Module 5 : Plane Waves at Media Interface

Lecture 32 : Plane Wave in Arbitrary Direction

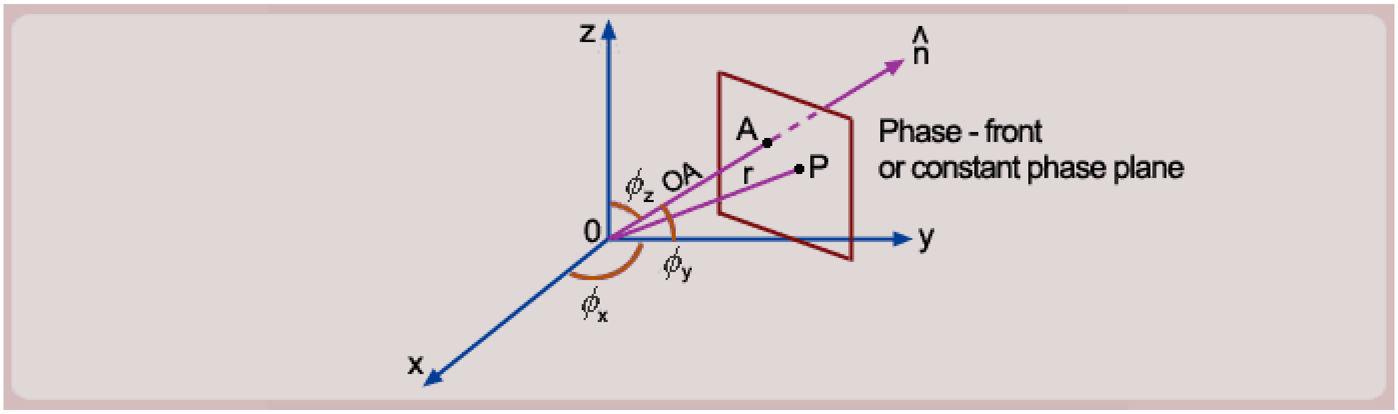
Objectives

In this course you will learn the following

- Wave Vector at Arbitrary Direction.
- Electric & Magnetic fields for a wave moving in direction $\hat{n}$.
- Phase Velocity and Wave length.

Wave Vector at Arbitrary Direction

- Let the wave be moving in direction making angles ϕ_x , ϕ_y , ϕ_z respectively with three axis x,y,z as shown in fig.



- The unit vector in the direction of the wave propagation is

$$\hat{\mathbf{n}} = \cos\phi_x\hat{\mathbf{x}} + \cos\phi_y\hat{\mathbf{y}} + \cos\phi_z\hat{\mathbf{z}}$$

where $\cos\phi_x$, $\cos\phi_y$, $\cos\phi_z$ are called the direction cosines of the vector $\hat{\mathbf{n}}$.

- The equation of a constant phase plane (the phase front) is given as

$$\hat{\mathbf{n}} \cdot \mathrm{OP} = \hat{\mathbf{n}} \cdot \mathbf{r} = constant$$

- Therefore, the phase of this constant phase plane is

$$\beta|\mathrm{O}A| = \beta\hat{\mathbf{n}} \cdot \mathbf{r}$$

Electric & Magnetic fields for a wave moving in direction $\hat{n}$

- The electric field of a plane wave travelling in direction $\hat{\mathbf{n}}$ can then be written as

$$\mathbf{E} = \mathbf{E}_o e^{-j\beta \hat{\mathbf{n}} \cdot \mathbf{r}}$$

Where $\mathbf{E_o}$ is a vector perpendicular to the unit vector $\hat{\mathbf{n}}$.

- Let us define the wave vector as

$$\mathbf{k} \equiv \beta \hat{n} \equiv k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$$

- The electric field then is

$$\bar{\mathbf{E}} = \mathbf{E}_0\, e^{-j\mathbf{k}.\mathbf{r}} = \mathbf{E}_0\, e^{-j\left(k_x x + k_y y + k_z z\right)}$$

We therefore get

$$\frac{\partial}{\partial x} = -jk_x$$

$$\frac{\partial}{\partial y} = -jk_y$$

$$\frac{\partial}{\partial z} = -jk_z$$

- The magnetic field is then obtained as

$$\mathbf{H} = -\frac{1}{j\omega\mu} \nabla \times \mathbf{E}$$

$$= -\frac{1}{j\omega\mu} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -jk_x & -jk_y & -jk_z \\ E_x & E_y & E_z \end{vmatrix}$$

$$= -\frac{1}{j\omega\mu}\{-j\mathbf{k} \times \mathbf{E}\} = \frac{1}{\omega\mu}\mathbf{k} \times \mathbf{E}$$

$$= \frac{\left(\hat{\mathbf{n}} \times \mathbf{E}_0\right) e^{-j\mathbf{k}.\mathbf{r}}}{\eta} \equiv \mathbf{H}_0\, e^{-j\mathbf{k}.\mathbf{r}}$$

- The $\mathbf{E_o}$, $\mathbf{H}_0$ and $\hat{\mathbf{n}}$ vectors are perpendicular to each other and

$$|\mathbf{E}_0| \big/ |\mathbf{H}_0| = \text{Interisic impedance of the medium } \eta$$

Phase Velocity and Wave length

- The electric field of a uniform plane wave travelling in a direction which makes angles ϕ_x, ϕ_y, and ϕ_z with three axis x, y and z respectively is written as

$$\begin{aligned}\mathbf{E} &= \mathbf{E_0} e^{-j\mathbf{k\cdot r}} \\ &= \mathbf{E_0} e^{-j\beta x\cos\phi_x} e^{-j\beta y\cos\phi_y} e^{-j\beta y\cos\phi_y}\end{aligned}$$

- Separating out the 'z' variation, we can write the electric field as

$$\mathbf{E} = \mathbf{E_0} e^{-j\beta(x\cos\phi_x + y\cos\phi_y)} e^{-j\beta z\cos\phi_z}$$

NOTE :

- In the xy plane (plane perpendicular to z-direction) the phase is not constant. So xy-plane is not a constant phase plane.
- The phase constant along z-direction is $k_z = \beta\cos\phi_z$.
- The phase velocity in the z-direction therefore is

$$v_{pz} = \frac{\omega}{k_z} = \frac{\omega}{\beta\cos\phi_z} = \frac{v_0}{\cos\phi_z}$$

- Similarly we can get the phase velocities along the x and y directions as

$$\begin{aligned}v_{px} &= \frac{\omega}{k_x} = \frac{\omega}{\beta\cos\phi_x} = \frac{v_0}{\cos\phi_x} \\ v_{py} &= \frac{\omega}{k_y} = \frac{\omega}{\beta\cos\phi_y} = \frac{v_0}{\cos\phi_y}\end{aligned}$$

- Since $|\cos\phi_x|$, $|\cos\phi_y|$, $|\cos\phi_z| \le 1$, the velocities v_{px}, v_{py}, v_{pz} are always greater than or equal to v_0. In fact when any of the angles ϕ_x, ϕ_y, ϕ_z $\rightarrow \pi/2$, the cosines of these angles tend to 0 and the corresponding velocities approach infinity. The bounds of the phase velocity therefore are

$$v_0 \le v_{px}, v_{py}, v_{pz} \le \infty$$

- The wavelength of the wave in x, y, z directions respectively are

$$\begin{aligned}\lambda_x &= \frac{v_{px}}{f} = \frac{\lambda_0}{\cos\phi_x} \\ \lambda_y &= \frac{v_{py}}{f} = \frac{\lambda_0}{\cos\phi_y} \\ \lambda_z &= \frac{v_{pz}}{f} = \frac{\lambda_0}{\cos\phi_z}\end{aligned}$$

Where $\lambda_0 = v_0/f$

Interesting observation

- If we consider the unbound medium as the free-space, the phase velocity of the wave is $v_0 = c$ (velocity of light in vaccum), we get

$$c \le v_{px}, v_{py}, v_{pz} \le \infty$$

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Recap

In this course you have learnt the following

- Wave Vector at Arbitrary Direction.
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