Probabilistic Odometric Localization for Differential Drive Robot

Based on Odometry Code

1 Robot Kinematics and Pose Update

Consider a differential drive robot with two rear wheels separated by distance L, each with radius r.

At time t, the robot pose is:

$$\mathbf{x}_t = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix}$$

where x_t, y_t represent the robot's position in the plane, and θ_t is the orientation (heading) angle in radians.

Let the wheels travel incremental distances Δd_L and Δd_R over the time interval $[t, t+1]$, computed from encoder ticks as:

$$\Delta d_L = r \cdot \Delta\theta_L, \quad \Delta d_R = r \cdot \Delta\theta_R$$

where $\Delta\theta_L$ and $\Delta\theta_R$ are the wheel rotations in radians.

Define:

$$\Delta d = \frac{\Delta d_R + \Delta d_L}{2} \quad \text{(translation)} \Delta\theta = \frac{\Delta d_R - \Delta d_L}{L} \quad \text{(rotation)}$$

To better model realistic motion (non-holonomic constraints and noise), the robot's motion is decomposed into three parts:

- Initial rotation $\Delta\theta_1$
- Translation Δd
- Final rotation $\Delta\theta_2$

The initial rotation $\Delta\theta_1$ is computed as:

$$\Delta\theta_1 = \arctan 2\big(\Delta d \sin(\Delta\theta), \Delta d \cos(\Delta\theta)\big) - \theta_t$$

The final rotation is:

$$\Delta\theta_2 = \Delta\theta - \Delta\theta_1$$

Then, the robot pose is updated as:

$$\begin{aligned} x_{t+1} &= x_t + \Delta d \cdot \cos(\theta_t + \Delta\theta_1) \\ y_{t+1} &= y_t + \Delta d \cdot \sin(\theta_t + \Delta\theta_1) \\ \theta_{t+1} &= \theta_t + \Delta\theta \end{aligned}$$

with θ_{t+1} normalized to $[-\pi, \pi]$.

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2 Probabilistic Motion Noise Model

The odometry measurements are subject to noise from multiple sources:

- α_1: Rotational noise proportional to rotation magnitude.
- α_2: Rotational noise proportional to translation magnitude.
- α_3: Translational noise proportional to translation magnitude.
- α_4: Translational noise proportional to total rotation magnitude.

The noise variances for each motion component are modeled as:

$$\begin{aligned}
\sigma^2_{\text{rot}_1} &= \alpha_1|\Delta\theta_1| + \alpha_2|\Delta d| \\
\sigma^2_{\text{trans}} &= \alpha_3|\Delta d| + \alpha_4(|\Delta\theta_1| + |\Delta\theta_2|) + \sigma^2_{\text{encoder}} \\
\sigma^2_{\text{rot}_2} &= \alpha_1|\Delta\theta_2| + \alpha_2|\Delta d|
\end{aligned}$$

where $\sigma^2_{\text{encoder}}$ models the encoder measurement noise.
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3 Covariance Propagation

We define the robot pose covariance matrix at time t as:

$$\Sigma_t = \text{Cov}(\mathbf{x}_t) = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{x\theta} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{y\theta} \\ \sigma_{x\theta} & \sigma_{y\theta} & \sigma_{\theta\theta} \end{bmatrix}$$

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3.1 Jacobian matrices

The motion update affects the covariance matrix. Define the Jacobians:

$$G_t = \frac{\partial \mathbf{x}_{t+1}}{\partial \mathbf{x}_t} = \begin{bmatrix} 1 & 0 & -\Delta d \sin(\theta_t + \Delta\theta_1) \\ 0 & 1 & \Delta d \cos(\theta_t + \Delta\theta_1) \\ 0 & 0 & 1 \end{bmatrix}$$

$$V_t = \frac{\partial \mathbf{x}_{t+1}}{\partial \mathbf{u}_t} = \begin{bmatrix} -\Delta d \sin(\theta_t + \Delta\theta_1) & \cos(\theta_t + \Delta\theta_1) & 0 \\ \Delta d \cos(\theta_t + \Delta\theta_1) & \sin(\theta_t + \Delta\theta_1) & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

where $\mathbf{u}_t = [\Delta\theta_1, \Delta d, \Delta\theta_2]^T$ represents the noisy control inputs.
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3.2 Control noise covariance

The covariance of the control noise is:

$$M_t = \begin{bmatrix} \sigma^2_{\text{rot}_1} & 0 & 0 \\ 0 & \sigma^2_{\text{trans}} & 0 \\ 0 & 0 & \sigma^2_{\text{rot}_2} \end{bmatrix}$$

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3.3 Covariance update equation

The covariance is updated as:

$$\Sigma_{t+1} = G_t \Sigma_t G_t^T + V_t M_t V_t^T$$

where the first term propagates previous uncertainty through the motion model, and the second term adds uncertainty from the control noise.

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4 Summary

The probabilistic odometry model updates the pose estimate and its uncertainty as follows:

$$\boxed{\begin{aligned} \mathbf{x}_{t+1} &= \begin{bmatrix} x_t + \Delta d \cos(\theta_t + \Delta\theta_1) \\ y_t + \Delta d \sin(\theta_t + \Delta\theta_1) \\ \theta_t + \Delta\theta \end{bmatrix} \\ \Sigma_{t+1} &= G_t \Sigma_t G_t^T + V_t M_t V_t^T \end{aligned}}$$

with noise parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ tuned to your robot and sensors.