Objective

In this course you will learn the following

- General Approach to Wave Guide Analysis.

- Transverse Magnetic Mode.

- Boundary conditions for the longitudinal electric field.

General Approach to Wave Guide Analysis

- In case of parallel wave guide the modal propagation was visualized as super position of multiply reflected plane wave from the two conducting sheets. This approach although provides better physical understanding of the modal propagation, becomes algebraically unmanagable for complicated waveguide in structure.

- In the following we develop a general framework for analyzing the wave guide structure like a **Rectangular Wave Guide**.

- A rectangular wave guide is a hollow metallic pipe with rectangular cross section. The electro magnetic energy propagates along the length of pipe. The net wave propagation therefore is along the length of the pipe.

- A direction along the length of the pipe is called the Longitudinal direction. Whereas any direction perpendicular to the wave propagation is called the **TRANSVERSE DIRECTION**.

- Let us choose the co-ordinate system such that the z - axis is along the longitudinal direction.

- You can note that in general there are six field components, three for electric field nad three for magnetic field which are related through Maxwell's equations. All the six components therefore cannot be independent. We can select to field components as an independent components and the remaining four components can be obtained from the Maxwell's equation.

- Since the z - direction is a special direction(direction of the net wave propagation) we choose the longitudinal electric and magnetic field components (E_z , H_z) as independent components and derive the transverse components in terms of the longitudinal components using the Maxwell's equation.

- Since the net wave is travelling in a $+z$ - direction any field component in the z - direction will be of the type $e^{j - j\beta z}$, where β is the phase constant of the net wave propagating along the z - direction.

- If we define a parameter h (transverse propagation constant) as

$$h^2 = \omega^2 \mu\epsilon - \beta^2$$

we can write transverse electric and magnetic field in term of the longitudinal field components (E_z , H_z) as

$$\mathbf{E}_\perp = \frac{-j\omega\mu}{h^2}\nabla_\perp \times (H_z\hat{\mathbf{z}}) - \frac{\gamma}{h^2}\nabla_\perp(E_z) \quad \text{---------- (6.28)}$$

$$\mathbf{H}_\perp = \frac{j\omega\epsilon}{h^2}\nabla_\perp \times (E_z\hat{\mathbf{z}}) - \frac{\gamma}{h^2}\nabla_\perp(H_z) \quad \text{---------- (6.29)}$$

Here ∇ transverse is defined by

$$\nabla_\perp = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} \quad \text{---------- (6.30)}$$

in Cartesians co-ordinate system.

- From the above equations we can make some important observations as follows :

(1) Transverse fields can exist provided at least one of the longitudinal components (E_z or H_z) is non-zero, except when h = 0. That is, in general there is no transverse electromagnetic wave propagation except when $h = 0$.

(2) For $h = 0$ we get $\beta = \omega\sqrt{\mu\epsilon}$ implying that the transverse electromagnetic wave can exist in a waveguide if its propagation constant is same as that of the unbound medium filling the waveguide.

(3) The fields corresponding to $E_z = 0$, have electric field transverse to the direction of wave propagation and hence

represent the Transverse electric (TE) wave.

The fields corresponding to H_z =0 have magnetic field transverse to the direction of wave propagation and hence represent Transverse magnetic (TM) wave.

For TM or TE case, E_z or H_z respectively is to be non zero, and h has to be non zero. Otherwise the transverse fields would become infinite. In other words, the TE and TM modes can not have the phase constant same as that of the unbound medium. The TE and TM modes then essentially have to be dispersive modes i.e., their phase velocity should vary as a function of frequency.

■ In Cartesian co-ordinate system we can explicitly write the transverse field component in terms of the longitudinal components as

$$E_x = -\frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} - \frac{j\beta}{h^2}\frac{\partial E_z}{\partial x} \qquad \text{---------- (6.31)}$$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} - \frac{j\beta}{h^2}\frac{\partial E_z}{\partial y} \qquad \text{---------- (6.32)}$$

$$H_x = \frac{j\omega\epsilon}{h^2}\frac{\partial E_z}{\partial y} - \frac{j\beta}{h^2}\frac{\partial H_z}{\partial x} \qquad \text{---------- (6.33)}$$
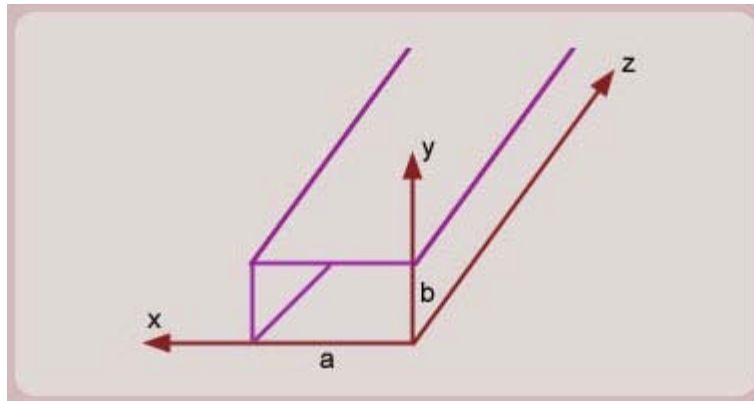
$$H_y = -\frac{j\omega\epsilon}{h^2}\frac{\partial E_z}{\partial x} - \frac{j\beta}{h^2}\frac{\partial H_z}{\partial y} \qquad \text{---------- (6.34)}$$

■ So, in the analysis of a wave guide first we obtain E_z or H_z which is consistent with the boundary conditions and then subsequently obtain the transverse component using the above equation.

Transverse Magnetic Mode

- A rectangular waveguide with cross-section $(a \times b)$ is shown in the figure below

- The walls of the waveguide are made of ideal conductior and the medium filling the waveguide is ideal dielectric.

- As a convention and the x-axis is oriented along the broader dimension of the wave guide, and the y-axis is oriented along the shorter dimension of the waveguide. The z-axis is oriented along the length of the waveguide and the waveguide is assumed to be infinite length.

- For transverse magnetic mode, we have $H_z = 0$ and $E_z \neq 0$. The transverse fields are therefore represented in terms of E_z components only.

- The wave equation is to be solved for E_z with appropriate boundary conditions. In Cartesian co-ordinates the wave equation for E_z can be written as

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + \omega^2 \mu \epsilon E_z = 0 \quad \text{-------- (6.35)}$$

- The equation can be solved by the separation of variables i.e by assuming that E_z is given as

$$E_z(x,y,z,t) = X(x)Y(y)Z(z) \times e^{j\omega t} \quad \text{-------(6.36)}$$

- The fields are assumed to be of sinosoidal nature with an angular frequency ω .

- Substituting for E_z from 6.36 into 6.35, we get

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} + \omega^2 \mu \epsilon = 0 \quad \text{-------- (6.37)}$$

- Note that in equation 6.37 the first term is a function of x, the second term is a function of y only, the third term is a function of z only and fourth term is a constant. Since the equation is to be satisfied for every value of x,y,z each term in equation 6.37 must be constant i.e

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -A^2 \qquad (say) \qquad \text{-------- (6.38)}$$

$$\frac{1}{Y}\frac{d^2 Y}{dy^2} = -B^2 \qquad (say) \qquad \text{-------- (6.39)}$$

$$and \qquad \frac{1}{Z}\frac{d^2 Z}{dz^2} = -\beta^2 \qquad (say) \qquad \text{-------- (6.40)}$$

A, B and β are real constants.

- From the physical understanding of reflection of waves from parallel conducting boundaries we expect a standing wave kind of behaviour in x and y directions and a travelling wave kind of behaviour in z direction. In any case, we expect a wave phenomenon in x, y, z direction which can be properly represented by putting a negative sign infront of the constant A^2, B^2 and β^2. Instead of negative sign if the positive sign was used the solutions will have real exponential functions which would not represent the wave phenomenon.

- The equations 6.38, 6.39 and 6.40 can be re-written as

$$\frac{d^2 X}{dx^2} + A^2 X = 0 \qquad \text{-------- (6.41)}$$

$$\frac{d^2 Y}{dy^2} + B^2 Y = 0 \qquad \text{-------- (6.42)}$$

$$and \qquad \frac{d^2 Z}{dz^2} + \beta^2 Z = 0 \qquad \text{-------- (6.43)}$$

These equations are identical to the transmission line equations.

- The solution to equations 6.41, 6.42 and 6.43 can be appropriately written as

$$X = C_1 \cos Ax + C_2 \sin Ax \quad \ldots\ldots \text{ Standing Wave} \qquad \text{-------- (6.44)}$$
$$Y = C_3 \cos By + C_4 \sin By \quad \ldots\ldots \text{ Standing Wave} \qquad \text{-------- (6.45)}$$
$$Z = C_5 e^{-j\beta z} + C_6 e^{+j\beta z} \quad \ldots\ldots \text{ Travelling Wave} \qquad \text{-------- (6.46)}$$

where $C_1, C_2, \ldots, C_6$ are arbitrary constants which are to be evaluated by boundary conditions.

- If we assume that waveguide is of infinite length, we can take only one travelling wave in $+z$-direction. We can then choose $C_6 \equiv 0$. Substituting for X, Y, Z from 6.44, 6.45 and 6.46 into equation 6.36 the general solution for E_z can be written as

$$E_z = C_5 (C_1 \cos Ax + C_2 \sin Ax)(C_3 \cos By + C_4 \sin By)e^{-j\beta z} \qquad \text{-------- (6.47)}$$

Boundary conditions for the longitudinal electric field component E_z

■

The E_z component is parallel to all the four walls of the waveguide. Since the tangential field of the electric field is zero at the conducting boundary, we get the boundary condition for E_z.

$$y = 0 \,, y = b \quad \text{---------- (6.48)}$$

■

The boundary conditions for $x = 0$ and $y = 0$ walls give $C_1 = 0$, $C_3 = 0$ and the boundary conditions for $x = a$, and $y = b$ give

$$Aa = m\pi \Rightarrow A = \frac{m\pi}{a} \quad \text{---------- (6.49)}$$

$$Bb = n\pi \Rightarrow B = \frac{n\pi}{b} \quad \text{---------- (6.50)}$$

where m and n are integers.

■

Substituting for A and B , we finally get the solution for E_z as

$$E_z = C \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z} \quad \text{---------- (6.51)}$$

Constant C is essentially a combination of C_2 , C_5 , C_4 which gives the amplitude of the longitudinal component of the electric field.

Transverse field component for TM mode

■
  Substituting for E_z from equation 6.51 and $H_z = 0$ in equations 6.31, 6.32, 6.33 and 6.34 we get the transverse electric and magnetic field components as

$$E_x = \frac{-j\beta}{h^2}\frac{\partial E_z}{\partial x} = \frac{-j\beta}{h^2}(\frac{m\pi}{a})C\cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})e^{-j\beta z} \qquad \text{--------- (6.52)}$$

$$E_y = -\frac{j\beta}{h^2}\frac{\partial E_z}{\partial y} = -\frac{j\beta}{h^2}(\frac{n\pi}{b})C\sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b})e^{-j\beta z} \qquad \text{--------- (6.53)}$$

$$H_x = \frac{j\omega\epsilon}{h^2}\frac{\partial E_z}{\partial y} = \frac{j\omega\epsilon}{h^2}(\frac{n\pi}{b})C\sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b})e^{-j\beta z} \qquad \text{--------- (6.54)}$$

$$H_y = \frac{-j\omega\epsilon}{h^2}\frac{\partial E_z}{\partial x} = \frac{-j\omega\epsilon}{h^2}(\frac{m\pi}{a})C\cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})e^{-j\beta z} \qquad \text{--------- (6.55)}$$

The integers m and n define the order of the mode and the mode is designated as TM_{mn}

■
  We can make the following observations regarding TM_{mn} mode :

(1)   Similar to that of the parallel plane waveguide the fields exists in the discrete electric and magnetic field pattern called modes of waveguide.

(2)   All field components are sinosoidally in x and y directions.

(3)   All transverse fields go to zero if either m or n is zero. In other words, both the indices m and n have to be non-zero for existence of the TM mode. That is, TM_{m0} and TM_{0n} modes can not exist. Consequently, the lowest order mode which can exist is TM_{11} mode.

■
  Substituting E_z from 6.51 into 6.35, we get what is called the dispersion relation for the mode as

$$\beta^2 = \omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 \qquad \text{--------- (6.56)}$$

The dispersion relation suggest that the phase constant for the mode β is different for different modes (for different values of m and n) and is no more proportional to ω.

■
  Implication of this would be discussed later along with the characteristics of the TE mode.

Recap

In this course you have learnt the following

- General Approach to Wave Guide Analysis.

- Transverse Magnetic Mode.

- Boundary conditions for the longitudinal electric field.