POLITECNICO
MILANO 1863

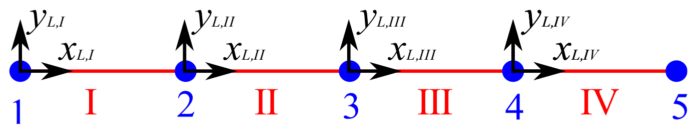
DEPARTMENT OF
MECHANICAL ENGINEERING

ADVANCED DYNAMICS OF MECHANICAL SYSTEMS

Finite Elements Method (FEM) in structural dynamics: software implementation in Matlab environment – part 2

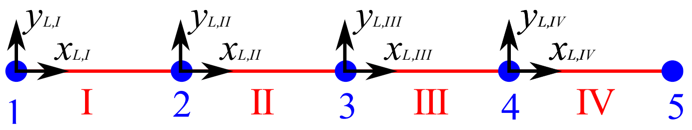
I. LA PAGLIA

EXAMPLE: MAIN CODE




```
clear all; close all; clc

L = 1.2;                % [m]
E = 68e9;               % [Pa]
b = 40e-3;              % [m]
h = 8e-3;               % [m]
r = 2700;               % [kg/m^3]
m = r*b*h;              % [kg/m]
J = 1/12*b*h^3;         % [m^4]
A = b*h;                % [m^2]
EA = E*A;               % [N]
EJ = E*J;               % [Nm^2]

Omax = 100*2*pi;
a = 1.5;
Lmax = sqrt(pi^2/a/Omax * sqrt(EJ/m));

% load the input file and assemble the structure
[file_i,xy,nnod,sizew,idb,ndof,incid,l,gamma,m,EA,EJ,posit,nbeam,pr]=loadstructure;

% draw the structure
dis_stru(posit,l,gamma,xy,pr,idb,ndof);

% assemble mass and stiffness matrices
[M,K]=assem(incid,l,m,EA,EJ,gamma,idb);
```

NATURAL FREQUENCIES AND MODE SHAPES



Once the M and K matrices have been assembled, natural frequencies and mode shapes of the system can be finally computed.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	2.2217e-04	0	0.0024	-1.6663e-04	0	0	0	0	0	0	0	0	0	0.0041	0
2	0	0.1728	0	0	0.0432	0	0	0	0	0	0	0	0.0432	0	0
3	0.0024	0	0.1925	0	0	0.0333	-0.0024	0	0	0	0	0	0	0.0333	0
4	-1.6663e-04	0	0	4.4434e-04	0	0.0024	-1.6663e-04	0	0	0	0	0	0	-0.0024	0
5	0	0.0432	0	0	0.1728	0	0	0.0432	0	0	0	0	0	0	0
6	0	0	0.0333	0.0024	0	0.1925	3.4694e-18	0	0.0333	-0.0024	0	0	0	0	0
7	0	0	-0.0024	-1.6663e-04	0	3.4694e-18	4.4434e-04	0	0.0024	-1.6663e-04	0	0	0	0	0
8	0	0	0	0	0.0432	0	0	0.1728	0	0	0.0432	0	0	0	0
9	0	0	0	0	0	0.0333	0.0024	0	0.1925	-4.3368e-18	0	-0.0024	0	0	0.0333
10	0	0	0	0	0	-0.0024	-1.6663e-04	0	-4.3368e-18	4.4434e-04	0	-1.6663e-04	0	0	0.0024
11	0	0	0	0	0	0	0	0.0432	0	0	0.0864	0	0	0	0
12	0	0	0	0	0	0	0	0	-0.0024	-1.6663e-04	0	2.2217e-04	0	0	-0.0041
13	0	0.0432	0	0	0	0	0	0	0	0	0	0	0.0864	0	0
14	0.0041	0	0.0333	-0.0024	0	0	0	0	0	0	0	0	0	0.0963	0
15	0	0	0	0	0	0	0	0	0.0333	0.0024	0	-0.0041	0	0	0.0963

M_{FF} M_{FC} M_{CF} M_{CC}

According to the boundary conditions, the mass matrix M can be partitioned as $[M] = \begin{bmatrix} [M_{FF}] & [M_{FC}] \\ [M_{CF}] & [M_{CC}] \end{bmatrix}$

- ➢ Similarly, also the stiffness matrix K is partitioned accordingly

NATURAL FREQUENCIES AND MODE SHAPES

Remember that

$$[M] = \begin{bmatrix} [M_{FF}] & [M_{FC}] \\ [M_{CF}] & [M_{CC}] \end{bmatrix}$$

$$[K] = \begin{bmatrix} [K_{FF}] & [K_{FC}] \\ [K_{CF}] & [K_{CC}] \end{bmatrix}$$

FF free dofs inertia/stiffness seen by free dofs

FC constrained dofs inertia/stiffness seen by free dofs

CF free dofs inertia/stiffness seen by constrained dofs

CC constrained dofs inertia/stiffness seen by constrained dofs

Natural frequencies and mode shapes are obtained by solving the following eigenvalue problem.

$$(-\omega^2\ [I] - [M_{FF}]^{-1}[K_{FF}])\ \underline{X} = \underline{0}$$

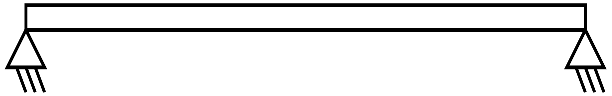
The **eig** matlab function can be used to this aim (NB: frequencies are not in increasing order!)

```
[modes omega2] = eig(inv(MFF)*KFF);
omega = diag(sqrt(omega2));

% Sort frequencies in ascending order
[omega_sorted omega_sorted_indices] = sort(omega);
% Sort mode shapes in ascending order
modes_sorted = modes(:,omega_sorted_indices);
```

eig returns a diagonal matrix «*omega2*» of eigenvalues and a matrix «*modes*» whose columns are the corresponding eigenvectors

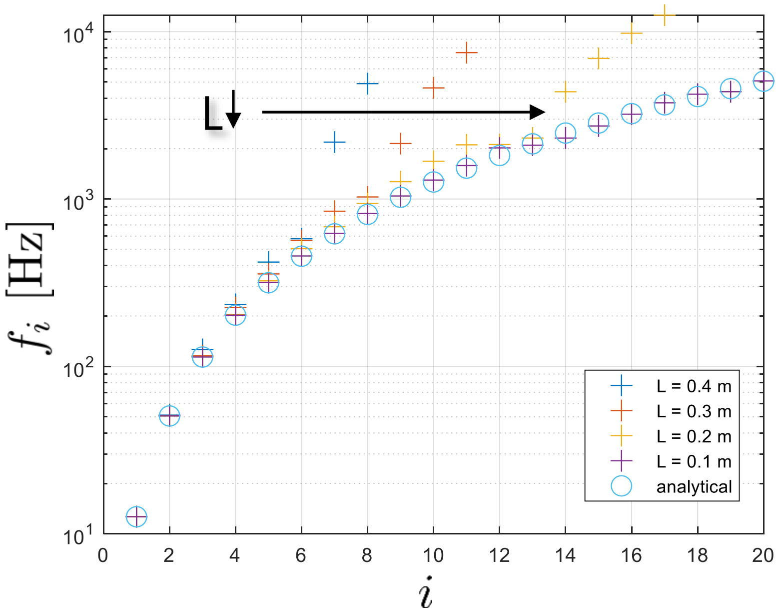
NATURAL FREQUENCIES AND MODE SHAPES

For the simple case of the pin-pin aluminium beam, the analytical solution can be considered as a reference for computing the natural frequencies of the system.



$$f_i = \frac{1}{2\pi}\left(\frac{i\,\pi}{L}\right)^2\sqrt{\frac{EJ}{m}}$$

Natural frequencies with different value of element length

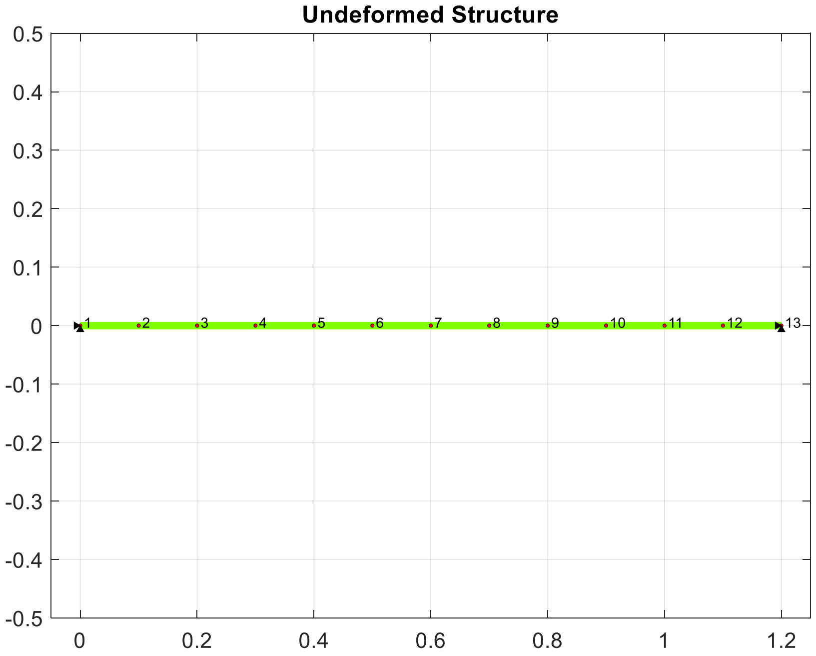


Mode	Analytical Freq [Hz]	FEM L 0.4m Freq [Hz]	FEM L 0.3m Freq [Hz]	FEM L 0.2m Freq [Hz]	FEM L 0.1m Freq [Hz]
1°	12.64	12.65	12.65	12.64	12.64
2°	50.57	51.17	50.77	50.61	50.57
3°	113.78	126.29	115.86	114.23	113.82
4°	202.28	234.82	224.51	204.67	202.51
5°	316.06	420.17	356.86	324.45	316.92

NATURAL FREQUENCIES AND MODE SHAPES

<u>Undeformed structure</u>

```
dis_stru(posit,l,gamma,xy,pr,idb,ndof);
```

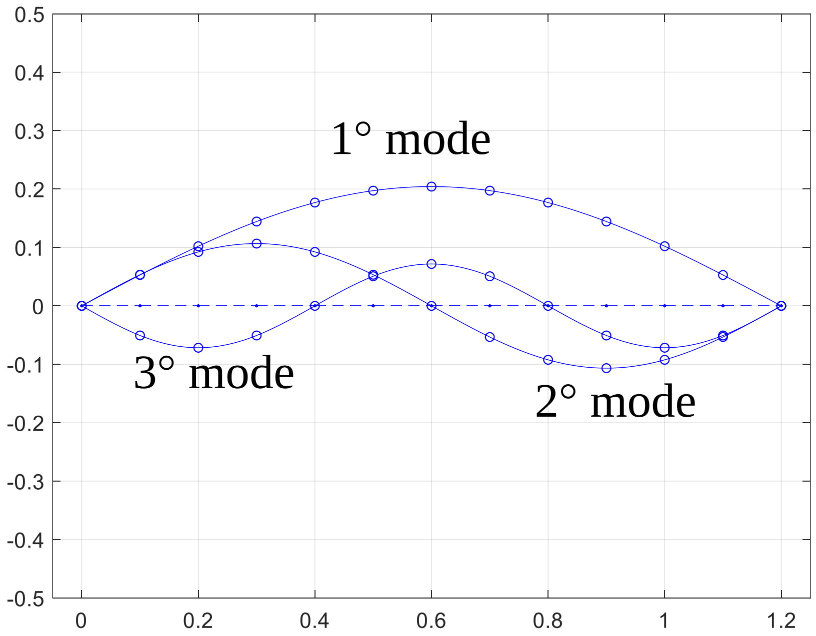


<u>Mode shapes</u>

```
scale_factor = 1.5;
mode = modes_sorted(:,ii);

figure();
diseg2(mode,scale_factor,incid,l,gamma,posit,idb,xy);
```

scale factor for the visualization

of the mode shape



DAMPING MATRIX

Assuming that damping ratios ξ_i have been identified by means of proper modal parameters identification techniques, the coefficients α and β necessary to compute the damping matrix can be identified.

$$[C_{FF}] = \alpha[M_{FF}] + \beta[K_{FF}] \quad \Rightarrow \quad \xi_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2} \quad \Rightarrow \quad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = ([A]^T\,[A])^{-1}[A]^T\underline{B}$$

where $$[A] = \begin{bmatrix} \frac{1}{2\omega_1} & \frac{\omega_1}{2} \\ \vdots & \vdots \\ \frac{1}{2\omega_i} & \frac{\omega_i}{2} \\ \vdots & \vdots \\ \frac{1}{2\omega_N} & \frac{\omega_N}{2} \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_i \\ \vdots \\ \xi_N \end{bmatrix}$$

Suppose to know the damping ratios of the first 4 modes

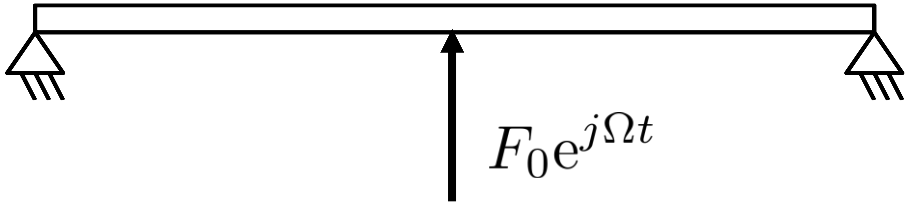
```
B = [0.01 0.015 0.0098 0.012]';
A = zeros(4,2);

for ii=1:4
    A(ii,:) = [1/(2*w(ii)) w(ii)/2];
End

ab = (A'*A)\A'*B;
ab = A\B;
CFF = ab(1)*MFF + ab(2)*KFF;
```

Pseudoinverse matrix → least square error

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1.79 \\ 2.17e^{-5} \end{bmatrix}$$

FREQUENCY RESPONSE FUNCTIONS

We want now to compute the frequency response function due to a concentrated force applied in the middle of the beam.

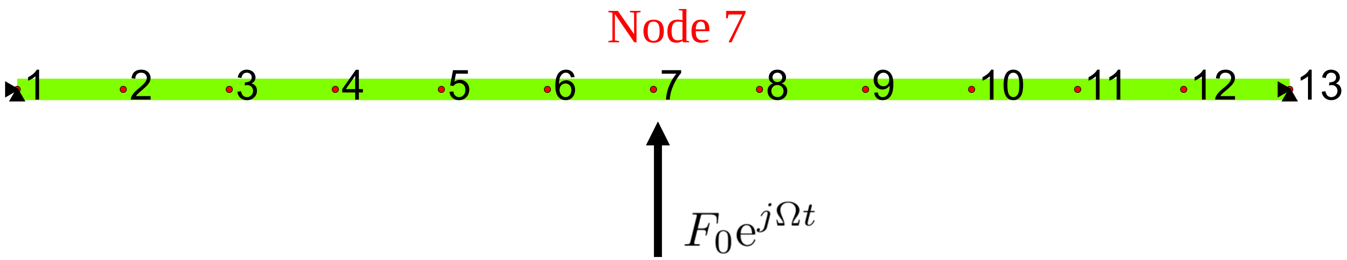


!

Pay attention to:

- have a node where the force is applied
- the frequency range of the input force

$$\Omega_{max} = 500 \cdot 2\pi \qquad \frac{\omega_k^{(1)}}{\Omega_{max}} \geq 1.5 \qquad \Rightarrow \qquad L_{max} = 155\ mm$$



$$idb = \begin{array}{ccc} 36 & 37 & 1 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \\ 11 & 12 & 13 \\ 14 & 15 & 16 \\ 17 & \boxed{18} & 19 \\ 20 & 21 & 22 \\ 23 & 24 & 25 \\ 26 & 27 & 28 \\ 29 & 30 & 31 \\ 32 & 33 & 34 \\ 38 & 39 & 35 \end{array}$$

Node 7
Y displacement
➢ 18° row

FREQUENCY RESPONSE FUNCTIONS

The Frequency Response Functions can thus be computed

18° row

$$(-\Omega^2[M_{FF}] + j\Omega[C_{FF}] + [K_{FF}])\underline{X}e^{j\Omega t} = \underline{F_0}e^{j\Omega t} \qquad \text{being} \qquad \underline{F_0} = \{\,0\ \dots\ 0\ 1\ 0\ \dots\ 0\,\}^T$$

```
F0 = zeros(ndof,1);
index = idb(7,2);
F0(index) = 1;

om = (0:1:500)*2*pi;

for ii=1:length(om)
    A = -om(ii)^2*MFF + 1i*om(ii)*CFF + KFF;
    X(:,ii) = A\F0;
end
```

We get as many FRFs as the number of dofs of the system.

As an example:

- y of node 4 $\rightarrow$ 9
- θ of node 4 $\rightarrow$ 10
- y of node 7 $\rightarrow$ 18 (colocated)

$$idb = \begin{matrix} 36 & 37 & 1 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & \boxed{9} & \boxed{10} \\ 11 & 12 & 13 \\ 14 & 15 & 16 \\ 17 & \boxed{18} & 19 \\ 20 & 21 & 22 \\ 23 & 24 & 25 \\ 26 & 27 & 28 \\ 29 & 30 & 31 \\ 32 & 33 & 34 \\ 38 & 39 & 35 \end{matrix}$$