Module 2 : Transmission Lines

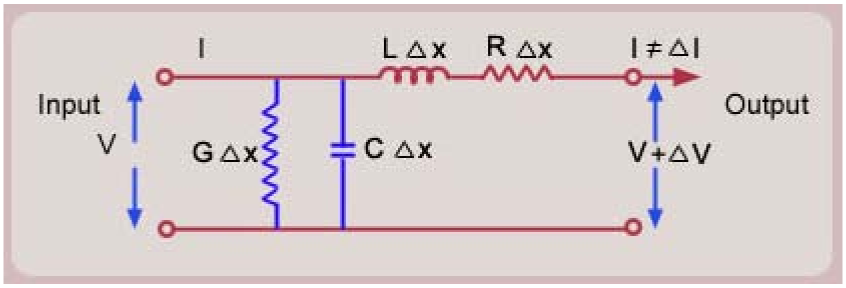
Lecture 3 : Transmission Line Analysis

Objectives

In this course you will learn the following

- Kirchoff's laws applied to an infinitesimal section of a line.
- Voltage and current equations for the transmission line exerted with time harmonic voltage and current.
- Solution of voltage and current equations
- Propagation constant of a line and its relation to the line parameters per unit length and frequency.
- Physical interpretation of voltage and current solutions.
- Existance of voltage and current waves on a transmission line.

Voltage & current equations for small section of a line



- Let us consider a small section of a transmission line of length Δx. Let the voltage at the input be V and current at the input be I.
- Due to voltage drop in the series arm, the output voltage will be different from the input voltage, say $V + \Delta V$.
- Similarly due to current through the capacitance and the conductance the output current will be different from the input through the current, say $I + \Delta I$. Then we can write

$$\Delta V = -(R\Delta x + j\omega L\Delta x)I$$
$$\Delta I = -(G\Delta x + j\omega C\Delta x)V$$

$$\frac{\Delta V}{\Delta x} = -(R + j\omega L)I$$
$$\frac{\Delta I}{\Delta x} = -(G + j\omega C)V$$

- Now if the lumped circuit model should be valid for arbitrarily high frequency (i.e. arbitrarily small λ), the analysis has to be carried out in the limit $\Delta x \longrightarrow 0$

$$\lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} \equiv \frac{dV}{dx} = -(R + j\omega L)I \quad \text{-------- (2.1)}$$

$$\lim_{\Delta x \to 0} \frac{\Delta I}{\Delta x} \equiv \frac{dI}{dx} = -(G + j\omega C)V \quad \text{-------- (2.2)}$$

Important

In general, the voltage and the current are not related through algebraic equations but are governed by differential equations.

Comment

The lines are essentially electromagnetic field problem. The simplified circuit analysis based on distributed circuit elements and the lumped circuit model gives the operating equation(in terms of the terminal quantities) as a one-dimensional wave equation is a proof that the equivalent circuit model is correct.

Solution of Voltage & Current equations of Transmission Line

- Differentiating eqn. 2.1 and substituting from eqn. 2.2 we get,

$$\frac{d^2V}{dx^2} = -(R + j\omega L)\frac{dI}{dx}$$

$$\frac{d^2V}{dx^2} = -(R + j\omega L)[-(G + j\omega C)V] = (R + j\omega L)(G + j\omega C)V$$

- Similarly differentiating eqn. 2.2 and substituting from 2.1 we get,

$$\frac{d^2I}{dx^2} = (R + j\omega L)(G + j\omega C)I$$

- Let us define a parameter γ as

$$\gamma^2 = (R + j\omega L)(G + j\omega C)$$

- The physical significance of γ will be explained later.
- However, γ is a parameter which depends upon the line parameters R, L, C and G and the frequency, ω .
- γ is called the propagation constant of the line, and is in general a complex quantity.

Solution of Voltage & Current equations of Transmission Line

- Both voltage and current are governed by the same second order differential equation i.e,

$$\frac{d^2V}{dx^2} = \gamma^2 V$$
$$\frac{d^2I}{dx^2} = \gamma^2 I$$

- The time harmonic function $e^{j\omega t}$ is implicit in these equations. The general solution to the differential equations with harmonic time function can be written as,

$$V(t) = V^+.\exp\{j\omega t - \gamma x\} + V^-.\exp\{j\omega t + \gamma x\}$$
$$I(t) = I^+.\exp\{j\omega t - \gamma x\} + I^-.\exp\{j\omega t + \gamma x\}$$

Where, V^+, V^-, I^+, I^- are the arbitrary complex constants which are to be evaluated from the boundary conditions.

- Since γ is in general a complex quantity let us write

$$\gamma = \alpha + j\beta$$ (Where α and β are real quantities)

- Substituting for $\gamma = \alpha + j\beta$, the voltage and current on any point of the line ' x ' at any instant, ' t ' can be written as

$$V(t) = V^+.\exp\{-\alpha x\}.\exp\{j\omega t - j\beta x\} + V^-.\exp\{\alpha x\}.\exp\{j\omega t + \beta x\}$$
$$I(t) = I^+.\exp\{-\alpha x\}.\exp\{j\omega t - j\beta x\} + I^-.\exp\{\alpha x\}.\exp\{j\omega t + \beta x\}$$

Physical Interpretation of Voltage & Current Solutions

- Let us now understand the phenomenon represented by the two terms of the voltage and current soutions.
- Let us consider the voltage solution. Take the first term of the solution

$$V^{+} . \exp\{-\alpha x\} . \exp\{j\omega t - j\beta x\}$$

Assuming that $V^{+} \equiv |V^{+}| e^{j\phi}$, the voltage due to this term at any point ' x ' on the line at any instant, ' t ' is

$$\mathrm{Re}\left\{V^{+} . \exp\{-\alpha x\} . \exp\{j\omega t - j\beta x\}\right\}$$
$$= \mathrm{Re}\left\{|V^{+}| . \exp\{j\phi\} . \exp\{-\alpha x\} . \exp\{j\omega t - j\beta x\}\right\}$$
$$= |V^{+}| . \exp\{-\alpha x\} \cos(\omega t - \beta x + \phi)$$

- The first term therefore represents a voltage whose amplitude reduces exponentially with distance, ' x ' and whose phase is a combination of space, ' x ' and time, ' t '.

The voltage is composite function of space and time.

Temporal variation of Voltage and Current

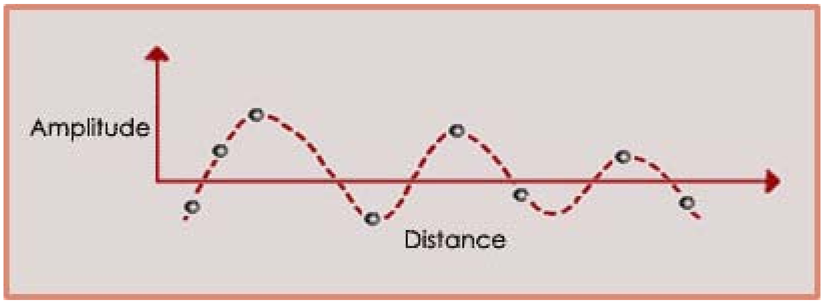
- For a given location on the line, βx is constant, and therefore voltage varies sinusoidally with time, with amplitude $|V^{+}| e^{-\alpha x}$ and frequency ' ω '. The phase of the voltage is $(\phi - \beta x)$.

Note and observe the following : Voltage at two locations

- The amplitudes at two locations are not the same.
- Due to differing phase differences, the voltages at two locations do not reach to the maximum at the same instant.

Spatial Variation of Voltage & Current

- On the other hand, for a given time, ωt is constant, and therefore the voltage has decaying Spatial Sinusoidal function with spatial frequency β and phase $\omega t + \phi$. That is, if we instantaneously look at the voltage along the line we see decaying sinusoidal function in the space (see figure)



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Recap

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