Module 4 : Uniform Plane Wave

Lecture 25 : Solution of Wave Equation in Homogeneous Unbound medium

Objectives

In this course you will learn the following

- Uniform plane wave solution to the wave equation.
- Propagation constant of the uniform plane wave.
- Relation between electric and magnetic field for uniform plane wave.
- Intrinsic impedence of the medium.
- Transverse nature of uniform plane wave.

Uniform Plane Wave

- The time varying fields which can exist in an unbound, homogeneous medium, are constant in a plane containing the field vectors and have wave motion perpendicular to the plane. This phenomenon is then called the `Uniform plane wave'.

- Let us take an x-directed $\mathbf{E}$ - field which is constant in the xy-plane. The field therefore is given as

$$\mathbf{E}(x,y,z,t) = E_x(z)e^{j\omega t}\hat{\mathbf{x}}$$

- Substituting $\mathbf{E}$ in the wave equation and noting that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$ (since $\mathbf{E}$ is function of Z only)

we get

$$\frac{d^2E_x(z)}{dz^2} = -\omega^2\mu\epsilon E_x(z) = \gamma^2 E_x(z)$$

- Note that since E_x is a function of z only, the partial derivative has been changed to full derivative. The propogation constant γ is defined as

$$\gamma = j\omega\sqrt{\mu\in} = j\beta$$

- The phase constant β for the medium therefore is $\omega\sqrt{\mu\epsilon}$ and the attenuation constant is zero. The solution of the wave equation now is

$$E_x = E_x^+ e^{-j\beta z} + E_x^- e^{j\beta z}$$

- The two terms on RHS represent the travelling waves moving in $+z$ and $-z$ directions respectively. Since in this case, γ is purely imaginary, (the attenuation constant for the medium is zero) , the two waves travel with constant amplitude and their amplitudes are E_x^+ and E_x^- respectively anywhere in the space.

Uniform Plane Wave(contd.)

$$E_x = E_x^+ e^{-j\beta z} + E_x^- e^{j\beta z}$$

- Substituting for E_x from the above equation in the Maxwell's equation we have

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -j\omega\mu\mathbf{H}$$

$$\begin{aligned} \Rightarrow \quad & \frac{\partial E_x}{\partial z}\hat{\mathbf{y}} = -j\omega\mu\mathbf{H} = -j\omega\mu\{H_x\hat{\mathbf{x}} + H_y\hat{\mathbf{y}} + H_z\hat{\mathbf{z}}\} \\ \Rightarrow \quad & -j\beta E_x^+ e^{-j\beta z} + j\beta E_x^- e^{j\beta z} = -j\omega\mu H_y \\ \Rightarrow \quad & H_y = \frac{\beta}{\omega\mu}E_x^+ e^{-j\beta z} - \frac{\beta}{\omega\mu}E_x^- e^{j\beta z} \end{aligned}$$

- We can note that the ratio of the electric field and the magnetic field for a wave travelling in $+z$ direction is

$$\frac{E_x}{H_y} = \frac{E^+}{\frac{\beta}{\omega\mu}E^+} = \frac{\omega\mu}{\beta}$$

and that travelling in $-z$ direction is

$$\frac{E_x}{H_y} = \frac{E^-}{-\frac{\beta}{\omega\mu}E^-} = -\frac{\omega\mu}{\beta}$$

- Let us define a quantity called the intrinsic impedance of the medium as

$$\eta = \frac{\omega\mu}{\beta} = \frac{\omega\mu}{\omega\sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}}$$

- The intrinsic impedance η for a medium serves the same purpose as the characteristic impedance Z_0 serves for a transmission line. The intrinsic impedance is a property of the medium. For the free-space (vacuum),

$$\mu = \mu_0 = 4\pi \times 10^{-7}\, H/m, \qquad \epsilon = \epsilon_0 = \frac{1}{36\pi} \times 10^{-9}\, F/m$$

- The intrinsic impedance of the free-space, η_0 is

$$\eta_0 = \sqrt{\frac{4\pi \times 10^{-7}}{\frac{1}{36\pi} \times 10^{-9}}} \approx 120\pi \quad \Omega$$

- For any other medium

$$\mu = \mu_0\mu_r \qquad \epsilon = \epsilon_0\epsilon_r$$

- Where μ_r and ϵ_r are the relative permeability and relative permittivity of the medium respectively, and the intrinsic impedance is

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r}} = \eta_0 \sqrt{\frac{\mu_r}{\epsilon_r}}$$

The uniform plane wave has following properties

- $\mathbf{E}$, $\mathbf{H}$ and direction of the wave are perpendicular to each other. Infact they form right handed orthogonal coordinate system when taken in that sequence. This wave, is therefore a 'Transvers Electromagnetic Wave' (TEM) Wave.

- The ratio $|E| / |H|$ for the wave is equal to the intrinsic impedance of the medicem, η.

- The Vector Magnetic Field for a uniform plane wave is completely defined if the Vector Electric Field and the direction of wave is known along with the medium parameters.

- We therefore discuss only the behaviour of the electric field of a plane wave.

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Recap

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