POLITECNICO
MILANO 1863

SCUOLA DI INGEGNERIA INDUSTRIALE
E DELL'INFORMAZIONE

ADMS - Assignment 1

LAUREA MAGISTRALE IN
MECHANICAL ENGINEERING

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Introduction

The aim of the project was to analyse the dynamic behaviour of a cantilever beam, by making reference to the standing wave solution of a slender beam in bending vibration, to then study the forced response of the system, by computing Frequency Response Function (for assigned input and output positions).

Assuming the theoretical FRFs to be representative of an experimental test, a multi-curve fitting based method was then applied to compute the modal parameters from the FRFs.
The following procedure was followed during the project:

- Create the analytical model of the cantilever beam
- Apply numerically the model with MATLAB
- Identify the modal parameters with a curve fitting method
- Compare the theoretical and numerical results

We have then been able to apply this process to a real experimental data set of FRFs measured on a light rail vehicle's wheel.

1. Part A - Cantilever beam

1.1. Data of the reference structure

The cantilever beam studied here is an aluminium bar with a rectangle cross-section, anchored at $x = 0$.

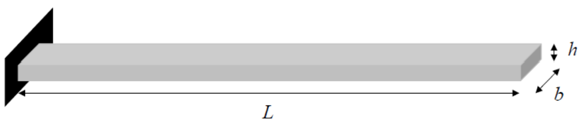


Figure 1: Cantilever beam

	symbol	unit	value
Length	L	mm	1200
Thickness	h	mm	8
Width	b	mm	40
Density	ρ	kg/m^3	2700
Young's Modulus	E	GPa	68

Table 1: Parameters of the beam

1.2. Natural frequencies of the cantilever beam

We know that the standing wave solution equation for a beam is the following:

$$w(x,t) = [A\cos(\gamma x) + B\sin(\gamma x) + C\cosh(\gamma x) + D\sinh(\gamma x)]\cos(\omega t + \psi)$$

We are only interested in the spatial part of the solution, so we can write:

$$\Phi(x) = A\cos(\gamma x) + B\sin(\gamma x) + C\cosh(\gamma x) + D\sinh(\gamma x)$$

The boundary conditions are:

1. At the fixed end ($x = 0$):
   - $w(x,t)|_{x=0} = 0 \implies \Phi(0) = 0$
   - $\frac{\partial w(x,t)}{\partial x}\Big|_{x=0} = 0 \implies \Phi'(0) = 0$
2. At the free end ($x = L$):
   - $M(L,t) = 0 \implies \frac{\partial^2 w(x,t)}{\partial x^2}\Big|_{x=L} = 0 \implies \Phi''(L) = 0$
   - $T(L,t) = 0 \implies \frac{\partial^3 w(x,t)}{\partial x^3}\Big|_{x=L} = 0 \implies \Phi'''(L) = 0$

From these conditions, we can derive the characteristic equation and write the following system of equations:

$$\begin{cases} A + C = 0 \\ B + D = 0 \\ -A\cos(\gamma L) - B\sin(\gamma L) + C\cosh(\gamma L) + D\sinh(\gamma L) = 0 \\ A\sin(\gamma L) - B\cos(\gamma L) + C\sinh(\gamma L) - D\cosh(\gamma L) = 0 \end{cases}$$

$$\iff \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\cos(\gamma L) & -\sin(\gamma L) & \cosh(\gamma L) & \sinh(\gamma L) \\ \sin(\gamma L) & -\cos(\gamma L) & \sinh(\gamma L) & -\cosh(\gamma L) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \iff [H(\gamma)]\underline{z} = \underline{0}$$

The matrix $H(\gamma)$ can be rewritten as $H(\omega)$ by replacing γ with the relation $\gamma = \sqrt{4\frac{m}{EJ}}\sqrt{\omega}$. The previous system can be then solved by finding the roots of the characteristic equation $\det[H(\omega)] = 0$. This equation is solved numerically in MATLAB but for the sake of verification, the characteristic equation was computed by hand:

$$\cos(\gamma L)^2 + \sin(\gamma L)^2 + \cosh(\gamma L)^2 - \sinh(\gamma L)^2 + 2\cos(\gamma L)\cosh(\gamma L) = 0$$

$$\iff \cos(\gamma L)\cosh(\gamma L) + 1 = 0$$

The roots are computed numerically by looking for the sign changes of the function, then refining with the MATLAB function `fzero`. With that, we find that the natural frequencies of the beam are:

$$\omega_i = [4.503, 28.225, 79.031, 154.868, ...]\, Hz$$

1.3. Mode shapes calculation

By replacing each mode in the equation, we can then find the corresponding mode shapes: $w_i \to [H(w_i)]\underline{z}^{(i)} = \underline{0} \to \Phi_i(x)$.

However, we need to set one of the unknown parameters to 1 to be able to find the other ones. We can set $A = 1$ and re-write the system as:

$$\begin{bmatrix} 1 \quad 0 \quad 1 \quad 0 \\ \underline{N} \quad \left[\hat{H}(\omega_i)\right] \end{bmatrix} \begin{bmatrix} 1 \\ \underline{\hat{z}}^{(i)} \end{bmatrix} = \underline{0} \implies \underline{N} + \left[\hat{H}(\omega_i)\right]\underline{\hat{z}}^{(i)} = \underline{0} \implies \underline{\hat{z}}^{(i)} = -\left[\hat{H}(\omega_i)\right]^{-1}\underline{N}$$

By numerically applying this method to all found natural frequencies, we obtained the corresponding modal vectors. The first 4 mode shapes are shown in Figure 2. We of course need to remember that the mode shapes are aside from a scaling factor, so the mode shapes are normalized to have a maximum amplitude of 1.

1.4. Frequency Response Functions

The Frequency Response Function (FRF) of the beam is computed by the following formula:

$$G_{jk}(\Omega) = \sum_{i=1}^{n} \frac{\Phi_i(x_j)\Phi(x_k)/m_i}{-\Omega^2 + j2\xi_i w_i \Omega + w_i^2}$$

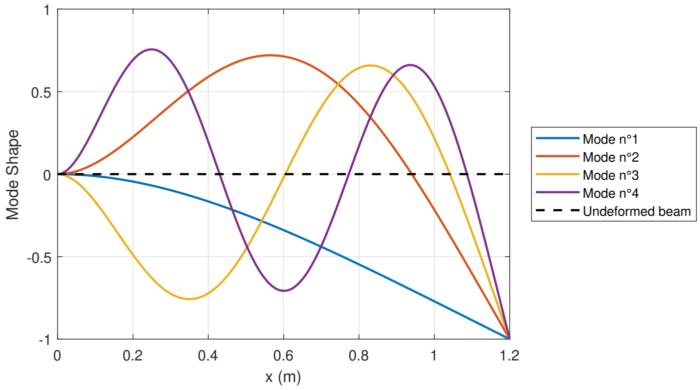


Figure 2: First 4 mode shapes of the cantilever beam

Figure 3: Input and sensor positions

Where x_j and x_k are the sensor and input force positions respectively (see Figure 3). ξ_i is the damping factor, that we decided to set to 1% for all modes. The modal mass m_i is computed as $m_i = \int_0^L m\Phi_i^2(x)dx$ where m is the linear mass of the beam.

This formula was implemented in MATLAB as a function for ease of use and to avoid code duplication. For verification, the FRF for $x_j = 0.2$m and $x_k = 1.2$m was computed and plotted in Figure 4.

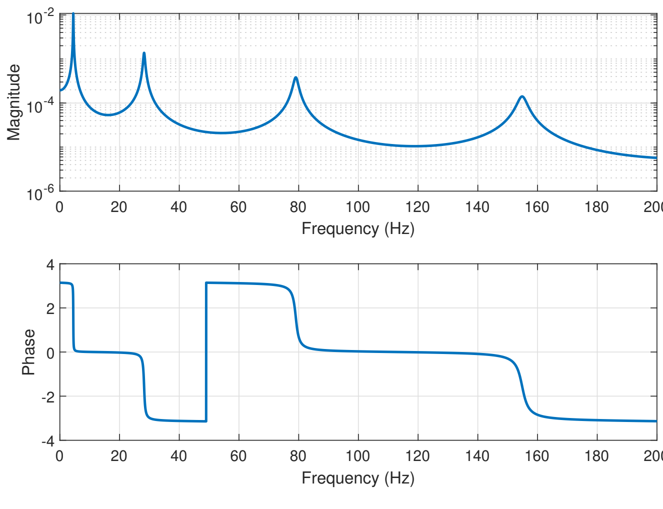


Figure 4: FRF for $x_j = 0.2$m and $x_k = 1.2$m

We are clearly able to see the peaks at the natural frequencies of the beam, and the phase behaves as expected. To constitute an experimental test, we considered 6 different sensor positions and one force input position: $x_k = 1.2$m and $x_j = 0.2, 0.3, 0.45, 0.66, 0.9, 1.13$m. The FRFs were computed for each of these positions and grouped in an $N \times M$ matrix, where N is the number of frequencies and M the number of sensor positions (Figure 5).

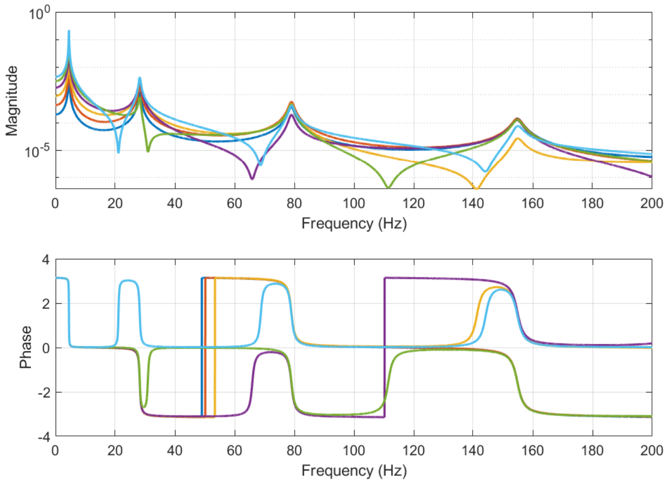


Figure 5: 6 FRFs for different sensor positions

1.5. Modal parameters identification

The "experimental" FRFs were then used to numerically identify the modal parameters of the beam. The method used was a multi-curve fitting method, where the FRFs are fitted around each natural frequency ω_i with the following formula:

$$G_{jk}^{NUM}(\Omega) = \frac{A_{jk}^{(i)}}{-\Omega^2 + 2j\xi_i\omega_i\Omega + \omega_i^2} + \frac{R_{jk}^L}{\Omega^2} + R_{jk}^H$$

Where ω_i, ξ_i, $A_{jk}^{(i)}$, R_{jk}^L and R_{jk}^H are the parameters to be identified. The fitting was done with the MATLAB function `lsqnonlin`. This function requires an initial guess for the parameters and an objective function to minimize.

- The guess for the mode frequencies was computed by finding the peaks of the FRFs using the MATLAB function `findpeaks`. The other parameters are then computed in a frequency range around each of those peaks.
- The damping factor is first computed by the phase derivative method, where

$$\xi_i = -\left(\omega_i \cdot \frac{\partial \Phi_{jk}}{\partial \Omega}\bigg|_{\Omega=\omega_i}\right)^{-1}$$

- The initial guesses for the amplitudes $A_{jk}^{(i)}$ was computed using the simplified formula

$$A_{jk}^{(i)} = \Re(G_{jk}^{EXP}(\Omega = \omega_i) \cdot (2j\xi_i\omega_i^2))$$

- The initial guesses for the low and high frequency residuals R_{jk}^L and R_{jk}^H were set to 0.

The objective function to minimize is a simple array of the difference between the experimental and numerical FRFs. This was chosen because the suggested method using the Frobenius norm of the difference of the matrices was not working as expected.

The results of the identification are shown in Figure 6. The identified parameters are then compared to the theoretical ones in Table 2.

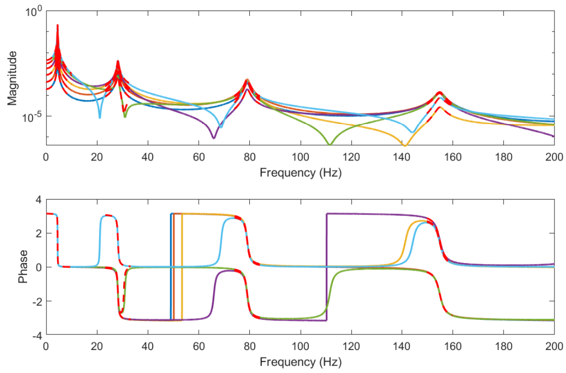


Figure 6: Identification of the modal parameters

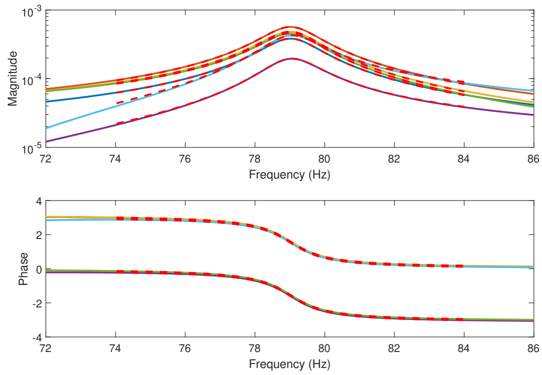


Figure 7: Focus on the third mode

	Mode 1	Mode 2	Mode 3	Mode 4	Damping
Theoretical	4.503 Hz	28.22 Hz	79.03 Hz	154.9 Hz	1.000%
Identified	4.503 Hz	28.22 Hz	79.03 Hz	154.9 Hz	1.002%

Table 2: Comparison of the identified and theoretical modal parameters

Now that we have the identified parameters, we can plot the mode shapes of the beam, as shown in Figure 8.

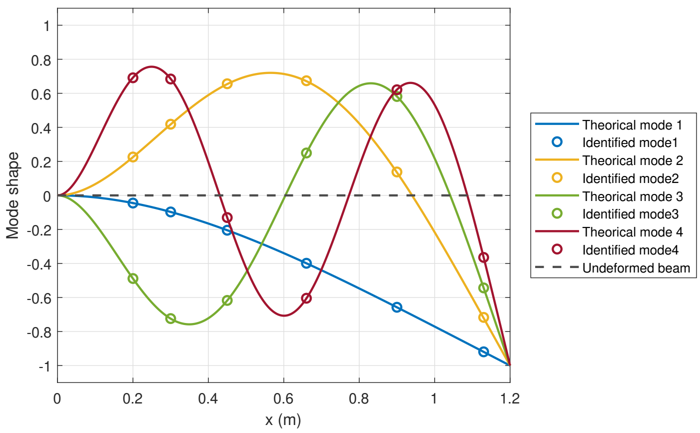


Figure 8: Mode shapes of the cantilever beam

As we can see, the identified parameters and the numerical FRFs are extremely close to the theoretical ones. The same can be said for the mode shapes that are pretty much aligned with the theoretical shapes. This shows that the method used is valid and can be applied to some real experimental data.

2. Part B - Light rail vehicle wheel

2.1. Data of the reference structure

Piezoelectric sensors were placed axially on the wheel of a light rail vehicle to measure the FRFs. 12 measurement points were used, separated by 15 degrees. The wheel was excited by a force applied at the 12 o'clock position (see Figure 9).
Those sensors were used to measure the 12 FRFs for the axial vibration of the wheel. The FRFs were then grouped in a $N \times M$ matrix, similar to the cantilever beam case.

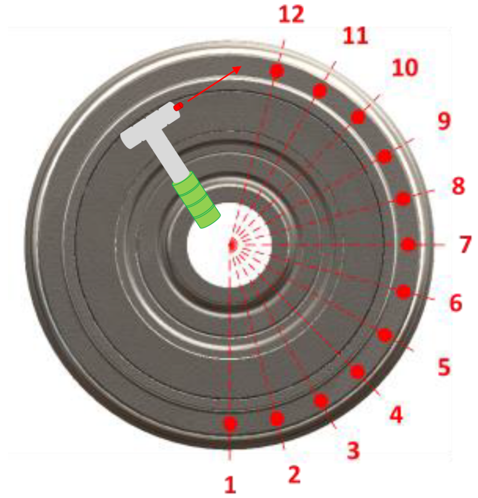


Figure 9: Sensor and input positions on the wheel

2.2. Modal parameters identification

The same method as for the cantilever beam was used to identify the modal parameters of the wheel. For this case, 4 axial modes are present in the given FRFs.

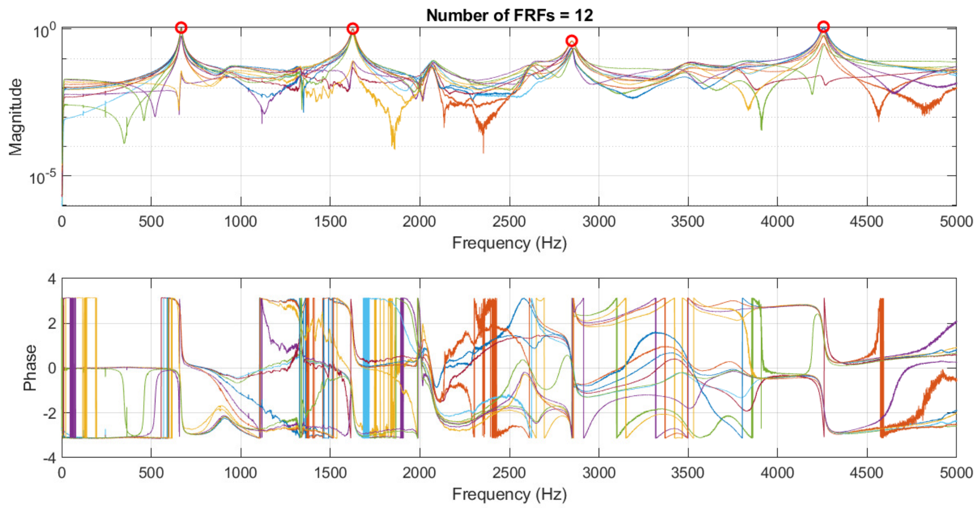


Figure 10: FRFs with identified modes

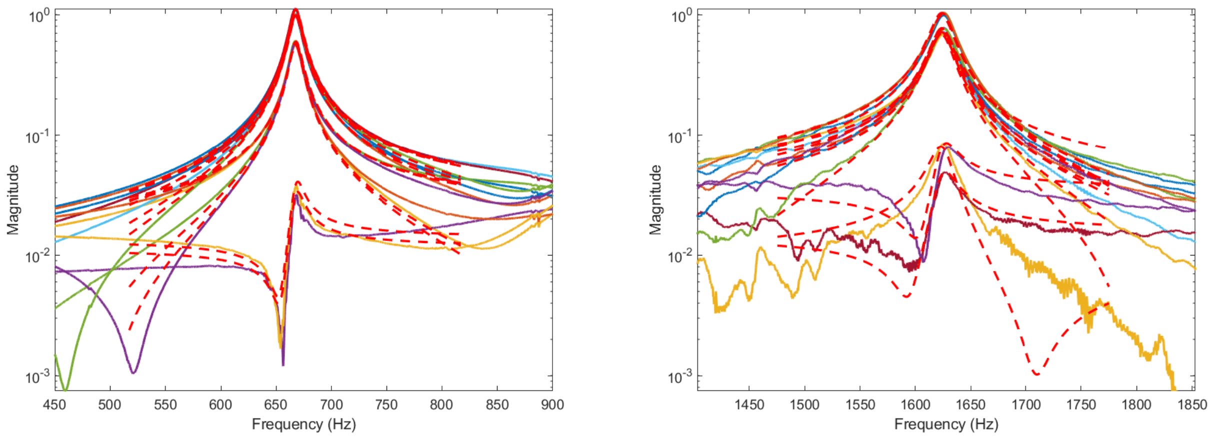
After running the numerical identification, the identified numerical FRFs were plotted and compared to the experimental ones.

As we can see in Figure 11, the visual quality is far from what we obtained for the cantilever beam, especially for the FRFs for which the sensor was close to a vibration node. However, the quality is still acceptable. The identified modal parameters are shown in Table 3 for the first two modes.

For a better visualisation of the modal vectors, we can plot them in a polar plot, as shown in Figure 12. However, the sensors allow us to identify the mode shapes only at the measurement points. To improve the readability of the plot, we interpolated those points with a spline function to obtain a continuous mode shape. In addition, thanks to the axial symmetry of the wheel, we can "expand" the mode shapes to the other half of the wheel by mirroring the mode shape across the line going through the impact point and the centre of the wheel.

We need to keep in mind that the mode shapes in the figure are represented as radial mode shapes, but the real mode shapes are axial mode shapes.

By judging visually the mode shapes, we can see that what we get is consistent with the expected mode shapes of a wheel, and of a quite good quality.



(a) Focus on mode 1

(b) Focus on mode 2

Figure 11: Comparison of the identified and experimental FRFs

	Mode 1	Mode 2
Mode frequency	667.7 Hz	1624 Hz
Mode Damping	0.814 %	0.599 %
Modal vector	[0.979, 0.894, 0.498, -0.027, -0.542, -0.908, -1.000, -0.866, -0.502, 0.033, 0.521, 0.895]	[-0.952, -0.676, 0.077, 0.748, 1.000, 0.692, -0.044, -0.746, -0.995, -0.660, 0.069, 0.735]

Table 3: Identified parameters of the wheel

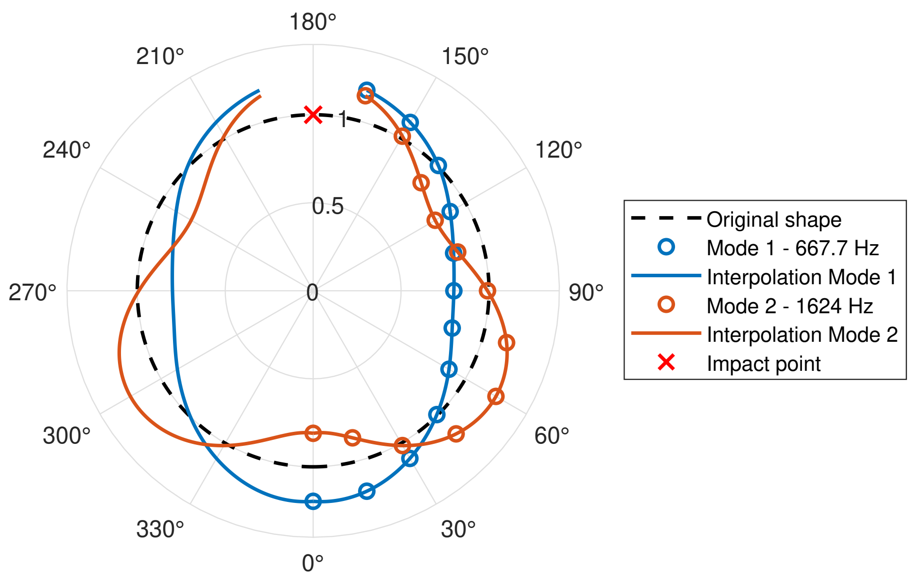


Figure 12: Polar plot of the modal vectors

3. Conclusions

Overall, the project was a success. We were able to identify the modal parameters of a cantilever beam and a light rail vehicle wheel using a multi-curve fitting method. The identified parameters were very close to the theoretical ones, and the mode shapes were consistent with the expected ones.