Module 4 : Uniform Plane Wave

Lecture 24 : Maxwell's Equations in Homogeneous Unbound Source Free Medium

Objectives

In this lecture you will learn the following

- Maxwell's equations for time varying fields.
- Interpendance of time varying, electric and magnetic fields.
- Derivation of wave equation in three dimensional space.
- Wave equation for time harmonic fields.
- Arguments for obtaining the solution to the wave equation.

Homogeneous Unbound Medium

Maxwell's Equations in Homogeneous Unbound Medium are as follows :

- $$\nabla \cdot \mathbf{B} = 0$$
  $$\nabla \cdot \mathbf{D} = 0$$
  $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
  $$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$
- From constitutive relations we have
  $$\mathbf{B} = \mu \mathbf{H}$$
  $$\mathbf{D} = \epsilon \mathbf{E}$$
- Since the medium is homogeneous and non-time varying, μ and ϵ are constants as a function of space and time. The Maxwell's equations therefore reduce to

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\mu \mathbf{H}) = \mu \nabla \cdot \mathbf{H} = 0$$
$$\Rightarrow \quad \nabla \cdot \mathbf{H} = 0$$
$$and \qquad \nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = \epsilon \nabla \cdot \mathbf{E} = 0$$
$$\Rightarrow \quad \nabla \cdot \mathbf{E} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial (\mu \mathbf{H})}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$
$$\nabla \times \mathbf{H} = \frac{\partial (\epsilon \mathbf{E})}{\partial t} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

From these equation we draw important conclusion that

- **Time varying Magnetic Field cannot exist without corresponding Electric Field and vice-versa.**
- **The Electric and Magnetic Field have to co-exist whenever they are time varying.**

Solution of the Maxwell's equations in a Unbound Medium

- Taking curl of equations

$$\nabla \times \nabla \times \mathbf{E} = -\mu \nabla \times \frac{\partial \mathbf{H}}{\partial t}$$
$$\nabla \times \nabla \times \mathbf{H} = \epsilon \nabla \times \frac{\partial \mathbf{E}}{\partial t}$$

- Interchanging space and time derivatives on the RHS, we get

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial (\nabla \times \mathbf{H})}{\partial t}$$
$$\nabla \times \nabla \times \mathbf{H} = \epsilon \frac{\partial (\nabla \times \mathbf{E})}{\partial t}$$

- Substituting for $(\nabla \times H)$ and $(\nabla \times E)$, we get

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}(\epsilon \frac{\partial \mathbf{E}}{\partial t}) = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$
$$\nabla \times \nabla \times \mathbf{H} = -\epsilon \frac{\partial}{\partial t}(\mu \frac{\partial \mathbf{H}}{\partial t}) = -\mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

- Using the vector identity $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, where $\mathbf{A}$ is any arbitrary vector, we get

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$
$$\nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = -\mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

- From equations $\nabla . \mathbf{H} = 0$ and $\nabla . \mathbf{E} = 0$, we get

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$
$$\nabla^2 \mathbf{H} = \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

- **These are the wave equations in Three Dimensional Space**
- **The phenomena governed by these equations is hence called '**Electromagnetic Wave**'**

Wave Equation for Time Harmonic Fields

- Let us consider the time harmonic fields with angular frequency ω . The fields in general can be written as

$$\mathbf{F}(x,y,z,t) = \mathbf{F}(x,y,z)e^{j\omega t}$$

- Now onwards we will assume that a vector $\mathbf{F}$ is only a function of space (x,y,z) and its time variation $e^{j\omega t}$ is implicit. The derivatives with respect to time are

$$\frac{\partial}{\partial t} \equiv j\omega$$
$$\frac{\partial^2}{\partial t^2} \equiv j\omega.\, j\omega = -\omega^2$$

- The wave equation becomes

$$\nabla^2\mathbf{E} = -\omega^2\mu\epsilon\mathbf{E}$$
$$\nabla^2\mathbf{H} = -\omega^2\mu\epsilon\mathbf{H}$$

- The each of the above equations, in fact, set of three equations, one for each vector component of $\mathbf{E}$ and $\mathbf{H}$. In other words each component of $\mathbf{E}$ and $\mathbf{H}$ satisfies the wave equation. Since the ∇^2 operator is a scalar operator,

$(\nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2})$ (in Cartiecian System)

the wave equation for each component is a scalar equation as

$$\nabla^2\psi = -\omega^2\mu\epsilon\psi$$

where, ψ could be E_x, E_y, E_z or H_x, H_y, H_z.

Solution of the Wave Equation

- It is not easy to find solution to the wave equation. We therefore try various funtions and see whether they are solution of the wave equation. We select the simplest solution which is consistent with the wave equation. We can then make following conclusions.
- A time varying electric or magnetic field which is uniform in the three-dimensional space, can not exist.
- A time varying field which is constant in a plane perpendicular to the field direction, also can not exist.
- The simplest form of field which can exist is the field which is constant in a plane containing the field vector, and consequently has spatial variation along the direction perpendicular to the constant field plane.
- **This solution is called '**Uniform Plane Wave**' solution. In Cartesian coordinate the wave equation therefore has 'Uniform Plane Wave' solution as the simplest possible solution.**

Recap

In this course you have learnt the following

- Maxwell's equations for time varying fields.
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- Derivation of wave equation in three dimensional space.
- Wave equation for time harmonic fields.
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