POLITECNICO
MILANO 1863

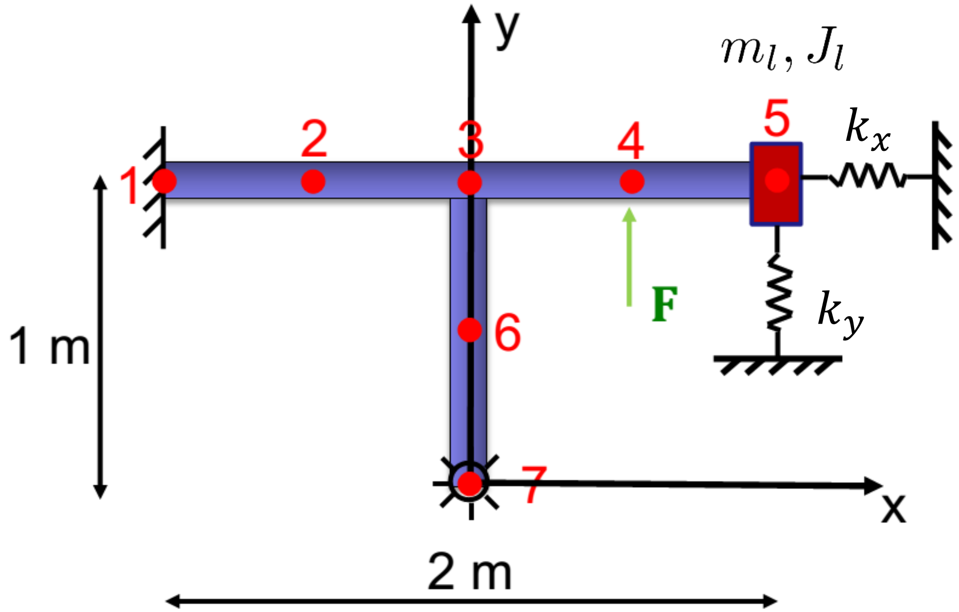
DEPARTMENT OF
MECHANICAL ENGINEERING

# ADVANCED DYNAMICS OF MECHANICAL SYSTEMS

## Finite Elements Method (FEM) in structural dynamics: software implementation in Matlab environment – part 3

I. LA PAGLIA

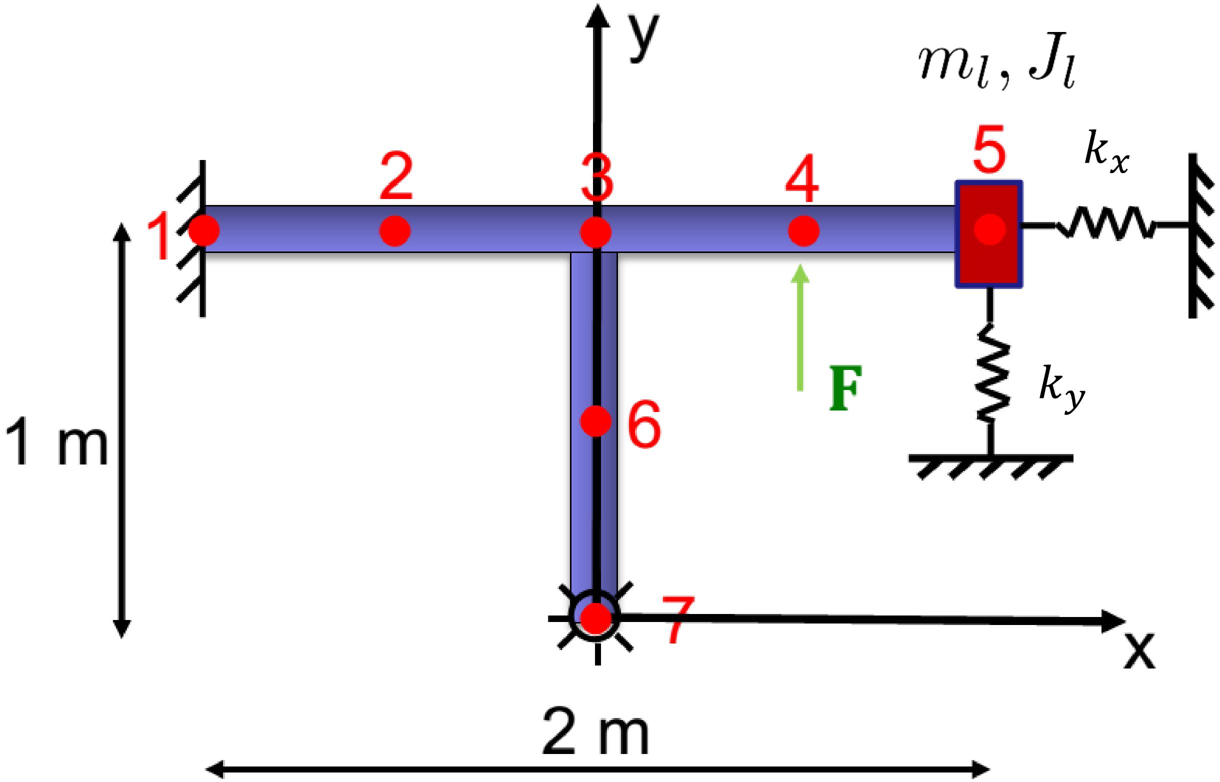
# ANOTHER EXAMPLE




| Property | symbol | value | unit |
|---|---|---|---|
| Linear Mass | $m$ | 9.75 | [kg/m] |
| Bending Stiffness | $EJ$ | 1.34e4 | [Nm$^2$] |
| Axial Stiffness | $EA$ | 2.57e7 | [N] |
| Lumped Mass | $m_l$ | 10 | [kg] |
| Lumped M.o.I | $J_l$ | 1 | [kgm$^2$] |
| Spring x | $k_x$ | 2e6 | [N/m] |
| Spring y | $k_y$ | 3e6 | [N/m] |
| Spring | $k$ | 4e6 | [N/m] |
| Damping ratio 1st mode (exp.) | $h_1$ | 0.01 | [-] |
| Damping ratio 2nd mode (exp.) | $h_2$ | 0.015 | [-] |
| Safety Factor | coef | 1.5 | [-] |

1. Verify that an element length of 0.5 m is consistent with a dynamic analysis up to 100 Hz
2. Write the input file for the structure
3. Compute the structure’s natural frequencies and vibration modes up to the 3$^{rd}$ one
4. Compute the damping matrix according to the experimental damping ratios provided
5. Compute the frequency response function of $y$ displacement of node 5

# CONCENTRATED ELEMENTS




➢ How to deal with concentrated elements?

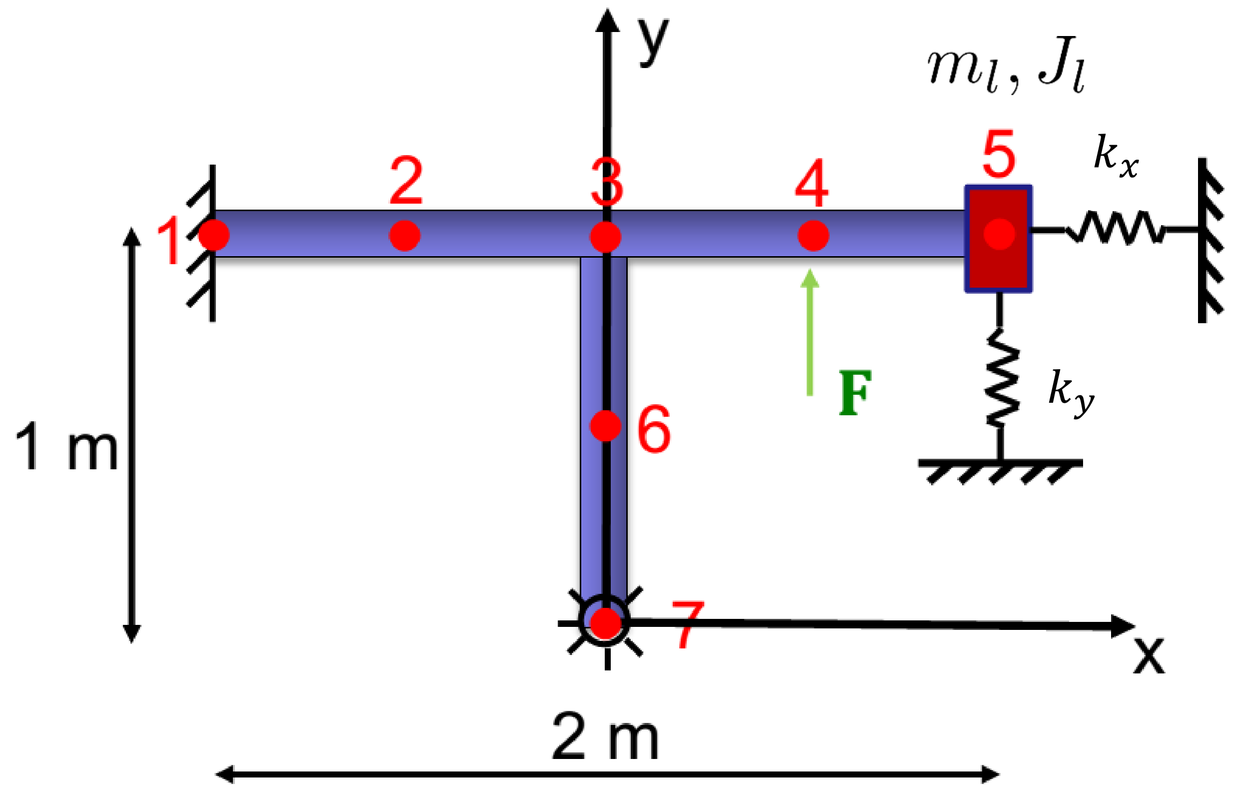
# CONCENTRATED ELEMENTS
## PROCEDURE

1. Build the input file (.inp) of the structure, neglecting the concentrated elements
2. Compute the mass matrix M and the stiffness matrix K
3. Add the concentrated elements contributions directly into the corresponding matrices
    - Masses
    - Springs
4. Compute the structural damping matrix C
5. Partition the matrices to get $M_{FF}$ , $K_{FF}$ and $C_{FF}$
6. If concentrated dampers are present, add their contributions directly into the corresponding matrix

# CONCENTRATED ELEMENTS

1. Build the input file (.inp) of the structure, neglecting concentrated elements

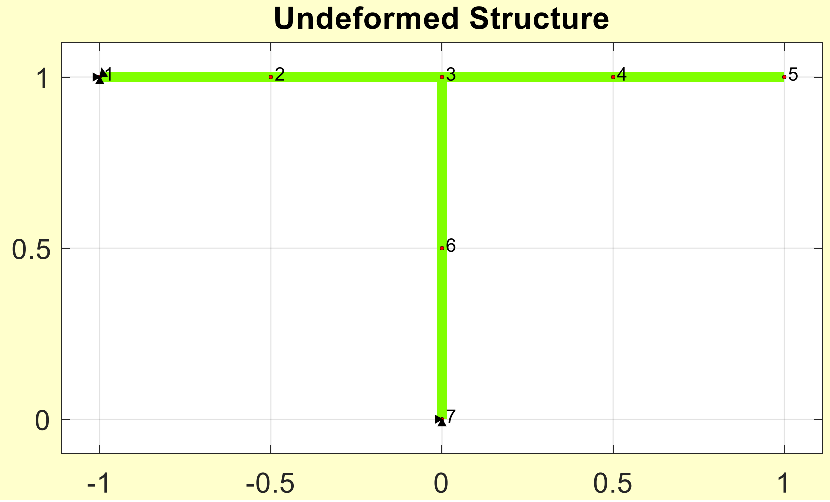
```
*NODES
1    1 1 1    -1.0 1.0
2    0 0 0    -0.5 1.0
3    0 0 0     0.0 1.0
4    0 0 0     0.5 1.0
5    0 0 0     1.0 1.0
6    0 0 0     0.0 0.5
7    1 1 0     0.0 0.0
*ENDNODES
*BEAMS
1    1 2        1
2    2 3        1
3    3 4        1
4    4 5        1
5    3 6        1
6    6 7        1
*ENDBEAMS
*PROPERTIES
1    9.75      2.57e7   1.34e4
*ENDPROPERTIES
```



# CONCENTRATED ELEMENTS

1. Build the input file (.inp) of the structure, neglecting concentrated elements

```
% load the input file and assemble the structure
[file_i,xy,nnod,sizew,idb,ndof,incid,l,gamma,m,EA,EJ,posit,nbeam,pr]=loadstructure;

% draw the structure
dis_stru(posit,l,gamma,xy,pr,idb,ndof);

% from the Command Window:
Number structure nodes 7
Number of structure d.o.f. 16 (+5 constrained)
Number of properties 1
Number of beam FE 6
```




2. Compute the mass matrix M and the stiffness matrix K

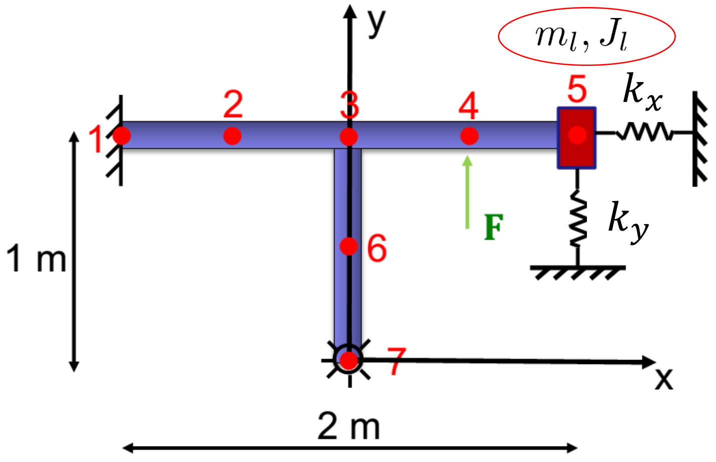
```
% assemble mass and stiffness matrices
[M,K]=assem(incid,l,m,EA,EJ,gamma,idb);
```

$$[M] = \begin{bmatrix} 3.25 & 0 & 0 & \dots \\ 0 & 3.62 & 0 & \dots \\ 0 & 0 & 0.02 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{21 \times 21}$$

- Symmetric sparse M matrix
- Similar results for K matrix

# CONCENTRATED MASSES

3. Add concentrated elements contributions directly into the corresponding matrices




The mass $m_l$ of the concentrated element is related to the node 5 displacement along $x$ and $y$ directions, while the inertia $J_l$ is associated to the rotation of node 5.

According to idb matrix

$$idb = \begin{bmatrix} 17 & 18 & 19 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \\ 13 & 14 & 15 \\ 20 & 21 & 16 \end{bmatrix}$$

Node 5

$$row(x_5) = 10$$
$$row(y_5) = 11$$
$$row(\theta_5) = 12$$

$$[\hat{M}_l] = \begin{bmatrix} m_l & 0 & 0 \\ 0 & m_l & 0 \\ 0 & 0 & J_l \end{bmatrix}$$

$$[M_l] = \underbrace{[E_{m_l}]^T}_{21 \times 3} \underbrace{[\hat{M}_l]}_{3 \times 3} \underbrace{[E_{m_l}]}_{3 \times 21}$$

$$[E_{m_l}] = \begin{bmatrix} \dots & 0 & 1 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 1 & 0 & \dots \end{bmatrix}$$

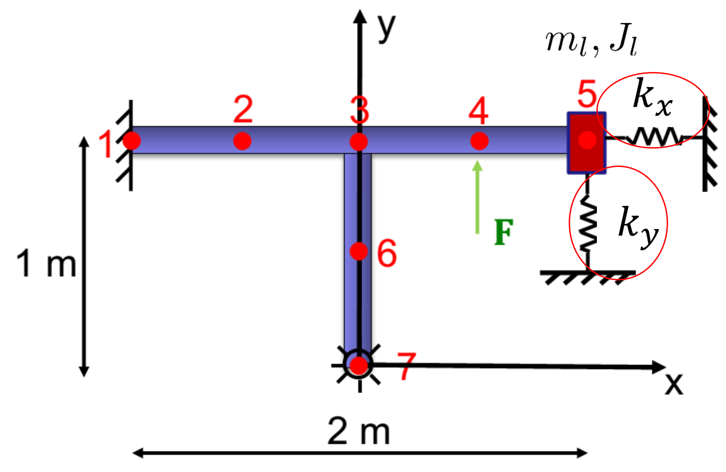
$$[M_{tot}] = [M] + [M_l]$$

# CONCENTRATED SPRINGS

3. Add concentrated elements contributions directly into the corresponding matrices

The elongation of the concentrated spring $k_x$ is given by the $x$ displacements of node 5, while the elongation of the spring $k_y$ is given by the $y$ displacements of node 5.

According to idb matrix




$$idb = \begin{bmatrix} 17 & 18 & 19 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \\ 13 & 14 & 15 \\ 20 & 21 & 16 \end{bmatrix}$$

Node 5

$$\underset{1\times 1}{[\hat{K}_x]} = [k_x]$$

$$[K_x] = \underset{21\times 1}{[E_{K_x}]^T}\underset{1\times 1}{[\hat{K}_x]}\underset{1\times 21}{[E_{K_x}]}$$

$$[E_{K_x}] = [\ldots \quad 0 \quad 1 \quad 0 \quad \ldots]$$

- According to the same procedure, also the contribution of the $k_y$ spring should be accounted for

$$[K_{tot}] = [K] + [K_x] + [K_y]$$

# CONCENTRATED ELEMENTS

4. Compute the structural damping matrix C

```
B = [0.01 0.015]';
A = zeros(2,2);

for ii=1:2
    A(ii,:) = [1/(2*w(ii)) w(ii)/2];
end

ab = A\B;
C  = ab(1)*Mtot + ab(2)*Ktot;
```

5. Partition the matrices to get $M_{FF}$ , $K_{FF}$ and $C_{FF}$

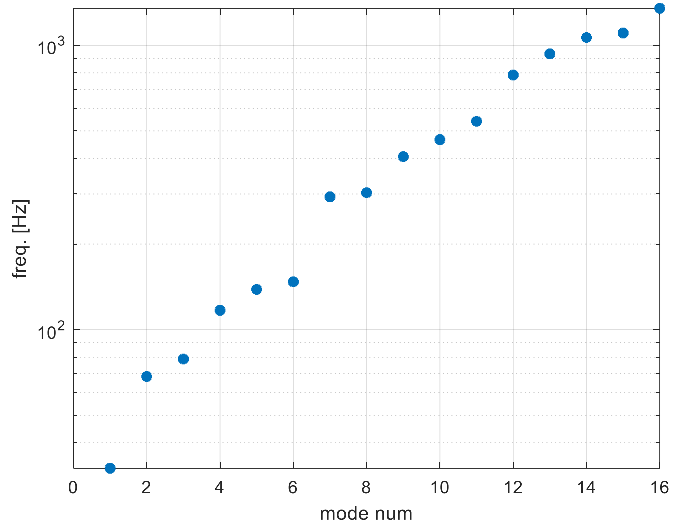
```
MFF = Mtot(1:ndof,1:ndof);
KFF = Ktot(1:ndof,1:ndof);
CFF = C(1:ndof,1:ndof);
```

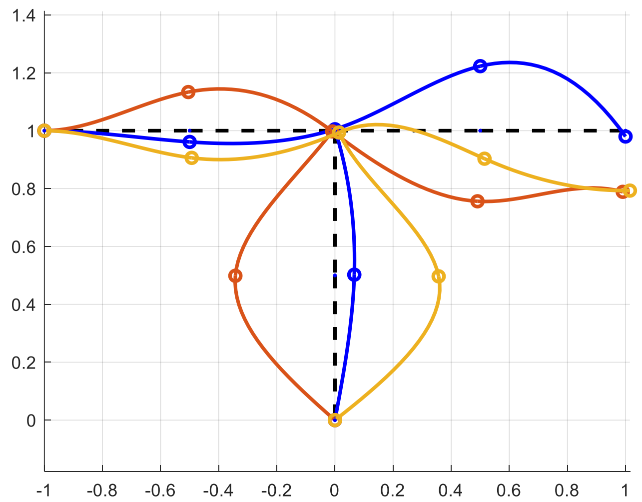
$$[M_{FF}] = \begin{bmatrix} 3.25 & 0 & 0 & \dots \\ 0 & 3.62 & 0 & \dots \\ 0 & 0 & 0.02 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{16 \times 16}$$

- Symmetric sparse $M_{FF}$ matrix
- Similar results also for $K_{FF}$ and $C_{FF}$ matrices

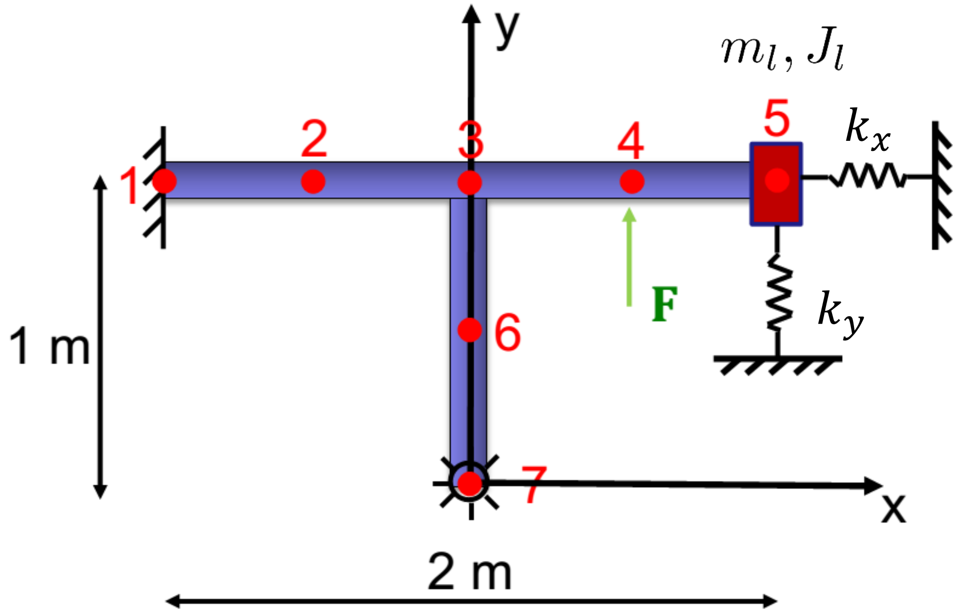
# NATURAL FREQUENCIES AND MODE SHAPES

```
Ml_h = diag([ml ml Jl]);
E_ml = zeros(3,21);
E_ml(:,10:12) = eye(3);
Ml   = E_ml'*Ml_h*E_ml;          21x21
Mtot = M + Ml;                   21x21
MFF  = Mtot(1:ndof,1:ndof);      16x16

E_kx     = zeros(1,21);
E_kx(10) = 1;
Kx       = E_kx'*kx*E_kx;        21x21
E_ky     = zeros(1,21);
E_ky(11) = 1;
Ky       = E_ky'*ky*E_ky;        21x21
Ktot     = K + Kx + Ky;          21x21
KFF      = Ktot(1:ndof,1:ndof);  16x16

[x0,w0] = eig(MFF\KFF);
w       = sqrt(diag(w0));
[w,ind] = sort(w);

f    = w/(2*pi);
mode = x0(:,ind);
```



# ANOTHER EXAMPLE

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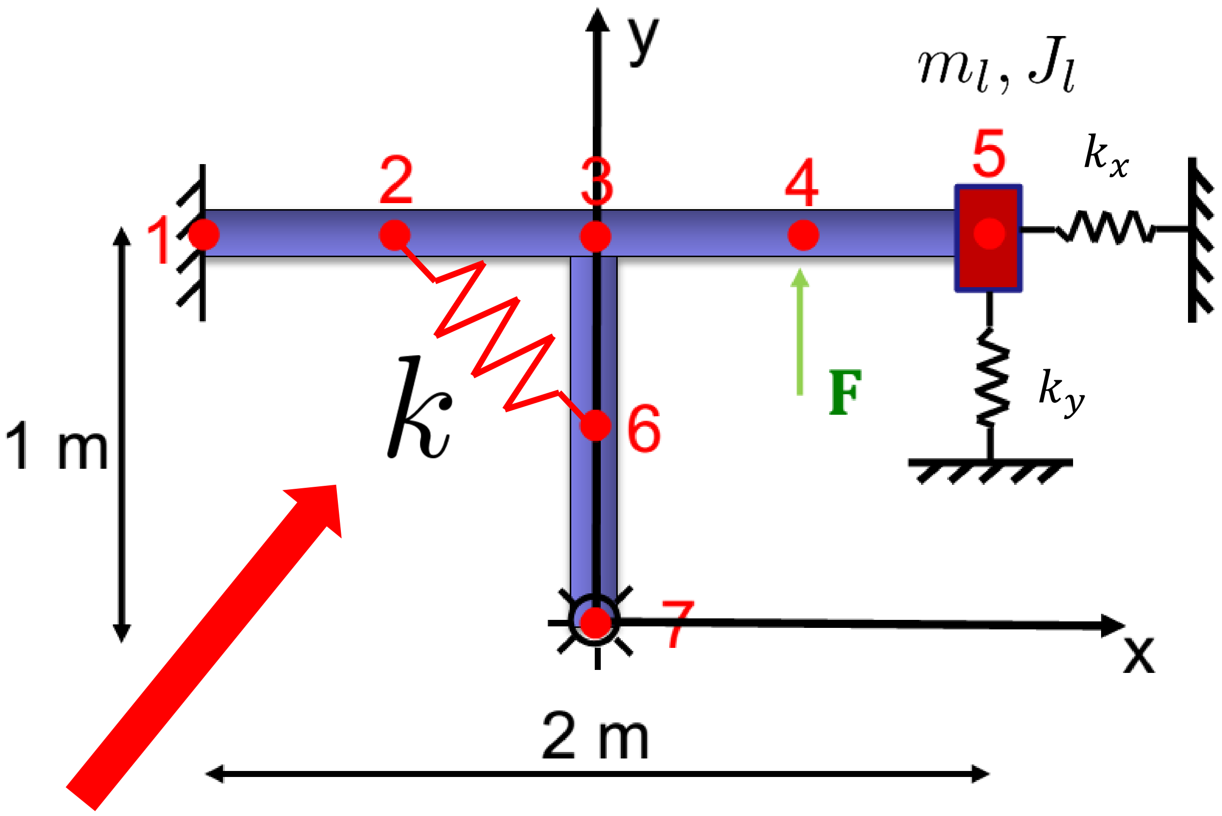
5. **Compute the frequency response function of *y* displacement of node 5**

**NOW IT'S YOUR TURN!**

# CONCENTRATED ELEMENTS

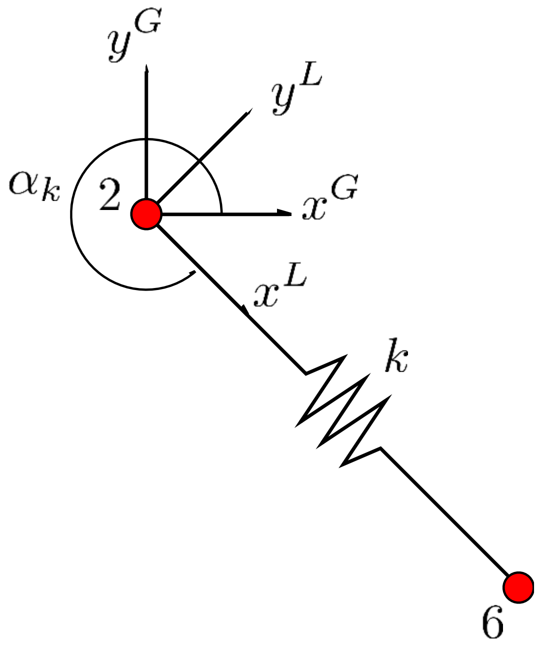



➢ How to deal with this concentrated element?

# CONCENTRATED ELEMENTS

The spring elastic potential energy reads like

$$V_k = \frac{1}{2}k\left(x_2^L - x_6^L\right)^2 = \frac{1}{2}\left(x_2^L - x_6^L\right)^T k\left(x_2^L - x_6^L\right)$$




If the finite element approach is considered

$$\underline{X}_k^L = \begin{bmatrix} x_2^L & y_2^L & \theta_2^L & x_6^L & y_6^L & \theta_6^L \end{bmatrix}^T$$

$$\left(x_2^L - x_6^L\right) = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \underline{X}_k^L$$

Thus

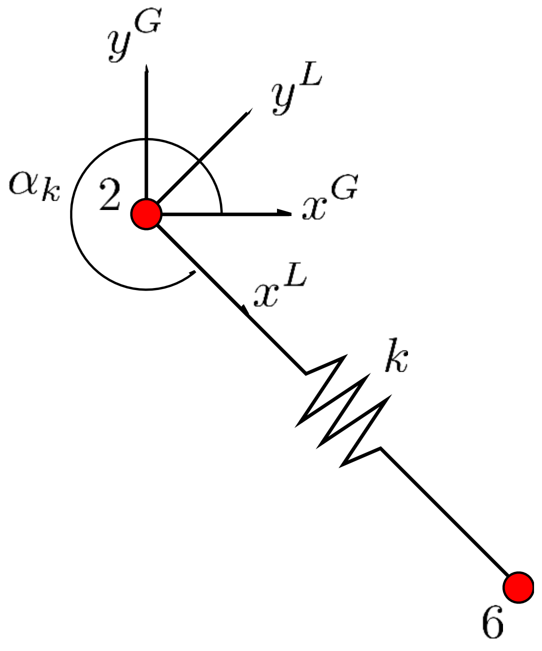
$$V_k = \frac{1}{2}\left(\underline{X}_k^L\right)^T \begin{bmatrix} k & 0 & 0 & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underline{X}_k^L = \frac{1}{2}\left(\underline{X}_k^L\right)^T [K_k^L] \underline{X}_k^L$$

- $[K_k^L]$ stiffness matrix of the $k$ spring in the local reference frame

# CONCENTRATED ELEMENTS

Projection onto the global reference frame

$$\underline{X}_k^L = [\Lambda_k]\underline{X}_k^G$$




with the rotation matrix $$[\Lambda_k] = \begin{bmatrix} [\lambda_k] & [0] \\ [0] & [\lambda_k] \end{bmatrix}$$

$$[\lambda_k] = \begin{bmatrix} \cos\alpha_k & \sin\alpha_k & 0 \\ -\sin\alpha_k & \cos\alpha_k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

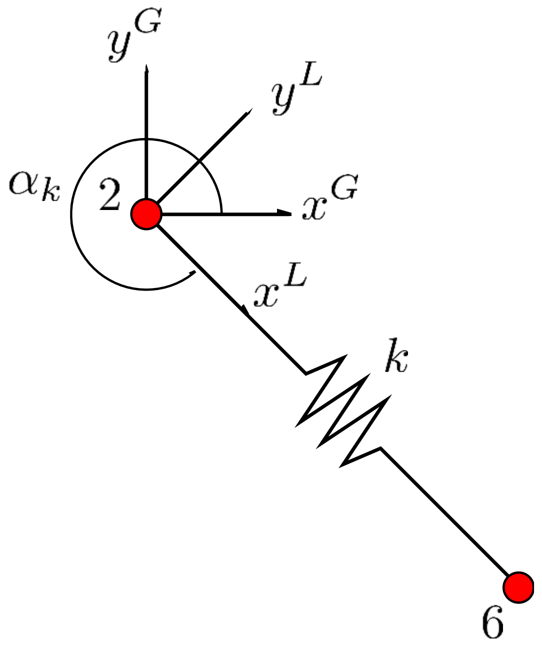
Thus
$$\begin{aligned} V_k &= \frac{1}{2}\left(\underline{X}_k^L\right)^T [K_k^L]\underline{X}_k^L \\ &= \frac{1}{2}\left(\underline{X}_k^G\right)^T [\Lambda_k]^T[K_k^L][\Lambda_k]\underline{X}_k^G \\ &= \frac{1}{2}\left(\underline{X}_k^G\right)^T [K_k^G]\underline{X}_k^G \end{aligned}$$

- $[K_k^G]$ stiffness matrix of the $k$ spring in the global reference frame

# CONCENTRATED ELEMENTS

Assembly into the global reference frame

$$\underset{6\times 21\ \ 21\times 1}{\underline{X}_k^G = [E_k]\underline{X}^G}$$




According to idb matrix $\quad idb = \begin{bmatrix} 17 & 18 & 19 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \\ 13 & 14 & 15 \\ 20 & 21 & 16 \end{bmatrix}$

$$[E_k](1:3, 1:3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

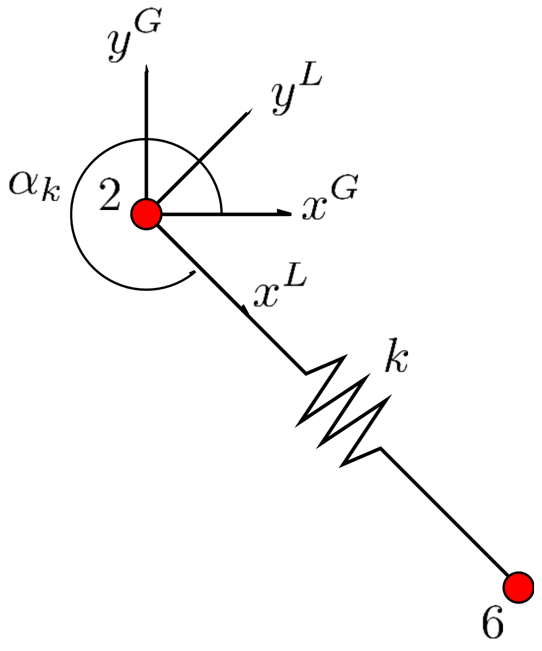
$$[E_k](4:6, 13:15) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus
$$\begin{aligned} V_k &= \frac{1}{2}\left(\underline{X}_k^G\right)^T [K_k^G]\underline{X}_k^G \\ &= \frac{1}{2}\left(\underline{X}^G\right)^T [E_k]^T[K_k^G][E_k]\underline{X}^G \\ &= \frac{1}{2}\left(\underline{X}^G\right)^T [K_k]\underline{X}^G \end{aligned}$$

- $[K_k]$ contribution of the $k$ spring to the overall stiffness matrix
- So that finally $\quad [K_{tot}] = [K] + [K_k]$

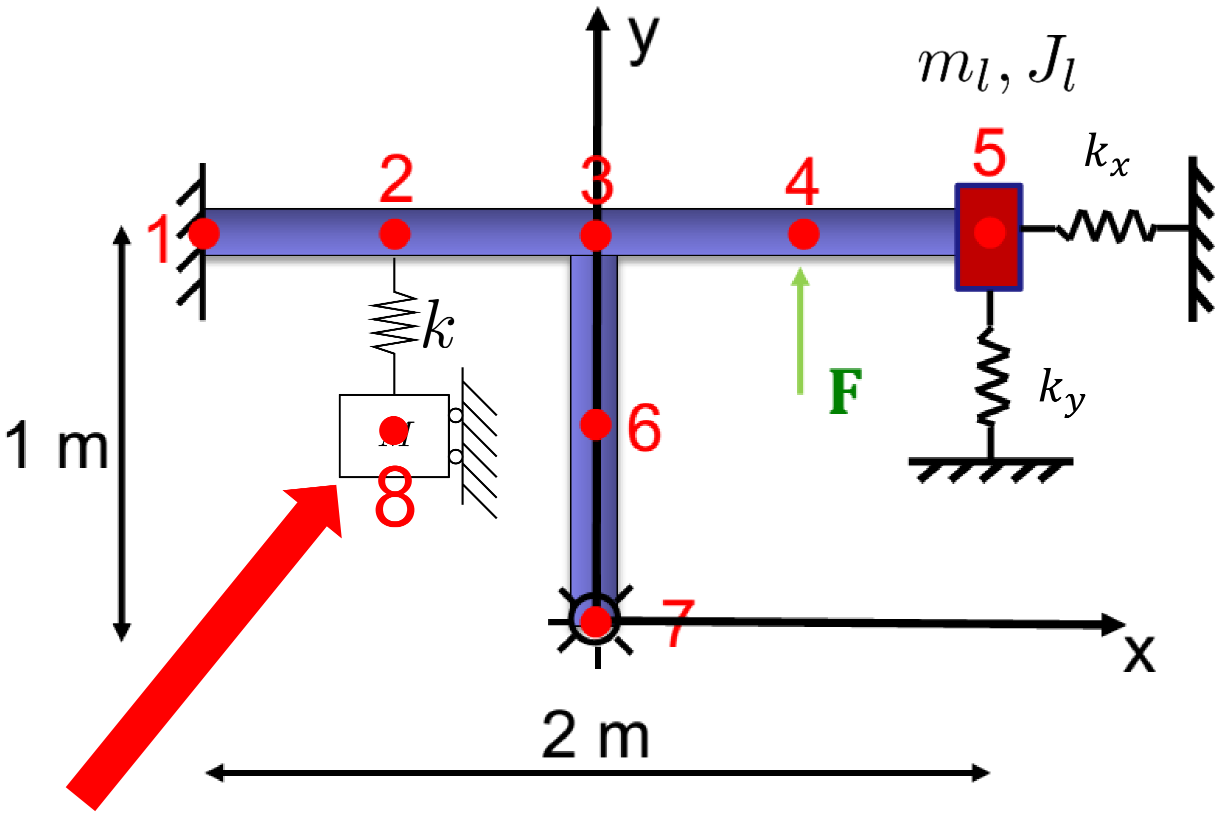
# CONCENTRATED ELEMENTS

```
K_k_L = [1 0 0 -1 0 0]' *k* [1 0 0 -1 0 0];

g = 7/4*pi;

lambda_k = [cos(g)    sin(g)   0
            -sin(g)   cos(g)   0
            0         0        1];
Lambda_k = [lambda_k   zeros(3,3)
            zeros(3,3) lambda_k ];

K_k_G  = Lambda_k' *K_k_L* Lambda_k;

idofn2 = idb(2,:);
idofn6 = idb(6,:);
idofk  = [idofn2 idofn6];

K(idofk, idofk) = K(idofk, idofk) + K_k_G;
```




Faster way to add the contribution of the *k* spring in the assembled stiffness matrix *K*

# CONCENTRATED ELEMENTS




- How to deal with these concentrated elements?

1. We need to place a node in correspondence of the concentrated mass, with the proper constrains
2. We need to properly modify the mass and stiffness matrices to account for the concentrated elements