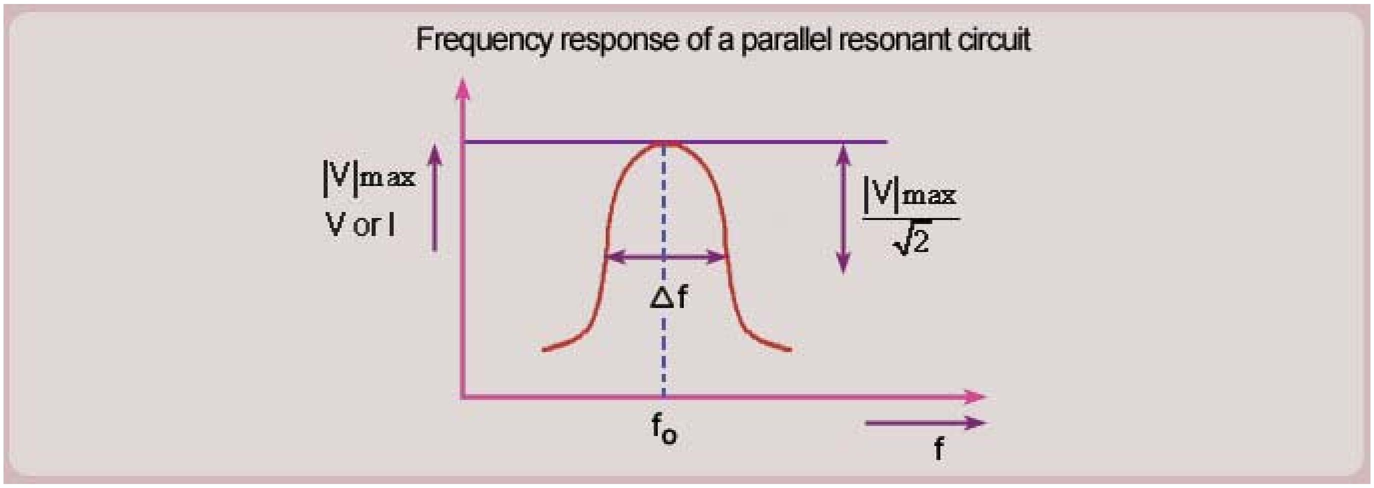
The Quality Factor of a Resonant Circuit



- The quality factor of a resonant circuit is given as

$$Q = 2\pi f_0 \frac{\text{Energy stored in the line}}{\text{Energy lost per second}}$$

where f_0 is the resonant frequency of the circuit.

'Q' of a Resonant Line

- The energy stored in a $n\lambda/4$ long section of the line is

U = Capacitive energy + Inductive energy

$$U = \frac{1}{2}C\int_0^{n\lambda/4}[V(l)]^2 dl + \frac{1}{2}L\int_0^{n\lambda/4}[I(l)]^2 dl$$
$$= \frac{1}{2}C\int_0^{n\lambda/4}[V_0 \sin\beta l]^2 dl + \frac{1}{2}L\int_0^{n\lambda/4}[\frac{V_0}{Z_0}\cos\beta l]^2 dl$$
$$= \frac{1}{4}CV_0^2(\frac{n\lambda}{4}) + \frac{1}{4}L\frac{V_0^2}{Z_0^2}(\frac{n\lambda}{4})$$

- Since $Z_0 = \sqrt{L/C}$, we have $L/Z_0^2 = C$. Hence the inductive and capacitive energies are equal, and the total energy is

$$U = \frac{1}{2}CV_0^2(\frac{n\lambda}{4})$$

- The energy lost per second is the power loss in the line. At parallel resonance, the line effectively appears like a resistance of value $Z_0/\alpha l$ $\left(l = n\lambda/4\right)$. The power loss in the line therefore is

$$P_{loss} = \frac{V_0^2}{(Z_0/\alpha l)} = \frac{V_0^2}{Z_0}.\alpha.\frac{n\lambda}{4}$$

- The quality factor of the line therefore is

$$Q = 2\pi f_0 \cdot \frac{U}{P_{loss}} = 2\pi f_0 \frac{\frac{1}{2}CV_0^2\left(\frac{n\lambda}{4}\right)}{\frac{V_0^2}{Z_0}\alpha\left(\frac{n\lambda}{4}\right)} = 2\pi f_0 \frac{Z_0 C}{2\alpha}$$

- Again noting that $Z_0 C = (\sqrt{L/C})C = \sqrt{LC}$, and $2\pi f_0 = \omega_0$. We get,

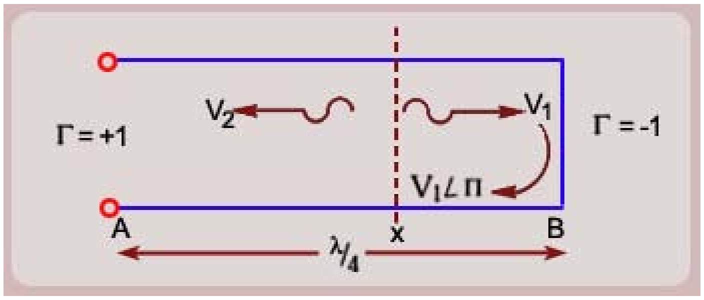
$$Q = \frac{W_0\sqrt{LC}}{2\alpha} = \frac{\beta}{2\alpha}$$

Where β is the phase constant at resonant frequency.

- One can note here that Q is independant of the length of the line as long as the loss is small.
- In practice, generally the lines have loss low enough to give a Q of few hundreds very easily.
- Since the $3dB$-bandwidth of a resonant circuit is f_0/Q, higher value of Q implies highly tuned or frequency selective circuits.
- The transmission line sections therefore act as excellent frequency selective circuits at high frequencies.

Voltage or Current Step-up Transformer

- Let us take a resonant transmission line of length $\lambda/4$. The line is open circuited at one end and short circuited at other as shown in the figure.



- Let us say there is a voltage source which induces a voltage in the line at some point 'X'. This induced voltage will send two voltage waves V_1 and V_2 with equal amplitudes. Consider now one of the waves, say V_1. This wave travels upto point B to encounter a short circuit. Since the reflection coefficient for a short circuit is -1, the wave gets fully reflected with a phase reversal. The wave after travelling a distance BA reachs to the open circuited end of the line and again gets fully reflected but with no phase reversal as $\Gamma = +1$ for the open circuit. After one round trip therefore when the wave V_1 reaches point X, its amplitude is same as its original value but its phase is changed by 2π, π due to reflection at point B and π due to propagation of a round trip distance of $\lambda/2$. This wave therefore adds up with the induced voltage in phase and the added up wave travels on the transmission line. The process is regenerative and the amplitude of the voltage wave V_1 goes on increasing. Exactly identical thing happens with the other wave V_2.

- Since the two waves travelling in the opposite directions identically grow in amplitude, the result is a continiously growing standing wave on the line with appropriate voltage maximum at A and voltage minimum at B.

- If the coupling of voltage is sustained, and the line is loss-less, there is no limit on the voltage and current, and the voltage and current eventually would grow to ∞.

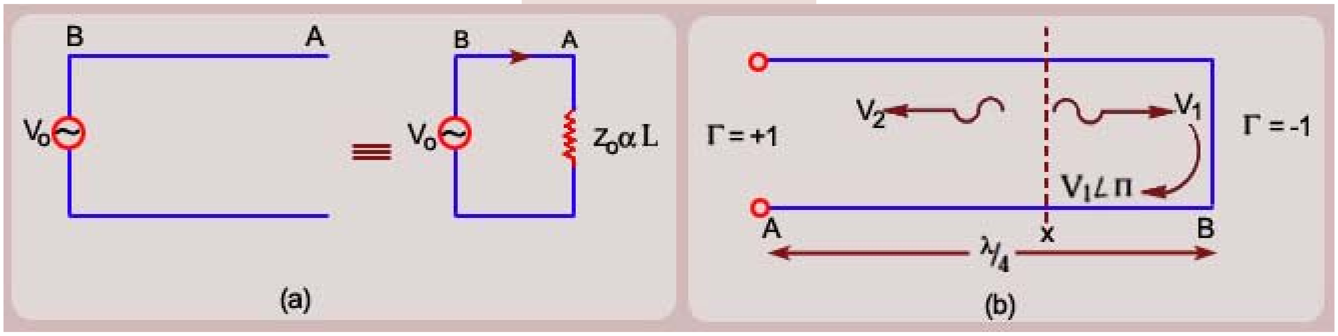
- However, if the line has a loss (no matter how small), then of course the voltage and current stabilize at some finite values. As the voltage/current increases the ohmic loss also increases and when the power lost in the line just equals the power supplied by the coupling source, the voltage/current stabilizes. It should be noted however that the maximum stabilized voltage or current on the line could be much higher than the coupling voltage or current.

- This suggests a possibility of using a resonant transmission line as a step-up voltage or current transformer.

Voltage Amplification on a Resonant Line

- As an illustrative example let us take a $\pi/4$ resonant section of a line, and instead of putting a short circuit put an ideal voltage source at point B as shown Figure (a). The open circuit at point A appears as almost short (for a low loss line) at point B. The impedance seen by the voltage source is $Z_0\alpha l$ and a current $V_0/Z_0\alpha l$ flows in terminal B. Since point B is a voltage minimum and current maximum, the source current $V_0/Z_0\alpha l$ is equal to the maximum current on the line I_{max}. The maximum voltage on the line then is $Z_0 I_{max}$ and it appears at point A. We therefore have

$$V_A = Z_0 \frac{V_0}{Z_0\alpha l} = \frac{V_0}{\alpha l}$$



- Since $\alpha l << 1$ for a low-loss line, we get $V_A >> V_0$. That is, the voltage at the open-circuited end of the resonant line is much higher compared to the excitation voltage V_0. That is, a section of a line can be used for stepping up a voltage.

- The Voltage step-up ratio is

$$\frac{V_A}{V_0} = \frac{1}{\alpha l}$$

- Taking $l = n\lambda/4$ (where n is an odd integer), and substituting $\lambda = 2\pi/\beta$, the voltage step-up ratio can be written as

$$\frac{V_A}{V_0} = \frac{2\beta}{n\pi\alpha}$$

$$\Rightarrow \quad \text{Volatge Amplification Ratio} = \frac{4Q}{n\pi}$$

- Since Q is typically few hundreds for a low-loss transmission line, a voltage amplification of few hundreds may occur in a resonant line.

Recap

In this course you have learnt the following

- Quality factor of a resonant circuit.
- Energy stored in a resonant section of a line.
- Quality factor of a resonant line.
- Transmission lines as step-up transformers.
- Relation of the step-up ratio with the quality factor.