EKF for Holonomic Odometry with Absolute-IMU Yaw

1 State, Inputs, and Measurements

State.

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix} \quad \text{(planar pose in the odom/map frame)}$$

Wheel encoder kinematics. Let the left/right wheel incremental rotations (rad) over one time step be

$$\Delta\theta_L = \frac{2\pi\,\Delta\mathrm{ticks}_L}{\mathrm{TICKS\_PER\_WHEEL\_REV}}, \qquad \Delta\theta_R = \frac{2\pi\,\Delta\mathrm{ticks}_R}{\mathrm{TICKS\_PER\_WHEEL\_REV}},$$

and the corresponding wheel arc lengths

$$\Delta s_L = R\,\Delta\theta_L, \qquad \Delta s_R = R\,\Delta\theta_R,$$

where R is the wheel radius. Then define the body motion over the step:

$$\Delta s = \tfrac{1}{2}(\Delta s_L + \Delta s_R), \qquad \Delta\psi = \frac{\Delta s_R - \Delta s_L}{B},$$

with B the wheelbase.

IMU measurement (absolute yaw). The IMU provides a heading measurement (converted to radians):

$$z_\psi \approx \psi + v, \quad v \sim \mathcal{N}(0,\, R_\psi).$$

2 Discrete Motion Model (Predict/Process Model)

Using the midpoint (second-order) unicycle update,

$$x_{k+1} = x_k + \Delta s\,\cos\!\big(\psi_k + \tfrac{1}{2}\Delta\psi\big), \tag{1}$$

$$y_{k+1} = y_k + \Delta s\,\sin\!\big(\psi_k + \tfrac{1}{2}\Delta\psi\big), \tag{2}$$

$$\psi_{k+1} = \psi_k + \Delta\psi, \tag{3}$$

with angle normalization $\psi \leftarrow \mathrm{wrap}_\pi(\psi)$ to $(-\pi, \pi]$.

2.1 Linearization (Jacobians)

Let

$$\alpha \;\triangleq\; \psi_k + \Delta\mathrm{rot}_1, \qquad \Delta\mathrm{rot}_1 \;=\; \begin{cases} \frac{1}{2}\Delta\psi, & \text{if } |\Delta s| > 0 \text{ and } |\Delta\psi| > 0, \\ 0, & \text{otherwise} \end{cases}$$

and $\Delta\mathrm{rot}_2$ the remainder (typically $\Delta\mathrm{rot}_2 = \frac{1}{2}\Delta\psi$ in the general case, or $\Delta\mathrm{rot}_2 = \Delta\psi$ for pure rotation, and $\Delta\mathrm{rot}_2 = 0$ for pure translation). The state Jacobian (w.r.t. $\boldsymbol{x}$) is

$$\boldsymbol{F} \;=\; \frac{\partial f}{\partial \boldsymbol{x}} \;=\; \begin{bmatrix} 1 & 0 & -\Delta s\,\sin\alpha \\ 0 & 1 & \Delta s\,\cos\alpha \\ 0 & 0 & 1 \end{bmatrix}.$$

Using the *rot1/trans/rot2* control parameterization

$$\boldsymbol{u} \;=\; \begin{bmatrix} \Delta\mathrm{rot}_1 \\ \Delta s \\ \Delta\mathrm{rot}_2 \end{bmatrix},$$

the input Jacobian (w.r.t. $\boldsymbol{u}$) is

$$\boldsymbol{V} \;=\; \frac{\partial f}{\partial \boldsymbol{u}} \;=\; \begin{bmatrix} -\Delta s\,\sin\alpha & \cos\alpha & 0 \\ \Delta s\,\cos\alpha & \sin\alpha & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

2.2 Process (Motion) Noise

Following the odometry noise model in *Probabilistic Robotics*, define the variances

$$\sigma^2_{\mathrm{rot}_1} = \alpha_1\,|\Delta\mathrm{rot}_1| + \alpha_2\,|\Delta s|, \tag{4}$$

$$\sigma^2_{\mathrm{trans}} = \alpha_3\,|\Delta s| + \alpha_4\,(|\Delta\mathrm{rot}_1| + |\Delta\mathrm{rot}_2|), \tag{5}$$

$$\sigma^2_{\mathrm{rot}_2} = \alpha_1\,|\Delta\mathrm{rot}_2| + \alpha_2\,|\Delta s|. \tag{6}$$

Optionally add an encoder-specific translational variance σ^2_{enc} (if modeled), e.g. $\sigma^2_{\mathrm{trans}} \leftarrow \sigma^2_{\mathrm{trans}} + \sigma^2_{\mathrm{enc}}$.

Collect these into

$$\boldsymbol{M} \;=\; \mathrm{diag}\big(\sigma^2_{\mathrm{rot}_1},\; \sigma^2_{\mathrm{trans}},\; \sigma^2_{\mathrm{rot}_2}\big).$$

2.3 Covariance Propagation

Let $\boldsymbol{P}_{k|k}$ be the covariance at time k after the last update. The predicted covariance is

$$\boxed{\boldsymbol{P}_{k+1|k} \;=\; \boldsymbol{F}\,\boldsymbol{P}_{k|k}\,\boldsymbol{F}^\top \;+\; \boldsymbol{V}\,\boldsymbol{M}\,\boldsymbol{V}^\top}.$$

3 Measurement Model (Absolute Yaw)

For an absolute yaw measurement,

$$h(\boldsymbol{x}) \;=\; \psi, \qquad \boldsymbol{H} \;=\; \frac{\partial h}{\partial \boldsymbol{x}} \;=\; \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

The (angle-wrapped) innovation is

$$\boxed{y \;=\; \mathrm{wrap}_\pi\big(z_\psi - \hat{\psi}_{k+1|k}\big)}\,.$$

The innovation covariance is

$$\boxed{S \;=\; \boldsymbol{H}\,\boldsymbol{P}_{k+1|k}\,\boldsymbol{H}^\top + R_\psi \;=\; P_{\psi\psi} + R_\psi}\,.$$

The Kalman gain is

$$\boxed{\boldsymbol{K} \;=\; \boldsymbol{P}_{k+1|k}\,\boldsymbol{H}^\top\,S^{-1} \;=\; \frac{1}{S}\begin{bmatrix} P_{x\psi} \\ P_{y\psi} \\ P_{\psi\psi} \end{bmatrix}}\,.$$

4 Update (Correction)

Update the mean (then wrap the angle):

$$\boxed{\hat{\boldsymbol{x}}_{k+1|k+1} \;=\; \hat{\boldsymbol{x}}_{k+1|k} \;+\; \boldsymbol{K}\,y, \qquad \hat{\psi}_{k+1|k+1} \leftarrow \mathrm{wrap}_\pi(\hat{\psi}_{k+1|k+1})}\,.$$

Update the covariance using the Joseph form (numerically stable):

$$\boxed{\boldsymbol{P}_{k+1|k+1} \;=\; (\boldsymbol{I} - \boldsymbol{K}\boldsymbol{H})\,\boldsymbol{P}_{k+1|k}\,(\boldsymbol{I} - \boldsymbol{K}\boldsymbol{H})^\top \;+\; \boldsymbol{K}\,R_\psi\,\boldsymbol{K}^\top}\,.$$

5 Outlier Rejection (Gating)

Reject the update if the (scalar) Mahalanobis distance exceeds a threshold γ:

$$\boxed{\frac{y^2}{S} \;\le\; \gamma^2 \quad (\text{e.g. } \gamma = 3)}\,.$$

6 Angle Normalization

Define wrapping to $(-\pi, \pi]$:

$$\mathrm{wrap}_\pi(\phi) \;=\; \big((\phi + \pi) \bmod 2\pi\big) - \pi.$$

Apply after the process update for ψ and again after the measurement update.

7 Initialization and Units

Initialize

$$\hat{\boldsymbol{x}}_{0|0} = \begin{bmatrix} x_0 \\ y_0 \\ \psi_0 \end{bmatrix}, \qquad \boldsymbol{P}_{0|0} = \mathrm{diag}\big(\sigma_x^2,\; \sigma_y^2,\; \sigma_\psi^2\big).$$

Units: All angles in *radians*. If an IMU reports yaw with standard deviation $\sigma_\psi^{(\circ)}$ in degrees,

$$R_\psi \;=\; \left(\sigma_\psi^{(\circ)} \cdot \frac{\pi}{180}\right)^2 \quad (\text{variance in rad}^2).$$

8 One EKF Cycle (Summary)

1. **Predict:** Compute $(\Delta s, \Delta\psi)$ from encoders; propagate $\hat{\boldsymbol{x}}_{k+1|k} = f(\hat{\boldsymbol{x}}_{k|k}, \cdot)$; build $\boldsymbol{F}, \boldsymbol{V}, \boldsymbol{M}$ and compute $\boldsymbol{P}_{k+1|k}$.
2. **Measure:** Acquire z_ψ; form $y = \mathrm{wrap}_\pi(z_\psi - \hat{\psi}_{k+1|k})$, $S = P_{\psi\psi} + R_\psi$.
3. **Gate (optional):** If $y^2/S > \gamma^2$, skip update.
4. **Update:** $\boldsymbol{K} = \boldsymbol{P}_{k+1|k}\boldsymbol{H}^\top S^{-1}$; $\hat{\boldsymbol{x}}_{k+1|k+1} = \hat{\boldsymbol{x}}_{k+1|k} + \boldsymbol{K}y$ (wrap ψ); $\boldsymbol{P}_{k+1|k+1} = (\boldsymbol{I} - \boldsymbol{K}\boldsymbol{H})\boldsymbol{P}_{k+1|k}(\boldsymbol{I} - \boldsymbol{K}\boldsymbol{H})^\top + \boldsymbol{K}R_\psi\boldsymbol{K}^\top$.

9 (Optional) Extension: Gyro-Rate With Bias

If using gyro z instead of absolute yaw, augment the state with a bias b_ω:

$$\boldsymbol{x} = \begin{bmatrix} x & y & \psi & b_\omega \end{bmatrix}^\top, \qquad b_{\omega,k+1} = b_{\omega,k} + w_b,$$

and use the measurement

$$z_\omega \approx \dot{\psi} + b_\omega \approx \frac{\Delta\psi_{\mathrm{odom}}}{\Delta t} + b_\omega + v, \quad \boldsymbol{H} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix},$$

with small process noise on b_ω. An occasional absolute yaw fixes global heading.