Objectives

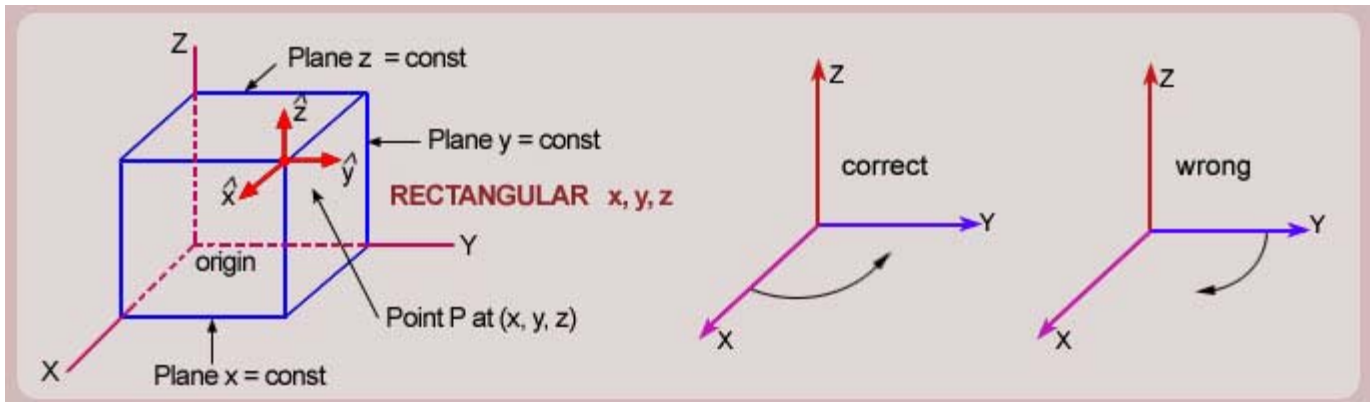**In this course you will learn the following**

- Co-ordinate Systems

- Vector Products

- Derivative Operations

- Important Vector Theorems

CO-ORDINATE SYSTEMS

There are three co-ordinate systems generally used in the analysis of the electromagnetic problems
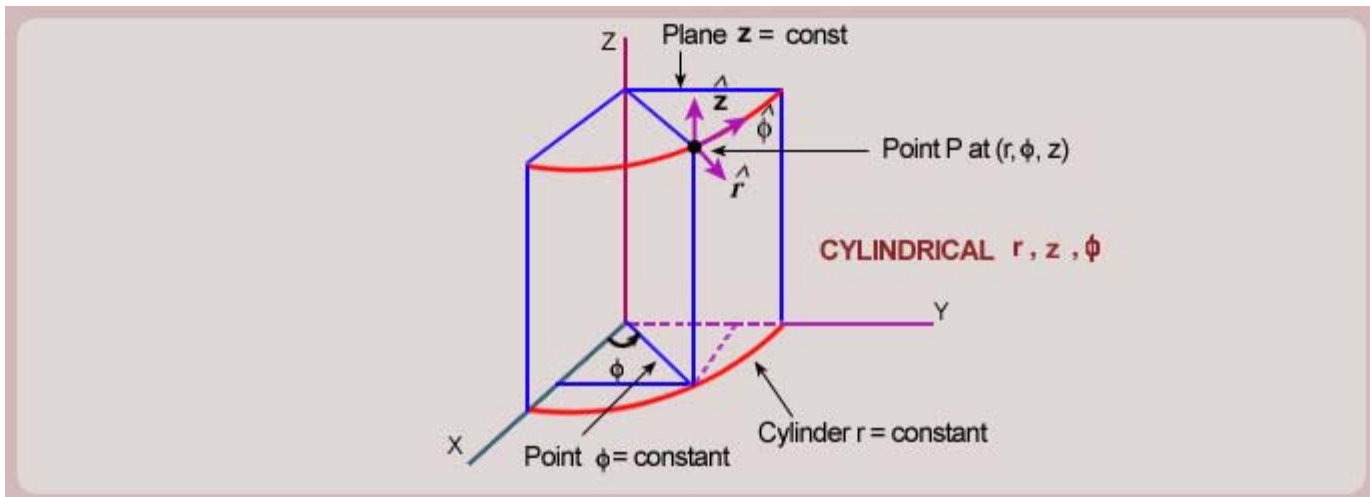
- **Cartesian Co-ordinate System**
  The 3-D space imagined like a box
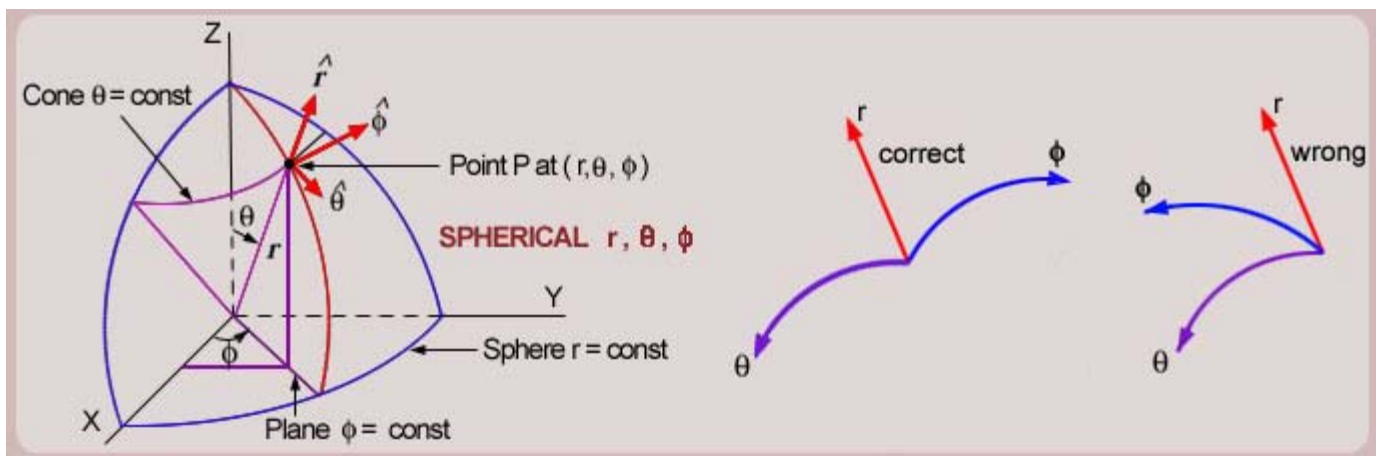

- Right handed convention has been followed.
- **Cylindrical Co-ordinate System**
  The 3-D space imagined like a cylinder

- **Spherical Co-ordinate System**
  The 3-D space imagined like a sphere

Cone θ = const

Point P at (r,θ, φ)

SPHERICAL r, θ, φ

Sphere r = const

Plane φ = const

correct

wrong

■ A vector **A** can be decomposed into the three components along the three axis in any co-ordinate system.

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

$$\text{or} \quad \mathbf{A} = A_r\hat{\mathbf{r}} + A_\phi\hat{\phi} + A_z\hat{\mathbf{z}}$$

$$\text{or} \quad \mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

Vector Products

- The scalar (dot) and vector (cross) products of the two vectors $\mathbf{A}$ and $\mathbf{B}$ are defined as

$$Dot\ Product\ :\ A \cdot B = A_i B_i + A_j B_j + A_k B_k$$

$$= \sum_{p=i,j,k} A_p B_p$$

$$Cross\ Product\ :\ A \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_i & A_j & A_k \\ B_i & B_j & B_k \end{vmatrix}$$

where i, j, k are the three axes of a co-ordinate system, and $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the corresponding unit vectors. The dot product of two vectors is a scalar quantity as where the cross-product of two vectors is a vector quantity. From the above two equations we can note that

$$as\ where,\ A \cdot B = B \cdot A$$

$$A \times B = - B \times A$$

Differential Operator ∇

- A very important differential operator which is used in the analysis of the electromagnetic problems is the operator ∇, pronounced as `del'. This operator is a vector operator and it has the dimension of length-inverse (L^{-1}). In cartesian co-ordinate system the operator can be written as

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

- In all algebric manipulations, the ∇ can be treated as a vector. The operator ∇ can operate on a scalar like the multiplication of a scalar and a vector, or it can operate on a vector like the dot or cross products of the two vectors.

Derivative Operations

Gradient of a Scalar Function

■
When the operator ∇ operates on a scalar function F the output is a vector and is called the gradient of function F. Physically this operations gives a vector whose magnitude is equal to the slope of the function F and whose direction is the direction of the maximum slope. In cartesian co-ordinate system the gradient can be written as

$$\nabla F = \frac{\partial F}{\partial x}\hat{x} + \frac{\partial F}{\partial y}\hat{y} + \frac{\partial F}{\partial z}\hat{z}$$

Divergence of a Vector Function

■
When the operator ∇ operates on a vector $\mathbf{A}(= A_x\hat{x} + A_y\hat{y} + A_z\hat{z})$ like a dot product the outcome is a scalar quantity

and is called the divergence of function $\mathbf{A}$. Physically this represents net outward flux per unit volume of the quantity

represented by the vector $\mathbf{A}$. In cartesian co-ordinate system the divergence can be written as

$$Div(\mathbf{A}) = \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

■ **Divergence in Cylindrical coordinates**

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}rA_r + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

■ **Divergence in Spherical coordinates**

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2 A_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Curl of a Vector Function

■
When a vector function $\mathbf{A}$ is operated by ∇ like a cross product the outcome is what is called the curl of $\mathbf{A}$. The

$curl(\mathbf{A})$ is a vector quantity and it is a measure of the rotation treated by the vector $\mathbf{A}$ per unit area. Mathematically we

can write $curl(\mathbf{A})$ as (in cartesian system)

$$curl(\mathbf{A}) = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$
$$+ \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

■ **Curl in Cylindrical coordinates**

$$\begin{vmatrix} \hat{r}\,\dfrac{1}{r} & \hat{\phi} & \hat{z}\,\dfrac{1}{r} \\[2mm] \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\[2mm] A_r & rA_\phi & A_z \end{vmatrix}$$

$$\nabla \times A = \hat{r}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{z}\,\frac{1}{r}\left(\frac{\partial}{\partial r}rA_\phi - \frac{\partial A_r}{\partial \phi}\right)$$

- **Curl in Spherical coordinates**

$$\nabla \times A = \hat{r}\,\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}\left(A_\phi \sin\theta\right) - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\theta}\,\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}rA_\phi\right) + \hat{\phi}\,\frac{1}{r}\left(\frac{\partial}{\partial}rA_\theta - \frac{\partial A_r}{\partial \theta}\right)$$

-

For the divergence, A_x is differentiated with respect to x, A_y is differentiated with respect to y and so on. Whereas

for the curl, A_x is differentiated with respect to other two variables y and z but not x. Same is true for A_y and A_z. This implies that for divergence to exist a function must vary along its direction wheareas, for curl to exist the function should vary perpendicular to its direction. It should be however noted that these are sufficient and not the essential conditions.

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If we take divergence of the curl of a vector, the result is always zero. That is,

$$\nabla \cdot (\nabla \times A) = 0$$

-

Similarly, we can show that the curl of the gradient of a scalar function is identically zero. That is,

$$\nabla \times (\nabla F) = 0$$

Derivative Operations (contd.)

The Laplacian Operator ∇^2

- The Laplacian operator ∇^2 is a scalar operator which can operate on a scalar or a vector. If it operates on a scalar, the outcome is a scalar quantity. Whereas, if it operates on a vector, the outcome is a vector quantity.

- Normally the operator ∇^2 is equal to $\nabla \cdot \nabla$ of a scalar function. In other words ∇^2 is equivalent to the divergence of the gradient of a scalar function, that is,

$$\nabla^2 = \nabla \cdot (\nabla F)$$

- In cartesian co-ordinate system we get.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- The use of ∇^2 however is not restricted to its operation on a scalar function. In fact ∇^2 can operate on a vector to give a vector. In this case, the Laplacian of a vector $\mathbf{A}$ would mean $\nabla^2 \mathbf{A}$ which if we write $\nabla \cdot (\nabla \mathbf{A})$ would not make any sense as $\nabla \mathbf{A}$ is not defined. The ∇^2 operator for a vector should be taken as the single operator and not as the divergence of the gradient. In the cartesian co-ordinate system the Laplacian of a vector $\mathbf{A}$ would be written as

$$
\begin{aligned}
\nabla^2 \mathbf{A} &= \frac{\partial^2 \mathbf{A}}{\partial x^2} + \frac{\partial^2 \mathbf{A}}{\partial y^2} + \frac{\partial^2 \mathbf{A}}{\partial z^2} \\
&= \hat{\mathbf{x}}\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right) + \hat{\mathbf{y}}\left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right) \\
&+ \hat{\mathbf{z}}\left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)
\end{aligned}
$$

Important Vector Theorems

Divergence Theorem

- This theorem relates Surface Integral to Volume Integral

- Let the volume enclosed by this surface be given by V. Then according to the Divergence theorem,

$$\oint_S \mathbf{A} \cdot \mathbf{da} = \iiint_V (\nabla . \mathbf{A}) dv$$

- Here the sign $\oint$ indicates the integral over a closed surface. The elemental area $\mathbf{da} = da\,\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ represents outward unit normal to the surface.

Stokes Theorem

- This theorem relates Contour Integral to Surface Integral
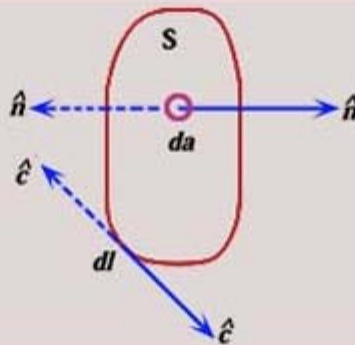
- Consider a closed curve C enclosing an area S in the vector field $\mathbf{A}$ as shown in the above figure. The Stokes theorem then states that

$$\oint_C \mathbf{A} \cdot \mathbf{dl} = \iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{da}$$

- The sign $\oint$ indicates line integral around a closed path and $\mathbf{dl} = dl\,\hat{\mathbf{c}}$ and $\mathbf{da} = da\,\hat{\mathbf{n}}$. $\hat{\mathbf{c}}$ is the unit vector along the line segment dl and $\hat{\mathbf{n}}$ is the unit vector normal to the surface. The directions of $\hat{\mathbf{c}}$ and $\hat{\mathbf{n}}$ should be consistent with the right handed system. If the line contour is traced in the anticlockwise direction (thick arrow for $\hat{\mathbf{c}}$) then the direction of $\hat{\mathbf{n}}$ vector will be pointing outward the plane of the paper. One can therefore chose $\hat{\mathbf{c}}$ and $\hat{\mathbf{n}}$ given either by the thick arrows or by dotted arrows.

Recap

In this course you have learnt the following

- Co-ordinate Systems

- Vector Products

- Derivative Operations

- Important Vector Theorems