCTIONS

1. Frequency Response Function

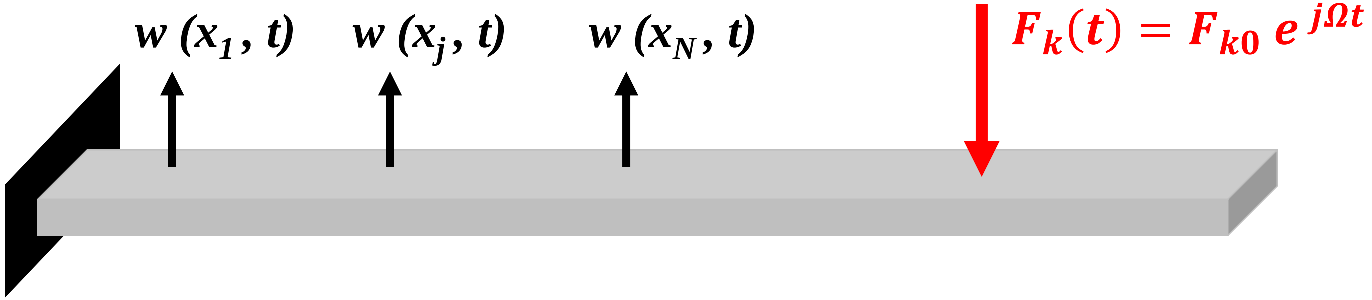
$$G_{jk}(\Omega) = \sum_{i=1}^{n} \frac{\Phi_i(x_j)\,\Phi_i(x_k)/m_i}{-\Omega^2 + j2\xi_i\omega_i\Omega + \omega_i^2}$$

2. Choose input and output positions (x_k and x_j respectively)
3. Define proper damping values (in the order of 1%) ξ_i
4. Compute the modal mass (hint: trapz Matlab function) $m_i = \int_0^L m\,\Phi_i^2(x)\,dx$
5. Plot the FRF

FREQUENCY RESPONSE FUNCTIONS



FREQUENCY RESPONSE FUNCTIONS



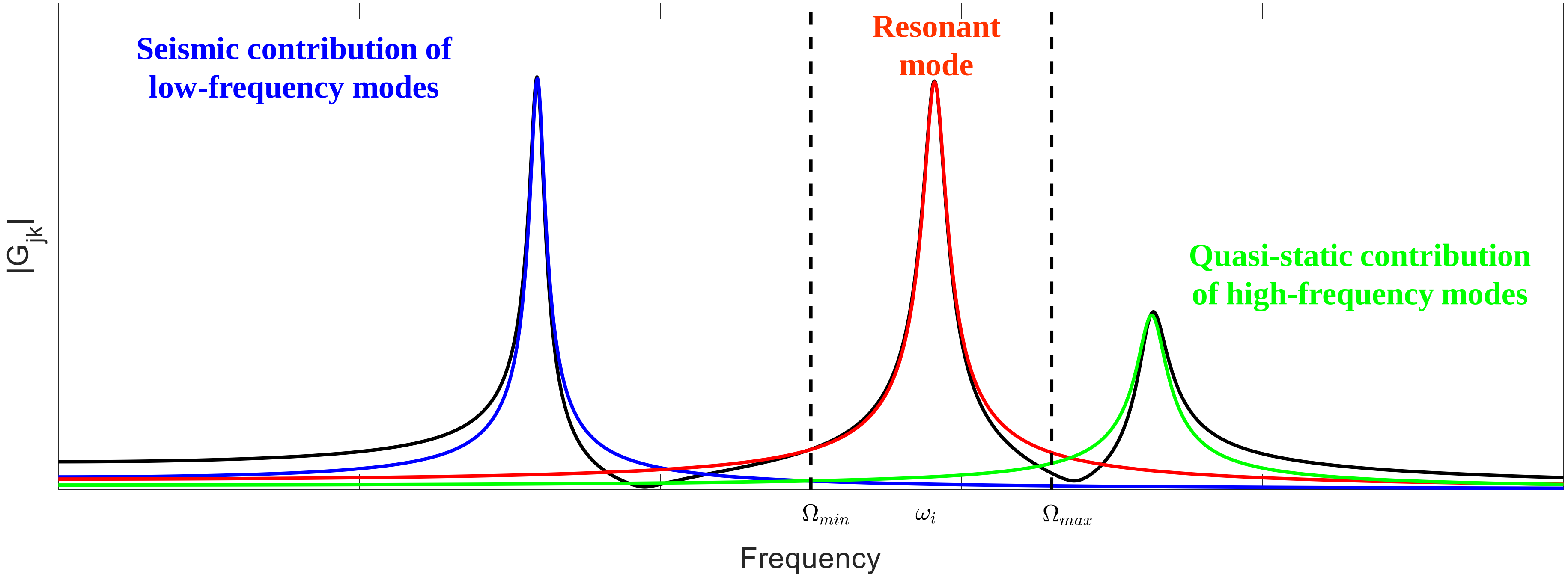
1. Consider the N FRFs numerically computed (for various input locations x_k and measuring positions x_j) to be representative of an experimental test. In the following, we will refer them to as $\left[G_{jk}^{EXP}\right]$
2. Identify the modal parameters according to the FRF-based multi-mode curve fitting method

$$G_{jk}^{NUM}(\Omega) = \sum_{i=1}^{n} \frac{A_{jk}^{(i)}}{-\Omega^2 + j2\xi_i\omega_i\Omega + \omega_i^2} + \frac{R_{jk}^{L}}{\Omega^2} + R_{jk}^{H}$$

MODAL PARAMETERS IDENTIFICATION

For lightly damped structures and well distinguished peaks, the FRF $G_{jk}^{NUM}(\Omega)$ can be approximated around a certain ω_i as:

$$G_{jk}^{NUM}(\Omega) = \frac{A_{jk}^{(i)}}{-\Omega^2 + j2\xi_i\omega_i\Omega + \omega_i^2} + \frac{R_{jk}^L}{\Omega^2} + R_{jk}^H \qquad \Omega_{min} < \omega_i < \Omega_{max}$$



MODAL PARAMETERS IDENTIFICATION

For a given set of experimental FRFs $\left[G_{jk}^{EXP}\right]$, a least squares minimization procedure can be implemented for the estimation of the modal parameters $(\omega_i, \xi_i, X^{(i)})$.

- Considering the experimental FRFs matrix $\left[G_{jk}^{EXP}\right]$, it shows:
  - M rows, corresponding to the length of the frequency vector (with the discrete Ω_s ranging from Ω_{min} to Ω_{max})
  - N columns, corresponding to the *j*-*k* pairs (i.e., of the available FRFs)

  ➢ $G_r^{EXP}(\Omega_s)$ is the generic element of the experimental FRFs matrix $\left[G_{jk}^{EXP}\right]$, corresponding to the r column (FRF) evaluated in correspondence of the frequency Ω_s

- Considering the expression of the FRF adopted to fit the data:

  ➢ $G_r^{NUM}(\Omega_s) = \frac{A_r^{(i)}}{-\Omega_s^2+j2\xi_i\omega_i\Omega_s+\omega_i^2} + \frac{R_r^L}{\Omega_s^2} + R_r^H$ is the numerical FRF estimation around a certain ω_i, corresponding to the r column (FRF) evaluated in correspondence of the frequency Ω_s

MODAL PARAMETERS IDENTIFICATION

The error function to be minimized is then:

$$\epsilon = \sum_{r=1}^{N} \sum_{s=1}^{M} \left(G_r^{EXP}(\Omega_s) - G_r^{NUM}(\Omega_s)\right) \left(G_r^{EXP}(\Omega_s) - G_r^{NUM}(\Omega_s)\right)^*$$

Since the error function ϵ non-linearly depends on the unknown parameters, an iterative minimization procedure is needed:

- lsqnonlin($\epsilon, \underline{x_0}, \dots$) Matlab function can be adopted

```
err = @(x) ...
options = optimoptions('lsqnonlin','Algorithm','levenberg-marquardt');
sol = lsqnonlin(@(x) err(x), X0, [], [], options);
```

To converge to the correct solution, the vector $\underline{x_0}$ of initial guesses is required, consisting of a preliminary estimate of:

- ω_i
- ξ_i
- $A_r^{(i)}$
- R_r^L and R_r^H

$\left.\begin{array}{l}\omega_i\\ \xi_i\\ A_r^{(i)}\end{array}\right\}$ Modal parameters to be identified

R_r^L and R_r^H Residuals

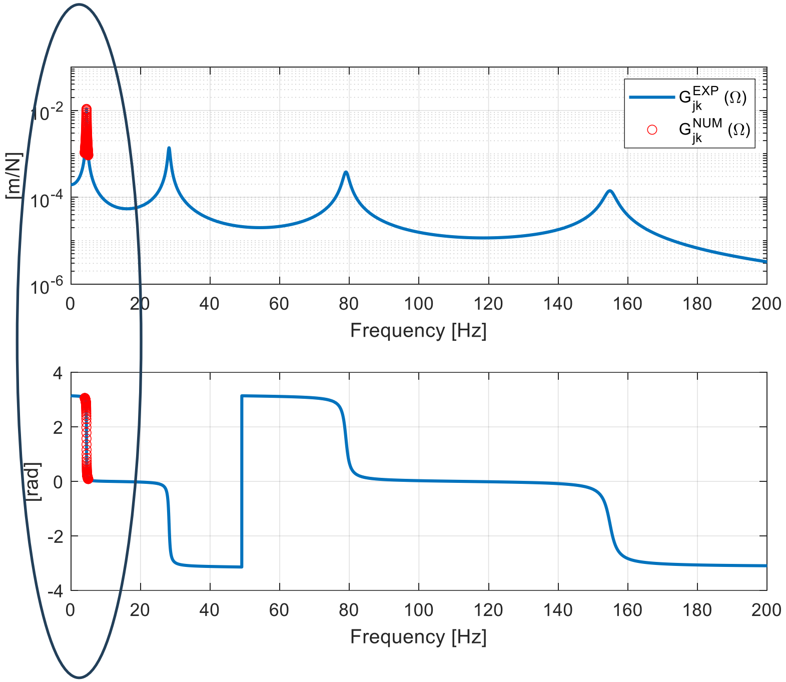
MODAL PARAMETERS IDENTIFICATION

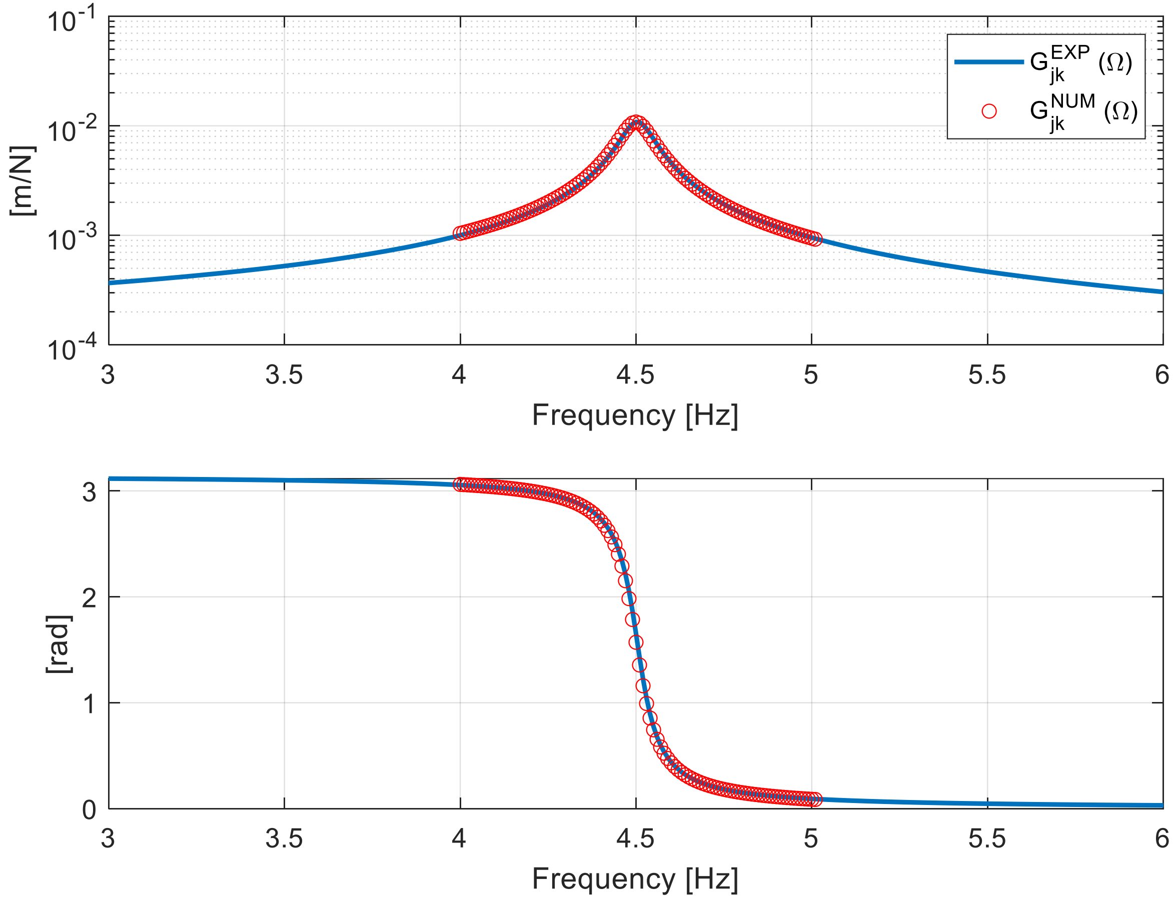
<u>Hints for the code implementation</u>

1. Before starting the identification, make sure that FRFs are arranged in **matrix form**
2. Remember that FRFs are **complex-valued** functions
3. Rely on the **simplified identification methods** to provide first attempt values for $\omega_i, \xi_i, A_r^{(i)}$
4. Assign null value as initial guesses for the other parameters
5. Rely on **function handles @(x)** to define the error to be minimized
6. The non-linear minimization procedure elaborates **simultaneously** the whole set of N FRFs
   → how many outputs **do you have to consider** for the lsqnonlin function, given N FRFs?
7. Residuals can be checked to verify the accuracy of the identification
8. The quality of the estimates can be visually assessed comparing in a plot the identified FRFs $\left[G_{jk}^{NUM}\right]$ and the experimental ones $\left[G_{jk}^{EXP}\right]$

MODAL PARAMETERS IDENTIFICATION

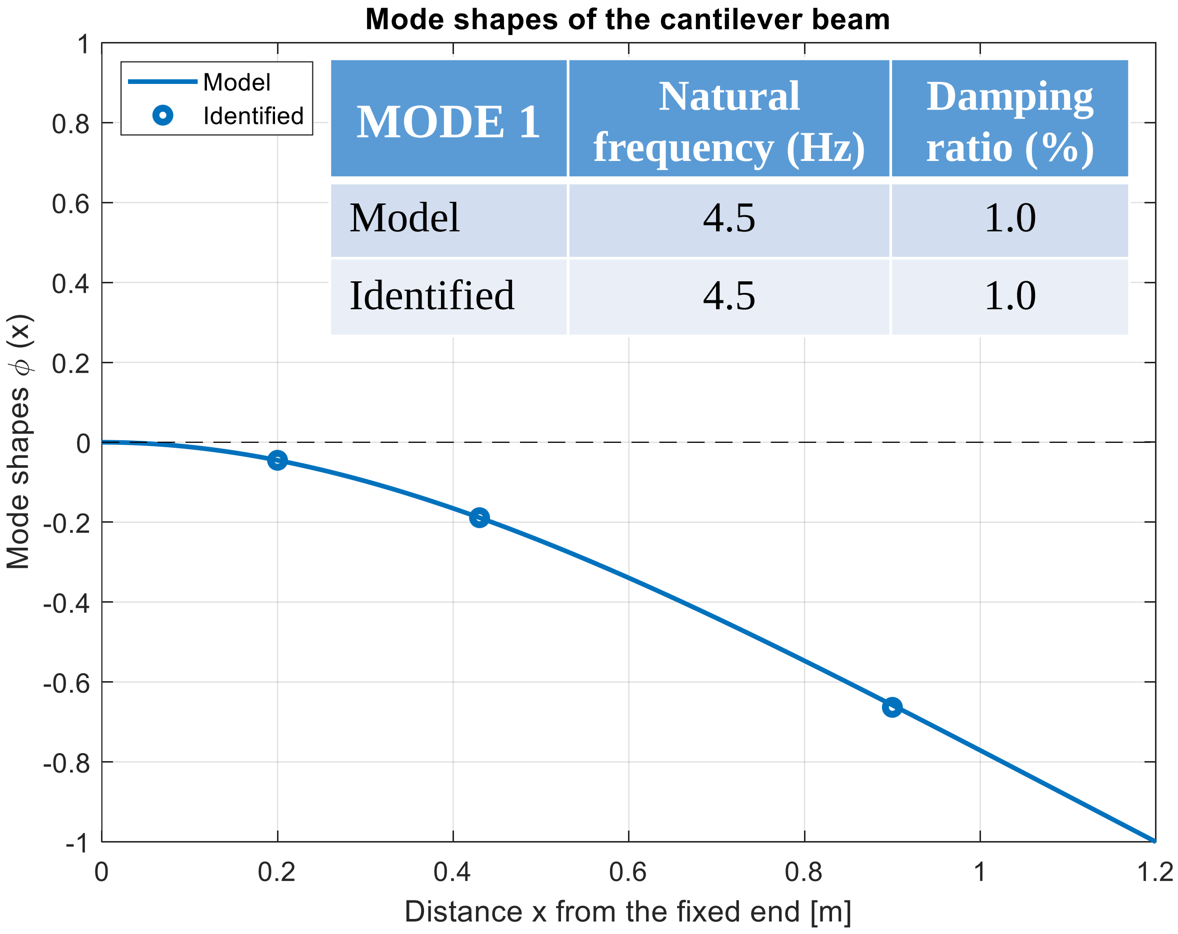
RESULTS OF THE IDENTIFICATION





MODAL PARAMETERS IDENTIFICATION

RESULTS OF THE IDENTIFICATION



MODE 1	Natural frequency (Hz)	Damping ratio (%)
Model	4.5	1.0
Identified	4.5	1.0

A common normalization is recommended for the visualization of the mode shapes comparison.

ASSIGNMENT 1 – PART A

Work out the following items and include the corresponding results in your report of Assignment 1.

1. Briefly describe the procedure followed for computing natural frequencies and mode shapes. Plot the mode shapes of the first four modes with the indication of the associated natural frequencies and provide comments to the results.
2. Compute the FRFs for some combinations of input and output positions. Comment the results.
3. Briefly describe the procedure followed for identifying the natural frequencies, damping ratios and mode shapes of the first four modes, relying on the FRF-based multi-mode curve fitting method ($n = 1$).
4. Check the quality of the identification comparing the identified FRFs and the ones numerically computed.
5. Compare the parameters defined at the simulation stage to the identified ones. Collect the results in table form and plot a diagram showing the comparison of the simulated and identified mode shapes.