POLITECNICO
MILANO 1863

DEPARTMENT OF
MECHANICAL ENGINEERING

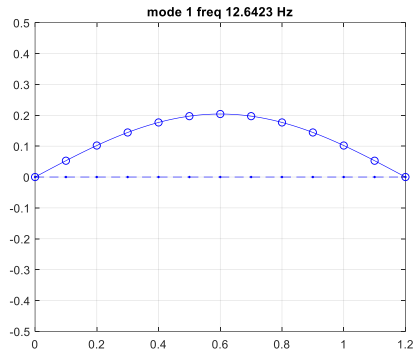
ADVANCED DYNAMICS OF MECHANICAL SYSTEMS

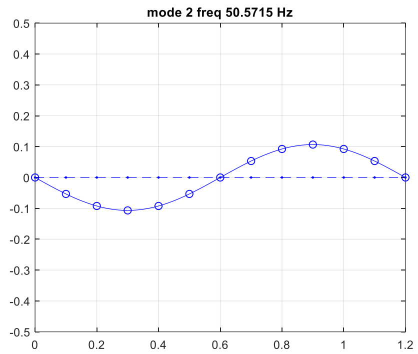
Finite Elements Method (FEM) in structural dynamics: software implementation in Matlab environment – part 4

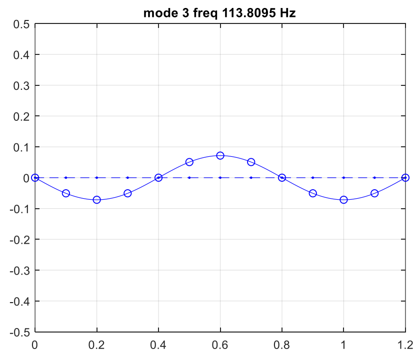
I. LA PAGLIA

MODAL SUPERPOSITION APPROACH

Suppose you want to compute the structure response considering the first 3 modes only.
The modal superposition approach can be used.







$$[\Phi] = \left[\underline{X}^{(1)} \quad \underline{X}^{(2)} \quad \underline{X}^{(3)}\right]_{Nx3}$$ Matrix collecting the modal vectors of modes 1, 2 and 3

$$\left[M_q\right] = [\Phi]^T \, [M] \, [\Phi]$$ Modal mass matrix

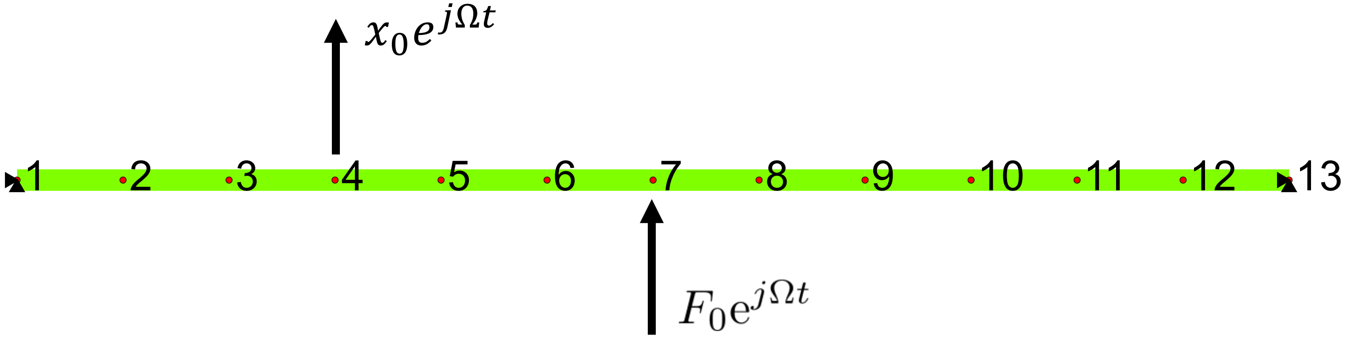
$$\left[K_q\right] = [\Phi]^T \, [K] \, [\Phi]$$ Modal stiffness matrix

$$\left[C_q\right] = [\Phi]^T \, [C] \, [\Phi]$$ Modal damping matrix

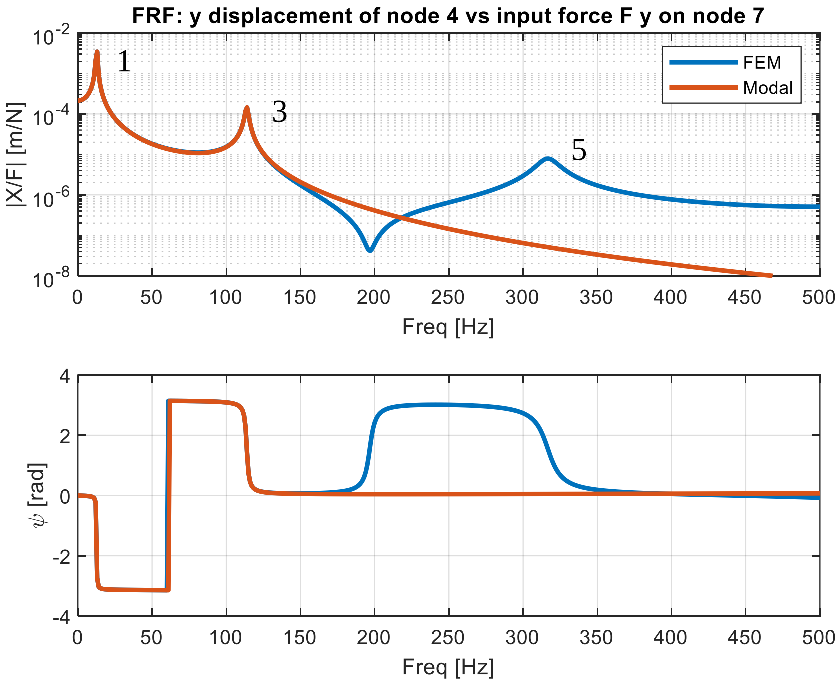
$$\left[Q_q\right] = [\Phi]^T \underline{F}$$ Generalized force vector

MODAL SUPERPOSITION APPROACH

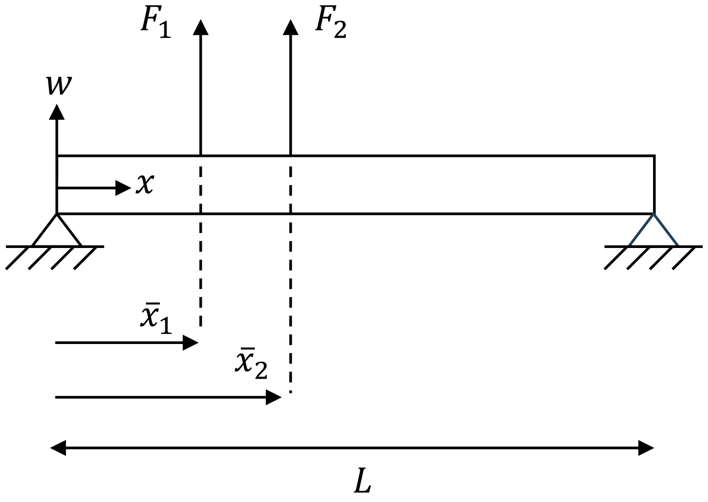
Input node: 7 y
Output node: 4 y




```
% Modal matrices
ii = 1:3              % first 3 mode shapes
Phi  = modes(:,ii);
Mmod = Phi'*MFF*Phi;
Kmod = Phi'*KFF*Phi;
Cmod = Phi'*CFF*Phi;
Fmod = Phi'*F0;

% FRF in modal superposition approach
for ii = 1:length(Om)
    xx_mod(:,ii) = (-Om(ii)^2*Mmod + 1i*Om(ii)*Cmod + Kmod) \ Fmod;
end
xx_2m = Phi * xx_mod;
%       35x3  3x501

FRF_mod = xx_2m(idb(4,2),:);
```



MODAL SUPERPOSITION APPROACH



Let us consider the case where two external forces are applied onto the system.

In general, $F_1(t)$ and $F_2(t)$ will not be synchronous (i.e., they will be associated to different frequencies), so that the superposition principle can be adopted (linear system).

Let us consider the case of two synchronous forces with frequency Ω

$$F_1(t) = F_{10}\, e^{j\Omega t}$$
$$F_2(t) = F_{20}\, e^{j\Omega t}$$

with F_{10} and F_{20} generally complex (different in amplitude and phase)

The virtual work can be computed as

$$\delta W = F_1\, \delta w(\bar{x}_1) + F_2\, \delta w(\bar{x}_2) = F_1\, \underline{\Phi}^T(\bar{x}_1)\, \delta \underline{q} + F_2\, \underline{\Phi}^T(\bar{x}_2)\, \delta \underline{q} =$$
$$= \left[F_{10}\, \underline{\Phi}^T(\bar{x}_1) + F_{20}\, \underline{\Phi}^T(\bar{x}_2)\right] e^{j\Omega t}\, \delta \underline{q} = \underline{Q_0}^T e^{j\Omega t}\, \delta \underline{q}$$

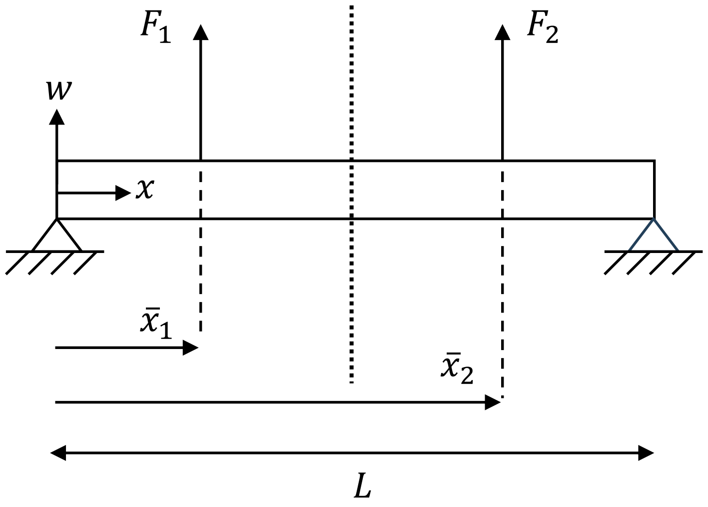
Finally, the generalized force reads like

$$\underline{Q}(t) = \underline{Q_0}\, e^{j\Omega t} = \left[F_{10}\, \underline{\Phi}(\bar{x}_1) + F_{20}\, \underline{\Phi}(\bar{x}_2)\right] e^{j\Omega t}$$

and the term associated to mode i is

$$Q_i(t) = [F_{10}\, \Phi_i(\bar{x}_1) + F_{20}\, \Phi_i(\bar{x}_2)]\ e^{j\Omega t} = Q_{0i}\, e^{j\Omega t}$$

MODAL SUPERPOSITION APPROACH



Suppose now that $\quad F_{10} = F_{20} \quad$ and $\quad \dfrac{\bar{x}_1 + \bar{x}_2}{2} = \dfrac{L}{2}$

that corresponds to the case of equal and in-phase external forces, applied in symmetric positions with respect to mid-span.

The generalized force can be expressed as

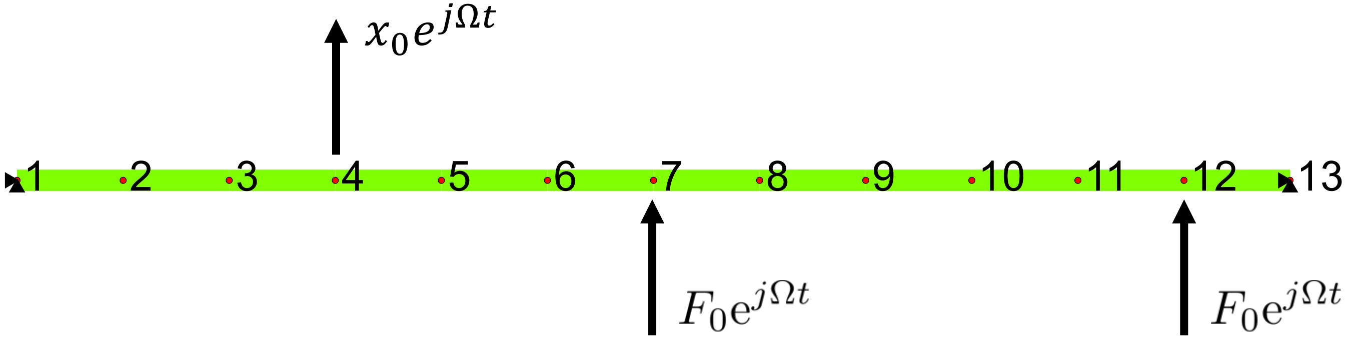
$$\underline{Q}(t) = \underline{Q_0}\, e^{j\Omega t} = \left[\, \underline{\Phi}(\bar{x}_1) + \underline{\Phi}(\bar{x}_2) \right] F_{10}\, e^{j\Omega t}$$

$$Q_i(t) = [\Phi_i(\bar{x}_1) + \Phi_i(\bar{x}_2)]\, F_{10}\, e^{j\Omega t} = Q_{0i}\, e^{j\Omega t} \begin{cases} \neq 0 & \text{for symmetric (odd-numbered) modes, provided that the considered mode is controllable } (\Phi_i(\bar{x}_1) = \Phi_i(\bar{x}_2) \neq 0) \\ = 0 & \text{for antisymmetric (even-numbered) modes} \end{cases}$$

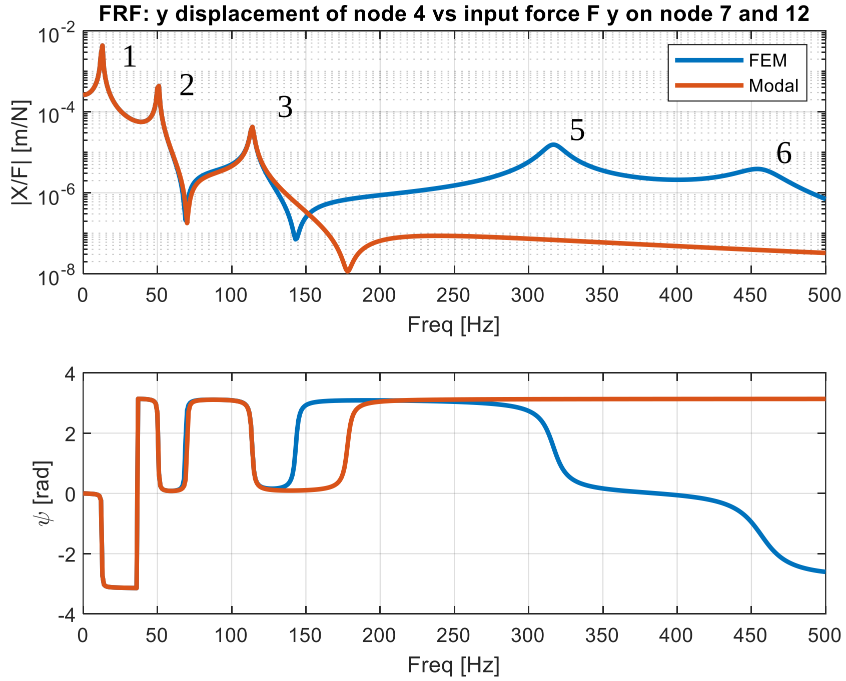
Note that opposite results will be achieved if the external forces are in anti-phase (i.e., $F_{10} = -F_{20}$)

MODAL SUPERPOSITION APPROACH

Input nodes: 7 y, 12 y
Output node: 4 y

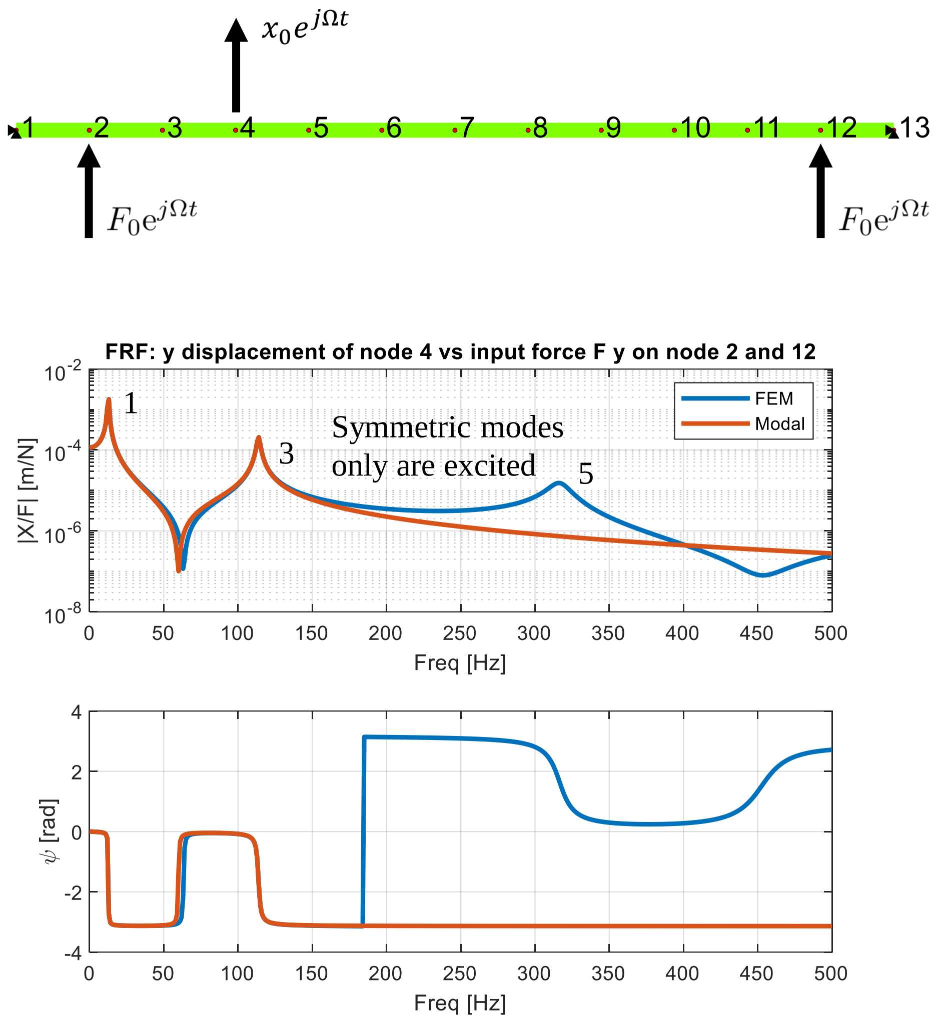


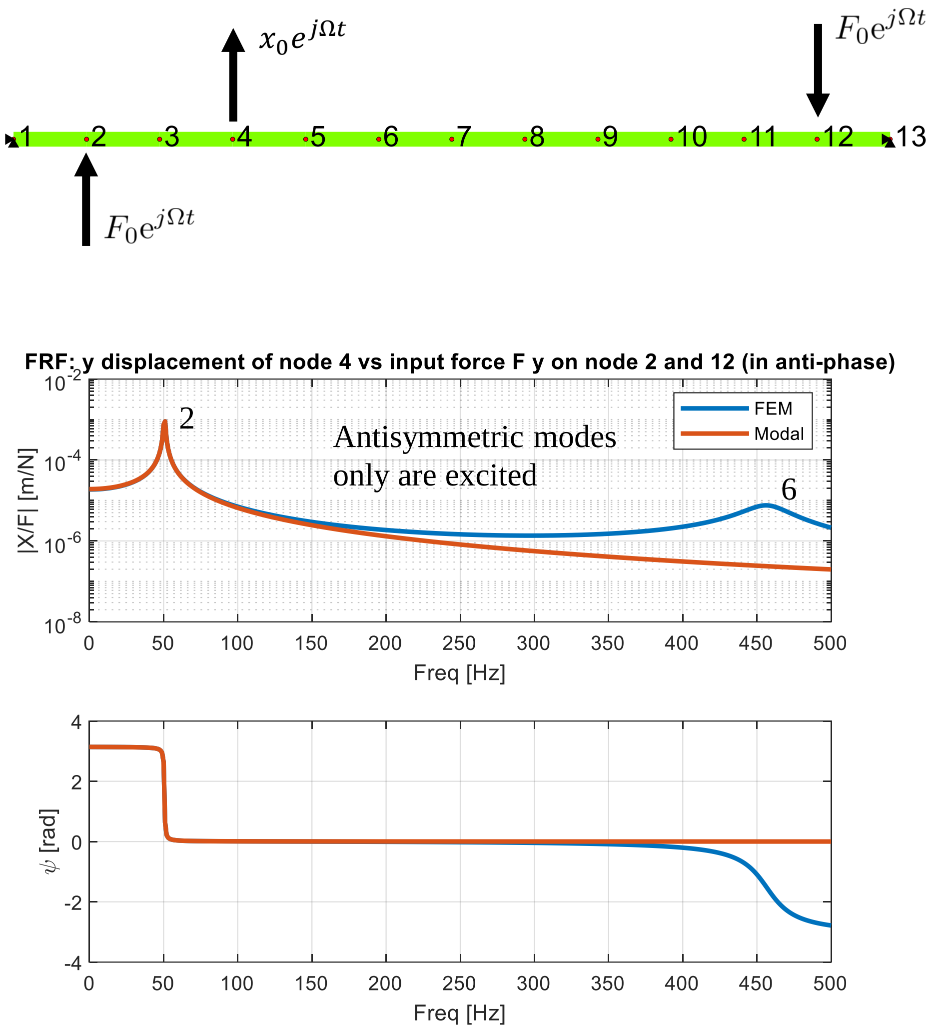

```
F0 = zeros(ndof,1);
F0(idb(7,2))  = 1;
F0(idb(12,2)) = 1;

Fmod = Phi'*F0;

for ii = 1:length(Om)
    x(:,ii)      = (-Om(ii)^2*MFF  + 1i*Om(ii)*CFF  + KFF)  \ F0;
    xx_mod(:,ii) = (-Om(ii)^2*Mmod + 1i*Om(ii)*Cmod + Kmod) \ Fmod;
end

FRF = x(idb(4,2),:);
xx_2m = Phi*xx_mod;
FRF_mod = xx_2m(idb(4,2),:);
```



MODAL SUPERPOSITION APPROACH

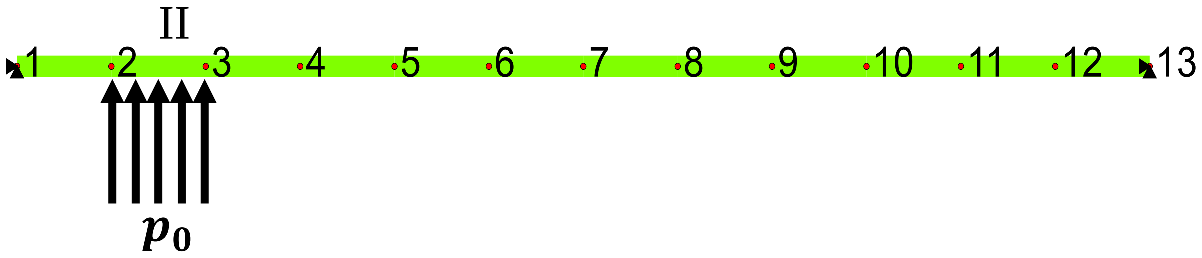
Input nodes: 2 y, 12 y
Output node: 4 y





DISTRIBUTED LOAD
WORK-EQUIVALENT NODAL FORCES/MOMENTS

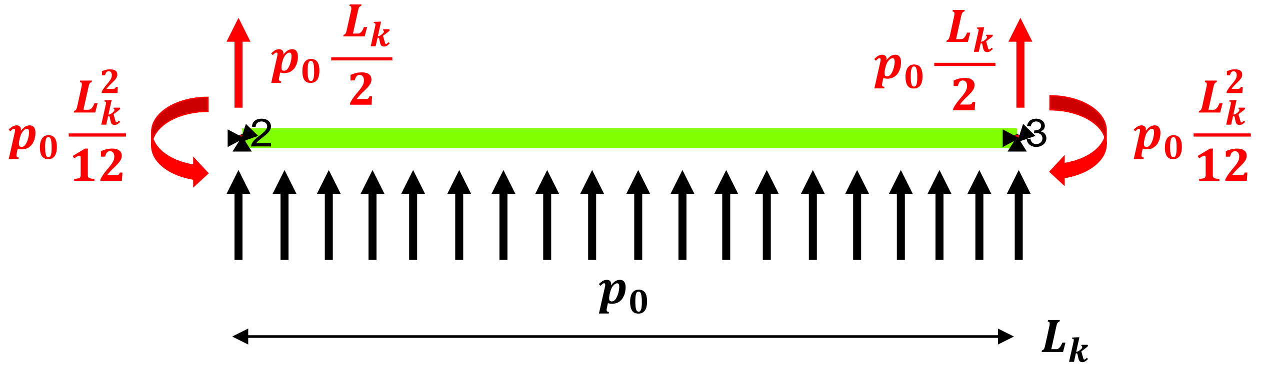
Reference example: a constant distributed load applied on element II (between nodes 2 and 3)



The distributed load on a beam can be replaced by the equivalent forces at the nodes.

At first, these can be computed solving the constraint forces of a fixed-fixed beam (3 times hyperstatic), subjected to the same load.

- Constraint bending moments are responsible for the beam deflection!



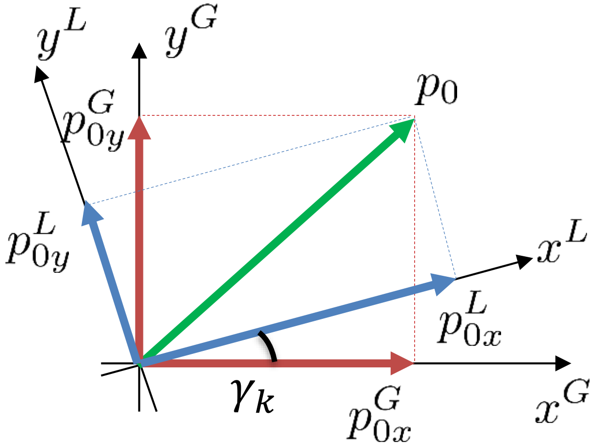
DISTRIBUTED LOAD
WORK-EQUIVALENT NODAL FORCES/MOMENTS

As an alternative, let us consider the work-equivalent nodal forces, computed according to the beam element shape functions.

$$\delta W_{ext,k} = (\delta \underline{x}_k^L)^T \left[\int_0^{L_k} \underline{f}_u(\xi) p_x(\xi) d\xi + \int_0^{L_k} \underline{f}_w(\xi) p_y(\xi) d\xi \right]$$

The load must be expressed in the <u>local reference frame</u>

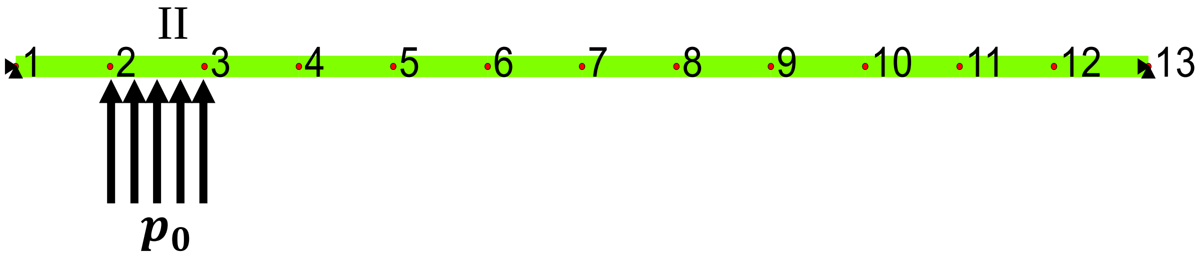
$$p_{0x}^G = \text{load absolute x component}$$
$$p_{0y}^G = \text{load absolute y component}$$
$$p_{0x}^L = p_{0x}^G \cos\gamma_k + p_{0y}^G \sin\gamma_k$$
$$p_{0y}^L = -p_{0x}^G \sin\gamma_k + p_{0y}^G \cos\gamma_k$$



Pay attention to the element orientation

DISTRIBUTED LOAD
WORK-EQUIVALENT NODAL FORCES/MOMENTS

Reference example: a constant distributed load applied on element II (between nodes 2 and 3)



Note that in the proposed example $p_{0x}^L = 0$

$$p_{0y}^L = p_0$$

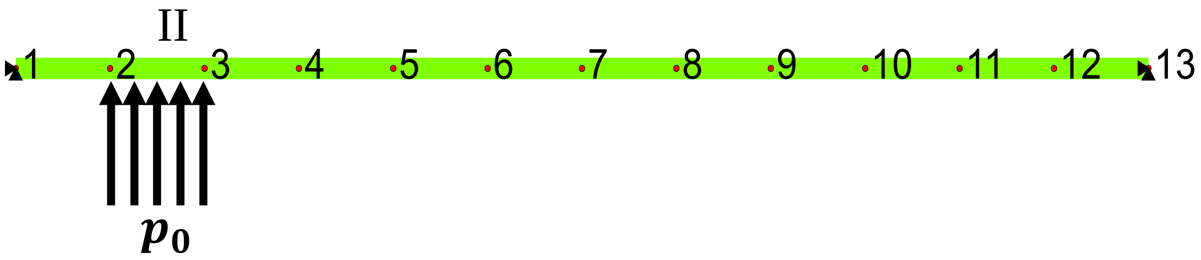
Thus

$$\underline{F}_k^L = p_0 \int_0^{L_k} \begin{bmatrix} 0 \\ 2\frac{\xi^3}{L_k^3} - 3\frac{\xi^2}{L_k^2} + 1 \\ \frac{\xi^3}{L_k^2} - 2\frac{\xi^2}{L_k} + \xi \\ 0 \\ -2\frac{\xi^3}{L_k^3} + 3\frac{\xi^2}{L_k^2} \\ \frac{\xi^3}{L_k^2} - \frac{\xi^2}{L_k} \end{bmatrix} d\xi = p_0 \begin{bmatrix} 0 \\ \frac{L_k}{2} \\ \frac{L_k^2}{12} \\ 0 \\ \frac{L_k}{2} \\ -\frac{L_k^2}{12} \end{bmatrix}$$

Same as before!

DISTRIBUTED LOAD
WORK-EQUIVALENT NODAL FORCES/MOMENTS

Reference example: a constant distributed load applied on element II (between nodes 2 and 3)



Nodal forces of beam element II in local and global reference frame ($\gamma_{II} = 0$)

Local reference frame

$$\underline{F}_k^L = p_0 \begin{array}{c} \begin{bmatrix} 0 \\ \frac{L_k}{2} \\ \frac{L_k^2}{12} \\ 0 \\ \frac{L_k}{2} \\ -\frac{L_k^2}{12} \end{bmatrix} \end{array} \begin{array}{c} x_2^L \\ y_2^L \\ \vartheta_2^L \\ x_3^L \\ y_3^L \\ \vartheta_3^L \end{array}$$

Rotation matrix

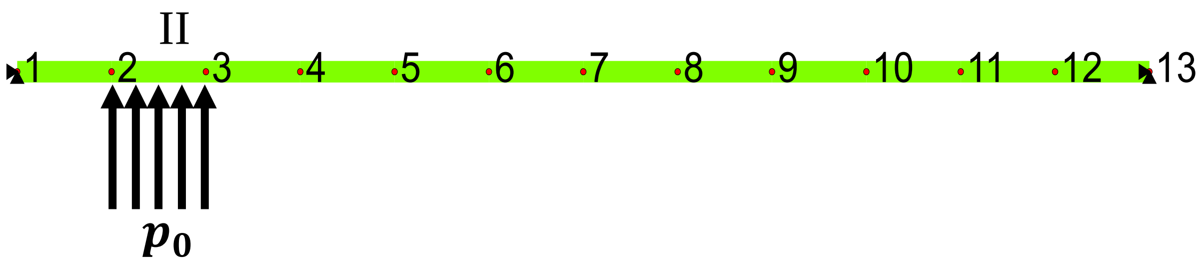
$$[\Lambda_k] = \begin{bmatrix} \cos\gamma_k & -\sin\gamma_k & 0 & 0 & 0 & 0 \\ \sin\gamma_k & \cos\gamma_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\gamma_k & -\sin\gamma_k & 0 \\ 0 & 0 & 0 & \sin\gamma_k & \cos\gamma_k & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Global reference frame

$$\underline{F}_k^G = [\Lambda_k]\underline{F}_k^L$$

DISTRIBUTED LOAD
WORK-EQUIVALENT NODAL FORCES/MOMENTS

Reference example: a constant distributed load applied on element II (between nodes 2 and 3)



Finally, the sum over all the elements according to the expansion matrix E_k is performed

$$\underline{x}_k^G = [E_k]\underline{x}^G \qquad \text{with} \qquad [E_k] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \end{bmatrix}$$

6 x (3xNnod) matrix
1 for dofs of nodes 2 and 3

$$\underline{F}^G = \sum_k [E_k]^T \underline{F}_k^G$$

DISTRIBUTED LOAD

WORK-EQUIVALENT NODAL FORCES/MOMENTS

Reference example: a constant distributed load applied on element II (between nodes 2 and 3)

```
p0  = 100;          %[N/m]
p0G = [0 p0]';
FG  = zeros(3*nnod,1);

for ii = 2     % cicle over elements involved: in this case only element II
    gammaii = gamma(ii);
    Lk  = l(ii);
    p0L = [cos(gammaii) sin(gammaii);
          -sin(gammaii) cos(gammaii)]*p0G;
    FkL = [Lk/2;  0   ;  0      ;  Lk/2;  0   ;   0      ]*p0L(1) + ...
          [0   ;  Lk/2;  Lk^2/12;  0   ;  Lk/2;  -Lk^2/12]*p0L(2);
    FkG = [cos(gammaii) -sin(gammaii) 0  0             0            0;
           sin(gammaii)  cos(gammaii) 0  0             0            0;
           0             0            1  0             0            0;
           0             0            0  cos(gammaii) -sin(gammaii) 0;
           0             0            0  sin(gammaii)  cos(gammaii) 0;
           0             0            0  0             0            1]*FkL;
    Ek  = zeros(6,3*nnod);
    for jj = 1:6
        Ek(jj,incid(ii,jj)) = 1;
    end
    FG = FG + Ek'*FkG;
end
```

$$\underline{F}^G = \begin{Bmatrix} 0 \\ \hdashline 0 \\ 5.0000 \\ 0.0833 \\ \hdashline 0 \\ 5.0000 \\ -0.0833 \\ \hdashline 0 \\ 0 \\ 0 \\ \hdashline \ldots \end{Bmatrix}$$

39 x 1

STATIC DEFLECTION

The static deflection of the structure is computed accounting only for constant forces and potential energy derivatives (i.e., the generalized component of constant conservative forces).

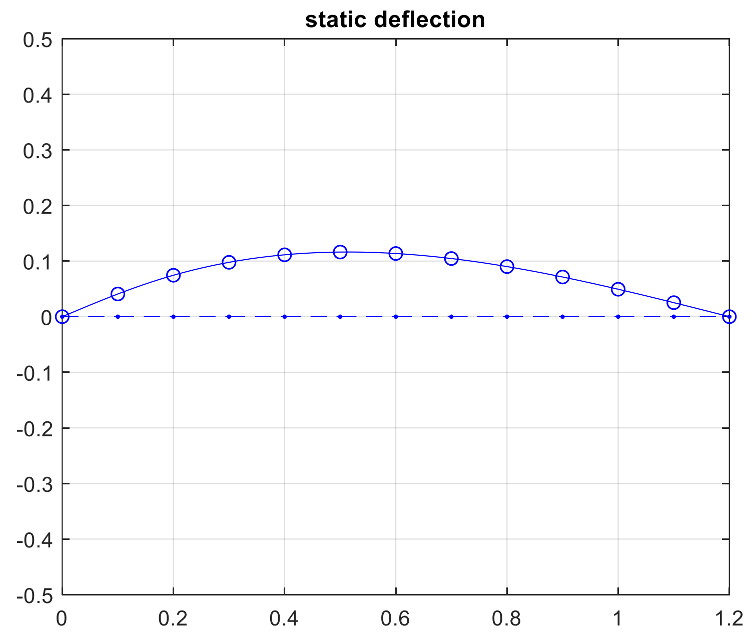
$$[M_{FF}]\underline{\ddot{x}}_F + [C_{FF}]\underline{\dot{x}}_F + [K_{FF}]\underline{x}_F = \underline{F}_F^G{}_{35x1}$$

In static conditions $\underline{\ddot{x}}_F = \underline{\dot{x}}_F = 0$

Thus $\underline{x}_F = [K_{FF}]^{-1}\underline{F}_F^G$

```
xF = KFF \ FG(1:ndof);

figure();
diseg2(xF,100,incid,l,gamma,posit,idb,xy)
title(['static deflection']);
```

Rely on the «diseg2» function to draw the static deflection



HINTS FOR THE ASSIGNMENT: STATIC RESPONSE

7. Compute the static response of the structure due to:

   a) the deck weight only;

   b) the weight of the entire structure.

   Plot the deformed shape of the structure compared to the undeformed configuration and compute the value of the maximum deflection.

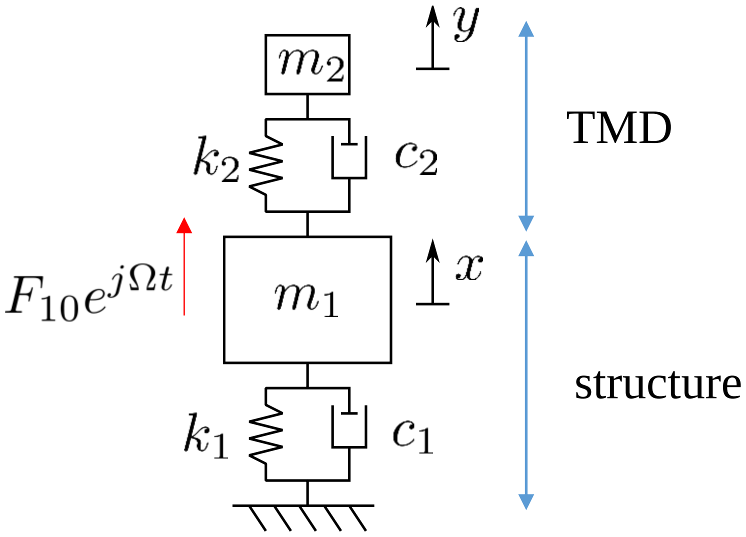
<u>Hints for the code implementation</u>

- ❑ Different strategies can be adopted to introduce the weight of the structure, namely distributed and concentrated loads
- ❑ Depending on your choice, properly build the external force vector

HINTS FOR THE ASSIGNMENT: TUNED MASS DAMPER

A tuned mass damper (TMD) is a device consisting of a mass m_2, a spring k_2 and a damper c_2. It is mounted on structures in order to reduce the vibration amplitude of a specific resonance.

To this end, its natural frequency is "tuned" into a specific structural mode, so that when it is excited by an external force, the damper will resonate out of phase so as to dissipate energy.



<u>Hints for the code implementation</u>

1. Tune the TMD on the *structure* natural frequency to be damped ($\omega_2 = \omega_1$) and investigate the effect of the TMD parameters over the system response. Rely on the **modal approach** reduced only to the dof of interest

2. Based on the parametric analysis, choose a proper value for the mass m_2 and derive k_2 (given ω_2); choose the damping ratio ξ_2 and derive c_2

3. Install the TMD on the *real structure* (truss bridge). Remember to account for the additional 1 dof of the TMD both in terms of matrices size and contributions of the lamped parameters

NB: do not change the overall system response! The FRF should change only in correspondence of the vibration mode of interest