POLITECNICO
MILANO 1863

SCUOLA DI INGEGNERIA INDUSTRIALE
E DELL'INFORMAZIONE

ADMS - Assignment 2

Laurea Magistrale in
Mechanical Engineering

Giosuè Maria Giovanni Landolfo 11099459, Maciej Rafał Kopeć 11069411, Virgile Cheminot 11034606

Introduction

The goal of the project is to create and analyse a Finite Element Model of a truss bridge. What we are interested in is the natural frequencies and mode shapes of the bridge, as well as the Frequency Response Functions (FRFs) at different points of the bridge.

The project is divided into 4 main parts: the creation of the FEM model of the bridge, the analysis of the model, the creation of a Tuned Mass Damper (TMD) to reduce the vibrations of the bridge, and a "real world" application of the FEM model.

1. Creation of the FEM model

The bridge is modelled as a truss structure in the vertical plane. It is composed of elements of 9 different properties depending on the position of the element on the bridge. The properties can be seen in Table 1 and the structure in Figure 1. The span of the bridge is simply supported.

	Colour	Mass [kg/m]	EA [N]	EJ [N·m^2]
Deck element	Pink	2.36×10^3	1.59×10^{10}	1.36×10^9
Diag. member K1	Light blue	4.42×10^2	1.18×10^{10}	2.80×10^8
Diag. member K2	Green	2.56×10^2	6.85×10^9	1.36×10^8
Diag. member K3	Cyan	2.75×10^2	7.35×10^9	1.51×10^8
Diag. member K4	Light green	1.75×10^2	4.68×10^9	4.48×10^7
Diag. member K5	Orange	1.45×10^2	3.89×10^9	6.13×10^7
Diag. member K6	Red	94.2	2.52×10^9	1.63×10^7
Upper chord G1	Blue	5.34×10^2	1.43×10^{10}	1.21×10^9
Upper chord G2	Purple	5.93×10^2	1.59×10^{10}	1.36×10^9

Table 1: Properties of the bridge elements

With this information, the input file of the FEM model was created using the `load_structure` function. A first version of the model was to consider each beam as seen in Figure 1 as a separate element, so 23 elements and

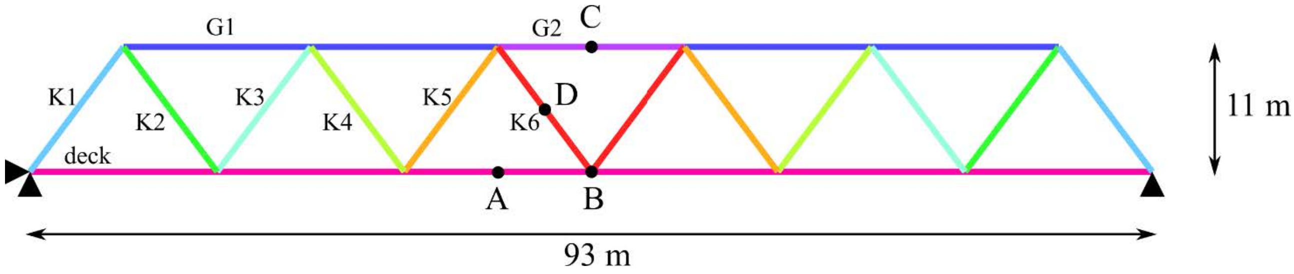


Figure 1: Truss structure of the bridge with the different properties

13 nodes. However, using this element does not allow to analyse the structure up to 7 Hz. In fact, none of the elements respect the ratio $\frac{\omega_{1,k}}{\Omega_{max}} \geqslant 1.5$. This ratio is used to ensure that every element stays in the quasi-static domain, with $\omega_{1,k}$ the first natural frequency of the element and Ω_{max} the maximum frequency of the analysis, which is 7 Hz in this case.
To solve this issue, each element was divided into 2 elements, so 46 elements and 36 nodes. This way, we are sure that the natural frequencies of the elements are above 7 Hz. The model can be seen in Figure 2 with all the node numbers.

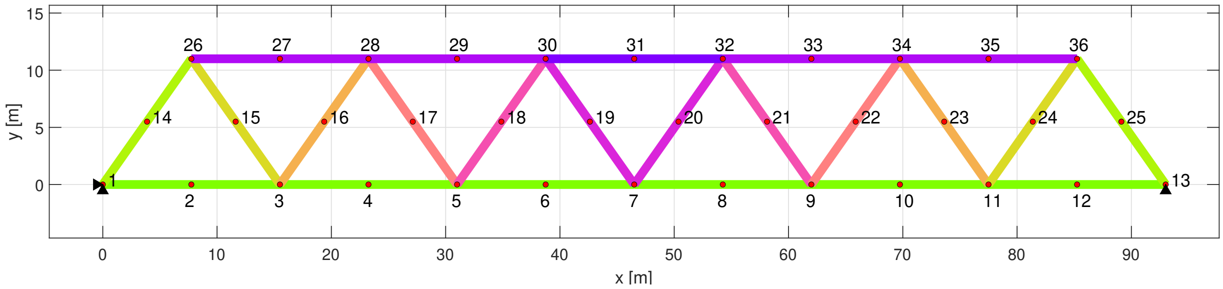


Figure 2: FEM model of the bridge

2. Analysis of the FEM model

2.1. Dynamic analysis

We can now compute the natural frequencies and mode shapes of the bridge using the FEM model. This is done by first computing the Mass and Stiffness matrices of the structure with `assem`, keeping only the free-free partition of the matrices. The first 6 modes are shown in Figure 3.

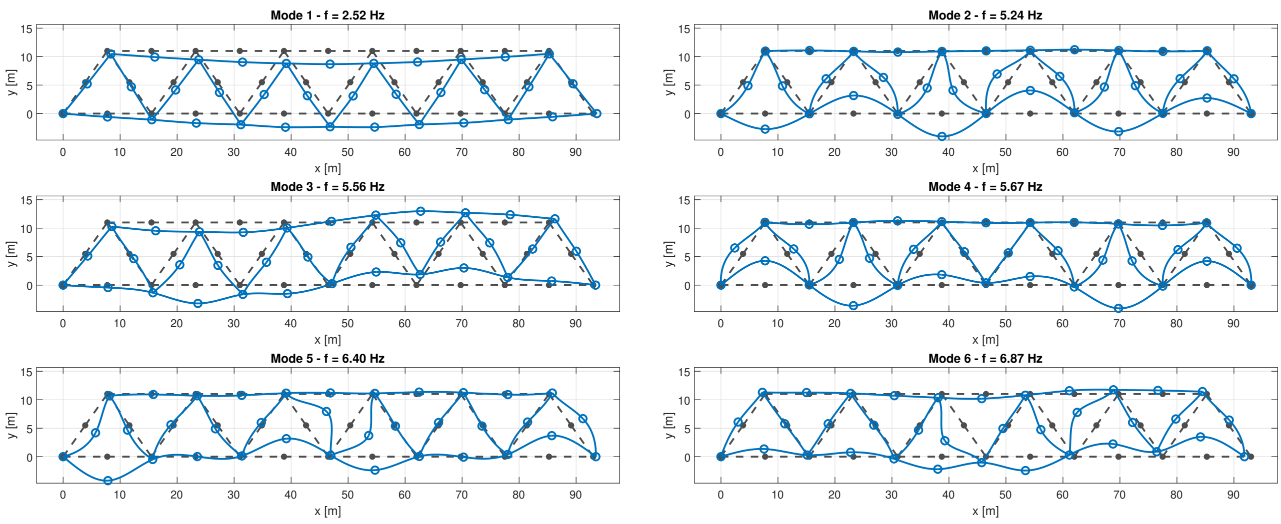
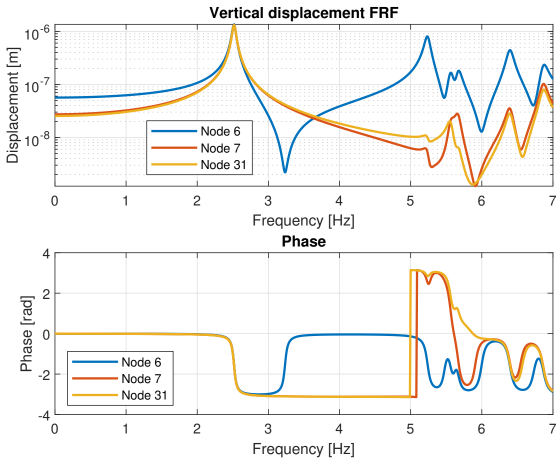


Figure 3: First 6 mode shapes of the bridge with the corresponding natural frequencies

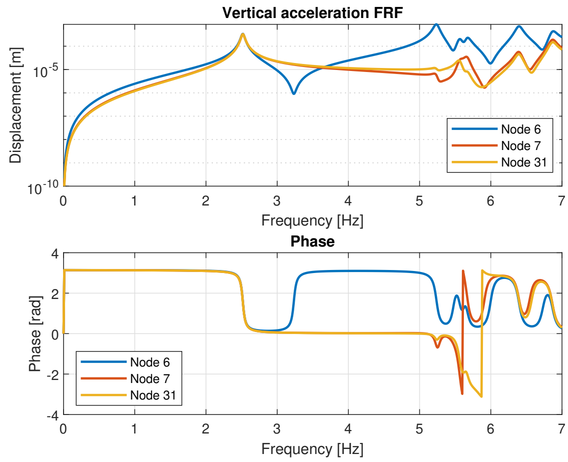
Once this is done, we need to compute the FRFs of the bridge at different points. To achieve that, we first need to compute the Damping matrix of the structure. We used the Rayleigh damping model $[C] = \alpha[M] + \beta[K]$, with :

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = (A^T A)^{-1} A^T \underline{B} \text{ where } A = \begin{bmatrix} \frac{1}{2\omega_1} & \frac{\omega_1}{2} \\ \frac{1}{2\omega_2} & \frac{\omega_2}{2} \end{bmatrix} \text{ and } \underline{B} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \text{ with } \zeta_1 = 1\% \text{ and } \zeta_2 = 0.75\%$$

We then choose to compute the FRFs for a vertical force at node A (Figure 1), which is node 6 in our model.



(a) Vertical displacement FRF



(b) Vertical acceleration FRF

Figure 4: FRFs of nodes A (co-located), B (7) and C (31)

The shear force, bending moment and axial force FRFs are computed at nodes C (31) and D (19). These internal forces are computed by using the shape functions of the elements and the displacement FRFs.

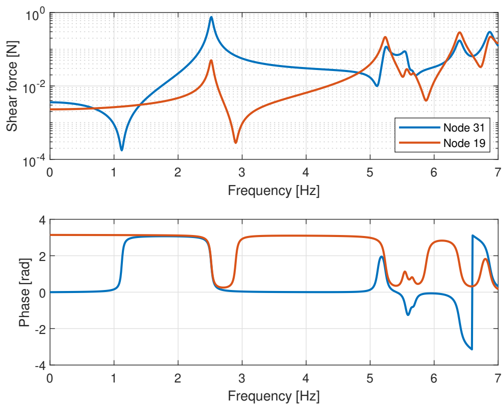


Figure 5: Shear force FRF

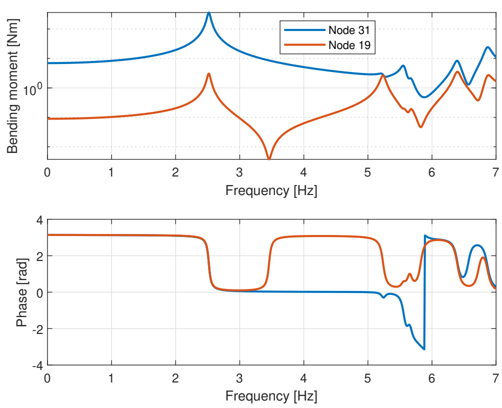


Figure 6: Bending moment FRF

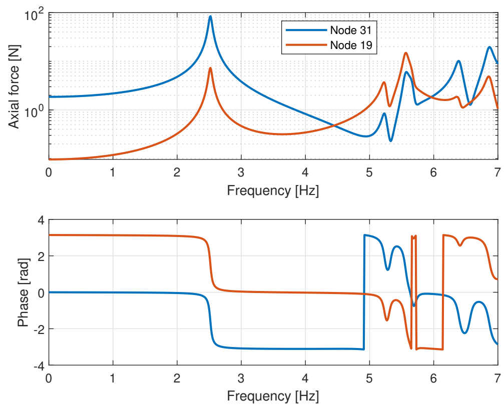


Figure 7: Axial force FRF

The reaction forces at the constrained nodes are computed as well using the constrained-free partition of the matrices. The results can be seen in Figure 8.

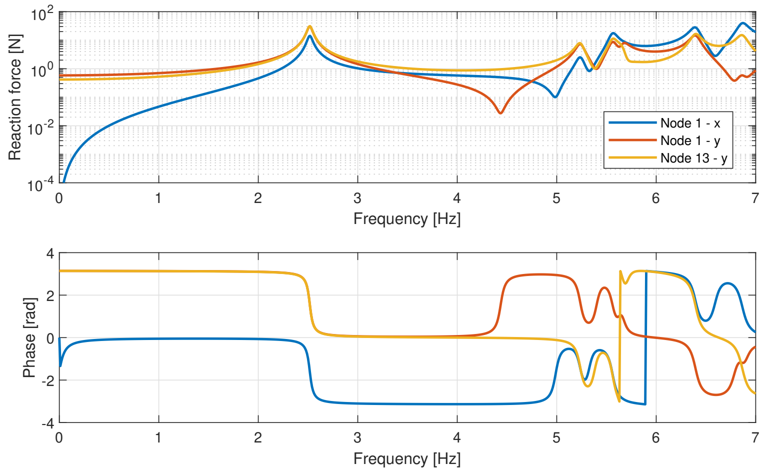


Figure 8: Reaction forces of the bridge

We now consider only the first 3 modes of the structure by using the modal superposition method. The modal vector matrix is created as such:

$$[\Phi] = [\underline{X}^{(1)}\underline{X}^{(2)}\underline{X}^{(3)}]_{105\times 3}$$

$$\text{So } [M_q] = [\Phi]^T[M][\Phi] \text{ ; } [K_q] = [\Phi]^T[K][\Phi] \text{ ; } [C_q] = [\Phi]^T[C][\Phi] \text{ and } [Q_q] = [\Phi]^T\underline{F}$$

Let's now compute the FRFs of the structure using the modal superposition method and then compare them to the FRFs computed with the full model. The results can be seen in Figure 9.

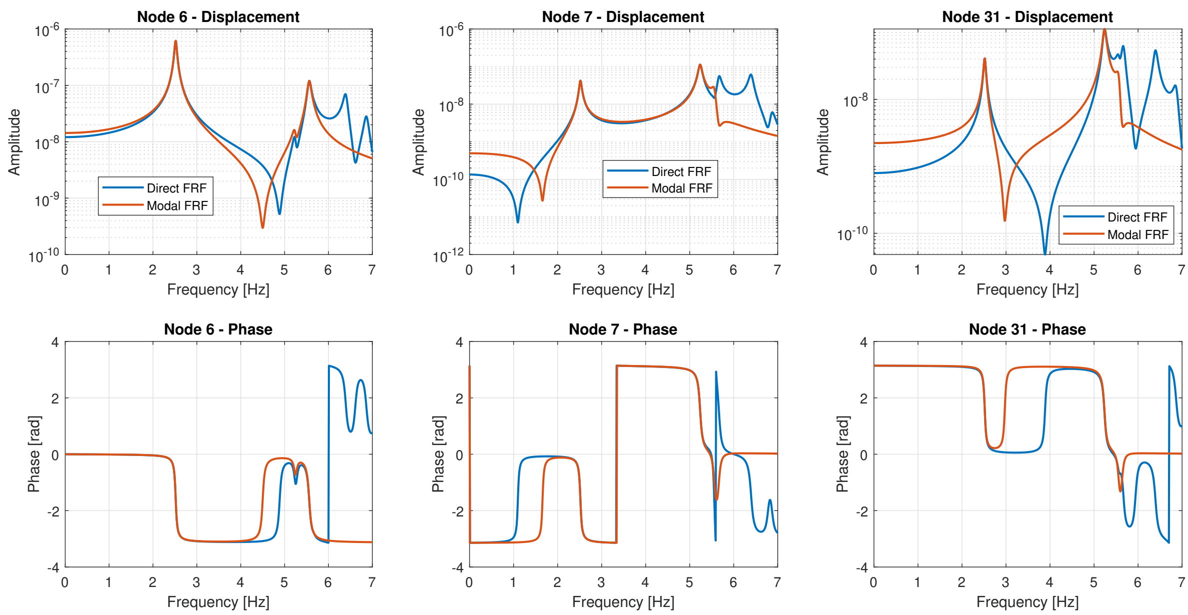


Figure 9: Comparison of the FRFs of the full model and the modal superposition method

As expected, the FRFs computed with the modal superposition method are very close to the FRFs computed with the full model when the frequency is around the 3 first natural frequencies of the structure. However, the anti-resonances are not well represented with this method.

2.2. Static analysis

In addition to the dynamic analysis, we also computed the static analysis of the bridge. The mass of the whole bridge is computed by summing the vertical contribution of each node in the mass matrix. The mass of the bridge is 299112 kg, which is not surprising considering that the length of the bridge is 93 m.

Let's now compute the vertical displacement of the bridge under its own weight. Knowing that the deck of the bridge has a linear mass density of 2.36×10^3 kg/m, which is the highest linear mass density by far compared to the other elements, we can expect that the displacement of the bridge will be mostly due to the deck. In fact, the deck's mass is $93 \times 2.36 \times 10^3 = 219480$ kg, which is more than 73% of the total mass of the bridge.

One first method to compute the force due to the weight of the bridge, and thus the deflection, is to use a distributed load on the selected elements. The force is computed using the shape functions of the elements. In our case, the structure is only composed of bar elements (1D elements), so the equivalent force is computed as such:

$$\underline{F}_k^L = \int_0^{L_k} \underline{N_u}^T(x) p_x^L(x) dx + \int_0^{L_k} \underline{N_w}^T(x) p_y^L(x) dx$$

with $\underline{N_u}$ and $\underline{N_w}$ the shape functions of the element, for the horizontal displacement, and vertical displacement and angle respectively. $p_x^L(x)$ and $p_y^L(x)$ are the horizontal and vertical distributed loads on the element. This force vector is in local coordinates, so it needs to be transformed into global coordinates with the rotation matrix of the element: $\underline{F}_k^G = [\Lambda_k]\underline{F}_k^L$. The force vector is then added to the global force vector to the corresponding degrees of freedom with an expansion matrix: $\underline{F}^G = \sum_k [E_k]^T \underline{F}_k^G$.
Inversely, the distributed load is converted from the global to the local coordinates with the inverse of the rotation matrix.
We apply this method to the deck elements of the bridge in the first time and then to all the elements of the bridge. The results can be seen in Figure 10 and Table 2. We choose to look at the vertical displacement of node 6 which is the node with the maximum deflection, and the horizontal displacement of node 13 which is the node at the right end of the bridge.

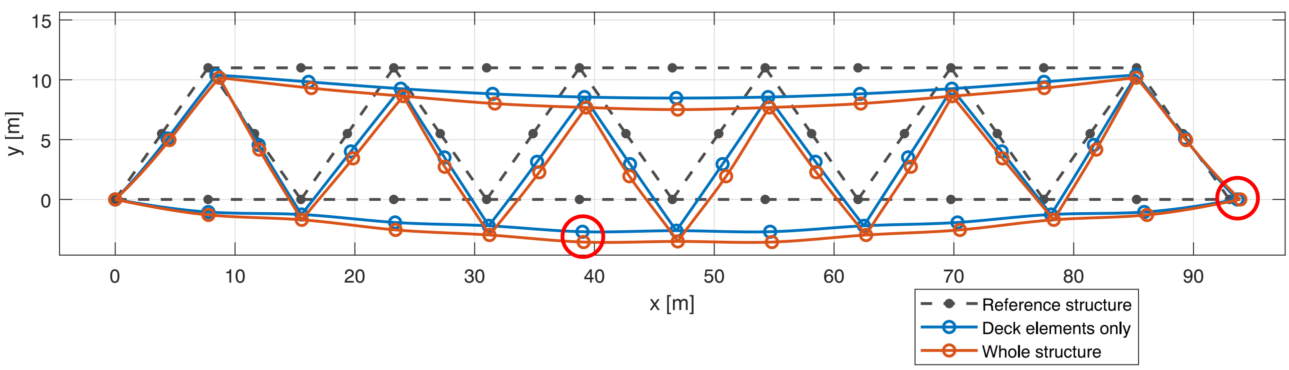


Figure 10: Static deflection of the bridge due to its own weight (scale factor of 75)

	Node 6 vertical disp.	Node 13 horizontal disp.
All elements	-0.0475 m	0.0121 m
Deck only	-0.0360 m (25.64% error)	0.0088 m (27.27% error)
Deck only compensated	-0.0490 m (1.28% error)	0.0120 m (0.83% error)

Table 2: Vertical displacement of the bridge due to its own weight (distributed load method)

The error between the vertical displacement of the bridge computed with the deck only and with all the elements is quite high. However, since the displacement is directly proportional to the weight of the elements, we can try to compensate the error by multiplying the vertical displacement of the deck only by the ratio of the total mass of the bridge and the mass of the deck $\frac{m_{tot}}{m_{deck}} = 1.363$. In our case, the error is reduced drastically, as seen in Table 2. This method is not perfect, but works quite well in this case since the geometry of the bridge is quite simple and the deck is the main contributor to the mass of the bridge. But in more complex structures, this method might not work as well.

Another method to compute the deflection of the bridge is to directly compute the weight of the bridge at each node and then compute the deflection. We can apply this the following way:

$$\underline{F}_k^G = [M]\underline{g} \text{ with } g(i) = \begin{cases} 9.81 \text{ if } i \text{ corresponds to a vertical DoF} \\ 0 \text{ otherwise} \end{cases}$$

We apply this method again to the deck elements of the bridge in the first time and then to all the elements of the bridge. As we can see in Table 3, the results are very close to the results obtained with the distributed load method. And especially considering all the elements, the result is basically identical.

	Node 6 vertical disp.	**Node 13 horizontal disp.**
All elements	-0.0475 m	0.0121 m
Deck only	-0.0370 m	0.0091 m

Table 3: Vertical displacement of the bridge due to its own weight (mass matrix method)

This method is way simpler than the distributed load method and gives quite good results. However, it is not as precise as the distributed load method, and this is even more true if the mesh is not fine enough.

3. Creation of a Tuned Mass Damper

The objective of of a TMD is to reduce the vibrations of the structure for a specific frequency. The TMD we want to create will be used to reduce the amplitude of the vertical vibration of the bridge at node A for the first natural frequency of the structure. The TMD is composed of a mass m_l, a spring k_l and a damper c_l. We need to find the optimal values for these parameters, whilst keeping the mass of the TMD within 2% of the mass of the structure, and the damping ratio under 30%.

Creating the TMD requires us to create a new input file for the FEM model with the TMD, adding a new node with its constraints (as seen in Figure 11). Nevertheless, the TMD is not yet connected to the structure. To do so, we need to add manually the stiffness and damping values associated with the TMD to the global stiffness and damping matrices of the structure.

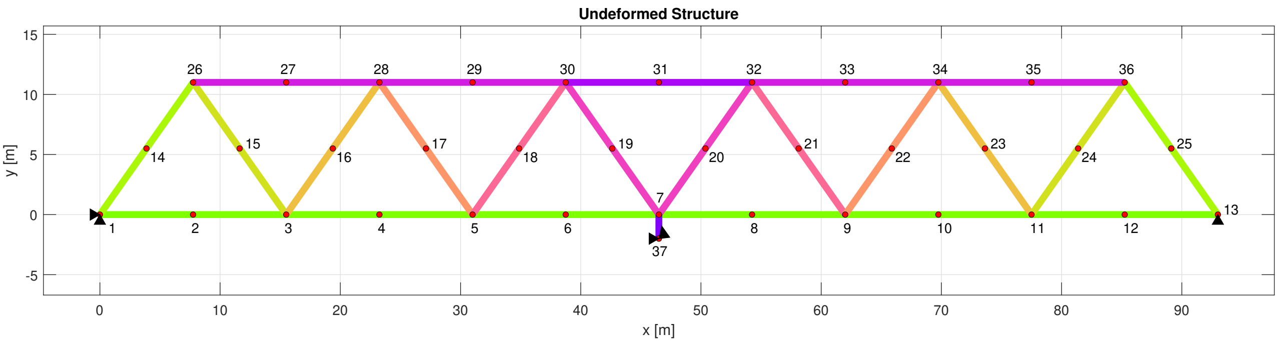


Figure 11: FEM model of the bridge with the TMD represented by node 37

Note that the element connecting the TMD to the structure (node 7 to node 37) is simply here to represent the connection between the TMD and the structure visually. It does not have any physical properties. In fact, its linear mass density, axial and bending stiffness have been set to 0.

The real challenge is to find the optimal values for the mass, spring and damper of the TMD. Since we want to damp the first mode of the structure, the spring stiffness and mass are related to the natural frequency of this node: $k_l = m_l\omega_1^2$. We then have two parameters to play with: the mass of the TMD m_l and the damping coefficient ζ_l. Using the modal method with only the first mode of the structure, we reduce the structure to a single degree of freedom system of mass m_{mod}, stiffness k_{mod} and damping c_{mod}.

We decided in the end to choose $m_l = 333$ kg and $\zeta_l = 0.2$. These values respect the constraints we set and reduce the amplitude of the vertical vibration of the bridge at node A by around 20% as seen in Figure 12. If we wanted to reduce the amplitude even more, we could use other configurations of the TMD. During tinkering with the TMD, we found that for a reduction of 50% of the amplitude, we could use $m_l = 999$ kg and $\zeta_l = 0.14$. The best configuration we found within the constraints we set was $m_l = 5982.2$ (2% of the mass of the structure) and $\zeta_l = 0.1204$, which reduces the amplitude by 82.5%. By tuning the natural frequency of the TMD slightly above or below the natural frequency of the structure, we can reduce the amplitude even more. We were able to reduce the amplitude by 86.2% with $m_l = 5982.2$ kg, $\zeta_l = 0.113$ and $\omega_l = 2.398$ Hz, which is slightly under the 2.52 Hz of the first mode of the structure.

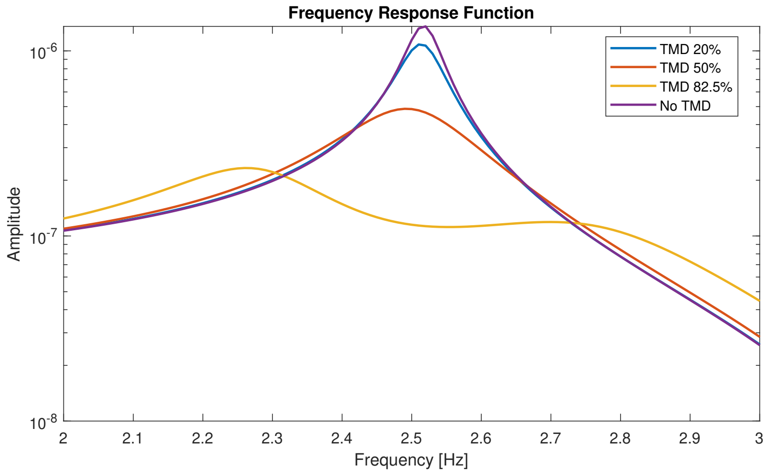


Figure 12: FRF of the bridge at node A with and without the TMD

4. Real world application

We now want to apply the FEM model to a "real world" application. Let's consider that the bridge is crossed by a long train. The train is modelled as a moving sequence of loads on the bridge, with the distance between the bogies of the train being $d = 27.5$ m.

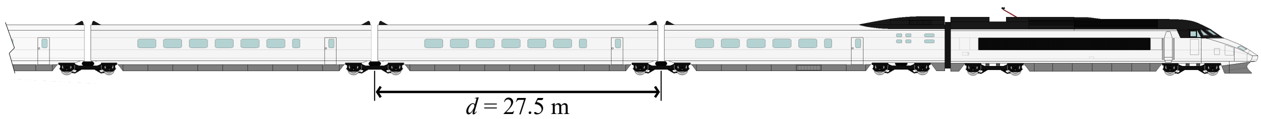


Figure 13: Model of the train

With $L = 93$ m the length of the bridge and V the speed of the train, we can compute the frequency of the loads on the bridge as $\Omega_0 = \frac{2\pi V}{d}$. Whenever an harmonic of this frequency is equal to the natural frequency of the bridge ω_i, the i-th mode of the bridge will be excited. We can write this condition as:

$$k\Omega_0 = \omega_i \Leftrightarrow k\frac{2\pi V_{ik}}{d} = \omega_i \Leftrightarrow V_{ik} = \frac{\omega_i d}{2\pi k}$$

with V_{ik} the critical speed for the mode i and harmonic k. Generally, the most dangerous situations correspond to the first modes of the structure and the lowest values of k ($k \leqslant 3$).

	k=1	**k=2**	**k=3**
Mode 1 (f = 2.52 Hz)	249.2 km/h	124.6 km/h	83.1 km/h
Mode 2 (f = 5.24 Hz)	518.5 km/h	259.2 km/h	172.8 km/h

Table 4: Critical speeds for the first 2 modes of the bridge

As we can see in Table 4, the critical speeds are well within the speed limits of trains. This is especially true for the first mode of the bridge with the first harmonic. In addition to being the most dangerous situation, a train travelling at 249.2 km/h is totally conceivable for high-speed rail networks.

5. Conclusions

This project allowed us to create and analyse a Finite Element Model of a truss bridge. We were able to compute the natural frequencies and mode shapes of the bridge, as well as the FRFs at different points of the bridge. We also created a Tuned Mass Damper to reduce the vibrations of the bridge and applied the FEM model to a "real world" application. The results were quite satisfying and allowed us to understand the process of creating and analysing a FEM model. Some improvements could be made, especially in the creation of the TMD, but overall the project was a success.