jensons inequality

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Jensen's Inequality

Statement. Let X be a random variable with expectation $\mu = E[X]$. If φ is convex and finite, then

$$E[\varphi(X)] \ \geq \ \varphi(\mu).$$

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Proof (finite-valued X)

Suppose X takes values $x_1, \dots, x_m$ with probabilities $p_1, \dots, p_m$, where $p_i \geq 0$ and $\sum_{i=1}^m p_i = 1$. Then

$$\mu = E[X] = \sum_{i=1}^{m} p_i x_i, \qquad E[\varphi(X)] = \sum_{i=1}^{m} p_i \varphi(x_i).$$

Base case ($m = 2$). By convexity of φ, for all $\lambda \in [0, 1]$,

$$\varphi(\lambda x_1 + (1 - \lambda)x_2) \ \leq \ \lambda\varphi(x_1) + (1 - \lambda)\varphi(x_2).$$

Taking $\lambda = p_1$ gives the desired inequality.

Induction step. Assume the inequality holds for $m-1$ points. For m points, write

$$\mu = \sum_{i=1}^{m} p_i x_i = (1 - p_m)\bar{x} + p_m x_m, \quad \bar{x} = \sum_{i=1}^{m-1} p_i 1 - p_m \, x_i.$$

By convexity (two-point case),

$$\varphi(\mu) \leq (1 - p_m)\varphi(\bar{x}) + p_m\varphi(x_m).$$

By the induction hypothesis,

$$\varphi(\bar{x}) \leq \sum_{i=1}^{m-1} p_i 1 - p_m \, \varphi(x_i).$$

Combining,

$$\varphi(\mu) \;\le\; \sum_{i=1}^{m} p_i \varphi(x_i) = E[\varphi(X)].$$

Thus $E[\varphi(X)] \ge \varphi(\mu)$.

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Alternative proof (differentiable φ)

If φ is convex and differentiable, then for any x_0,

$$\varphi(y) \;\ge\; \varphi(x_0) + \varphi'(x_0)(y - x_0), \quad \forall y.$$

Choose $x_0 = \mu$. Taking expectations:

$$E[\varphi(X)] \;\ge\; \varphi(\mu) + \varphi'(\mu)\, E[X - \mu].$$

Since $E[X - \mu] = 0$, this yields

$$E[\varphi(X)] \;\ge\; \varphi(\mu).$$

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Equality conditions

Equality holds if X is almost surely constant, or if φ is affine (linear plus constant) on the convex hull of the support of X.